



Summing free unitary Brownian motions with applications to quantum information

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Abstract

Motivated by quantum information theory, we introduce a dynamical random density matrix built out of the sum of $k \geq 2$ independent unitary Brownian motions. In the large size limit, its spectral distribution equals, up to a normalising factor, that of the free Jacobi process associated with a single self-adjoint projection with trace $1/k$. Using free stochastic calculus, we extend this equality to the radial part of the free average of k free unitary Brownian motions and to the free Jacobi process associated with two self-adjoint projections with trace $1/k$, provided the initial distributions coincide. In the single projection case, we derive a binomial-type expansion of the moments of the free Jacobi process which extends to any $k \geq 3$ the one derived in Demni et al. (Indiana Univ Math J 61:1351–1368, 2012) in the special case $k = 2$. Doing so give rise to a non normal (except for $k = 2$) operator arising from the splitting of a self-adjoint projection into the convex sum of k unitary operators. This binomial expansion is then used to derive a pde satisfied by the moment generating function of this non normal operator and for which we determine the corresponding characteristic curves. As an application of our results, we compute the average purity and the entanglement entropy of the large-size limiting density matrix.

Keywords Bell states · Reduced density matrix · Unitary Brownian motion · Free Jacobi process · Partial differential equation

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1 Introduction and motivation

1.1 Random matrices in quantum information theory

Randomness lies at the heart of Shannon's pioneering work on classical information theory (see the expository paper [29]). It also plays a key role in quantum information theory through the use of techniques from random matrix theory. Actually, the latter open the way to choose typical random subspaces in large-size quantum systems which violate additivity conjectures for minimum output Rényi and von Neumann entropies (see [5] and references therein). Here, typicality is taken with respect to the uniform measure in the compact complex Grassmann manifold or equivalently with respect to the Haar distribution in the group of unitary matrices. Note that this distribution together with Ginibre random matrices also served in [6] to generate random density matrices induced from states in bipartite systems (see [22] for similar constructions of quantum channels).

A natural dynamical version of the Haar distribution in the group of unitary matrices is the so-called unitary Brownian motion [24]. This stochastic process was used in [27] where the authors introduced and studied a random state drawn from the Brownian motion on the complex projective space (the row vector of a unitary Brownian motion up to a phase). There, the main problem was to write explicitly the joint distributions of tuples formed by the moduli of the state coordinates. This problem was entirely solved in [8] using spherical harmonics in the unitary group. To the best of our knowledge, [8, 27] are the only papers where the unitary Brownian motion is used as a random model in quantum information theory, in contrast to the high occurrence of Haar-distributed unitary matrices [5]. Moreover, it is tempting and challenging as well to prove finite-time analogues of important results in quantum information theory proved using Haar unitary matrices and their Weingarten Calculus (as summarized in [5]).

In this paper, we appeal once more to the unitary Brownian motion in order to introduce a stochastic process valued in the space of density matrices [see (1) below]. The large time limit of this process was already constructed in [6] by partially tracing a pure random state in a bipartite quantum system.

1.2 The dynamical density matrix

Let $N \geq 1$ be a positive integer and consider a bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A, \mathcal{H}_B$, are complex N -dimensional Hilbert spaces. If $(e_j^A)_{j=1}^N, (e_j^B)_{j=1}^N$, are the canonical basis of \mathcal{H}_A and \mathcal{H}_B respectively, then

$$\psi := \frac{1}{\sqrt{N}} \sum_{j=1}^N e_j^A \otimes e_j^B$$

is referred to as the Bell or maximally-entangled state. Now, consider $k \geq 2$ Haar-distributed unitary matrices $U_\infty^1(N), \dots, U_\infty^k(N)$, and define the vector $\psi^k \in \mathcal{H}_A \otimes \mathcal{H}_B$ by:

$$\psi^k(N) := \frac{1}{\sqrt{N}} \sum_{m=1}^k \sum_{j=1}^N \left(U_\infty^m(N) e_j^A \right) \otimes e_j^B = \frac{1}{\sqrt{N}} \sum_{j=1}^N \left(\sum_{m=1}^k U_\infty^m(N) e_j^A \right) \otimes e_j^B.$$

Then the partial trace with respect to \mathcal{H}_B of the pure state associated with $\psi^k(N)$ yields the following reduced state:

$$\tilde{W}_\infty^k(N) := \frac{(U_\infty^1(N) + \dots + U_\infty^k(N))(U_\infty^1(N) + \dots + U_\infty^k(N))^*}{\text{tr}[(U_\infty^1(N) + \dots + U_\infty^k(N))(U_\infty^1(N) + \dots + U_\infty^k(N))^*]}$$

where tr is the trace operator on the space $\mathbb{M}_N(\mathbb{C})$ of $N \times N$ complex matrices. Since the Haar distribution is the stationary distribution of the unitary Brownian motion, it is then natural to introduce the following stochastic process valued in the space of density matrices:

$$0 \leq t \mapsto \tilde{W}_t^k(N) := \frac{(U_t^1(N) + \dots + U_t^k(N))(U_t^1(N) + \dots + U_t^k(N))^*}{\text{tr}[(U_t^1(N) + \dots + U_t^k(N))(U_t^1(N) + \dots + U_t^k(N))^*]}, \tag{1}$$

where $(U_t^j(N))_{t \geq 0}, 1 \leq j \leq k$, are k independent unitary Brownian motions. In particular, $(\tilde{W}_t^k(N))_{t \geq 0}$ interpolates between the completely mixed state Id_N/N at $t = 0$ (Id_N being the identity matrix of size N) and the stationary state $\tilde{W}_\infty^k(N)$. Note also that since $(U_t^j(N))_{t \geq 0}, 1 \leq j \leq k$, are Lévy processes in the unitary group with identical distributions, then

$$\begin{aligned} \mathbb{E} \left\{ \text{tr}[(U_t^1(N) + \dots + U_t^k(N))(U_t^1(N) + \dots + U_t^k(N))^*] \right\} &= kN + k(k-1)\mathbb{E}[\text{tr}(U_{2t}^1)] \\ &= kN[1 + (k-1)e^{-Nt}], \end{aligned}$$

where the second equality follows from [23], Example 3.3.

As we shall now explain, introducing this model is not simply a matter of replacing Haar-distributed matrices by unitary Brownian motions. Indeed, the large-size limit of $\tilde{W}_t^2(N)$ for fixed time t bears a close connection to an instance of the so-called free Jacobi process.

1.3 The large size limit of $\tilde{W}_\infty^k(N)$ and the free Jacobi process

Recall that independent random matrices behave in the large-size limit, under additional law-invariance assumptions, as \star -free operators (in Voiculescu’s sense) in a tracial non commutative probability space, say (\mathcal{A}, τ) [26]. For instance, independent Haar-distributed unitary matrices converge strongly and almost surely as $N \rightarrow \infty$ to Haar-distributed unitary operators (see [4] and references therein). Consequently, the norm of the operator $\tilde{W}_\infty^k(N)$ converges almost surely as $N \rightarrow \infty$ to

$$\tilde{W}_\infty^k := \frac{(U_\infty^1 + \dots + U_\infty^k)(U_\infty^1 + \dots + U_\infty^k)^\star}{\tau[(U_\infty^1 + \dots + U_\infty^k)(U_\infty^1 + \dots + U_\infty^k)^\star]}$$

where $\{U_\infty^j, 1 \leq j \leq k\}$ is a k -tuple of Haar unitary operators which are \star -free in (\mathcal{A}, τ) . Note that

$$\tilde{W}_\infty^k = \frac{(U_\infty^1 + \dots + U_\infty^k)(U_\infty^1 + \dots + U_\infty^k)^\star}{k},$$

since $\tau(U_\infty^j) = 0$ and since \star -freeness entails $\tau(U_\infty^j(U_\infty^m)^\star) = \tau(U_\infty^j)\tau((U_\infty^m)^\star) = 0$ for any $1 \leq j \neq m \leq k$. In particular, when $k = 2$, the invariance of the Haar distribution shows further that $\tilde{W}_\infty^2/2$ is equally distributed as:

$$\frac{(\mathbf{1} + U_\infty^1)(\mathbf{1} + U_\infty^1)^\star}{4} = \frac{2\mathbf{1} + U_\infty^1 + (U_\infty^1)^\star}{4},$$

where $\mathbf{1}$ stands for the unit of \mathcal{A} . The spectral distribution of this Hermitian operator is known to be the arcsine distribution [18] and coincides also with an instance of the stationary (i.e. $t = +\infty$) distribution of the so-called free Jacobi process [7]. At any time $t > 0$, the latter is the squared radial part of the compression PU_tQ of a free unitary Brownian motion $(U_s)_{s \geq 0}$ by two orthogonal (self-adjoint) projections P and Q in (\mathcal{A}, τ) which are \star -free from $(U_s)_{s \geq 0}$ [7]. Besides, it was shown in [12] that the coincidence alluded to above holds even at any time $t > 0$: if $P = Q$ and if $\tau(P) = 1/2$ then the spectral distribution of the free Jacobi process $PU_tPU_t^\star P$ (in the compressed algebra $(\mathcal{A}, 2\tau)$) coincides with that of

$$\frac{2\mathbf{1} + U_{2t} + U_{2t}^\star}{4} \in \mathcal{A}. \tag{2}$$

Since $(U_s)_{s \geq 0}$ is a unitary free Lévy process with respect to the free multiplicative convolution on the unit circle [18], then the spectral distribution of (2) coincides with

that of

$$\frac{(U_t^1 + U_t^2)(U_t^1 + U_t^2)^*}{4}, \tag{3}$$

where $(U_s^1)_{s \geq 0}$ and $(U_s^2)_{s \geq 0}$ are two free copies of $(U_s)_{s \geq 0}$.

