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Superspace realizations of the Bannai–Ito algebra

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Abstract
A model of the Bannai–Ito algebra in a superspace is introduced. It is obtained from the threefold tensor product of the basic realization of the Lie superalgebra $\mathfrak{osp}(1|2)$ in terms of operators in one continuous and one Grassmanian variable. The basis vectors of the resulting Bannai–Ito algebra module involve Jacobi polynomials.

Keywords Superspace · Super Lie algebra · Jacobi polynomials

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1 Introduction

This paper offers a realization of the Bannai–Ito algebra in superspace. Models of algebras with operators acting on spaces of functions are especially relevant in mathematical physics and allow also to enrich the understanding of special functions [1].

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This is the basic motivation here. The Bannai–Ito algebra [2] encodes the bispectral properties of the eponymous polynomials which were first introduced in the context of P- and Q- polynomial association schemes [3]. Their characterization as eigenfunctions of shift operators of Dunkl type was given in [4].

The Bannai–Ito polynomials have explicit expressions as combinations of two Racah polynomials [5]. Moreover, the Racah algebra [6] that is abstracted from the relations verified by the difference and recurrence operators corresponding to these last polynomials can be embedded in the Bannai–Ito algebra [7] much like the even Lie algebra part of a superalgebra can be obtained from quadratic expressions in the odd generators.

As a matter of fact, it was also observed that the Racah algebra can be directly embedded in the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$ [8]. Calling upon the Bargmann realization of $\mathfrak{sl}(2)$ in terms of differential operators in one variable [9], this last embedding thus immediately provides a realization of the Racah algebra in terms of differential operators of the hypergeometric type.

Given that the Racah polynomials arise in the $6-j$ recoupling coefficients of the $\mathfrak{sl}(2)$ representations, it is not surprising that a centrally extended version of the Racah algebra arises by taking as generators the intermediate Casimir in a threefold product of representations [10]. Here, the Bargmann realization produces a model of the Racah algebra in terms of differential operators in three variables. The dimensional reduction in this three-variable realization through separation of variables has been examined in [11] and shown to yield the one-variable model of the Racah algebra mentioned before. It has been shown in this spirit how the higher rank Racah algebras [12] constructed in this fashion using the n variables associated with an n-fold product of irreducible $\mathfrak{sl}(2)$ representations can be reduced to a model involving differential operators in only $n-2$ variables [13].

The Bannai–Ito algebra can also be viewed as the commutant of a diagonal embedding of an algebra module in a threefold tensor product of representations. In this case the underlying algebraic structure is the Lie superalgebra $\mathfrak{osp}(1|2)$ and it is hence understood that the Bannai–Ito polynomials are essentially the Racah coefficients of this superalgebra [14]. Now one extension to $\mathfrak{osp}(1|2)$ of the Bargmann model of $\mathfrak{sl}(2)$ involves thickening the one-dimensional space by the addition of a real Grassmann variable; this leads to using operators acting on functions defined on the resulting superspace [15]. Considering the tensor product of three irreducible (discrete series) representations of $\mathfrak{osp}(1|2)$ [16] in this picture therefore brings three continuous variables and three Grasmannian ones. In view of the corresponding studies of the Racah realizations mentioned before, natural questions we address in this paper are: What is the Bannai–Ito algebra model that the irreducible decomposition of this tensor product representation brings? To how many variables does this reduction yield? What are the special functions involved?

The remainder of the paper is organized as follows. The fundamental superspace model of $\mathfrak{osp}(1|2)$ is recalled in Sect. 2. The construction of the Bannai–Ito algebra generated by the intermediate Casimir operators of three copies of this $\mathfrak{osp}(1|2)$ realization is described in Sect. 3. This module decomposes into an even and an odd part each with four components. The dimensional reduction that occurs upon requesting that the representation space be bounded from below is discussed in Sect. 4. It will be shown that the kernel of the total lowering operator is such that its even and odd part only have two components. The action of the intermediate Casimir operators in each of these 4 parts will be given in Sect. 5. The construction of the irreducible representations is carried out by considering in turn the even subspace in Sect. 6 and the odd one in Sect. 7. In each case, first, the concrete representation basis will be obtained by diagonalizing the total Casimir operator and one of the intermediate Casimir elements and second, the tridiagonal action of the other independent intermediate Casimir element will be computed. The Jacobi polynomials will be seen to appear in the expression of these basis elements. The concluding section will offer summary, remarks and outlook.

