Letters in Mathematical Physics (2024) 114:14
https://doi.org/10.1007/s11005-023-01757-w

Almost multiplicity free subgroups of compact Lie groups and polynomial integrability of sub-Riemannian geodesic flows

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Received: 12 August 2023 / Revised: 1 November 2023 / Accepted: 11 December 2023 /
Published online: 6 January 2024

Abstract

We classify almost multiplicity free subgroups K of compact simple Lie groups G. The problem is related to the integrability of Riemannian and sub-Riemannian geodesic flows of left-invariant metrics defined by a specific extension of integrable systems from T^*K to T^*G.

Keywords Invariant polynomials · Gel'fand-Cetlin systems · Multiplicity of Hamiltonian action · (almost) multiplicity free spaces

Mathematics Subject Classification 37J35 · 17B63 · 17B80 · 53D20

1 Introduction

Let G be a compact connected Lie group with the Lie algebra $\mathfrak{g}$ and $K \subset G$ a connected subgroup with the Lie algebra $\mathfrak{k}$. We fix an invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. The same notation is used to denote the restriction of the scalar product on the subalgebra $\mathfrak{k}$. By the use of $\langle \cdot, \cdot \rangle$, we identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{k} \cong \mathfrak{k}^*$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ and let $\mathrm{pr}_\mathfrak{p}$ and $\mathrm{pr}_\mathfrak{k}$ be the orthogonal projections onto $\mathfrak{p}$ and $\mathfrak{k}$, respectively.

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It is well known that the symplectic leaves of the Lie–Poisson bracket

$$\{f, g\}|_x = -\langle x, [\nabla f(x), \nabla g(x)]\rangle \tag{1}$$

on $\mathfrak{g}$ are adjoint orbits $\mathcal{O}(a) = \mathrm{Ad}_G(a)$, $a \in \mathfrak{g}$. With the above sign convention, the Hamiltonian equation of the Hamiltonian H with respect to the Lie-Poisson bracket reads

$$\dot{x} = X_H(x), \qquad X_H(x) = [x, \nabla H(x)]. \tag{2}$$

Therefore, since the vector field corresponding to the infinitesimal adjoint action of $\xi \in \mathfrak{g}$ on x is given by $[\xi, x]$, the adjoint action of K on an orbit $\mathcal{O}$ is Hamiltonian with the momentum mapping

$$\Phi(x) = -\,\mathrm{pr}_{\mathfrak{k}}(x), \qquad x \in \mathcal{O}.$$

Recall that the Hamiltonian action of a compact connected Lie group K on a $2m$-dimensional symplectic manifold M^{2m} with the equivariant momentum mapping $\Phi : M^{2m} \to \mathfrak{k}^* \cong \mathfrak{k}$ is *multiplicity free* if the algebra of K–invariant functions is commutative [16]. More generally, the Hamiltonian action of a compact connected Lie group K on a symplectic manifold M^{2m} has *multiplicity* $2\mathbf{c}(M^{2m}, K)$ (or *complexity* $\mathbf{c}(M^{2m}, K)$) if the Poisson algebra of K-invariant functions has exactly $2\mathbf{c}$ additional independent functions, besides Casimir functions. In other words, the dimension of a generic symplectic leaf in the "singular Poisson manifold" M^{2m}/K is $2\mathbf{c}$. In the case of a free action, the multiplicity is the dimension of a generic symplectic reduced space $\Phi^{-1}(\mu)/G_\mu$, $\mu \in \Phi(M^{2m})$.

In our case, the complexity $\mathbf{c}(\mathcal{O}, K)$ of Ad_K-action on a generic G-adjoint orbit $\mathcal{O} \subset \mathfrak{g}$ will be denoted by $\mathbf{c}(\mathfrak{g}, \mathfrak{k})$. If $\mathbf{c}(\mathfrak{g}, \mathfrak{k}) = 0$, K is called a *multiplicity free subgroup* of G. There are equivalent definitions of multiplicity free subgroups within the framework of the representation theory, and the classification of multiplicity free subgroups K of compact Lie groups G is given by Krämer [22] (see also Heckman [17]). If G is a simple group, the pair of corresponding Lie algebras $(\mathfrak{g}, \mathfrak{k})$ is

$$(B_n, D_n), \quad (D_n, B_{n-1}), \quad \text{or} \quad (A_n, A_{n-1} \oplus u(1)).$$

Definition 1 We say that K is an *almost multiplicity free subgroup* of G if the adjoint action of K on a generic G–adjoint orbit $\mathcal{O} \subset \mathfrak{g}$ has the multiplicity two, i.e., $\mathbf{c}(\mathfrak{g}, \mathfrak{k}) = 1$.

In this paper we prove the statement announced in [20] that the pair of Lie algebras $(\mathfrak{g}, \mathfrak{k})$ corresponding to the almost multiplicity free subgroups $K \subset G$ belongs to the following list:

$$\begin{aligned} &(A_n, A_{n-1}), \quad (A_3, A_1 \oplus A_1 \oplus \mathfrak{u}(1)), \quad (B_2, \mathfrak{u}(2)), \\ &(B_2, B_1 \oplus \mathfrak{u}(1)), \quad (B_3, \mathfrak{g}_2), \quad (\mathfrak{g}_2, A_2). \end{aligned}$$

In Sect. 2 the problem is related to the integrability of the Euler equations of left-invariant metrics on G, obtained by a specific extension of left invariant metrics on K with integrable geodesic flows (Proposition 2). The proof of the classification of almost multiplicity free subgroups is given in Sect. 3 (Theorem 4). Finally, in Sect. 4, we apply the results in the problem of integrability of sub-Riemannian geodesic flows on Lie groups.

2 Integrability of Euler equations related to the pairs (G, K)

2.1 Collective complete integrability

In the framework of Arnold–Liouville integrability, the complexity $\mathbf{c}(M^{2m}, K)$ can be characterized as follows. Consider a K–invariant function F and the corresponding Hamiltonian equation

$$\dot{x} = X_F \tag{3}$$

on M^{2m}. According to the Noether theorem, the momentum mapping is conserved along the flow of X_F. In other words, we have $\{F, \Phi \circ q\} = 0$, where q is a function on $\mathfrak{k}$. Besides, if q_1 and q_2 commute with respect to the Lie-Poisson bracket on $\mathfrak{k}$, then $Q_1 = \Phi \circ q_1$ and $Q_2 = \Phi \circ q_2$ Poisson commute on M^{2m} as well.