On the other hand, it was proved in [1] that $(U_s)_{s \geq 0}$ is the large-size limit of the time-rescaled Brownian motion in the group of unitary matrices. As a matter of fact, (3) is the large-size limit $N \rightarrow \infty$ of

$$\frac{(U_{t/N}^1(N) + U_{t/N}^2(N))(U_{t/N}^1(N) + U_{t/N}^2(N))^*}{4}.$$

Up to a scalar random factor, this Hermitian random matrix is nothing else but $(1/2)\tilde{W}_{t/N}^2(N)$ which converges almost surely and strongly to

$$\frac{(U_t^1 + U_t^2)(U_t^1 + U_t^2)^*}{2[2 + \tau(U_t^1(U_t^1)^*) + \tau(U_t^2(U_t^2)^*)]} = \frac{(U_t^1 + U_t^2)(U_t^1 + U_t^2)^*}{4(1 + e^{-t})}, \tag{4}$$

where the second equality follows from $\tau(U_t^1(U_t^1)^*) = \tau(U_t^1)\tau((U_t^1)^*) = e^{-t}$ [1]. In a nutshell, the free Jacobi process associated with an orthogonal projection with trace $1/2$ is, up to a normalising factor, the large-size limit of $\tilde{W}_{t/N}^2(N)$.

The above picture extends to any integer $k \geq 2$ as follows. On the one hand, $\tilde{W}_{t/N}^k(N)$ converges strongly and almost surely as $N \rightarrow \infty$ to the self-adjoint and unit-trace operator [4]:

$$\tilde{W}_t^k := \frac{(U_t^1 + \dots + U_t^k)(U_t^1 + \dots + U_t^k)^*}{\tau[(U_t^1 + \dots + U_t^k)(U_t^1 + \dots + U_t^k)^*]} = \frac{(U_t^1 + \dots + U_t^k)(U_t^1 + \dots + U_t^k)^*}{k[1 + (k - 1)e^{-t}]},$$

where $(U_s^j)_{s \geq 0}$, $1 \leq j \leq k$ are free copies of $(U_s)_{s \geq 0}$ in (\mathcal{A}, τ) . In particular, the following diagram commute:

$$\begin{array}{ccc} \tilde{W}_{t/N}^k(N) & t \longrightarrow \infty & \tilde{W}_\infty^k(N) \\ N \rightarrow \infty \downarrow & & \downarrow N \rightarrow \infty \\ \tilde{W}_{t/N}^k & t \longrightarrow \infty & \tilde{W}_\infty^k \end{array}$$

On the other hand, if

$$G_t^k := U_t^1 + \dots + U_t^k, \quad t \geq 0,$$

and if $\tau(P) = 1/k$, then Nica and Speicher’s boxed convolution [28] implies that the \star -moments of G_t^k in (\mathcal{A}, τ) coincide with those of PU_tP in the compressed space

$(P\mathcal{A}P, k\tau)$. Consequently, their corresponding Brown measures coincide [26] and so do the spectral distributions of their squared radial parts

$$\frac{W_t^k}{k^2} := \frac{G_t^k(G_t^k)^*}{k^2} = \frac{(U_t^1 + \dots + U_t^k)}{k} \frac{(U_t^1 + \dots + U_t^k)^*}{k} \in (\mathcal{A}, \tau),$$

and

$$(PU_tP)(PU_tP)^* = PU_tPU_t^*P \in (P\mathcal{A}P, k\tau).$$

1.4 Main results

The choice $P = Q$ is not a restriction and is rather a matter of simplicity. Indeed, since we are dealing with time dynamics (of Burgers-type) instead of stationary ($t = +\infty$) regimes, we have to match initial data at $t = 0$ in order to obtain equalities between spectral distributions at any fixed $t > 0$. For instance, the operator displayed in (2) and $PU_tQU_t^*P$ share the same spectral distribution in their corresponding probability spaces when $\tau(P) = \tau(Q) = 1/2$ and provided that their moment sequences at $t = 0$ coincide [20]. More generally, we shall prove using free stochastic calculus that for any $t > 0$, the moment sequences of W_t^k/k^2 in (\mathcal{A}, τ) and of $PU_tQU_t^*P$ in $(P\mathcal{A}P, k\tau)$ satisfy the same recurrence relation when $\tau(P) = \tau(Q) = 1/k, k \geq 2$. We shall also prove that the moment sequence of W_t^k/k^2 converge as $k \rightarrow \infty$ to $(e^{-nt})_{n \geq 0}$ for any fixed time t , which contrasts the weak convergence of W_∞^k/k to the Marchenko–Pastur distribution [6]. This contrast is due to the high complexity of the structure of the \star -cumulants of U_t in comparison with those of U_∞ [9].

Back to the case $P = Q$, the equality between the spectral distributions of W_t^k/k^2 and of $PU_tPU_t^*P$ under the assumption $\tau(P) = 1/k$ opens the way to compute the moments of the former by studying those of the latter. Indeed, for any $n \geq 1, \tau[(W_t^k)^n]$ is a linear combination of k^{2n} factors of the form

$$\tau[U_t^{i_1}(U_t^{i_2})^* \dots (U_t^{i_{2m-1}})^* U_t^{i_{2m}}], \quad 1 \leq m \leq n, \quad i_j \in \{1, \dots, k\}.$$

Apart from constant factors, those where any index i_j occurs at most once may be computed using the multiplicative Lévy property of the free unitary Brownian motions. However, to the best of our knowledge, the contributions of the remaining factors may be only computed using the freeness property. In this respect, the complexity of $\tau[(W_t^k)^n]$ increase rapidly even for small orders. For that reason, we rather focus on the moments of $PU_tPU_t^*P$ and our main result (Theorem 2 below) establishes for any $n \geq 1$ a binomial-type expansion of

$$k\tau[(PU_tPU_t^*P)^n]$$

as a linear combination of the moments

$$\tau[(T_k U_t T_k U_t^*)^j], \quad 0 \leq j \leq n,$$

where $T_k := kP - \mathbf{1} = T_k^*$ satisfies $\tau(T_k) = 0$. This expansion extends to any integer $k \geq 3$ the one proved in [12] for $k = 2$ for which $T_2 = 2P - \mathbf{1}$ is unitary and self-adjoint, which in turn implies that $T_2 U_t T_2 U_t^*$ is distributed as U_{2t} . However, for any $k \geq 3$, T_k is not even normal: it is the sum of $(k - 1)$ unitary operators and satisfies the relation

$$(T_k)^2 = (k - 2)T_k + (k - 1)\mathbf{1}.$$

Of course, the constant term corresponding to $j = 0$ in the obtained binomial-type expansion is nothing else but the n th moment of the spectral distribution of $PU_\infty PU_\infty^* P$, and may be expressed for instance as a weighted sum of Catalan numbers. Surprisingly, the higher order coefficients split as

$$\frac{k(k - 1)^{n-j}}{k^{2n}} \binom{2n}{n - j}, \quad 1 \leq j \leq n,$$

and are derived after a careful and tricky analysis of several inductive relations. Nonetheless, it would be interesting to seek a combinatorial proof explaining both the splitting of these higher order coefficients and the occurrence of the same binomial coefficients as in the $k = 2$ case.

Once the binomial-type expansion derived, we turn it into a relation between the moment generating functions of the free Jacobi process at time $t > 0$ and of $T_k U_t T_k U_t^*$. Using the partial differential equation (hereafter pde) satisfied by the former and derived in [7] for arbitrary traces $\tau(P)$ and $\tau(Q)$, we derive the one satisfied by the latter and determine its characteristic curves.

At the quantum information theoretical side, our results allow to compute large-size asymptotics of relevant quantities such as the average purity and the entanglement entropy of the dynamical density matrix $\tilde{W}_{t/N}^k(N)$. In the stationary regime $t = +\infty$, the former was computed in [6] (see section III.C. there) and amounts to compute the second moment of the spectral distribution of \tilde{W}_∞^k/k . For any fixed time $t > 0$, the second moment $\tau[(\tilde{W}_t^k)^2]$ is readily computed from Theorem 2 together with the freeness property. As a by-product, we recover the large-time limit of the asymptotic average purity for fixed k and we derive its large- k limit at any fixed time $t > 0$. Still in the large-size limit, we derive formulas for the entanglement entropy both in the stationary and in fixed time regimes. Both expressions involve Gauss hypergeometric polynomials and the second one expresses the difference between the entanglement entropy in both regimes as an infinite series of the moments $\tau[(T_k U_t T_k U_t^*)^j]$, $j \geq 0$.

The paper is organized as follows. In the next section, we discuss the relation between the \star -moments of $(G_t^k)/k$ and those of the compression $PU_t P$ when $\tau(P) = 1/k$. There, we also prove that the moment sequences of the radial parts $(W_t^k)/k^2$ and of $PU_t Q$ with $\tau(P) = \tau(Q) = 1/k$ satisfy the same recurrence relation and that their limits as $k \rightarrow \infty$ is the Dirac mass at e^{-t} . In the third section, we prove the binomial-type formula for the moments of J_t then turn it into a relation between moment generating functions. Once we do, we deduce a pde for the moment generating function of $T_k U_t T_k U_t^*$ and determine its characteristic curves. The fourth section is devoted to formulas for the average purity and for the average entanglement entropy

of the density matrix in the large-size limit. We also include two appendices where we prove two formulas which we could not find in literature and which we think are of independent interest. The first formula has the merit to express the moments of the stationary distribution of the free Jacobi process corresponding to $\tau(P) = 1/k$ as a perturbation of those corresponding to $\tau(P) = 1/2$. In particular, it involves a family of polynomials with integer coefficients in the variable $(k - 2)$ and its derivation relies on special properties of the Gauss hypergeometric function. As to the second formula, it expresses the free cumulants of a self-adjoint projection with arbitrary rank as a difference of two Legendre polynomials.

2 Relating G_k^t and compressions of U_t

2.1 Compression by a free projection and Brown measure

Given a collection of operators (a_1, \dots, a_n) in a non commutative probability space (\mathcal{A}, τ) , their joint distribution μ_{a_1, \dots, a_n} is the linear functional which assigns to any polynomial P in n non commuting indeterminates its trace $\tau(P(a_1, \dots, a_n))$. In this respect, the Nica–Speicher generalized R -transform [28] allows to relate the joint distribution of the compressed collection (Pa_1P, \dots, Pa_nP) by a free self-adjoint projection P in the compressed algebra to μ_{a_1, \dots, a_n} . In particular, when $n = 2$ and if $a_1 = a, a_2 = a^*$ then μ_{a, a^*} is given by all the \star -moments of a and we shall simply refer to it as the distribution of a . The following result shows that if $\tau(P) = 1/k, k \geq 2$, then the compression of (U_t, U_t^*) by P amounts to summing k free copies of (U_t, U_t^*) up to dilation. Though we expect that this result is known among the free probability community, we did not find it written anywhere and we include it here for the reader’s convenience. Note also that it reduces to the Nica–Speicher convolution semi-group when a is self-adjoint.