2 The fundamental realization of $\mathfrak{osp}(1|2)$ on superspace

Consider the operators

$$A_- = \theta\partial_x + \partial_\theta,$$

$$A_0 = 2x\partial_x + \theta\partial_\theta + 2\nu,$$

$$A_+ = x\theta\partial_x + x\partial_\theta + 2\nu\theta,$$

where θ is a Grassmann variable $\theta^2 = 0$ commuting with x. It is easy to check that the operators satisfy the following relations

$$[A_0, A_\pm] = \pm A_\pm \qquad \text{and} \qquad \{A_+, A_-\} = A_0, \tag{2.1}$$

where as usual $[a, b] = ab - ab$ denotes the commutator of the operators a and b, and $\{a, b\} = ab + ba$ denotes the anticommutator. Using the identity

$$[B, CD] = [B, C]D + C[B, D] \tag{2.2}$$

and (2.1) we see that

$$[A_0, A_\pm^2] = [A_0, A_\pm]A_\pm + A_\pm[A_0, A_\pm] = \pm 2A_\pm^2. \tag{2.3}$$

Next, we use

$$[B, CD] = \{B, C\}D - C\{B, D\} \tag{2.4}$$

and (2.1) to deduce that

$$[A_\mp, A_\pm^2] = \{A_\mp, A_\pm\}A_\pm - A_\pm\{A_\mp, A_\pm\} = [A_0, A_\pm] = \pm A_\pm. \tag{2.5}$$

Finally, using the last equation, applying again (2.2) and then (2.1) yields

$$[A_+^2, A_-^2] = [A_+^2, A_-]A_- + A_-[A_+^2, A_-] = -\{A_+, A_-\} = -A_0. \tag{2.6}$$

If we set

$$H=\frac{1}{2}A_0, \qquad F^{\pm}=\frac{1}{2}A_{\pm}, \quad \text{and} \quad E^{\pm}=\pm A_{\pm}^2, \tag{2.7}$$

then we can rewrite Eqs. (2.1), (2.3), (2.5), (2.6) and the last formula as follows:

$$\begin{aligned}
&[H,F^{\pm}]=\pm\frac{1}{2}F^{\pm}, && \{F^+,F^-\}=\frac{1}{2}H,\\
&[H,E^{\pm}]=\pm E^{\pm}, && [E^{\pm},F^{\mp}]=-F^{\pm},\\
&[E^+,E^-]=2H, && \{F^{\pm},F^{\pm}\}=\pm\frac{1}{2}E^{\pm},
\end{aligned}$$

which are precisely the relations of the superalgebra $\mathfrak{osp}(1|2)$ in the standard notations, see for instance [16, Section 52]. Therefore, the operators above define a representation of the superalgebra $\mathfrak{osp}(1|2)$ and one can easily check that

$$A_-^2=\partial_x.$$

Let P be the operator acting on functions $f(x,\theta)$ by

$$P(f(x,\theta))=f(x,-\theta).$$

Clearly, P is an involution, i.e. $P^2=1$ and

$$[P,A_0]=0, \qquad \{P,A_{\pm}\}=0.$$

This operator P is realized as follows:

$$P=1-2\theta\partial_\theta.$$

The Casimir operator

$$Q=(A_0-2A_+A_--1/2)P$$

commutes with A_0, $A_{\pm}$ and P. One can also easily check that in our current realization it holds that

$$Q=2\nu-1/2.$$

3 Three copies and the Bannai–Ito algebra

Let $\theta_1,\theta_2,\theta_3$ be anticommuting Grassmann variables, i.e.

$$\theta_i\theta_j=-\theta_j\theta_i \quad \text{for } i\neq j.$$

We can naturally extend this relation to differential operators by setting

$$\partial_{\theta_i}\theta_j = -\theta_j\partial_{\theta_i} \quad \text{and} \quad \partial_{\theta_i}\partial_{\theta_j} = -\partial_{\theta_j}\partial_{\theta_i} \quad \text{for } i \neq j.$$

For $j = 1, 2, 3$, let

$$\begin{aligned} A_-^{(j)} &= \theta_j\partial_{x_j} + \partial_{\theta_j}, \\ A_0^{(j)} &= 2x_j\partial_{x_j} + \theta_j\partial_{\theta_j} + 2\nu_j, \\ A_+^{(j)} &= x_j\theta_j\partial_{x_j} + x_j\partial_{\theta_j} + 2\nu_j\theta_j, \end{aligned}$$

be the operators discssued in the previous section. Note that for $i \neq j$ we have

$$[A_0^{(i)}, A_0^{(j)}] = 0, \qquad [A_0^{(i)}, A_\pm^{(j)}] = 0, \qquad \{A_\pm^{(i)}, A_\pm^{(j)}\} = 0, \qquad \{A_\pm^{(i)}, A_\mp^{(j)}\} = 0.$$

In particular, these commutativity relations imply that for every nonempty $S \subset \{1, 2, 3\}$ we obtain another representation of $\mathfrak{osp}(1|2)$ with operators defined by

$$A_0^{(S)} = \sum_{i\in S} A_0^{(i)}, \qquad A_\pm^{(S)} = \sum_{i\in S} A_\pm^{(i)}, \qquad P^{(S)} = \prod_{i\in S} P^{(i)},$$

and associated Casimir operator

$$Q^{(S)} = (A_0^{(S)} - 2A_+^{(S)}A_-^{(S)} - 1/2)P^{(S)}.$$

Note that $Q^{(S)}$, usually called intermediate Casimir operator, commutes with $A_0^{(T)}$, $A_\pm^{(T)}$ and $P^{(T)}$ for every $T \supset S$ and $Q^{(i)} = 2\nu_i - 1/2$.