The complexity $\mathbf{c}$ a minimal number of Poisson commuting K-invariant functions on M^{2m} we need to add to the Noether functions $\Phi^*(\mathbb{R}[\mathfrak{k}^*])$ to obtain a complete commutative set of m independent functions on M^{2m}. In the case of multiplicity free spaces, all K-invariant Hamiltonian flows are completely integrable by means of Noether integrals - so called *collective complete integrability* [14]. There exist m Lie-Poisson commuting functions $q_1, \dots, q_m$ on $\mathfrak{k}$, such that $Q_1 = \Phi \circ q_1, \dots, Q_m = \Phi \circ q_m$ are independent Poisson commuting functions on M. Also, then we have that all K–invariant functions are functionally dependent on Noether functions.

Similarly, in the case of a multiplicity two action, if we take a K-invariant function F, which is not functionally dependent on Noether functions, there exist Lie-Poisson commuting functions $q_1, \dots, q_{m-1}$ on $\mathfrak{k}$, such that $F, Q_1 = \Phi \circ q_1, \dots, Q_{m-1} = \Phi \circ q_{m-1}$ is a complete set of commuting functions on M^{2m}. Thus, the Hamiltonain flow of (3) is completely integrable. It has zero topological entropy as well, see [27].

2.2 Extension of integrable geodesic flows from K to G

The invariant polynomials $\mathbb{R}[\mathfrak{g}]^G$ are Casimir functions with respect to the Lie-Poisson bracket and generic adjoint orbits $\mathcal{O}$ are the regular level sets of the basic invariant polynomials $p_1, \dots, p_r$, $r = \operatorname{rank} G$. Thus, for the integrability of the Euler equations (2) we need, in addition, $\frac{1}{2}\dim \mathcal{O} = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$ independent commuting functions. Then, according to the Arnold–Liouville theorem, a generic motion is a quasi-periodic winding over δ_0–dimensional invariant tori, where

$$\delta_0 = \dim \mathfrak{g} - \mathbf{a}(\mathfrak{g}), \qquad \mathbf{a}(\mathfrak{g}) := \frac{1}{2}\big(\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g}\big).$$

Here, by $\mathbf{a}(\mathfrak{g})$ we denoted the maximal number of independent Poisson commuting functions on $\mathfrak{g}$.

In the study of integrable systems related to filtration of Lie algebras, Bogoyavlenski [3] considered the following natural problem which we slightly reformulate for the filtration $\mathfrak{k} \subset \mathfrak{g}$. For $x \in \mathfrak{g}$, we denote $x = x_0 + x_1$, $x_0 = \mathrm{pr}_{\mathfrak{k}}(x)$, $x_1 = \mathrm{pr}_{\mathfrak{p}}(x)$. Then the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is reductive:

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{4}$$

Let $A_0 \colon \mathfrak{k} \to \mathfrak{k}$ be a positive definite symmetric operator and $H_0 = \frac{1}{2}\langle A_0(x_0), x_0\rangle$ the Hamiltonian of the corresponding left-invariant metric on K. Assume that the Euler equation

$$\dot{x}_0 = [x_0, \nabla H_0(x_0)] = [x_0, A_0(x_0)] \tag{5}$$

is completely integrable with a complete set of commuting first integrals (including invariants) $q_1, \dots, q_{\mathbf{a}(\mathfrak{k})}$. Bogoyavlenski conjectured that the Euler equation (2) with the Hamiltonian of the form

$$H(x) = H(x_0 + x_1) = \frac{1}{2}\langle A_0(x_0), x_0\rangle + s\frac{1}{2}\langle x_1, x_1\rangle$$

are completely integrable as well [3]. Here s is a real parameter, greater then zero for left-invariant Riemannian metrics.

Due to the relation (4), the corresponding Hamiltonian system can be rewritten into the form

$$\dot{x}_0 = [x_0, A_0(x_0)], \quad \dot{x}_1 = [x_1, A_0(x_0) - s x_0]. \tag{6}$$

In [20] the following statement was proved (see [20, Theorem 2] with $n = 1$ and assume that the integrals in the first step are commutative).

Theorem 1 *The equations (6) are integrable in a noncommutative sense by means of commuting integrals* $Q_1 = \mathrm{pr}_{\mathfrak{k}}^* q_1, \dots, Q_{\mathbf{a}(\mathfrak{k})} = \mathrm{pr}_{\mathfrak{k}}^* q_{\mathbf{a}(\mathfrak{k})}$ *and the set of* Ad_K*–invariants* $\mathbb{R}[\mathfrak{g}]^K$ *on* $\mathfrak{g}$. *A generic motion is a quasi-periodic winding over* δ*–dimensional invariant tori, where*

$$\delta = \mathbf{a}(\mathfrak{k}) - \operatorname{rank} \mathfrak{g} + \dim \mathrm{pr}_{\mathfrak{p}}(\mathfrak{g}(x)) \le \delta_0 = \dim \mathfrak{g} - \mathbf{a}(\mathfrak{g}). \tag{7}$$

Here we take a generic element $x \in \mathfrak{g}$ and $\mathfrak{g}(x) = \{\xi \in \mathfrak{g} \,|\, [\xi, x] = 0\}$ is the isotropy algebra of x.

Mishchenko and Fomenko stated the conjecture that noncommutative integrability implies the Liouville integrability by means of an algebra of integrals that belong to the same functional class as the original one [25]. The conjecture is solved in a smooth category [6] and in polynomial category when noncommuting integrals form finite dimensional Lie algebras (see [4, 29, 31]). Note that from a point of view of the dynamics, noncommutative integrability is stronger than the Liouville one. Isotropic tori, level sets of noncommutative integrals, can be reorganized into Lagrangian tori, level sets of commuting integrals, in a many different ways. Thus, Lagrangian tori are resonant and not an intrinsic property of the system.