Proposition 1 *Let P be a self-adjoint projection freely independent from $\{U_t, U_t^*\}_{t \geq 0}$ with $\tau(P) = 1/k, k \geq 2$, and recall the non normal operator:*

$$G_t^k = U_t^1 + \dots + U_t^k.$$

Then, the distribution of PU_tP in $(P\mathcal{A}P, k\tau)$ coincides with that of G_t^k/k in (\mathcal{A}, τ) .

Proof Given an operator $a \in \mathcal{A}$, let $R(\mu_{a, a^*})$ be its generalized R -transform ([28], section 3.9) and recall that it entirely determines the distribution of a . Then, one has on the one hand:

$$R\left(\mu_{\frac{1}{k}U_t^1 + \dots + \frac{1}{k}U_t^k, (\frac{1}{k}U_t^1)^* + \dots + (\frac{1}{k}U_t^k)^*}\right) = kR\left(\mu_{\frac{1}{k}U_t, \frac{1}{k}U_t^*}\right)$$

due to the \star -freeness of $(U_t^j)_{j=1}^k$ [28]. On the other hand, [28, Application 1.11] entails

$$R\left(\mu_{PU_tP, PU_t^*P}\right) = kR\left(\mu_{\frac{1}{k}U_t, \frac{1}{k}U_t^*}\right),$$

where the distribution μ_{PU_tP, PU_t^*P} is considered in the compressed space $(P\mathcal{A}P, k\tau)$. □

The Brown measure of a non normal operator plays a key role in random matrix theory since it supplies a candidate for the limiting empirical distribution of a non normal matrix ([26], chapter XI). It is defined through the Fuglede–Fuglede–Kadison determinant given for any $a \in \mathcal{A}$ by:

$$\Delta(a) := \exp[\tau(\log(|a|))] = \exp \int_{\mathbb{R}} \log(u) \mu_{|a|}(du) \in [0, +\infty[,$$

where $|a| = (a^*a)^{1/2}$ is the radial part of a and $\mu_{|a|}$ is its spectral measure. The Brown measure μ_a of a is then defined by:

$$\mu_a := \frac{1}{2\pi} \nabla^2 \log \Delta(a - \lambda \mathbf{1}),$$

where ∇^2 denotes the Laplacian taken in the distributional sense, and is uniquely determined among all compactly-supported measure by its logarithmic potential:

$$\int_{\mathbb{C}} \log(|\lambda - z|) \mu_a(dz) = \log \Delta(a - \lambda \mathbf{1}).$$

In a tracial non commutative probability space, the Brown measure is fully determined by \star -moments and one immediately deduces from the previous proposition that the Brown measures of G_t^k/k and of PU_tP coincide when $\tau(P) = 1/k$. In general, the description of the Brown measure of PU_tP is a quite difficult problem: the main result proved in [10] already provides a Jordan domain containing its support. As a matter of fact, Proposition 1 offers another way to compute the Brown measure of PU_tP in the particular case $\tau(P) = 1/k$ relying on operator-valued free probability as explained in [3]. However, it turns out that the computations are already tedious even for $k = 2$ and as such, we postpone them to a future research work.

In the stationary regime $t = +\infty$, the fact that the R -diagonal operator $PU_\infty P$ and the average of k free Haar unitaries share the same Brown measure is transparent from Haagerup–Laarsen results ([17], examples 5.3 and 5.5) though not being explicitly pointed out there. Indeed, this measure is radial and absolutely continuous with density given by [17]:

$$\frac{k - 1}{\pi(1 - |\lambda|^2)^2} \mathbf{1}_{(0, 1/\sqrt{k})}(|\lambda|) d\lambda,$$

with respect to Lebesgue measure $d\lambda$. Another approach relying on the so-called quaternion free probability may be found in [21].

2.2 Radial parts and beyond

If we consider the radial parts of PU_tP and of G_t^k/k , then the equality between the moments of $PU_tPU_t^*P$ and of W_t^k/k^2 may be readily deduced from the moment-cumulant formula for the compression by a free projection (see Theorem 14.10, [29]). Actually, if (a_1, \dots, a_m) is a collection of operators in \mathcal{A} which is free from P , then

$$\frac{1}{\tau(P)}\tau(Pa_{i_1}Pa_{i_2}P \dots Pa_{i_n}P) = \sum_{\pi \in NC(n)} \kappa_\pi[a_{i_1}, \dots, a_{i_n}][\tau(P)]^{n-|\pi|}, \tag{5}$$

for any indices $1 \leq i_1, \dots, i_n, \leq m$. Here $NC(n)$ is the lattice of non crossing partitions, $|\pi|$ is the number of blocks of the partition $\pi \in NC(n)$ and κ_π is the multiplicative functional of free cumulants of blocks of π (see Lectures 10 and 11 in [29] for more details). Specializing (5) with $(a_{i_{2j+1}}, a_{i_{2j+2}}) = (U_t, U_t^*)$, $0 \leq j \leq n-1$, and $\tau(P) = 1/k$, we get:

$$k\tau(PU_tPU_t^*P \dots PU_tPU_t^*P) = \sum_{\pi \in NC(2n)} \kappa_\pi[\underbrace{U_t, U_t^*, \dots, U_t, U_t^*}_{2n}]k^{|\pi|-2n}. \tag{6}$$

On the other hand, the moment-cumulant formula (11.8) in [29] entails:

$$\frac{1}{k^{2n}}\tau[(W_t^k)^n] = \frac{1}{k^{2n}} \sum_{\pi \in NC(2n)} \kappa_\pi[G_t, (G_t^k)^\star, \dots, G_t, (G_t^k)^\star] \tag{7}$$

where we recall that $G_t^k = U_t^1 + \dots + U_t^k$ and $W_t^k = G_t^k(G_t^k)^\star$. But if V is a block of π then κ_V is the sum of terms of the form

$$\kappa_{|V|}[(U_t^{j_1})^{\epsilon(1)}, (U_t^{j_2})^{\epsilon(2)}, \dots, (U_t^{j_{2n}})^{\epsilon(2n)}],$$

where $\epsilon(1), \dots, \epsilon(2n) \in \{1, \star\}$ and $1 \leq j_1, \dots, j_{2n}, \leq k$. All these terms vanish due to the \star -freeness of $(U_t^j)_{j=1}^k$ except those of the form

$$\kappa_{|V|}[(U_t^j)^{\epsilon(1)}, (U_t^j)^{\epsilon(2)}, \dots, (U_t^j)^{\epsilon(2n)}],$$

for a single index $1 \leq j \leq k$. There are k such terms and all give the same contribution

$$\kappa_{|V|}[(U_t)^{\epsilon(1)}, (U_t)^{\epsilon(2)}, \dots, (U_t)^{\epsilon(2n)}],$$

since U_t^1, \dots, U_t^k have the same spectral distribution as U_t . Consequently, the RHS of (7) and (6) are equal.

More generally, we shall prove below that given two orthogonal projections P and Q which are \star -free from $(U_t)_{t \geq 0}$, the moments of $PU_tQU_t^*P$ and those of W_t^k/k^2 coincide provided that $\tau(P) = \tau(Q) = 1/k$. Our main tool is free stochastic calculus and we refer to [2, 15] for further details on this calculus. To proceed, recall from

[1] the stochastic differential equation satisfied by the free unitary Brownian motion $(U_t)_{t \geq 0}$:

$$dU_t = iU_t dX_t - \frac{U_t}{2} dt, \quad U_0 = \mathbf{1},$$

where $(X_t)_{t \geq 0}$ is a free additive Brownian motion. Hence, there exists a k -tuple free additive Brownian motions $(X_t^j)_{t \geq 0}$, $1 \leq j \leq k$, which are free in \mathcal{A} and such that

$$dU_t^j = iU_t^j dX_t^j - \frac{U_t^j}{2} dt, \quad U_0^j = \mathbf{1}. \tag{8}$$

With the help of the free Itô formula [1], we shall prove:

Theorem 1 For any $n \geq 1, t > 0$, set¹

$$s_n(t) := \tau[(W_t^k)^n].$$

Then,

$$\partial_t s_n(t) = -n s_n(t) + n k s_{n-1}(t) + n k \sum_{j=0}^{n-2} s_{n-j-1}(t) s_j(t) - \frac{n}{k} \sum_{j=0}^{n-2} s_{n-j-1}(t) s_{j+1}(t), \tag{9}$$

where an empty sum is zero.

Proof Using (8), we get

$$dG_t^k = i \sum_{j=1}^k U_t^j dX_t^j - \frac{G_t^k}{2} dt$$

whence

$$dW_t^k = d[G_t^k (G_t^k)^*] = dG_t^k (G_t^k)^* + G_t^k (dG_t^k)^* + (dG_t^k)((dG_t^k)^*),$$

where $(dG_t^k)((dG_t^k)^*)$ stands for the bracket of the semimartingales dG_t^k and $(dG_t^k)^*$. Since $(X_t^j)_{t \geq 0}$ are assumed free then

$$(dX_t^j)(dX_t^m) = \delta_{jm} dt, \quad 1 \leq j, m \leq k,$$

so that

$$dW_t^k = \sum_{j=1}^k \left[(iU_t^j) dX_t^j (G_t^k)^* + G_t^k dX_t^j (iU_t^j)^* \right] + (k - W_t^k) dt.$$

¹ We omit the dependence on k for sake of clarity.