The Casimir operators $Q^{(S)}$ provide a realization of the Bannai–Ito algebra i.e. they satisfy the following relations

$$\{Q^{(12)}, Q^{(23)}\} = Q^{(13)} + 2Q^{(1)}Q^{(3)} + 2Q^{(2)}Q^{(123)}, \tag{3.1}$$

$$\{Q^{(12)}, Q^{(13)}\} = Q^{(23)} + 2Q^{(2)}Q^{(3)} + 2Q^{(1)}Q^{(123)}, \tag{3.2}$$

$$\{Q^{(13)}, Q^{(23)}\} = Q^{(12)} + 2Q^{(1)}Q^{(2)} + 2Q^{(3)}Q^{(123)}. \tag{3.3}$$

In this realization of the Bannai–Ito algebra, the following relation also holds

$$\left(Q^{(12)}\right)^2 + \left(Q^{(13)}\right)^2 + \left(Q^{(23)}\right)^2 + \frac{1}{4} = \left(Q^{(123)}\right)^2 + \left(Q^{(1)}\right)^2 + \left(Q^{(2)}\right)^2 + \left(Q^{(3)}\right)^2.$$

4 Dimensional reduction

Below we work with the superspace

$$\mathbb{V} = \mathbb{C}[x_1, x_2, x_3]\langle\theta_1, \theta_2, \theta_3\rangle.$$

We can decompose $\mathbb{V}$ as the direct sum of the odd subspace $\mathbb{V}^{\mathrm{o}}$ and the even subspace $\mathbb{V}^{\mathrm{e}}$ as follows:

$$\mathbb{V} = \mathbb{V}^{\mathrm{o}} \oplus \mathbb{V}^{\mathrm{e}}, \tag{4.1}$$

where

$$\begin{aligned}\mathbb{V}^{\mathrm{o}} &= \mathbb{C}[x_1, x_2, x_3]\theta_1 \oplus \mathbb{C}[x_1, x_2, x_3]\theta_2 \oplus \mathbb{C}[x_1, x_2, x_3]\theta_3 \oplus \mathbb{C}[x_1, x_2, x_3]\theta_1\theta_2\theta_3\\ \mathbb{V}^{\mathrm{e}} &= \mathbb{C}[x_1, x_2, x_3] \oplus \mathbb{C}[x_1, x_2, x_3]\theta_1\theta_2 \oplus \mathbb{C}[x_1, x_2, x_3]\theta_1\theta_3 \oplus \mathbb{C}[x_1, x_2, x_3]\theta_2\theta_3.\end{aligned}$$

These subspaces can be characterized as the spaces consisting of skew-symmetric and symmetric functions with respect to the involution $P^{(123)}$ on $\mathbb{V}$. Since $P^{(123)}$ anti commutes with $A_-^{(123)}$, we can decompose the $\ker(A_-^{(123)})$ on $\mathbb{V}$ as a direct sum of odd and even functions

$$\ker(A_-^{(123)}) = \left(\ker(A_-^{(123)}) \cap \mathbb{V}^{\mathrm{o}}\right) \oplus \left(\ker(A_-^{(123)}) \cap \mathbb{V}^{\mathrm{e}}\right).$$

To describe each component, we fix

$$u = x_1 - x_2 \qquad \text{and} \qquad v = x_2 - x_3,$$

and we consider the linear transformations

$$\begin{aligned}\mathcal{O}_1, \mathcal{O}_2 &: \mathbb{C}[u, v] \to \mathbb{V}^{\mathrm{o}}\\ \mathcal{E}_1, \mathcal{E}_2 &: \mathbb{C}[u, v] \to \mathbb{V}^{\mathrm{e}}\end{aligned}$$

defined as follows:

$$\begin{aligned}\mathcal{O}_1(h(u, v)) &= h(u, v)(\theta_1 - \theta_2) + h_v(u, v)\theta_1\theta_2\theta_3,\\ \mathcal{O}_2(h(u, v)) &= h(u, v)(\theta_2 - \theta_3) - h_u(u, v)\theta_1\theta_2\theta_3,\\ \mathcal{E}_1(h(u, v)) &= h(u, v)(\theta_1\theta_2 - \theta_1\theta_3 + \theta_2\theta_3),\\ \mathcal{E}_2(h(u, v)) &= h(u, v) + h_u(u, v)\theta_1\theta_2 + h_v(u, v)\theta_2\theta_3,\end{aligned}$$

where $h(u, v) \in \mathbb{C}[u, v]$.

Proposition 4.1 *Let $F \in \mathbb{V}$. Then F solves the equation*

$$A_-^{(123)} F = 0, \tag{4.2}$$

if and only if F can be written as

$$F = F_o + F_e, \tag{4.3}$$

where

$$F_o = \mathcal{O}_1(h_1(u, v)) + \mathcal{O}_2(h_2(u, v)) \tag{4.4}$$
$$F_e = \mathcal{E}_1(g_1(u, v)) + \mathcal{E}_2(g_2(u, v)), \tag{4.5}$$

for some $h_1(u, v), h_2(u, v), g_1(u, v), g_2(u, v) \in \mathbb{C}[u, v]$.