In our case, we assume that q_i are polynomials and the polynomial conjecture reduces to the construction of a Lie-Poisson commutative set $P_1, \dots, P_{\mathbf{b}}$ of Ad_K–invariant polynomials,

$$\mathbf{b} = \mathbf{b}(\mathfrak{g}, \mathfrak{k}) := \mathbf{a}(\mathfrak{g}) - \mathbf{a}(\mathfrak{k}) = \frac{1}{2}(\dim \mathfrak{p} + \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k}), \tag{8}$$

independent of the polynomials on $\mathfrak{k}$ (see [20]):

$$\mathbf{b} = \dim \mathrm{pr}_{\mathfrak{p}} \operatorname{span} \{\nabla P_i(x) \mid i = 1, \dots, \mathbf{b}\}, \qquad \text{for a generic} \qquad x \in \mathfrak{g}.$$

The simplest situation is the case when the algebra of Ad_K–invariants is already commutative. Then the inequality (7) becomes the equality:

$$\delta = \mathbf{a}(\mathfrak{k}) - \operatorname{rank} \mathfrak{g} + \dim \mathrm{pr}_{\mathfrak{p}}(\mathfrak{g}(x)) = \dim \mathfrak{g} - \mathbf{a}(\mathfrak{g}) = \delta_0,$$

and for the additional integrals we can take the invariants $p_1, \dots, p_r$ (some of the integrals can be dependent since r can be different from $\mathbf{b}(\mathfrak{g}, \mathfrak{k})$). The system (6) is integrable in the usual Arnold–Liouville (or commutative) sense. Since the momentum mapping of the K-action on the adjoint orbit $\mathcal{O} \subset \mathfrak{g}$ is given by $\Phi = -\mathrm{pr}_{\mathfrak{k}} \circ \imath$ ($\imath$ is the inclusion), it is obvious that this condition is equivalent to the fact that the Ad_K-action on a generic orbit $\mathcal{O}$ is multiplicity free.

For example, $SO(n-1)$ and $U(n-1)$ are multiplicity free subgroups of $SO(n)$ and $U(n)$, respectively. Therefore, considering the chains of subalgebras

$$\mathfrak{so}(2) \subset \mathfrak{so}(3) \subset \cdots \subset \mathfrak{so}(n-1) \subset \mathfrak{so}(n),$$
$$\mathfrak{u}(1) \subset \mathfrak{u}(2) \subset \cdots \subset \mathfrak{u}(n-1) \subset \mathfrak{u}(n),$$

and taking the lifts of the invariants from $\mathfrak{so}(k)$ and $\mathfrak{u}(k)$ ($k \le n$), by induction, we obtain a complete commutative set of polynomials on the Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{u}(n)$ (see Thimm [30]). Since the paper [15], the corresponding integrable systems are refereed as Gel'fand-Cetlin systems on $\mathfrak{so}(n)$ and $\mathfrak{u}(n)$. Namely, Gel'fand and Cetlin constructed canonical bases for a finite-dimensional representation of the orthogonal and unitary groups by the decomposition of the representation by a chain of subgroups [11, 12]. The corresponding integrable systems on the adjoint orbits can be seen as a symplectic version of the Gelfand-Cetlin construction [15], which motivated Guillemin and Sternberg to introduce an important notion of multiplicity free Hamiltonain actions [14, 16].

The next natural step is to consider a subgroup $K \subset G$ when apart from Ad_G–invariants, for a complete commutative set of Ad_K–invariants we can take arbitrary

Ad_K–invariant polynomial P, which is not in the center of $\mathbb{R}[\mathfrak{g}]^K$, that is, when K is an *almost multiplicity free subgroup* of G. Then $\delta = \delta_0 - 1$ in (7) and among commuting polynomials $P, p_1, \dots, p_r, Q_1, \dots, Q_{\mathbf{a}(\mathfrak{k})}$ there are $\mathbf{a}(\mathfrak{g})$ independent ones providing the usual Arnold–Liouville (or commutative) integrability of the equations (6). For a polynomial integral P we can take some of the Bogoyavlensky integrals

$$p_{i,\lambda}(x) = p_i(x_0 + \lambda x_1), \qquad i = 1\dots, r = \operatorname{rank}\mathfrak{g}, \tag{9}$$

where λ is a real paremeter (see [3, 20]).

Proposition 2 *Let (G, K) be a pair from the list (14). Assume that the Euler equation (5) is completely integrable with a complete set of commuting polynomial first integrals. Then the extended system (6) is completely integrable with a complete set of commuting polynomial first integrals as well.*

In general, the construction of commuting polynomials P_i is still an open problem (see [20]). It is solved in the case when $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair ($[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$) by Mikityuk [23]. A similar open problem can be formulated for the integrability of G-invariant geodesic flows on homogeneous spaces G/K (see [7]).

3 Multiplicity free and almost-multiplicity free subgroups of compact Lie groups

Recall that $\mathbf{b}(\mathfrak{g}, \mathfrak{k})$ is the difference of maximum numbers of independent Poisson commuting functions on $\mathfrak{g}$ and $\mathfrak{k}$ (see (8)). We can state the following algebraic condition for the pair (G, K) in order to have (almost) multiplicity free K adjoint action on a generic orbit $\mathcal{O} \subset \mathfrak{g}$.

Proposition 3 *The multiplicity of the K-action on $\mathcal{O}$ is given by*

$$2\mathbf{c}(\mathfrak{g}, \mathfrak{k}) = 2\mathbf{b}(\mathfrak{g}, \mathfrak{k}) - 2\dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x)),$$

for a generic $x \in \mathfrak{g}$. Therefore

(i) *K is a multiplicity free subgroup of G if and only if*

$$\dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x)) = \mathbf{b}(\mathfrak{g}, \mathfrak{k}).$$

(ii) *K is an almost multiplicity free subgroup of G if and only if*

$$\dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x)) = \mathbf{b}(\mathfrak{g}, \mathfrak{k}) - 1.$$

Proof Recall that the Hamiltonian action of a compact connected Lie group K on a symplectic manifold M has the *multilicity* $2\mathbf{c} = \mathbf{c}(M, K)$ if the Poisson algebra of K-invariant functions $C^\infty_K(M)$, besides Casimir functions, has exactly $2\mathbf{c}$ additional independent functions:

$$\begin{aligned} 2\mathbf{c}(M,K) &= \operatorname{ddim} C^\infty_K(M) - \operatorname{dind} C^\infty_K(M) \\ &= \big(\dim M - (\dim K - \dim K_x)\big) - (\dim K_\mu - \dim K_x) \\ &= \dim M - \dim K + 2\dim K_x - \dim K_\mu, \end{aligned} \tag{10}$$

where K_x and K_μ are isotropic subgroups of a generic $x \in M$ and $\mu = \Phi(x)$ ($\operatorname{ddim} C^\infty_K(M)$ denotes the number of independent K–invariant functions and $\operatorname{dind} C^\infty_K(M)$ denotes the dimension of the kernel of the Poisson bracket restricted to the space spanned by the differentials of the K–invariant functions, see [6, 20]).