Now, borrowing the terminology and the notations of [2], we introduce the bi-processes:

$$F_t^j := (iU_t^j) \otimes (G_t^k)^\star + (G_t^k) \otimes (iU_t^j)^\star, \quad 1 \leq j \leq k,$$

and write:

$$dW_t^k = \sum_{j=1}^k F_t^j \sharp dX_t^j + (k - W_t^k)dt.$$

Consequently, for any $n \geq 1$, Proposition 4.3.2 in [2] entails:

$$\begin{aligned} d[(W_t^k)^n] = & \text{Martingale part} + \sum_{j=0}^{n-1} (W_t^k)^j \otimes (W_t^k)^{n-1-j} \sharp (k - W_t^k)dt \\ & - \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} (W_t^k)^l U_t^j (G_t^k)^\star (W_t^k)^{n-m-l-2} \tau[(W_t^k)^m U_t^j (G_t^k)^\star] dt \\ & - \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} (W_t^k)^l G_t^k (U_t^j)^\star (W_t^k)^{n-m-l-2} \tau[(W_t^k)^m G_t^k (U_t^j)^\star] dt \\ & + \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} \left\{ (W_t^k)^{n-m-2} \tau[(W_t^k)^{m+1}] + (W_t^k)^{n-m-1} \tau[(W_t^k)^m] \right\}. \end{aligned}$$

Taking the expectation with respect to τ of both sides and differentiating with respect to the variable t ,² we get:

$$\begin{aligned} \partial_t s_n(t) = & -ns_n(t) + nks_{n-1}(t) \\ & - \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} \tau[(W_t^k)^{n-m-2} U_t^j (G_t^k)^\star] \tau[(W_t^k)^m U_t^j (G_t^k)^\star] \\ & - \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} \tau[(W_t^k)^{n-m-2} G_t^k (U_t^j)^\star] \tau[(W_t^k)^m G_t^k (U_t^j)^\star] \\ & + \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} \left\{ \tau[(W_t^k)^{n-m-2}] \tau[(W_t^k)^{m+1}] + \tau[(W_t^k)^{n-m-1}] \tau[(W_t^k)^m] \right\}. \end{aligned}$$

² The state τ is tracial and all the processes are continuous in the strong topology.

The last (triple) sum yields the following contribution (the summands there do not depend on the indices j, l):

$$\begin{aligned}
 &k \sum_{m=0}^{n-2} (n-m-1) \tau[(W_t^k)^{n-m-2}] \tau[(W_t^k)^{m+1}] \\
 &\quad + k \sum_{m=0}^{n-2} (n-m-1) \tau[(W_t^k)^{n-m-1}] \tau[(W_t^k)^m] \\
 &= nk \sum_{m=0}^{n-2} \tau[(W_t^k)^{n-m-1}] \tau[(W_t^k)^m],
 \end{aligned}$$

where the last equality follows from the index change $m \mapsto n - m - 2$. Finally, the summands

$$\tau[(W_t^k)^{n-m-2} U_t^j (G_t^k)^*] \tau[(W_t^k)^m U_t^j (G_t^k)^*], \quad 1 \leq j \leq k,$$

do not depend on j since W_t^k and G_t^k are symmetric (invariant under permutations) and since the unitary operators $U_t^j, 1 \leq j \leq k$, are free and have identical distributions. As a result,

$$\begin{aligned}
 S_1 &:= \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} \tau[(W_t^k)^{n-m-2} U_t^j (G_t^k)^*] \tau[(W_t^k)^m U_t^j (G_t^k)^*] dt \\
 &= \sum_{j=1}^k \sum_{m=0}^{n-2} (n-m-1) \tau[(W_t^k)^{n-m-2} U_t^j (G_t^k)^*] \tau[(W_t^k)^m U_t^j (G_t^k)^*] dt \\
 &= \frac{1}{k} \sum_{m=0}^{n-2} (n-m-1) \sum_{j,l=1}^k \tau[(W_t^k)^{n-m-2} U_t^j (G_t^k)^*] \tau[(W_t^k)^m U_t^l (G_t^k)^*] dt \\
 &= \frac{1}{k} \sum_{m=0}^{n-2} (n-m-1) \tau[(W_t^k)^{n-m-1}] \tau[(W_t^k)^{m+1}].
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 S_2 &:= \sum_{j=1}^k \sum_{\substack{m,l \geq 0 \\ m+l \leq n-2}} \tau[(W_t^k)^{n-m-1} G_t^k (U_t^j)^*] \tau[(W_t^k)^m G_t^k (U_t^j)^*] dt \\
 &= \frac{1}{k} \sum_{m=0}^{n-2} (n-m-1) \tau[(W_t^k)^{n-m-1}] \tau[(W_t^k)^{m+1}].
 \end{aligned}$$

Performing the index change $m \mapsto n - m - 2$ in S_2 , we end up with:

$$S_1 + S_2 = \frac{n}{k} \sum_{m=0}^{n-2} \tau[(W_t^k)^{n-m-1}] \tau[(W_t^k)^{m+1}].$$

Gathering all the contributions above, we obtain (9). □

Setting $r_n(t) := s_n(t)/k^{2n} = \tau[(W_t^k/k^2)^n]$, we readily infer from (9):

Corollary 1 For any $n \geq 1$,

$$\partial_t r_n(t) = -nr_n(t) + \frac{n}{k} r_{n-1}(t) + \frac{n}{k} \sum_{j=0}^{n-2} r_{n-j-1}(t)[r_j(t) - r_{j+1}(t)]. \tag{10}$$

The moment relation (10) is an instance of the one derived in Corollary 6.1 in [7]. More precisely, let

$$J_t := PU_tQU_t^*P$$

be the free Jacobi process associated with the self-adjoint projections (P, Q) . Viewed as an operator in the compressed algebra $(P\mathcal{A}P, \tau/\tau(P))$, its moments

$$m_n(t) = \frac{\tau(J_t^n)}{\tau(P)}, \quad n \geq 1, \quad m_0(t) = 1,$$

satisfy the following differential system:

$$\partial_t m_n(t) = -nm_n(t) + n\theta m_{n-1}(t) + n\lambda\theta \sum_{j=0}^{n-2} m_{n-j-1}(t)[m_j(t) - m_{j+1}(t)], \tag{11}$$

where $\tau(P) = \lambda\theta \in (0, 1]$, $\tau(Q) = \theta \in (0, 1]$. Consequently, if $\lambda = 1, \theta = 1/k$ then (10) and (11) coincide and in turn both moment sequences coincide provided that $m_n(0) = r_n(0)$ for all $n \geq 0$.

2.3 Limit as $k \rightarrow \infty$

Let $U_\infty \in \mathcal{A}$ be a Haar unitary operator and assume that U_∞ is free with $\{P, Q\}$. If $\tau(P) = \tau(Q) = 1/k$ then the spectral distribution of

$$J_\infty := PU_\infty QU_\infty^*P$$

in the compressed algebra $(P\mathcal{A}P, \tau/\tau(P))$ is given by (see e.g. [7], p. 130):

$$\tilde{\mu}_\infty^k(dx) = \frac{1}{2\pi} \frac{\sqrt{4(k-1)x - k^2x^2}}{x(1-x)} \mathbf{1}_{[0, 4(k-1)/k^2]}(x) dx.$$

Its pushforward under the dilation $x \mapsto kx$ is readily computed as

$$\mu_\infty^k(dx) = \frac{1}{2\pi} \frac{\sqrt{4k(k-1)x - k^2x^2}}{kx - x^2} \mathbf{1}_{[0, 4(k-1)/k]}(x) dx,$$

and converges weakly to the Marchenko–Pastur distribution of parameter one [6]:

$$\nu_{MP}(du) := \frac{1}{2\pi} \sqrt{\frac{4-u}{u}} \mathbf{1}_{[0,4]}(u) du.$$

If we denote $W_\infty^k := (U_\infty^1 + \dots + U_\infty^k)(U_\infty^1 + \dots + U_\infty^k)^\star$ then we can rephrase the weak convergence above as follows: for any $n \geq 0$,

$$\lim_{k \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\tau[(W_t^k)^n]}{k^n} = \lim_{k \rightarrow \infty} \frac{\tau[(W_\infty^k)^n]}{k^n} = \int_0^1 u^n \nu_{MP}(du) = \frac{4^n (1/2)_n}{(n+1)!}. \tag{12}$$

The normalization by k^n may be guessed from the moment-cumulant expansion:

$$\tau[(W_\infty^k)^n] = \sum_{\pi \in NC(2n)} \kappa_\pi \underbrace{[U_\infty, U_\infty^\star, \dots, U_\infty, U_\infty^\star]_{2n}}_{2n} k^{|\pi|},$$

since partitions $\pi \in NC(2n)$ with more than $(n + 1)$ blocks have zero contribution. Indeed, in any such partition, at least one block admits an odd number of elements in which case the corresponding free \star -cumulant vanishes (see [29], Proposition 15.1).

For fixed time $t > 0$, the situation becomes different since the free \star -cumulants of U_t admit a considerably more complicated structure compared with those of U_∞ [9]. In this respect, we can prove the following limiting result under the stronger normalization k^2 , which shows that reversing the order of the (k, t) limits in (12) does not lead to a finite limit.

Proposition 2 *For any $n \geq 0, t \geq 0$,*

$$\lim_{k \rightarrow \infty} \tau \left(\frac{(W_t^k)^n}{k^{2n}} \right) = [\tau(U_t)]^{2n} = e^{-nt}.$$

In particular, the free Jacobi process $(PU_tPU_t^\star)_{t \geq 0}$ with $\tau(P) = 1/k$ converges weakly as $k \rightarrow \infty$ to the constant e^{-t} in the compressed algebra.

Proof From (6), we readily see that the limit as $k \rightarrow \infty$ of $k\tau(J_t^n)$ is given by the (non crossing) partition with $2n$ blocks. Therefore,

$$\lim_{k \rightarrow \infty} \tau \left(\frac{(W_t^k)^n}{k^{2n}} \right) = \underbrace{c_1(U_t)c_1(U_t^\star) \dots c_1(U_t)c_1(U_t^\star)}_{2n \text{ terms}}.$$

Since $c_1(U_t) = c_1(U_t^\star) = \tau(U_t) = e^{-t/2}$ (see e.g. [12] and references therein), the proposition follows. □

3 Analysis of the moments of the free Jacobi process

3.1 Moments binomial-type formula

For sake of simplicity, we restrict our study from now on to the free Jacobi process $(J_t)_{t \geq 0}$ associated with a single projection P . For those interested in a more general treatment of the free Jacobi process associated with a pair of projections with arbitrary traces, we recommend referring to [14–16]. Recall from [12] that when $\tau(P) = 1/2$, the moments of the free Jacobi process are linear combinations of those of U_{2t} . Indeed, it was observed there that

$$2\tau(J_t^n) = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{2}{2^{2n}} \sum_{j=1}^n \binom{2n}{n-j} \tau[(SU_t SU_t^*)^j], \tag{13}$$

where $S = 2P - \mathbf{1}$ satisfies $S = S^* = S^{-1}$. Moreover, Lemme 3.8 in [17] together with the semi-group property of $(U_t)_{t \geq 0}$ show that the spectral distributions of $SU_t SU_t^*$ and of U_{2t} coincide. More generally, write:

$$P = \frac{1}{k} \sum_{j=0}^{k-1} S_{j,k}, \quad S_{j,k} := e^{2i\pi j(1-P)/k}.$$

Then $S_{j,k}$ is a unitary operator satisfying $(S_{j,k})^k = \mathbf{1}$ and

$$S_{j,k} = \mathbf{1} + (\omega_{j,k} - 1)(\mathbf{1} - P),$$

where $\omega_{j,k} = e^{2i\pi j/k}$ is the k th root of unity. Set

$$T_k := kP - \mathbf{1} = \sum_{j=1}^{k-1} S_{j,k}.$$

Then T_k is self-adjoint and $(T_k)^2 = k(k - 2)P + \mathbf{1} = (k - 2)T_k + (k - 1)\mathbf{1}$. In this respect, we shall prove the following generalization of (13):

Theorem 2 *For any $k \geq 2$ and any $n \geq 1$,*

$$m_n(t) = k\tau(J_t) = m_n(\infty) + \frac{k}{k^{2n}} \sum_{j=1}^n (k - 1)^{n-j} \binom{2n}{n-j} \tau[(T_k U_t T_k U_t^*)^j],$$

where $m_n(\infty)$ is the n th moment of J_∞ in $(P \mathscr{A} P, k\tau)$, given by (21) and (34).