Proof Since

$$\left(A_-^{(123)}\right)^2 F = \left(\partial_{x_1} + \partial_{x_2} + \partial_{x_3}\right) F = 0,$$

we see that the x dependence in F is only through the variables u and v, i.e. we can write F as

$$F = \sum_{i_1,i_2,i_3=0}^{1} G_{i_1,i_2,i_3}(u, v)\theta_1^{i_1}\theta_2^{i_2}\theta_3^{i_3},$$

where $G_{i_1,i_2,i_3}(u, v) \in \mathbb{C}[u, v]$. Substituting this into (4.2) and equating the coefficients in the different powers of $\theta_1, \theta_2, \theta_3$ shows that F will satisfy (4.2) if and only if the representation in (4.3) holds. □

5 Action of the intermediate Casimir operators on ker$(A_-^{(123)})$

Since for every nonempty $S \subset \{1, 2, 3\}$ the Casimir operator $Q^{(S)}$ commutes with $A_-^{(123)}$, it follows that $Q^{(S)}$ preserves the space of solutions of equation (4.2). Below we compute the action of the intermediate Casimir operators $Q^{(12)}$, $Q^{(13)}$ and $Q^{(23)}$ on the basis of solutions of (4.2) described in Proposition 4.1. For $i \neq j$ we set $\nu_{ij} = \nu_i + \nu_j$.

5.1 Action of $Q^{(12)}$

We have

$$\begin{aligned}
Q^{(12)} \circ \mathcal{O}_1 =& -\mathcal{O}_1 \circ (2u\partial_u + 2\nu_{12} + 1/2) - \mathcal{O}_2 \circ (2u\partial_v),\\
Q^{(12)} \circ \mathcal{O}_2 =& \mathcal{O}_1 \circ (2u\partial_u + 4\nu_1) + \mathcal{O}_2 \circ (2u\partial_u + 2\nu_{12} - 1/2),\\
Q^{(12)} \circ \mathcal{E}_1 =& -\mathcal{E}_1 \circ (2u\partial_u + 2\nu_{12} + 1/2) + \mathcal{E}_2 \circ (2u),\\
Q^{(12)} \circ \mathcal{E}_2 =& -\mathcal{E}_1 \circ (2u\partial_u\partial_v + 4\nu_1\partial_v) + \mathcal{E}_2 \circ (2u\partial_u + 2\nu_{12} - 1/2).
\end{aligned}$$

5.2 Action of $Q^{(13)}$

We have

$$\begin{aligned}
Q^{(13)}\circ\mathcal{O}_1 &= -\,\mathcal{O}_1\circ(2\nu_1-2\nu_3-1/2)-\mathcal{O}_2\circ(2(u+v)\partial_v+4\nu_3),\\
Q^{(13)}\circ\mathcal{O}_2 &= -\,\mathcal{O}_1\circ(2(u+v)\partial_u+4\nu_1)-\mathcal{O}_2\circ(2\nu_1-2\nu_3+1/2),\\
Q^{(13)}\circ\mathcal{E}_1 &= -\,\mathcal{E}_1\circ(2\nu_{13}-3/2)-\mathcal{E}_2\circ(2u+2v),\\
Q^{(13)}\circ\mathcal{E}_2 &= -\,\mathcal{E}_1\circ(2(u+v)\partial_u\partial_v+4\nu_3\partial_u+4\nu_1\partial_v)+\mathcal{E}_2\circ(2\nu_{13}-1/2).
\end{aligned}$$

5.3 Action of $Q^{(23)}$

We have

$$\begin{aligned}
Q^{(23)}\circ\mathcal{O}_1 &= \mathcal{O}_1\circ(2v\partial_v+2\nu_{23}-1/2)+\mathcal{O}_2\circ(2v\partial_v+4\nu_3),\\
Q^{(23)}\circ\mathcal{O}_2 &= -\,\mathcal{O}_1\circ(2v\partial_u)-\mathcal{O}_2\circ(2v\partial_v+2\nu_{23}+1/2),\\
Q^{(23)}\circ\mathcal{E}_1 &= -\,\mathcal{E}_1\circ(2v\partial_v+2\nu_{23}+1/2)+\mathcal{E}_2\circ(2v),\\
Q^{(23)}\circ\mathcal{E}_2 &= -\,\mathcal{E}_1\circ(2v\partial_u\partial_v+4\nu_3\partial_u)+\mathcal{E}_2\circ(2v\partial_v+2\nu_{23}-1/2).
\end{aligned}$$