Let

$$\mathfrak{k}(x) = \{\xi \in \mathfrak{k}, \mid [\xi, x] = 0\} = \mathfrak{g}(x) \cap \mathfrak{k}$$

and let $\iota : \mathcal{O} \hookrightarrow \mathfrak{g}$ be the inclusion. Consider the momentum mapping $\Phi = -\operatorname{pr}_{\mathfrak{k}} \circ \iota$ of the K-action on $\mathcal{O}$. From $\mu = \Phi(x) = -x_0$ we have

$$\begin{aligned} &\dim K_x = \dim \mathfrak{k}(x) = \dim \mathfrak{g}(x) - \dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x)) = \operatorname{rank}\mathfrak{g} - \dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x))), \\ &\dim K_\mu = \dim \mathfrak{k}(x_0) = \operatorname{rank}\mathfrak{k}, \end{aligned}$$

for a generic $x \in \mathcal{O}$. By plugging the above relations into (10) we obtain

$$2\mathbf{c}(\mathfrak{g},\mathfrak{k}) = (\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g}) - \dim\mathfrak{k} + 2(\operatorname{rank}\mathfrak{g} - \dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x))) - \operatorname{rank}\mathfrak{k},$$

which proves the statement. Note that (i) and (ii) also follow from the identities $\delta = \delta_0$ and $\delta = \delta_0 - 1$ in the inequality (7), respectively. □

Remark 1 Since $\mathfrak{g}(x)$ is spanned by the gradients of the invariant polynomials $p_1, \dots, p_r$, for a generic $x \in \mathfrak{g}$, Proposition 3 is related to the fact that among invariant polynomials $p_1, \dots, p_r$, we have $\dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x))$ independent ones from functions on $\mathfrak{k}$. Further, since $\dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x)) \le \operatorname{rank}\mathfrak{g}$, from Proposition 3 we have

$$\dim\mathfrak{p} \le \operatorname{rank}\mathfrak{g} + \operatorname{rank}\mathfrak{k} + 2\mathbf{c}(\mathfrak{g},\mathfrak{k}). \tag{11}$$

The classification of multiplicity free subgroups K of compact Lie groups G is given by Krämer [22] (see also Heckman [17]). If G is a simple group, the pairs of corresponding Lie algebras $(\mathfrak{g}, \mathfrak{k})$ are

$$(B_n, D_n), \quad (D_n, B_{n-1}), \quad \text{or} \quad (A_n, A_{n-1} \oplus u(1)).$$

Note that K does not have to be a multiplicity free subgroup in order to have $\mathbf{c}(\mathcal{O}, K) = 0$ for a singular adjoint orbit $\mathcal{O}$ (see [32]).

Example 1 Multiplicity free pairs are:

$$\begin{aligned} &(SU(n), S(U(1)\times U(n-1))),\ (SU(n), U(n-1)),\ (SU(4), Sp(2)), \\ &(SO(n), SO(n-1)),\ (SO(4), U(2)),\ (SO(4), SU(2)), \\ &(SO(6), U(3)),\ (SO(8), Spin(7)),\ (Spin(7), SU(4)). \end{aligned}$$

Table 1 Rank and number of positive roots of the classical Lie algebras

$\mathfrak{g}$	$\mathfrak{su}(n)$	$\mathfrak{so}(2n+1)$	$\mathfrak{sp}(n)$	$\mathfrak{so}(2n)$	$\mathfrak{g}_2$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
rank $\mathfrak{g}$	$n-1$	n	n	n	2	4	6	7	8
card $R_{\mathfrak{g}}^+$	$\frac{n(n-1)}{2}$	n^2	n^2	$n(n-1)$	6	24	36	63	120

Recall that there are low dimensional isomorphisms of Lie algebras $\mathfrak{su}(2) = \mathfrak{so}(3) = \mathfrak{sp}(1)$ $(A_1 = B_1 = C_1)$, $\mathfrak{so}(5) = \mathfrak{sp}(2)$ $(B_2 = C_2)$, $\mathfrak{so}(6) = \mathfrak{su}(4)$ $(D_3 = A_3)$.

By using the inequality (11) and modifying Krämer's proof, we obtain the following statement.

Theorem 4 *Let G be a compact simple Lie group and $K \subset G$ a connected subgroup, such that* Ad_K*-action on a generic G-adjoint orbit is almost multiplicity free. Then the pair of corresponding Lie algebras* $(\mathfrak{g}, \mathfrak{k})$ *belongs to the following list:*

$$(A_n, A_{n-1}), \quad (A_3, A_1 \oplus A_1 \oplus \mathfrak{u}(1)), \quad (B_2, \mathfrak{u}(2)),$$
$$(B_2, B_1 \oplus \mathfrak{u}(1)), \quad (B_3, \mathfrak{g}_2), \quad (\mathfrak{g}_2, A_2).$$

Proof Denote by $R_{\mathfrak{g}}^+$ and $R_{\mathfrak{k}}^+$ the numbers of positive roots of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Since $R_{\mathfrak{g}}^+ = \frac{1}{2}(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})$ for any semi-simple Lie algebra $\mathfrak{g}$, the condition (11) takes the form:

$$\operatorname{card} R_{\mathfrak{g}}^+ \leq \operatorname{card} R_{\mathfrak{k}}^+ + \operatorname{rank} \mathfrak{k} + \mathbf{c}(\mathfrak{g}, \mathfrak{k}). \tag{12}$$

For $\mathbf{c}(\mathfrak{g}, \mathfrak{k}) = 0$, the inequality (12) is obtained in [17, 22] and used in the classification of multiplicity free subgroups. If the pair $(\mathfrak{g}, \mathfrak{k})$ is almost multiplicity free, then $\mathbf{c}(\mathfrak{g}, \mathfrak{k}) = 1$.