The proof of this Theorem relies on the following four key lemmas.

Lemma 1 *Let $a, b \in \mathcal{A}$ be two operators satisfying $a^2 = (k - 2)a + (k - 1)\mathbf{1}$, $b^2 = (k - 2)b + (k - 1)\mathbf{1}$. Then, the expansion of $[(a + 1)(b + 1)]^n$ is uniquely written as:*

$$\begin{aligned}
 [(\mathbf{1} + a)(\mathbf{1} + b)]^n &= m_n \mathbf{1} + \sum_{j=1}^n c_{n,j} (ab)^j + \sum_{j=1}^{n-1} d_{n,j} (ba)^j \\
 &\quad + \sum_{j=0}^{n-1} e_{n,j} (ab)^j a + \sum_{j=0}^{n-1} f_{n,j} (ba)^j b
 \end{aligned} \tag{14}$$

for some integer sequences $m_n, (c_{n,j}), (d_{n,j}), (e_{n,j}), (f_{n,j})$ satisfying

$$\begin{aligned}
 m_n &= f_{n,0} = e_{n,0}, \\
 c_{n,j} &= f_{n,j-1} = e_{n,j-1}, \quad 1 \leq j \leq n, \\
 d_{n,j} &= c_{n,j+1}, \quad 1 \leq j \leq n - 1.
 \end{aligned}$$

Proof For sake of clarity, we shall omit the notation $\mathbf{1}$ in front of the constant terms. Firstly, the uniqueness follows from the fact that the expansion is a reduced expression. Now, since $a(a + \mathbf{1}) = (k - 1)(a + \mathbf{1})$ then

$$\begin{aligned}
 (k - 1)[(a + \mathbf{1})(b + \mathbf{1})]^n &= a[(\mathbf{1} + a)(\mathbf{1} + b)]^n \\
 &= m_n a + \sum_{j=1}^n c_{n,j} a (ab)^j + \sum_{j=1}^{n-1} d_{n,j} a (ba)^j + \sum_{j=0}^{n-1} e_{n,j} a (ab)^j a + \sum_{j=0}^{n-1} f_{n,j} a (ba)^j b \\
 &= m_n a + (k - 2) \sum_{j=1}^n c_{n,j} (ab)^j + (k - 1) \sum_{j=0}^{n-1} c_{n,j+1} (ba)^j b + \sum_{j=1}^{n-1} d_{n,j} (ab)^j a \\
 &\quad + \sum_{j=0}^{n-1} e_{n,j} [(k - 2)(ab)^j a + (k - 1)(ba)^j] + \sum_{j=1}^n f_{n,j-1} (ab)^j \\
 &= (k - 1)e_{n,0} + \sum_{j=1}^n [(k - 2)c_{n,j} + f_{n,j-1}] (ab)^j + (k - 1) \sum_{j=1}^{n-1} e_{n,j} (ba)^j \\
 &\quad + (m_n + (k - 2)e_{n,0})a + \sum_{j=1}^{n-1} [(k - 2)e_{n,j} + d_{n,j}] (ab)^j a + (k - 1) \sum_{j=0}^{n-1} c_{n,j+1} (ba)^j b.
 \end{aligned}$$

Multiplying (14) by $(k - 1)$ and using the uniqueness of the coefficients, we readily get:

$$m_n = e_{n,0}, \quad c_{n,j} = f_{n,j-1}, \quad 1 \leq j \leq n, \quad e_{n,j} = d_{n,j}, \quad 1 \leq j \leq n - 1. \tag{15}$$

Similarly, $b(b + 1) = (k - 1)(b + 1)$ so that

$$\begin{aligned}
 (k - 1)[(a + 1)(b + 1)]^n &= [(1 + a)(1 + b)]^n b \\
 &= m_n b + \sum_{j=1}^n c_{n,j} (ab)^j b + \sum_{j=1}^{n-1} d_{n,j} (ba)^j b + \sum_{j=0}^{n-1} e_{n,j} (ab)^j ab + \sum_{j=0}^{n-1} f_{n,j} (ba)^j b^2 \\
 &= m_n b + (k - 2) \sum_{j=1}^n c_{n,j} (ab)^j + (k - 1) \sum_{j=0}^{n-1} c_{n,j+1} (ab)^j a + \sum_{j=1}^{n-1} d_{n,j} (ba)^j b \\
 &\quad + \sum_{j=1}^n e_{n,j-1} (ab)^j + (k - 2) \sum_{j=0}^{n-1} f_{n,j} (ba)^j b + (k - 1) \sum_{j=0}^{n-1} f_{n,j} (ba)^j \\
 &= (k - 1) f_{n,0} + \sum_{j=1}^n [(k - 2) c_{n,j} + e_{n,j-1}] (ab)^j + (k - 1) \sum_{j=1}^{n-1} f_{n,j} (ba)^j \\
 &\quad + (m_n + (k - 2) f_{n,0}) b + (k - 1) \sum_{j=0}^{n-1} c_{n,j+1} (ab)^j a + \sum_{j=1}^{n-1} [(k - 2) f_{n,j} + d_{n,j}] (ba)^j b.
 \end{aligned}$$

The uniqueness property again yields:

$$m_n = f_{n,0}, \quad c_{n,j} = e_{n,j-1}, \quad 1 \leq j \leq n, \quad f_{n,j} = d_{n,j}, \quad 1 \leq j \leq n - 1. \tag{16}$$

Combining (15) and (16), the lemma is proved. □

According to Lemma 1, we only need to focus on the sequences $(m_n)_n, (c_{n,j})_{1 \leq j \leq n}$. The former is closely related to the moment sequence $m_n(\infty)$ of $\tilde{\mu}_\infty$. As to the latter, it satisfies the following relations:

Lemma 2 For any $2 \leq j \leq n - 1$,

$$c_{n+1,j} = (k - 1)c_{n,j} + c_{n,j-1} + (k - 1)^2 e_{n,j} + (k - 1)e_{n,j-1}, \tag{17}$$

while

$$\begin{cases} c_{n+1,n+1} = c_{n,n} = 1 \\ c_{n+1,n} = c_{n,n-1} + (k - 1) + (k - 1)e_{n,n-1} \\ c_{n+1,1} = (k - 1)c_{n,1} + (k - 1)^2 e_{n,1} + (k - 1)e_{n,0} + m_n \end{cases} . \tag{18}$$

Proof Follows readily from

$$[(1 + a)(1 + b)]^{n+1} = [(1 + a)(1 + b)]^n (1 + a + ab),$$

together with the identities:

$$\begin{aligned}
 (ab)^j &= (ab)^{j-1}(ab), \\
 (ab)^j b &= (k-2)(ab)^j + (k-1)(ab)^{j-1}a, \\
 ((ab)^{j-1}a)b &= (ab)^j, \\
 ((ab)^j a)a &= (k-2)(ab)^j a + (k-1)(ab)^j, \\
 ((ab)^j a)ab &= (k-2)(ab)^{j+1} + (k-1)(ab)^j b \\
 &= (k-2)(ab)^{j+1} + (k-1)(k-2)(ab)^j + (k-1)^2(ab)^{j-1}a.
 \end{aligned}$$

□

Note that Lemma (1) allows to rewrite (17) and (18) as

$$\begin{cases} c_{n+1,j} = 2(k-1)c_{n,j} + c_{n,j-1} + (k-1)^2c_{n,j+1}, & 2 \leq j \leq n+1, \\ c_{n+1,1} = (2k-1)c_{n,1} + (k-1)^2c_{n,2}, \end{cases} \tag{19}$$

where we set $c_{n,j} = 0, j > n$. Next, we need the following routine computations to prove Lemma 4 below and which give our first formula for $m_n(\infty)$:

Lemma 3 *For any $n \geq 1$, we have*

$$m_n(\infty) - m_{n+1}(\infty) = \frac{(k-1)^{n+1}}{k^{2n+1}} C_n, \tag{20}$$

where C_n is the n th Catalan number. In particular,

$$m_n(\infty) = 1 - \sum_{j=0}^{n-1} \frac{(k-1)^{j+1}}{k^{2j+1}} C_j. \tag{21}$$

Proof

$$\begin{aligned}
 m_n(\infty) - m_{n+1}(\infty) &= \frac{1}{2\pi} \int x^{n-1/2} \sqrt{4(k-1) - k^2x} \mathbf{1}_{[0,4(k-1)/k^2]}(x) dx \\
 &= \frac{2^{2n+2}(k-1)^{n+1}}{2\pi k^{2n+1}} \int x^{n-1/2} \sqrt{1-x} \mathbf{1}_{[0,1]}(x) dx \\
 &= \frac{2^{2n}(k-1)^{n+1}}{\sqrt{\pi} k^{2n+1}} \frac{\Gamma(n+1/2)}{(n+1)!} \\
 &= \frac{(k-1)^{n+1}}{k^{2n+1}} \frac{(2n)!}{(n+1)n!} = \frac{(k-1)^{n+1}}{k^{2n+1}} C_n.
 \end{aligned}$$

The expression of $m_n(\infty)$ follows.