6 Diagonalization of $Q^{(123)}$ and $Q^{(12)}$ in the odd subspace

6.1 Spectral equations for $Q^{(123)}$

Consider an odd element

$$f(\mathbf{x};\theta)=\mathcal{O}_1(h(u,v))+\mathcal{O}_2(g(u,v))\in\ker A_-^{(123)}, \tag{6.1}$$

where $\mathbf{x}=(x_1,x_2,x_3)$ and $\theta=(\theta_1,\theta_2,\theta_3)$. Then the spectral equation

$$Q^{(123)}f(\mathbf{x};\theta)=\mu f(\mathbf{x};\theta) \tag{6.2}$$

is equivalent to the equations

$$\begin{aligned}
uh_u+vh_v &= -(\nu_{123}+\mu/2+1/4)h,\\
ug_u+vg_v &= -(\nu_{123}+\mu/2+1/4)g,
\end{aligned}$$

where $\nu_{123}=\nu_1+\nu_2+\nu_3$. The last two equations are satisfied if and only if h and g are homogeneous in u and v of the same degree N, where

$$N=-(\nu_{123}+\mu/2+1/4),$$

or equivalently

$$\mu = -(2N + 2\nu_{123} + 1/2).$$

Since $Q^{(123)}$ commutes with each of the operators $Q^{(12)}$, $Q^{(13)}$, $Q^{(23)}$, we can look at their restrictions on the space of solutions of (6.2).

6.2 Spectral equations for $Q^{(12)}$

Consider now the equation

$$Q^{(12)} f(\mathbf{x};\theta) = \lambda f(\mathbf{x};\theta), \tag{6.3}$$

where $\lambda \in \mathbb{C}$ and $f(\mathbf{x};\theta)$ is the element in $\ker A_{-}^{(123)}$ in (6.1). The coefficient of θ_2 shows that

$$(1 + 2\lambda + 4\nu_1 - 4\nu_2)g = (1 + 2\lambda + 4\nu_{12})h + 4uh_u - 4uh_v. \tag{6.4}$$

It is straightforward to check that if g and h are homogeneous polynomials in u and v of the same degree satisfying (6.3), then $1 + 2\lambda + 4\nu_1 - 4\nu_2$ can be zero only when $\nu_1 = \nu_2 = 0$. Thus for generic ν_1, ν_2 we can assume that $1 + 2\lambda + 4\nu_1 - 4\nu_2 \neq 0$ and therefore (6.4) determines g uniquely from h as follows:

$$g = \frac{(1 + 2\lambda + 4\nu_{12})h + 4uh_u - 4uh_v}{1 + 2\lambda + 4\nu_1 - 4\nu_2}. \tag{6.5}$$

Substituting the last formula into (6.3), we see that (6.3) holds if and only if $h(u, v)$ satisfies the equation

$$\Big(u^2\partial_u^2 - u^2\partial_u\partial_v + (2\nu_{12} + 1)u\partial_u - (2\nu_1 + 1)u\partial_v + (\nu_{12}^2 - (2\lambda + 1)^2/16)\Big)h = 0. \tag{6.6}$$

If $h(u, v)$ is homogeneous of degree N we can write it as

$$h(u, v) = u^N \phi(v/u), \tag{6.7}$$

and substituting the last equation in (6.6), we obtain the following equation for $\phi(z)$:

$$\Big(z(1 + z)\partial_z^2 - ((2\nu_{12} + 2N - 1)z + 2\nu_1 + N)\partial_z + (N + \nu_{12})^2 - (2\lambda + 1)^2/16\Big)\phi(z) = 0. \tag{6.8}$$

If the last equation has a solution which is a polynomial of degree $k \leq N$, then the coefficient of z^k yields

$$(N-k+\nu_{12})^2 = \frac{(2\lambda+1)^2}{16}.$$

This leads to

$$\frac{2\lambda+1}{4} = \pm(N-k+\nu_{12}), \qquad \text{or equivalently} \quad \lambda = \pm 2(N-k+\nu_{12}) - \frac{1}{2} \tag{6.9}$$

and for these values of λ, Eq. (6.8) reduces to

$$\Big(z(1+z)\partial_z^2 - ((2\nu_{12}+2N-1)z + 2\nu_1 + N)\partial_z + k(2N+2\nu_{12}-k)\Big)\phi(z) = 0. \tag{6.10}$$

Note that this equation is the hypergeometric differential equation with three regular singular points: 0, -1 and ∞. For generic parameters ν_1, ν_2, up to a constant factor, it has a unique polynomial solution of degree k given in terms of the Jacobi polynomial by

$$\phi(z) = P_k^{(-N-2\nu_1-1,-N-2\nu_2)}(1+2z). \tag{6.11}$$

Let us recall that the Jacobi polynomials are expressed as follows in terms of the hypergeometric functions

$$P_k^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_k}{k!}\,{}_2F_1\left(\begin{matrix}-k,\ k+\alpha+\beta+1\\ \alpha+1\end{matrix};\ \frac{1-x}{2}\right). \tag{6.12}$$

Summarizing the above computations we obtain the following theorem describing the common eigenfunctions of the operators $Q^{(123)}$ and $Q^{(12)}$ which can be simultaneously diagonalized on $\ker(A_-^{(123)}) \cap \mathbb{V}^{\mathrm{o}}$.