Similar to Krämer's proof of [22, Proposition 3], we consider the classical Lie algebras $A_n = \mathfrak{su}(n-1)$, $B_n = \mathfrak{so}(2n+1)$, $C_n = \mathfrak{sp}(n)$, $D_n = \mathfrak{so}(2n)$ and the exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ and their maximal subalgebras, since the condition (12) requires "large" subalgebras.

Maximal subalgebras of compact simple Lie algebras are roughly divided in the following classes (see [8, 10]): maximal non-simple reducible subalgebras embedded in the standard way, maximal non-simple irreducible subalgebras represented as tensor products of vector representations and maximal simple subalgebras.

Case 1. Let $\mathfrak{g} = \mathfrak{su}(n)$, $n \geq 2$. Maximal, non-simple, reducible subalgebras are of the form $\mathbb{R} \oplus \mathfrak{su}(p) \oplus \mathfrak{su}(q)$, $n = p + q$, $p \geq q \geq 1$. In that case the inequality (12) becomes

$$\frac{n(n-1)}{2} \leq 1 + \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + (p-1) + (q-1) + 1.$$

Using the relation $n = p + q$, after simple calculation, this is equivalent to $n \geq pq$. This is fulfilled only for $p = n - 1,\ q = 1$ (which corresponds to the multiplicity free case) and $n = 4,\ p = q = 2$.

We have to show that the pair $(\mathfrak{su}(4), \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ is indeed almost multiplicity free. Denote by e_{ij} the standard basis of algebra $\mathfrak{so}(n)$. Consider a regular element

$$x = e_{12} + e_{23} + e_{34} \in \mathfrak{su}(4) = \mathfrak{so}(4) \cap \mathfrak{gl}_4(\mathbb{C})$$

with the isotropy subalgebra $\mathfrak{su}(4)_x$ spanned by x, ix^2, x^3. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k} = \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with respect to the Killing form. Since

$$\mathrm{pr}_{\mathfrak{p}}(x) = e_{23},\ \ \mathrm{pr}_{\mathfrak{p}}(ix^2) = ie_{13} + ie_{22},\ \ \mathrm{pr}_{\mathfrak{p}}(x^3) = 3e_{23} + e_{14},$$

are linearly independent, the pair $(\mathfrak{su}(4), \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ is almost multiplicity free.

The only non-simple, maximal, irreducible subalgebras of $\mathfrak{su}(n)$ are $\mathfrak{su}(p) \oplus \mathfrak{su}(q)$, $pq = n,\ p, q \geq 2$, acting via tensor product representation. In this case inequality (12) cannot be fulfilled for any $p, q \geq 2$.

The non-simple, the non-maximal subalgebras: the direct sum $\mathbb{R}^{l-1} \oplus \bigoplus_{k=1}^{l} \mathfrak{su}(p)$ $(n = pl,\ l \geq 3,\ p \geq 2)$ and the Cartesian product $\prod_{k=1}^{l} \mathfrak{su}(p)$ $(n = p^l,\ l \geq 3,\ p \geq 2)$ obviously cannot satisfy (12) for any $l \geq 3,\ p \geq 2$.

Now, let us consider simple non-maximal subalgebras. If $\mathfrak{k} = \mathfrak{su}(p) \subset \mathfrak{su}(n)$, then the inequality (12) becomes $n(n-1) \leq p(p+1)$. Hence, (12) is satisfied only for $p = n - 1$ and the pair $(\mathfrak{su}(n), \mathfrak{su}(n-1))$ is a candidate for almost multiplicity free pair. Using a technique similar to the case $(\mathfrak{su}(4), \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2))$ one can show that $(\mathfrak{su}(n), \mathfrak{su}(n-1))$ is a multiplicity free pair.

If $\mathfrak{k} = \mathfrak{so}(p)$, the only examples are the trivial $\mathfrak{so}(2) \subset \mathfrak{su}(2)$ and $\mathfrak{so}(3) = \mathfrak{su}(2) \subset \mathfrak{su}(3)$ which we already covered. Finally, in case $\mathfrak{k} = \mathfrak{sp}(p)$ or $\mathfrak{k}$ is exceptional, there is no pair $(\mathfrak{su}(n), \mathfrak{k})$ satisfying (12) since the number of positive roots of $\mathfrak{k}$ is significantly smaller than the number of roots of $\mathfrak{su}(n)$.

Case 2. Let $\mathfrak{g} = \mathfrak{so}(n),\ n \neq 4$. This requires more detailed analysis than the previous case.

First, let us exclude the case $\mathfrak{so}(n-1) \subset \mathfrak{so}(n)$ which is multiplicity free and consider non-simple, reducible, maximal subalgebras $\mathfrak{k} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$, $n = p + q,\ p \geq q \geq 2$. By using the identification $\mathfrak{so}(2) = \mathbb{R} = \mathfrak{u}(1)$, here we are also considering the case of reducible subalgebra $\mathbb{R} \oplus \mathfrak{so}(p),\ p \geq 3$. If both p and q are odd, inequality (12) reduces to $pq \leq p + q + 1$, that is never fulfilled for $p, q \geq 3$. Hence, there are no candidates for almost multiplicity free pairs.

If q is even, we find two possible pairs $(\mathfrak{so}(5), \mathfrak{so}(3) \oplus \mathfrak{so}(2))$ and $(\mathfrak{so}(6), \mathfrak{so}(4) \oplus \mathfrak{so}(2))$. The second one is isomorphic to $(\mathfrak{su}(4), \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2))$, we have already found to be almost multiplicity free. Using a similar method, one can show that the pair $(\mathfrak{so}(5), \mathfrak{so}(3) \oplus \mathfrak{so}(2))$ is almost multiplicity free as well.

For irreducible, maximal subalgebras $\mathfrak{k} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$, $n = pq$, $p \geq q \geq 3$, $p, q \neq 4$ and $\mathfrak{k} = \mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$, $n = 4pq \neq 4$, given by tensor product of vector representations, no example exists.

Now we consider reducible subalgebras. If $\mathfrak{k} = \mathfrak{u}(p) \subset so(n)$, $n = 2p$, $p \geq 2$. The inequality (12) becomes $p^2 - 3p - 2 \leq 0$. Therefore, the only examples are multiplicity free pairs $(\mathfrak{so}(4), \mathfrak{u}(2))$ and $(\mathfrak{so}(6), \mathfrak{u}(3))$.