□

Remark 1 Taking the expectation in (14), we infer that $m_n(\infty) = m_n/k^{2n-1}$. Consequently, the last relation may be written as

$$k^2m_n - m_{n+1} = (k - 1)^n C_n,$$

or equivalently,

$$c_{n,1} - c_{n,2} = \frac{(k - 1)^{n-1}}{n + 1} \binom{2n}{n}. \tag{22}$$

This elementary identity will be used in the proof of Lemma 4 below.

Now, set

$$K_{n,0} := 2(k - 1) \left(c_{n,1} + (k - 2) \sum_{l=2}^n (k - 1)^{l-2} c_{n,l} \right), \quad n \geq 1, \tag{23}$$

where an empty sum is zero. Then

Lemma 4 For any $n \geq 1$, we have

$$K_{n,0} = (k - 1)^n \binom{2n}{n}. \tag{24}$$

Proof We proceed by induction: $K_{1,0} = 2(k - 1)c_{1,1} = 2(k - 1)$. Next, assume the result is valid up to order n and write (we recall that $c_{n,j} = 0, j > n$):

$$\begin{aligned} K_{n+1,0} &= 2(k - 1) \left(c_{n+1,1} + (k - 2) \sum_{l=2}^{n+1} (k - 1)^{l-2} c_{n+1,l} \right) \\ &= 2(k - 1) \left((2k - 1)c_{n,1} + (k - 1)^2 c_{n,2} + (k - 2) \sum_{l=2}^{n+1} (k - 1)^{l-2} [2(k - 1)c_{n,l} \right. \\ &\quad \left. + c_{n,l-1} + (k - 1)^2 c_{n,l+1}] \right) \\ &= 2(k - 1) \left((2k - 1)c_{n,1} + (k - 1)^2 c_{n,2} + 2(k - 2) \sum_{l=2}^n (k - 1)^{l-1} c_{n,l} \right. \\ &\quad \left. + (k - 2) \sum_{l=1}^n (k - 1)^{l-1} c_{n,l} + (k - 2) \sum_{l=3}^n (k - 1)^l c_{n,l+1} \right) \\ &= 2(k - 1) \left(3(k - 1)c_{n,1} + (k - 1)c_{n,2} + 4(k - 2) \sum_{l=2}^n (k - 1)^{l-1} c_{n,l} \right) \\ &= 4(k - 1)K_{n,0} - 2(k - 1)^2(c_{n,1} - c_{n,2}). \end{aligned}$$

Appealing to the induction hypothesis and to the identity (22), we end up with

$$\begin{aligned}
 K_{n+1,0} &= 4(k-1)(k-1)^n \binom{2n}{n} - 2(k-1)^2 \frac{(k-1)^{n-1}}{n+1} \binom{2n}{n} \\
 &= (k-1)^{n+1} \binom{2n+2}{n+1},
 \end{aligned}$$

as desired. □

We are now ready to prove Theorem 2.

Proof of Theorem 2 We apply Lemma 1 to $a = T_k$ and $b = U_t T_k U_t^*$ and take the expectation with respect to τ . By the trace property and the fact that $\tau(T_k) = \tau(U_t T_k U_t^*) = 0$, we have

$$\tau((ab)^j) = \tau((ba)^j), \quad \tau((ab)^j a) = (k-2) \sum_{l=1}^j (k-1)^{j-l} \tau((ab)^l) = \tau((ba)^j b),$$

whence

$$\tau([(1 + T_k)(1 + U_t T_k U_t^*)]^n) = m_n + \sum_{j=1}^n K_{n,j} \tau((T_k U_t T_k U_t^*)^j) \tag{25}$$

where

$$K_{n,j} = \begin{cases} c_{n,j} + d_{n,j} + (k-2) \sum_{l=j}^{n-1} (k-1)^{l-j} (e_{n,l} + f_{n,l}), & 1 \leq j \leq n-1 \\ 1, & j = n \end{cases} . \tag{26}$$

Equivalently, Lemma 1 again entails:

$$\begin{aligned}
 K_{n,j} &= c_{n,j} + c_{n,j+1} + 2(k-2) \sum_{l=j}^{n-1} (k-1)^{l-j} c_{n,l+1} \\
 &= c_{n,j} + c_{n,j+1} + 2(k-2) \sum_{l=j+1}^n (k-1)^{l-(j+1)} c_{n,l}, \quad 1 \leq j \leq n.
 \end{aligned}$$

Appealing further to (19), we obtain

$$K_{n+1,j} = (k-1)^2 K_{n,j+1} + 2(k-1) K_{n,j} + K_{n,j-1}, \quad 1 \leq j \leq n+1, \tag{27}$$

with the convention $K_{n,j} = 0, j > n$ and with $K_{n,0}$ given by (23). Finally, (27) is satisfied by the sequence

$$(k-1)^{n-j} \binom{2n}{n-j}$$

as readily seen from the identity

$$\binom{n}{j} + \binom{n}{j+1} = \binom{n+1}{j+1}.$$

Moreover, Lemma 4 and the obvious value $K_{n,n} = 1, n \geq 1$, show that the boundary conditions coincide, whence we deduce:

$$K_{n,j} = (k-1)^{n-j} \binom{2n}{n-j}.$$

Noting that

$$\tau[(PU_tPU_t^*)^n] = \frac{1}{k^{2n}} \tau[(\mathbf{1} + T_k)(\mathbf{1} + U_tT_kU_t^*)^n],$$

we are done. □

Remark 2 (Combinatorial approach) Applying the moment formula with product as entries, it follows that:

$$\tau(J_t^n) = \sum_{NC(2n)} \kappa_\pi(\underbrace{P, \dots, P}_{2n}) \tau_{K(\pi)}(U_t, U_t^*, \dots, U_t, U_t^*).$$

When $\tau(P) = 1/2$, it is known that the free cumulants of P are given by ([29], Exercise 11.35):

$$\tau_{2j+1}(P) = \frac{\delta_{j0}}{2}, \quad \tau_{2j}(P) = \frac{(-1)^{j-1}}{2^{2j}} C_{j-1}. \tag{28}$$

It would be interesting to recover (13) using these formulas together with properties of non crossing partitions. More generally, we can prove (see ‘‘Appendix B’’) that if P is a self-adjoint projection with $\tau(P) = \alpha$, then

$$\kappa_1(P) = \alpha, \quad \kappa_n(P) = \frac{1}{2(2n-1)} [P_{n-2}(1-2\alpha) - P_n(1-2\alpha)], \quad n \geq 2. \tag{29}$$

where $(P_n)_{n \geq 0}$ is the family of Legendre polynomials defined through the Gauss hypergeometric function by:

$$P_n(x) = {}_2F_1\left(-n, n+1, 1; \frac{1-x}{2}\right), \quad x \in [-1, 1].$$

3.2 pde for the moment generating function

Let

$$M_{t,k}(z) := k \sum_{n \geq 0} \tau(J_t^n) z^n, \quad \rho_{t,k} := \sum_{n \geq 1} \tau[(T_k U_t T_k U_t^*]^n \frac{z^n}{(k-1)^n},$$

be the moment generating functions of J_t in the compressed space $(P\mathcal{A}P, k\tau)$ and of $T_k U_t T_k U_t^*$ in (\mathcal{A}, τ) . Both series have positive convergence radii since the corresponding operators are bounded. From Theorem 2, we deduce the following relation:

Corollary 2 For any $k \geq 2$ and any $t > 0$,

$$M_{t,k}(z) = M_{\infty,k}(z) + \frac{k^2}{\sqrt{k^2 - 4(k-1)z}} \rho_{t,k} \left(\alpha \left[\frac{4(k-1)z}{k^2} \right] \right),$$

where

$$M_{\infty,k}(z) := \sum_{n \geq 0} m_n(\infty) z^n = \frac{2 - k + \sqrt{k^2 - 4(k-1)z}}{2(1-z)}, \quad |z| < 1, \quad (30)$$

is the moment generating function of J_∞ and

$$\alpha(z) = \frac{1 - \sqrt{1-z}}{1 + \sqrt{1-z}}, \quad z \in \mathbb{C} \setminus [1, \infty[.$$

Proof It is obvious from Theorem 2 that

$$M_{t,k}(z) = M_{\infty,k}(z) + k \sum_{n \geq 1} \frac{((k-1)z)^n}{k^{2n}} \sum_{j=1}^n \binom{2n}{n-j} \frac{\tau[(T_k U_t T_k U_t^*)^j]}{(k-1)^j}.$$

The expression of $M_{\infty,k}$ is already known (see e.g. section 5 in [7]). Now, recall the following result ([25], p.357): if $(a_n)_{n \geq 0}, (b_n)_{n \geq 0}$ are two real sequences satisfying

$$b_n = \sum_{j=0}^n \binom{2n}{n-j} a_j,$$

then

$$\sum_{n \geq 0} b_n \frac{w^n}{4^n} = \frac{1 + \alpha(w)}{1 - \alpha(w)} \sum_{n \geq 0} a_n [\alpha(w)]^n$$

whenever both series converge absolutely. Applying this result with

$$a_0 = b_0 = 0, \quad a_j = \frac{\tau[(T_k U_t T_k U_t^*)^j]}{(k-1)^j}, \quad j \geq 1, \quad w = \frac{4(k-1)z}{k^2},$$

and noting that

$$\frac{1 + \alpha(w)}{1 - \alpha(w)} = \frac{1}{\sqrt{1-w}}, \quad w \in \mathbb{C} \setminus [1, \infty[,$$

conclude the proof. □

From this corollary, we can derive a pde for $\rho_{t,k}$:

Proposition 3 *The moment generating function $\rho_{t,k}(z)$ satisfies the pde:*

$$\partial_t \rho_{t,k}(z) = -z \partial_z \left[\rho_{t,k}(z) + \frac{4(k-1) - k^2 \alpha^{-1}(z)}{4(k-1)(1 - \alpha^{-1}(z))} \rho_{t,k}^2(z) \right], \tag{31}$$

in a neighborhood of the origin with the initial condition:

$$\rho_{0,k}(z) = \frac{(k-1)z(1-z)}{(k-1-z)(1+z-kz)}.$$

Proof Let

$$G_{t,k}(z) := \frac{1}{z} M_{t,k} \left(\frac{1}{z} \right), \quad |z| > 1,$$

be the Cauchy–Stieltjes transform of the free Jacobi process $PU_t PU_t^* P$ with $\tau(P) = 1/k$, and recall from [7] that it satisfies the pde:

$$\partial_t G_{t,k}(z) = \frac{1}{k} \partial_z [(k-2)z G_{t,k}(z) + z(z-1)G_{t,k}^2(z)].$$