Theorem 6.1 *For $N \in \mathbb{N}_0$ and $k \in \{0, \ldots, N\}$, let*

$$h_{k,N}(u,v) = u^N P_k^{(-N-2\nu_1-1,-N-2\nu_2)}\left(1+2\frac{v}{u}\right), \tag{6.13a}$$

$$g_{k,N}^+(u,v) = \frac{2N+2\nu_{12}-k}{N+2\nu_1-k}\, u^N P_k^{(-N-2\nu_1,-N-2\nu_2)}\left(1+2\frac{v}{u}\right), \tag{6.13b}$$

$$g_{k,N}^-(u,v) = -u^N P_{k-1}^{(-N-2\nu_1,-N-2\nu_2)}\left(1+2\frac{v}{u}\right). \tag{6.13c}$$

Then

$$\left\{f_{k,N}^+(\mathbf{x};\theta),\ f_{k,N}^-(\mathbf{x};\theta) : N \in \mathbb{N}_0, 0 \leq k \leq N\right\}$$

where

$$f_{k,N}^{\pm}(\mathbf{x};\theta)=\mathcal{O}_1(h_{k,N}(u,v))+\mathcal{O}_2(g_{k,N}^{\pm}(u,v)),$$

form a basis of $\ker(A_-^{(123)})\cap\mathbb{V}^o$ *and*

$$Q^{(123)}\,f_{k,N}^{\pm}(\mathbf{x};\theta)=-(2N+2\nu_{123}+1/2)\,f_{k,N}^{\pm}(\mathbf{x};\theta), \tag{6.14a}$$

$$Q^{(12)}\,f_{k,N}^{\pm}(\mathbf{x};\theta)=(\pm 2(N-k+\nu_{12})-1/2)\,f_{k,N}^{\pm}(\mathbf{x};\theta). \tag{6.14b}$$

Proof The result for $h_{k,N}(u,v)$ follows directly from the discussion preceding the theorem. Using this result for $h_{k,N}(u,v)$ and the eigenvalues $\lambda=\pm 2(N-k+\nu_{12})-1/2$ in (6.5), one gets the following expressions for $g_{k,N}^{\pm}(u,v)$:

$$g_{k,N}^{+}(u,v)=\frac{(N-k+2\nu_{12})h_{k,N}(u,v)+u(\partial_u h_{k,N}(u,v)-\partial_v h_{k,N}(u,v))}{N-k+2\nu_1},$$

$$g_{k,N}^{-}(u,v)=\frac{(N-k)h_{k,N}(u,v)-u(\partial_u h_{k,N}(u,v)-\partial_v h_{k,N}(u,v))}{N-k+2\nu_2}.$$

Note that

$$\text{if}\quad h=u^N\phi(v/u),\quad \text{then}\quad (u\partial_u-u\partial_v)h=u^N[(N-(t+1)\partial_t)\phi(t)]|_{t=v/u}.$$

Using the above formula and the identity

$$(-k+(t+1)\partial_t)\,P_k^{(\alpha,\beta)}(1+2t)=(k+\beta)\,P_{k-1}^{(\alpha+1,\beta)}(1+2t)$$

we obtain the explicit formulas (6.13b)–(6.13c) for $g_{k,N}^{\pm}(u,v)$. □

Using the fact that the intermediate Casimir operators satisfy the Bannai–Ito algebra and knowing the spectrum of $Q^{(12)}$ and $Q^{(123)}$, one can deduce that the action of $Q^{(23)}$ is tridiagonal as follows:

$$Q^{(23)}\,f_{k,N}^{+}(\mathbf{x};\theta)=\alpha_k^{+}\,f_{k,N}^{-}(\mathbf{x};\theta)+\beta_k^{+}\,f_{k,N}^{+}(\mathbf{x};\theta)+\gamma_k^{+}\,f_{k+1,N}^{-}(\mathbf{x};\theta), \tag{6.15a}$$

$$Q^{(23)}\,f_{k,N}^{-}(\mathbf{x};\theta)=\alpha_{k-1}^{-}\,f_{k-1,N}^{+}(\mathbf{x};\theta)+\beta_k^{-}\,f_{k,N}^{-}(\mathbf{x};\theta)+\gamma_k^{-}\,f_{k,N}^{+}(\mathbf{x};\theta). \tag{6.15b}$$

The coefficients are also constrained by the algebra as follows:

$$\alpha_k^{-}\gamma_k^{+}=\frac{4(k+1)(N-k)(2\nu_{12}+2N-k)(2\nu_{12}+N-k-1)}{(2\nu_{12}+2N-2k-1)^2}, \tag{6.16}$$

$$\alpha_k^{+}\gamma_k^{-}=\frac{(2\nu_1+N-k)(2\nu_2+N-k)(2\nu_3+k)(2\nu_{123}+2N-k)}{(\nu_{12}+N-k)^2}, \tag{6.17}$$

$$\beta_k^{\pm}=\pm\frac{(\nu_1-\nu_2)(\nu_{12}+2\nu_3+N)}{\nu_{12}+N-k}\mp\frac{(2\nu_{12}-1)(2\nu_{12}+2N+1)}{2(2\nu_{12}+2N-2k\mp 1)}. \tag{6.18}$$

The exact expressions of these coefficients are given in the following corollary.