Note that $\mathfrak{k} = \mathfrak{u}(p)$ is also the subalgebra of $\mathfrak{so}(2p+1)$ and it satisfies (12) for $p = 2$. It can be shown that the pair $(\mathfrak{so}(5), \mathfrak{u}(2))$ is almost multiplicity free.

Also, non-maximal non-simple subalgebras $\bigoplus_{k=1}^{l} \mathfrak{so}(p)$ $(n = pl,\ p, l \geq 3)$, $\prod_{k=1}^{l} \mathfrak{so}(p)$ $(n = p^l,\ p, l \geq 3,\ p \neq 4)$, $\prod_{k=1}^{l} \mathfrak{sp}(p)$ $(n = (2p)^l,\ p \geq 1,\ l \geq 4,\ l\text{–even})$ and $\mathfrak{so}(p) \times \mathfrak{sp}(2) \times \mathfrak{sp}(2)$ $(p \geq 3,\ p \neq 4)$ cannot be part of the almost multiplicity free pair.

The simple subalgebra $\mathfrak{so}(p) \subset \mathfrak{so}(n)$ satisfies (12) only for $p = n - 1$, i.e. only if the pair is multiplicity free. Next, subalgebra $\mathfrak{su}(p) \subset \mathfrak{so}(2p)$ is multiplicity free for $p = 2$ and almost multiplicity free for $p = 3$ (we already considered this as $\mathfrak{so}(6) = \mathfrak{su}(4)$). Note that we could also consider inclusion $\mathfrak{su}(p) \subset \mathfrak{so}(2p + 1)$. However, the inequality (12) holds only for $p = 1$ which corresponds to the already examined case $\mathfrak{su}(1) \subset \mathfrak{so}(3) = \mathfrak{su}(2)$.

One can easily verify that the last simple classical Lie algebra $\mathfrak{k} = \mathfrak{sp}(p)$ cannot fulfill the condition (12) for any p. The same is true for all exceptional Lie algebras except $\mathfrak{g}_2 \subset \mathfrak{so}(7)$.

Let us examine the pair $(\mathfrak{so}(7), \mathfrak{g}_2)$ in more details. $\mathfrak{g}_2$ has rank 2, the number of positive roots is 6, and the number of positive roots of $\mathfrak{so}(7)$ is 9. Thus, in (12) the equality holds. As above, let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{g}_2$ with respect to the Killing form and set $(\mathfrak{g}_2)_x = \{y \in \mathfrak{g}_2 \,|\, [x, y] = 0\}$. The condition that a generic Cartan's subalgebra and its projection onto $\mathfrak{p}$ have the same dimensions is equivalent to the condition that $(\mathfrak{g}_2)_x$ is trivial for a generic $x \in \mathfrak{so}(7)$. Denote by $\left\{e_{ij} \,\middle|\, 1 \leq i < j \leq 7\right\}$ the standard basis of $\mathfrak{so}(7)$. It is well known that

$$\begin{aligned}
&P_0 = e_{32} + e_{67}, \quad P_1 = e_{13} + e_{57}, \quad P_2 = e_{21} + e_{74}, \quad P_3 = e_{14} + e_{72}, \\
&P_4 = e_{51} + e_{37}, \quad P_5 = e_{17} + e_{35}, \quad P_6 = e_{61} + e_{43}, \\
&Q_0 = e_{45} + e_{67}, \quad Q_1 = e_{46} + e_{57}, \quad Q_2 = e_{56} + e_{74}, \quad Q_3 = e_{36} + e_{72}, \\
&Q_4 = e_{26} + e_{37}, \quad Q_5 = e_{24} + e_{35}, \quad Q_6 = e_{25} + e_{43}
\end{aligned} \tag{13}$$

constitute the basis of the Lie algebra $\mathfrak{g}_2$. One can show that for $x = e_{12} + e_{34} + e_{56} \in \mathfrak{so}(7)$ none of the elements $\sum_i a_i P_i + b_j Q_j \in \mathfrak{g}_2$ commutes with it. Thus, $(\mathfrak{g}_2)_x$ is trivial for a generic $x \in \mathfrak{so}(7)$ and $(\mathfrak{so}(7), \mathfrak{g}_2)$ is indeed an almost multiplicity free pair.

Case 3. In the case $\mathfrak{g} = \mathfrak{sp}(n)$, $n \geq 3$ the analysis similar to the previous cases shows that $\mathfrak{sp}(n)$ cannot contribute to the list of almost multiplicity free pairs.

Case 4. Let us consider the exceptional Lie algebra $\mathfrak{g} = \mathfrak{g}_2$. According to [8] it has two subalgebras as maximal subalgebras with maximal rank 2. The algebra $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ doesn't satisfy (12). For the maximal subalgebra $\mathfrak{k} = \mathfrak{su}(3)$ the inequality (12) is satisfied so let us examine if the pair $(\mathfrak{g}_2, \mathfrak{su}(3))$ is almost multiplicity free. Let the basis of the Lie algebra $\mathfrak{g}_2$ be given by (13), then the basis of $\mathfrak{su}(3)$ is $\{P_0, Q_0, \dots, Q_6\}$. For example, set $x = P_0 + P_1$. Then it is an easy exercise to show that the element from $\mathfrak{su}(3)$ commuting with it does not exist. Hence, the pair $(\mathfrak{g}_2, \mathfrak{su}(3))$ is almost multiplicity free.

Case 5. Finally, by the results of [10] and [13] the maximal subalgebra $\mathfrak{k}$ of one of the the exceptional Lie algebras $\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ is isomorphic to one in the following list:

$\mathfrak{f}_4$: $\mathfrak{su}(2) \oplus \mathfrak{sp}(3)$, $\mathfrak{su}(3) \oplus \mathfrak{su}(3)$, $\mathfrak{so}(9)$;
$\mathfrak{e}_6$: $\mathfrak{su}(2) \oplus \mathfrak{su}(6)$, $\mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3)$, $\mathfrak{so}(10) \oplus T^1$;
$\mathfrak{e}_7$: $\mathfrak{su}(2) \oplus \mathfrak{so}(12)$, $\mathfrak{su}(3) \oplus \mathfrak{so}(10)$, $\mathfrak{su}(8)$, $\mathfrak{e}_6 \oplus T^1$;
$\mathfrak{e}_8$: $\mathfrak{su}(2) \oplus \mathfrak{e}_7$, $\mathfrak{su}(3) \oplus \mathfrak{e}_6$, $\mathfrak{su}(5) \oplus \mathfrak{su}(5)$, $\mathfrak{so}(16)$, $\mathfrak{su}(9)$;

where T^1 denotes the 1-dimensional center of the subalgebra. From Table 1 it can be seen that the number of positive roots of the exceptional Lie algebra is much larger than the number of positive roots of all its subalgebras. Therefore, no new examples can occur. □