Then the variable change $z \mapsto 1/z$ shows that

$$\partial_t M_{t,k}(z) = -\frac{z}{k} \partial_z [(k-2)M_{t,k}(z) + (1-z)M_{t,k}^2(z)]. \tag{32}$$

Now set $R_{t,k} := M_{t,k} - M_{\infty,k}$, then we further get:

$$\begin{aligned} \partial_t R_{t,k}(z) &= -\frac{z}{k} \partial_z [(k-2)R_{t,k}(z) + 2(1-z)R_{t,k}(z)M_{\infty,k} + (1-z)R_{t,k}^2(z)] \\ &= -\frac{z}{k} \partial_z [\sqrt{k^2 - 4(k-1)z} R_{t,k}(z) + (1-z)R_{t,k}^2(z)], \end{aligned}$$

where the first equality follows from the fact that $M_{\infty,k}$ is a stationary solution of the pde (32):

$$\partial_z[(k - 2)M_{\infty,k}(z) + (1 - z)M_{\infty,k}^2(z)] = 0,$$

and the second one follows from (30). Noting that

$$\sqrt{k^2 - 4(k - 1)z}R_{t,k}(z) = k^2\rho_{t,k}\left(\alpha\left[\frac{4(k - 1)z}{k^2}\right]\right),$$

it follows that the map $(t, z) \mapsto \rho_{t,k}(\alpha(z))$ satisfies:

$$\partial_t\rho_{t,k}(\alpha(z)) = -z\sqrt{1 - z}\partial_z\left[\rho_{t,k}(\alpha(z)) + \frac{4(k - 1) - k^2z}{4(k - 1)(1 - z)}\rho_{t,k}^2(\alpha(z))\right].$$

Next, α is a one-to-one holomorphic map in $\mathbb{C} \setminus [1, \infty[$ onto the open unit disc with inverse given by:

$$\alpha^{-1}(z) = \frac{4z}{(1 + z)^2}.$$

Moreover

$$\alpha'(z) = \frac{\alpha(z)}{z\sqrt{1 - z}}$$

whence

$$[\alpha^{-1}]'(z) = \frac{\alpha^{-1}(z)\sqrt{1 - \alpha^{-1}(z)}}{z}.$$

The sought pde satisfied by $\rho_{t,k}(z)$ follows after few computations. Finally,

$$\rho_{0,k}(z) = \sum_{n \geq 1} \tau[(T_k)^{2n}] \frac{z^n}{(k - 1)^n}.$$

Letting $h_n := \tau[(T_k)^n]$, $n \geq 0$, then we can easily prove using the relation $T_k^2 = (k - 2)T_k + (k - 1)\mathbf{1}$ that

$$h_{n+2} = (k - 2)h_{n+1} + (k - 1)h_n, \quad h_0 = 1, h_1 = 0.$$

This is a generalized Fibonacci sequence for which a Binet formula already exists [19]:

$$h_n = \frac{(k - 1)^n + (-1)^n(k - 1)}{k}, \quad n \geq 0.$$

As a result,

$$\rho_{0,k}(z) = \sum_{n \geq 1} h_{2n} \frac{z^n}{(k-1)^n} = \frac{(k-1)z(1-z)}{(k-1-z)(1+z-kz)}.$$

The proposition is proved. □

Setting $\eta_{t,k}(z) := \rho_{t,k}(e^t z)$ then

$$\partial_t \eta_{t,k} = \partial_t \rho_{t,k}(e^t z) + e^t z \partial_z \rho_{t,k}(e^t z),$$

yielding

$$\partial_t \eta_{t,k} = -z \partial_z \left[\frac{4(k-1) - k^2 \alpha^{-1}(e^t z)}{4(k-1)(1 - \alpha^{-1}(e^t z))} \eta_{t,k}^2(z) \right]. \tag{33}$$

In particular, if $k = 2$ then

$$\partial_t \eta_{t,2}(z) = -z \partial_z \left[\eta_{t,2}^2(z) \right],$$

while

$$\eta_{0,2}(z) = \frac{z}{1-z}.$$

In this case, it is known that $\eta_{t,2}$ is the moment generating function of the free unitary Brownian motion $e^t U_{2t}$ [12]:

$$\eta_{t,2}(z) = \sum_{n \geq 1} \frac{z^n}{n} L_{n-1}^{(1)}(2nt)$$

where $L_{n-1}^{(1)}$ is the $(n - 1)$ th Laguerre polynomial of parameter one. More generally (i.e. for $k \geq 3$), the analysis of the characteristic curves of (31) is quite involved as shown in the following paragraph.

3.3 Characteristic curves of the pde

Denote

$$\lambda_k(z) := \frac{4(k-1) - k^2 \alpha^{-1}(z)}{4(1 - \alpha^{-1}(z))},$$

so that the pde (31) reads:

$$\partial_t \rho_{t,k}(z) = -z \partial_z \left[\rho_{t,k}(z) + \frac{\lambda_k(z)}{(k-1)} \rho_{t,k}^2(z) \right].$$

Elementary transformations show that $\tilde{\rho}_{t,k}(z) := [\rho_{t,k}(z)]/(k - 1)$ satisfies

$$\partial_t \tilde{\rho}_{t,k}(z) = -z \partial_z \left[\tilde{\rho}_{t,k}(z) + \lambda_k(z) \tilde{\rho}_{t,k}^2(z) \right].$$

Let z be fixed in a neighborhood of the origin. Then a characteristic curve starting at z is locally the unique solution of the Cauchy problem:

$$z'_k(t) = z(t)[1 + 2\lambda_k(z_k(t))f_k(t)], \quad z_k(0) = z,$$

where we set $f_k(t) := \tilde{\rho}_{t,k}(z_k(t))$. Along such curve, it holds that:

$$\begin{aligned} (f_k)'(t) &= -z(t)(\lambda_k)'(z_k(t))f_k^2(t), \\ f_k(0) &= \tilde{\rho}_{t,k}(z) = \frac{z(1 - z)}{(k - 1 - z)(1 + z - kz)}. \end{aligned}$$

Now, set

$$H(u) := \frac{u + 1}{u - 1}$$

and note that H is an involution ($H^{-1} = H$), $H'(u) = -(H(u) - 1)^2/2$ and

$$\lambda_k(z) = \frac{1}{4}[k^2 - (k - 2)^2 H^2(z)].$$

Then the curve defined by $y_k(t) := H(z_k(t))$ solves locally around -1 the Cauchy problem:

$$\begin{aligned} y'_k(t) &= \frac{1 - y_k^2(t)}{2} \left[1 + \frac{k^2 - (k - 2)^2 y_k^2(t)}{2} f_k(t) \right], \\ y_k(0) &= \frac{z + 1}{z - 1} := y. \end{aligned}$$

Besides,

$$\begin{aligned} (f_k)'(t) &= -\frac{(k - 2)^2}{4} y_k(t) H(y_k(t)) (y_k(t) - 1)^2 f_k^2(t) \\ &= \frac{(k - 2)^2 (1 - y_k^2(t))}{4} y_k(t) f_k^2(t). \end{aligned}$$

Consequently,

$$y'_k(t) y_k(t) \frac{(k - 2)^2 (1 - y_k^2(t))}{4} f_k^2(t) = \frac{1 - y_k^2(t)}{2} \left[1 + \frac{k^2 - (k - 2)^2 y_k^2(t)}{2} f_k(t) \right] (f_k)'(t),$$

or equivalently

$$\frac{(k - 2)^2}{4} \left[(y_k^2)'(t) f_k^2(t) + y_k^2(t) (f_k^2)'(t) \right] = \left[1 + \frac{k^2}{2} f_k(t) \right] (f_k)'(t).$$

This equation is integrable and yields:

$$\frac{(k - 2)^2}{4} \left[(y_k^2)(t) f_k^2(t) - y_k^2(0) (f_k^2)(0) \right] = \left[\frac{k^2}{4} f_k^2(t) + f_k(t) - \frac{k^2}{4} f_k^2(0) - f_k(0) \right].$$

Written differently leads to the functional equation:

$$\tilde{\lambda}_k(y(t)) f_k^2(t) + f_k(t) - \tilde{\lambda}_k(y) f_k^2(0) - f_k(0) = 0,$$

where we simply wrote

$$\tilde{\lambda}_k(y) := \frac{1}{4} [k^2 - (k - 2)^2 y^2] = \lambda_k(z).$$

Setting $g_k(0) := \tilde{\lambda}_k(y) f_k^2(0) + f_k(0)$, then one has locally:

$$f_k(t) = \frac{-1 + \sqrt{1 + 4g_k(0)\tilde{\lambda}_k(y_k(t))}}{2\tilde{\lambda}_k(y_k(t))},$$

where the principal determination of the square root is considered. It follows that:

$$\begin{aligned} y_k'(t) &= \sqrt{1 + 4g_k(0)\tilde{\lambda}_k(y_k(t))} \frac{1 - y_k^2(t)}{2} \\ &= \frac{1 - y_k^2(t)}{2} \sqrt{1 + k^2 g_k(0) - (k - 2)^2 g_k(0) (y_k^2(t))}. \end{aligned}$$

Now, consider the indefinite integral

$$I_{A,B}(u) = 2 \int^u \frac{du}{(1 - u^2)\sqrt{A - Bu^2}}$$

for two indeterminates (A, B) independent of the variable u . Then

$$I_{A,B}(u) = \frac{1}{\sqrt{A - B}} \log \left[\frac{(\sqrt{A - Bu^2} + \sqrt{A - Bu})^2}{A(1 - u^2)} \right],$$

provided the square root is well-defined (we can take any determination of the logarithm). Taking $A = 1 + k^2 g_k(0)$, $B = (k - 2)^2 g_k(0)$, one gets

$$I_{A,B}(y(t)) - I_{A,B}(y) = t,$$

or after exponentiating this identity:

$$\frac{(\sqrt{A - B(y_k(t))^2} + \sqrt{A - B}y_k(t))^2}{A(1 - y_k(t)^2)} = e^{\sqrt{A-B}t} \frac{(\sqrt{A - By^2} + \sqrt{A - B}y)^2}{A(1 - y^2)}.$$