Corollary 6.2 *The coefficients in relations* (6.15) *are*

$$\alpha_k^+ = \frac{(2\nu_2+N-k)(2\nu_{123}+2N-k)}{\nu_{12}+N-k},$$
$$\alpha_k^- = \frac{2(2\nu_1+N-k+1)(2\nu_{12}+N-k)}{2\nu_{12}+2N-2k+1},$$
$$\gamma_k^+ = \frac{2(N-k)(k+1)(2\nu_{12}+2N-k)}{(2\nu_1+N-k)(2\nu_{12}+2N-2k-1)},$$
$$\gamma_k^- = \frac{(2\nu_3+k)(2\nu_1+N-k)}{\nu_{12}+N-k}.$$

Proof The coefficients of θ_1 in the relations (6.15), computed by using the explicit formulas for $f_{k,N}^{\pm}(\mathbf{x};\theta)$, provide constraints between the coefficients $\alpha^{\pm}$, $\beta^{\pm}$ and $\gamma^{\pm}$. Combing these constraints with the expression (6.18) of $\beta_k^{\pm}$, one gets the expressions in the corollary. □

The values of these coefficients are compatible with the relations (6.16)–(6.17).

7 Diagonalization of $Q^{(123)}$ and $Q^{(12)}$ in the even subspace

The computations for the even subspace are similar to the ones in the previous section by considering the following even element

$$f(\mathbf{x};\theta) = \mathcal{E}_1(h(u,v)) + \mathcal{E}_2(g(u,v)) \in \ker A_-^{(123)}, \tag{7.1}$$

instead of the odd element (6.1). We formulate the main results for this case omitting the proofs. The analogue of Theorem 6.1 can be stated as follows:

Theorem 7.1 *For* $N \in \mathbb{N}_0$ *and* $k \in \{0,\dots,N\}$*, let*

$$g_{k,N}(u,v) = u^N P_k^{(-N-2\nu_1,-N-2\nu_2)}\left(1+2\frac{v}{u}\right), \tag{7.2a}$$
$$h_{k,N}^+(u,v) = u^{N-1}(N+2\nu_1-k)P_{k-1}^{(-N-2\nu_1,-N-2\nu_2+1)}\left(1+2\frac{v}{u}\right), \tag{7.2b}$$
$$h_{k,N}^-(u,v) = (k+1-2N-2\nu_{12})\,u^{N-1}P_k^{(-N-2\nu_1,-N-2\nu_2+1)}\left(1+2\frac{v}{u}\right). \tag{7.2c}$$

Then

$$\{f_{k,N}^+(\mathbf{x};\theta) : N\in\mathbb{N}_0,\ 0\le k\ \le N\}\cup\{f_{k,N}^-(\mathbf{x};\theta) : N\in\mathbb{N}_0,\ 0\le k< N\}$$

where

$$f_{k,N}^{\pm}(\mathbf{x};\theta) = \mathcal{E}_1(h_{k,N}^{\pm}(u,v)) + \mathcal{E}_2(g_{k,N}(u,v)),$$

form a basis of $\ker(A_-^{(123)}) \cap \mathbb{V}^e$ *and*

$$Q^{(123)}\, f_{k,N}^{\pm}(\mathbf{x};\theta) = (2N + 2\nu_{123} - 1/2)\, f_{k,N}^{\pm}(\mathbf{x};\theta), \tag{7.3a}$$

$$Q^{(12)}\, f_{k,N}^{\pm}(\mathbf{x};\theta) = (\pm 2(N - k + \nu_{12} - 1/2) + 1/2)\, f_{k,N}^{\pm}(\mathbf{x};\theta). \tag{7.3b}$$

The action of $Q^{(23)}$ is also tridiagonal as follows:

$$Q^{(23)}\, f_{k,N}^{+}(\mathbf{x};\theta) = \alpha_k^{+}\, f_{k-1,N}^{-}(\mathbf{x};\theta) + \beta_k^{+}\, f_{k,N}^{+}(\mathbf{x};\theta) + \gamma_k^{+}\, f_{k,N}^{-}(\mathbf{x};\theta), \tag{7.4a}$$

$$Q^{(23)}\, f_{k,N}^{-}(\mathbf{x};\theta) = \alpha_k^{-}\, f_{k,N}^{+}(\mathbf{x};\theta) + \beta_{k+1}^{-}\, f_{k,N}^{-}(\mathbf{x};\theta) + \gamma_{k+1}^{-}\, f_{k+1,N}^{+}(\mathbf{x};\theta), \tag{7.4b}$$

with

$$\alpha_k^{+} = \frac{(2\nu_1 + N - k)(2\nu_2 + N - k)}{\nu_{12} + N - k},$$

$$\alpha_k^{-} = \frac{2(2\nu_{12} + N - k - 1)(2\nu_{123} + 2N - k - 1)}{2\nu_{12} + 2N - 2k - 1},$$

$$\gamma_k^{+} = \frac{2(2\nu_3 + k)(N - k)}{2\nu_{12} + 2N - 2k - 1},$$

$$\gamma_k^{-} = \frac{k(2\nu_{12} + 2N - k)}{\nu_{12} + N - k}.$$

8 Conclusion

Summing up, we have provided a model for the Bannai–Ito algebra on a superspace with $\mathbb{C}^2$ as body and with soul generated by three anticommuting Grassmann variables. The even and odd basis vectors were found to each have two components and to be realized in terms of Jacobi polynomials. The tridiagonal action of one Bannai–Ito generator in the eigenbasis of the other was explicitly calculated. The formulas thus obtained reflect contiguous relations of the Jacobi polynomials.