Example 2 Almost multiplicity free pairs of Lie groups are:

$$\begin{aligned}
&(SU(n), SU(n-1)),\ (SU(4), S(U(2)\times U(2)),\ (SU(3), SO(3)),\\
&(SO(5), SO(3)\times SO(2)),\ (Sp(2), Sp(1)\times U(1)),\ (SO(5), U(2)),\\
&(Sp(2), U(2)),\ (SO(6), SO(4)\times SO(2)),\ (SO(6), SU(3)),\\
&(Spin(7), G_2),\ (G_2, SU(3)),\ (SO(3)\times SO(4), SO(3)).
\end{aligned} \tag{14}$$

Note that in Examples 1 and 2 we consider natural inclusions $SU(n) \subset U(n) \subset SO(2n)$, $Sp(n) \subset SU(2n)$, and $SO(3)$ is diagonally embedded into $SO(3) \times SO(4)$. Semi-simple examples with $G = SO(4)$ and $G = SO(3) \times SO(4)$ are also given.

4 Examples: integrable sub-Riemannian flows

4.1 $\mathfrak{p} \oplus \mathfrak{k}$–sub-Riemannian problem

In [18] we used the chains of subalgebras in order to construct integrable nonholonomic and sub-Riemannian flows with left-invariant metrics and left-invariant nonholonomic distributions on compact Lie groups. While the nonholonomic problem is not Hamiltonian, the sub-Riemannian is, and can be described as follows.

With the above notation, assume that the symmetric operator A_0 in (5) has a non-empty zero-eigenvalue subspace $\mathfrak{h} \subset \mathfrak{k}$. Let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{v}$ be the orthogonal decomposition of $\mathfrak{k}$ with respect to the bi-invariant scalar product. We further assume that the restriction

$A_0 : \mathfrak{v} \to \mathfrak{v}$ is positive definite and that $\mathfrak{d} = \mathfrak{v} \oplus \mathfrak{p}$ generates Lie algebra $\mathfrak{g}$. Then the left-invariant distribution $\mathcal{D}$ of TG induced by $\mathfrak{d}$ is completely nonholonomic. For $s > 0$, on $\mathcal{D}$ we have left-invariant metric $ds^2_{\mathcal{D}}$ defined on $\mathfrak{d}$ by defined by left-translation of the scalar product

$$(\xi, \eta)_{\mathfrak{d}} = \langle A_0^{-1}(\mathrm{pr}_{\mathfrak{v}}(\xi)) + s^{-1}\,\mathrm{pr}_{\mathfrak{p}}(\xi), \eta\rangle, \qquad \xi, \eta \in \mathfrak{d}.$$

Then the Euler equations (6), together with the kinematic equation

$$\dot{g} = d(L_g)(\omega), \qquad \omega = A_0(x_0) + s x_1 \in \mathfrak{d},$$

describe the sub-Riemannian geodesic flow of the metric $ds^2_{\mathfrak{d}}$ on the left-trivialization of the cotangent bundle $T^*G \cong G \times \mathfrak{g}\{g, x\}$.

Thus, according to Theorem 1, if the Euler equation (2) are integrable by means of polynomial integrals, then the extended Euler equations (6) are integrable in a non-commutative sense by polynomial integrals as well.

In particular, the simplest situation is when $\mathfrak{d} = \mathfrak{p}$ ($A_0 = 0$). The sub-Riemannian metric $ds^2_{\mathfrak{d}}$ is then simply the restriction of a bi-invariant metric to $\mathcal{D}$. The Euler and the kinematic equations take the form

$$\dot{x}_0 = 0, \qquad \dot{x}_1 = s[x_0, x_1] \tag{15}$$

$$\dot{g} = d(L_g)(s x_1). \tag{16}$$

The Euler equations can be solved easily:

$$x_0 \equiv \bar{x}_0, \qquad x_1(t) = \mathrm{Ad}_{\exp(st\bar{x}_0)}(\bar{x}_1), \tag{17}$$

where $\bar{x}_0 = x_0(0)$, $\bar{x}_1 = x_1(0)$. The solution of the kinematic equation (16) is given by Agrachev [1] and Brockett [9]:

$$g(t) = \bar{g}\exp(t(s\bar{x}_0 + \bar{x}_1))\exp(-ts\bar{x}_0), \qquad g(0) = \bar{g}. \tag{18}$$

The sub-Riemannian problem (15), (16) is usually called a $\mathfrak{p} \oplus \mathfrak{k}$–problem on a Lie group G (see [28]).

From the point of view of the geometry of integrable systems, the equations (15) are integrable in a noncommutative sense. The complete set of polynomial integrals are linear functions on $\mathfrak{k}$ together with all Ad_K–invariant polynomials $\mathbb{R}[\mathfrak{g}]^K$ on $\mathfrak{g}$. A generic motion given by (17) is a quasi-periodic winding over

$$\delta = \operatorname{rank} \mathfrak{k} - \operatorname{rank} \mathfrak{g} + \dim \mathrm{pr}_{\mathfrak{p}}(\mathfrak{g}(x))$$

dimensional invariant tori on $\mathfrak{g}$ (see [20, Theorem 2] with $n = 1$ and assume that the Euler equations in the first step are trivial). Considered on T^*G we also have Noether integrals - the right invariant functions, obtained from linear functions on $\mathfrak{g}$. In total, the system (15), (16) is integrable in a non-commutative sense on T^*G with an invariant tori of dimension $\Delta = \delta + \operatorname{rank} \mathfrak{g}$ (see [25][1]).

We can summarize the above consideration in the following statement.

Theorem 5 *The $\mathfrak{p} \oplus \mathfrak{k}$–sub-Riemannian problem (15), (16) on a Lie group G is completely integrable in a non-commutative sense by means of integrals polynomial in momenta. A generic motion given by (17), (18) is a quasi-periodic winding over*

$$\Delta = \operatorname{rank} \mathfrak{k} + \dim \operatorname{pr}_{\mathfrak{p}}(\mathfrak{g}(x))$$

dimensional invariant isotropic tori in T^*G.

Let $Q_1 = \operatorname{pr}_{\mathfrak{k}}^* q_1, \dots, Q_{\mathbf{a}(\mathfrak{k})} = \operatorname{pr}_{\mathfrak{k}}^* q_{\mathbf{a}(\mathfrak{k})}$, where $q_1, \dots, q_{\mathbf{a}(\mathfrak{k})}$ is arbitrary complete set of commuting polynomials on $\mathfrak{k}$ (see [4, 24]). Let, as above, $p_1, \dots, p_r$ be independent invariant polynomials on $\mathfrak{g}$, $r = \operatorname{rank} \mathfrak{g}$.

If K is a multiplicity free subgroup of G then $Q_1, \dots, Q_{\mathbf{a}(\mathfrak{k})}, p_1, \dots, p_r$ is a complete commutative set polynomials on $\mathfrak{g}$ (some of them can be dependent). The left translations of $Q_1, \dots, Q_{\mathbf{a}(\mathfrak{k})}, p_1, \dots, p_r$ and the right translations of $Q_1, \dots, Q_{\mathbf{a}(\mathfrak{k})}$ form a complete commutative set of integrals of the $\mathfrak{p} \oplus \mathfrak{k}$–sub-Riemannian flow on T^*G. If K is an almost multiplicity free subgroup of G then we need to add left and right translations of one of the Bogoyavlensky integrals (9). Hence, the pairs given in Examples 1 and 2 define examples of polynomial Arnold–Liouville integrable sub-Riemannian geodesic flows.

Proposition 6 *Multiplicity free and almost multiplicity free pairs of Lie groups define $\mathfrak{p} \oplus \mathfrak{k}$–sub-Riemannian geodesic flows that are Arnold–Liouville integrable by means of integrals polynomial in momenta.*

4.2 Integrable sub-Riemannian geodesic flows related to filtrations of groups

By induction, we can use the above construction to consider the chain of connected compact Lie subgroups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n = G$$

and the corresponding filtration of the Lie algebra $\mathfrak{g} = Lie(G)$

$$\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}. \tag{19}$$

Let $\mathfrak{p}_i$ be the orthogonal complement of $\mathfrak{g}_{i-1}$ in $\mathfrak{g}_i$. Then $\mathfrak{g}_i = \mathfrak{g}_0 \oplus \mathfrak{p}_0 \oplus \cdots \oplus \mathfrak{p}_i$. Assume that $\mathfrak{d} = p_1 \oplus \cdots \oplus p_n$ generate $\mathfrak{g}$ by commutations. Then the corresponding left-invariant distribution is completely nonholonomic For $s_1, \dots, s_n > 0$, on $\mathcal{D}$ we

[1] Alternatively, we can use a following general statement. Assume that compact group G acts freely in a Hamiltonian way on a symplectic manifold M. If a G–invariant Hamiltoniam system is integrable in a noncommutative sense on the reduced space M/G with δ–dimensional invariant tori, then the original system on M is also integrable with $(\delta + \operatorname{rank} G)$–dimensional invariant tori (see [19, 33])

have left-invariant metric $ds^2_{\mathcal{D}}$ defined by the scalar product

$$(\xi,\eta)_{\mathfrak{d}} = \langle \sum_{i=1}^{n} s_i^{-1} \operatorname{pr}_{\mathfrak{p}_i}(\xi), \eta\rangle, \qquad \xi,\eta\in\mathfrak{d}.$$

The corresponding sub-Riemannian geodesic flow is given by (see [3, 18, 20]):

$$\dot{x}_0 = 0, \tag{20}$$
$$\dot{x}_i = [s_i x_0 - (s_1 - s_i)x_1 - \cdots - (s_{i-1} - s_i)x_{i-1}, x_i], \quad i = 1,\dots,n, \tag{21}$$
$$\dot{g} = d(L_g)(s_1 x_1 + \dots s_n x_n), \tag{22}$$

where we presented $x \in \mathfrak{g}$ as the sum

$$x = x_0 + x_1 + \cdots + x_n, \quad x_0 \in \mathfrak{g}_0, \quad x_i \in \mathfrak{p}_i, \quad i = 1,\dots n.$$

The problem of the integrability of the Euler equations (20), (21) is studied by Bogoyavlensky (see [3]). From [20, Theorem 2], as above, we get.

Theorem 7 *The Euler equations* (20)*,* (21) *are integrable in a noncommutative sense on* $\mathfrak{g}$*. A generic motion is a quasi-periodic winding over*

$$\delta = \operatorname{rank}\mathfrak{g}_0 - \operatorname{rank}\mathfrak{g} + \sum_{i=1}^{n} \dim \operatorname{pr}_{\mathfrak{p}_i}(\mathfrak{g}_i(x_0 + \cdots + x_i))$$

dimensional invariant tori.[2] *The sub-Riemannian geodesic flow* (20)*,* (21)*,* (22) *on* $T^*G \cong G \times \mathfrak{g}$ *is completely integrable in a non-commutative sense by means of integrals polynomial in momenta. A generic motion is a quasi-periodic winding over*

$$\Delta = \operatorname{rank}\mathfrak{g}_0 + \sum_{i=1}^{n} \dim \operatorname{pr}_{\mathfrak{p}_i}(\mathfrak{g}_i(x_0 + \cdots + x_i))$$

dimensional invariant isotropic tori.

Additional examples of integrable sub-Riemannian geodesic flows on Lie groups and homogeneous spaces can be found in e.g. [2, 5, 7, 21, 26, 28].

Acknowledgements The research was supported by the Project no. 7744592 MEGIC "Integrability and Extremal Problems in Mechanics, Geometry and Combinatorics" of the Science Fund of Serbia. The authors would like to thank the reviewers for their thorough reviews and constructive comments, which greatly improved the quality of the paper.

Data availability statement We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

[2] Here we take $x_i \in \mathfrak{p}_i$, $i = 1,\dots,n$, such that the dimensions of the isotropy algebras $\mathfrak{g}_i(x_0 + \cdots + x_i)$ and $\mathfrak{g}_{i-1}(x_0 + \cdots + x_i)$ are minimal.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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