It is worth mentioning that the Bannai–Ito algebra is isomorphic to the degenerate double affine Hecke algebra (DAHA) [17] which has thus been modelled here in superspace by the same token.

For the sake of completeness, let us mention that an embedding of the Bannai–Ito algebra into $\mathfrak{osp}(1|2)$ was presented in [18] where an analytic realization of $\mathfrak{osp}(1|2)$ in terms of Dunkl operators was used to connect to some -1 - polynomials. It should be stressed that the Bannai–Ito model that results from combining the realization of $\mathfrak{osp}(1|2)$ used throughout in this paper with this embedding is clearly not the one that is attained from the dimensional reduction performed here.

Finally, it would definitely be of interest to develop along the lines pursued here realizations on superspaces of the higher rank Bannai–Ito algebras that have been constructed from the intermediate Casimir operators arising in manifold tensor products of $\mathfrak{osp}(1|2)$ [19].

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Miller, W.: Lie theory and special functions, vol. 43. Academic Press, New York (1968)
2. De Bie, H., Vincent, G.X., Tsujimoto, S., Vinet, L., Zhedanov, A.: The Bannai-Ito algebra and some applications. J. Phys: Conf. Ser. **597**, 012001 (2015)
3. Bannai, E., Bannai, E., Ito, T., Tanaka, R.: Algebraic combinatorics. De Gruyter, In Algebraic Combinatorics (2021)
4. Tsujimoto, S., Vinet, L., Zhedanov, A.: Dunkl shift operators and Bannai–Ito polynomials. Adv. Math. **229**(4), 2123–2158 (2012)
5. Koekoek, R, Lesky, P.A., Swarttouw, R.F.: Hypergeometric orthogonal polynomials and their *q-analogues*. Springer Monographs in Mathematics. Springer, Berlin. With a foreword by Tom H. Koornwinder (2010)
6. Genest, V.X., Vinet, L., Zhedanov, A.: The Racah algebra and superintegrable models. J. Phys.: Conf. Series **512**, 012011 (2014)
7. Genest, V.X., Vinet, L., Zhedanov, A.: Embeddings of the Racah algebra into the Bannai–Ito algebra. SIGMA Symm. Integr. Geomet.: Methods Appl. **11**, 1050 (2015)
8. Gao, S., Wang, Y., Hou, B.: The classification of Leonard triples of Racah type. Linear Algebra Appl. **439**(7), 1834–1861 (2013)
9. Perelomov, A.: Generalized coherent states and their applications. Springer, Berlin (2012)
10. Genest, V.X., Vinet, L., Zhedanov, A.: Superintegrability in two dimensions and the Racah–Wilson algebra. Lett. Math. Phys. **104**, 931–952 (2014)
11. Genest, V.X., Vinet, L., Zhedanov, A.: The equitable Racah algebra from three algebras. J. Phys. A: Math. Theoret. **47**(2), 025203 (2013)
12. De Bie, H., Genest, V.X., van de Vijver, W., Vinet, L.: A higher rank Racah algebra and the Laplace-Dunkl operator. J. Phys. A: Math. Theor. **51**(2), 025203 (2017)
13. De Bie, H., Iliev, P., Vinet, L.: Bargmann and Barut-Girardello models for the Racah algebra. J. Math. Phys. **60**(1), 011701 (2019)
14. Genest, V., Vinet, L., Zhedanov, A.: The Bannai-Ito polynomials as Racah coefficients of the $\mathfrak{sl}_{-1}(2)$ algebra. Proceed. Am. Math. Soci. **142**(5), 1545–1560 (2014)
15. DeWitt, B.: Supermanifolds. Cambridge University Press, Cambridge (1992)
16. Frappat, L., Sciarrino, A., Sorba, P.: Dictionary on Lie algebras and superalgebras. pages xxii+410,: With 1 CD-ROM. Windows, Macintosh and UNIX (2000)
17. Genest, V., Vinet, L., Zhedanov, A.: The non-symmetric Wilson polynomials are the Bannai–Ito polynomials. Proceed. Am. Math. Soci. **144**(12), 5217–5226 (2016)
18. Baseilhac, P., Genest, V.X., Vinet, L., Zhedanov, A.: An embedding of the Bannai-Ito algebra in $\mathcal{U}(\mathfrak{osp}(1,2)))$ and -1-polynomials. Lett. Math. Phys. **108**, 1623–1634 (2018)
19. De Bie, H., Genest, V.X., Vinet, L.: The $\mathbb{Z}_2^n$ Dirac-Dunkl operator and a higher rank Bannai-Ito algebra. Adv. Math. **303**, 390–414 (2016)