Noting that

$$f_k(0) = -\frac{1 + y}{2\tilde{\lambda}_k(y)}, \quad 4g_k(0) = \frac{y^2 - 1}{\tilde{\lambda}_k(y)},$$

then $A - By^2 = 1 + 4g_k(0)\tilde{\lambda}_k(y) = y^2$ which in turn entails:

$$\begin{aligned} \frac{(\sqrt{A - B(y_k(t))^2} + \sqrt{A - B}y_k(t))^2}{A(1 - y_k(t)^2)} &= e^{\sqrt{A-B}t} \frac{y^2(1 - \sqrt{A - B})^2}{A(1 - y^2)} \\ &= e^{\sqrt{A-B}t} \frac{1 - \sqrt{A - B}}{1 + \sqrt{A - B}}. \end{aligned}$$

If we denote the LHS of the second equality $\xi_{2t}(\sqrt{A - B})$, then lengthy computations yield:

$$\begin{aligned} y_k^2(t) &= \frac{A(1 - \xi_{2t}(\sqrt{A - B}))^2}{A(1 + \xi_{2t}(\sqrt{A - B}))^2 - 4B\xi_{2t}(\sqrt{A - B})} \\ &= \frac{A}{(A - B)H^2[\xi_{2t}(\sqrt{A - B})] + B} \\ &= \frac{1 + k^2g_k(0)}{(1 + 4(k - 1)g_k(0))[H^2[\xi_{2t}(\sqrt{1 + 4(k - 1)g_k(0)})] - 1] + (1 + k^2g_k(0))}. \end{aligned}$$

The map $(-\xi_{2t})$ is the inverse of the Herglotz transform $1 + 2\eta_{t,2}$ of the spectral distribution of U_{2t} in a neighborhood of $u = 1$. As a matter of fact, the map

$$u \mapsto \frac{1 + k^2u}{(1 + 4(k - 1)u)[H^2[\xi_{2t}(\sqrt{1 + 4(k - 1)u})] - 1] + (1 + k^2u)}$$

is a locally invertible in a neighborhood of the origin $u = 0$. Let ζ_{2t} be its inverse then

$$g_k(0) = \zeta_{2t}(y_k^2(t)),$$

whence we end up with:

$$\begin{aligned} f_k(t) &= \frac{-1 + \sqrt{1 + 4\zeta_{2t}(y_k^2(t))\tilde{\lambda}_k(y_k(t))}}{2\tilde{\lambda}_k(y_k(t))} \\ &= 2 \frac{\zeta_{2t}[H^2(z_k(t))]}{1 + \sqrt{1 + \zeta_{2t}[H^2(z_k(t))][k^2 - (k - 2)^2 H^2(z_k(t))]} = \tilde{\rho}_{t,k}(z_k(t)). \end{aligned}$$

Remark 3 If $k = 2$ then

$$y_2^2(t) = \frac{1}{H^2[\xi_{2t}(\sqrt{1+4g_2(0)})]} = \frac{1}{H^2[\xi_{2t}(-y)]} = H^2[-\xi_{2t}(-y)].$$

Thus, it holds locally:

$$y_2(t) = H[-\xi_{2t}(-y)] \Rightarrow z_2(t) = (-\xi_{2t}) \left(\frac{1+z}{1-z} \right) = z e^{t(1+z)/(1-z)}.$$

For fixed $t > 0$, the map $z \mapsto z_2(t)$ is known as the Σ -transform of the spectral distribution of $U_{2t} [1]$.

4 Application to quantum information: asymptotics of the average purity and of the entanglement entropy

Starting from a pure state ψ in a bipartite quantum system $\mathcal{H}_A \otimes \mathcal{H}_B$, the reduced state corresponding to \mathcal{H}_A , say $\rho_A \in \mathbb{M}_N(\mathbb{C})$, is in general mixed. The average purity of ρ_A is then defined by $\text{tr}(\rho_A^2)$ and ranges in the interval $[1/N, 1]$ whose endpoints correspond respectively to the completely mixed and to pure states.

As far as the dynamical density matrix $\tilde{W}_t^k(N)$ is concerned, we can use Theorem (2) together with the freeness property to compute the large-size asymptotic of its average purity (after time-rescaling $t \mapsto t/N$ in order to make the matrix model converge). Straightforward computations then yield the following expression:

$$\begin{aligned} \mathbb{E} \left\{ \text{tr} \left([\tilde{W}_{t/N}^k(N)]^2 \right) \right\} &\approx \frac{1}{N} \tau[(W_t^k)^2], \quad N \rightarrow +\infty, \\ &= \frac{k^2}{N[1 + (k-1)e^{-t}]} m_2(t) \\ &= \frac{(2k-1) + 4(k-1)^2 e^{-t} + (k-1)[k^2 - (k-1)(2t+3)] e^{-2t}}{kN[1 + (k-1)e^{-t}]^2}. \end{aligned}$$

Letting $t \rightarrow +\infty$, we recover the stationary asymptotic average purity $[2 - (1/k)]/N$ computed in [6]. Fixing $t > 0$ and letting $k \rightarrow +\infty$, we rather get:

$$e^{-2t} = \lim_{k \rightarrow +\infty} m_2(t).$$

The entanglement entropy of the bipartite system is defined as the von Neumann entropy:

$$-\text{tr}(\rho_A \ln[\rho_A]),$$

where $\ln(\rho)$ is defined through functional calculus. Since $\tilde{W}_{t/N}^k(N)$ converges in operator norm to \tilde{W}_t^k then we define the asymptotic entanglement entropy of $\tilde{W}_{t/N}^k(N)$ as

$N \rightarrow \infty$ by:

$$-\tau \left(\widetilde{W}_t^k \ln \left[\widetilde{W}_t^k \right] \right),$$

where we use again the functional calculus. To the best of our knowledge, the asymptotic entanglement entropy of \widetilde{W}_∞^k has never been computed though it is given by the integral:

$$\begin{aligned} - \int x \ln(x) \mu_\infty^k(dx) &= -\frac{1}{2\pi} \int \ln(x) \frac{\sqrt{4k(k-1)x - k^2x^2}}{k-x} \mathbf{1}_{[0,4(k-1)/k]}(x) dx \\ &= \frac{k}{2\pi} \int -\ln(x) \frac{\sqrt{4(k-1)x - k^2x^2}}{1-x} \mathbf{1}_{[0,4(k-1)/k^2]}(x) dx \end{aligned}$$

Expanding

$$-\frac{\ln(x)}{1-x} = \sum_{n \geq 0} \frac{(1-x)^n}{n+1},$$

one obtains (using Tonelli Theorem) the following expression of the asymptotic entanglement entropy in the stationary regime $t = +\infty$:

$$\begin{aligned} - \int x \ln(x) \mu_\infty^k(dx) &= \frac{[4(k-1)]^2}{2k^2\pi} \sum_{n \geq 0} \frac{1}{n+1} \int \left(1 - \frac{4(k-1)}{k^2}x\right)^n \sqrt{x(1-x)} \mathbf{1}_{[0,1]}(x) dx \\ &= \frac{(k-1)^2}{k^2} \sum_{n \geq 0} \frac{1}{n+1} {}_2F_1 \left(-n, \frac{3}{2}, 3; \frac{4(k-1)}{k^2} \right), \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function and the second equality follows from its Euler integral representation ([13], p. 59). In particular, when $k = 2$, the corresponding entropy reduces to (one either uses the Beta integral or equivalently the Gauss hypergeometric Theorem):

$$- \int x \ln(x) \mu_\infty^2(dx) = \frac{\Gamma(3)}{4\Gamma(3/2)} \sum_{n \geq 0} \frac{\Gamma(n + (3/2))}{(n+1)\Gamma(n+3)} = \frac{1}{\sqrt{\pi}} \sum_{n \geq 0} \frac{\Gamma(n + (3/2))}{(n+1)\Gamma(n+3)}.$$

For fixed time $t > 0$, the spectral distribution of J_t (and in turn of \widetilde{W}_t^k) is far from being as simple as in the stationary regime, unless $k = 2$. Nonetheless, we can compute the asymptotic entanglement entropy of \widetilde{W}_t^k appealing to Theorem 2 and using the expansion

$$-x \ln(x) = 1 - x + \sum_{m \geq 2} \frac{1}{m(m-1)} (1-x)^m, \quad 0 \leq x \leq 1,$$

which follows readily from

$$\ln(x) = - \sum_{m \geq 1} \frac{(1-x)^m}{m}.$$

To this end, we need the following lemma:

Lemma 5 For any $m \geq 2$,

$$\begin{aligned} \tau[(\mathbf{1} - \tilde{W}_t^k)^m] &= \tau \left[\left(\mathbf{1} - \frac{\tilde{W}_\infty^k}{1 + (k-1)e^{-t}} \right)^m \right] + k \sum_{j=1}^m \frac{(-1)^j}{k^{2j}} \binom{m}{j} \\ &\quad {}_2F_1 \left(j - m, j + \frac{1}{2}, 2j + 1; \frac{4(k-1)}{k^2} \right) \tau[(T_k U_t T_k U_t^*)^j]. \end{aligned}$$

Proof We use the binomial Theorem together with Theorem 2 to write

$$\begin{aligned} \tau[(\mathbf{1} - \tilde{W}_t^k)^m] &= 1 + \sum_{n=1}^m \binom{m}{n} (-1)^n \tau[(\tilde{W}_t^k)^n] \\ &= 1 + \sum_{n=1}^m \binom{m}{n} \frac{(-k)^n}{[1 + (k-1)e^{-t}]^n} m_n(t) \\ &= \tau \left[\left(\mathbf{1} - \frac{\tilde{W}_\infty^k}{1 + (k-1)e^{-t}} \right)^m \right] \\ &\quad + k \sum_{n=1}^m \binom{m}{n} \frac{(-1)^n}{k^{2n}} \sum_{j=1}^n (k-1)^{n-j} \binom{2n}{n-j} \tau[(T_k U_t T_k U_t^*)^j] \\ &= \tau \left[\left(\mathbf{1} - \frac{\tilde{W}_\infty^k}{1 + (k-1)e^{-t}} \right)^m \right] \\ &\quad + k \sum_{j=1}^m \tau[(T_k U_t T_k U_t^*)^j] \sum_{n=j}^m \frac{(-1)^n}{k^{2n}} (k-1)^{n-j} \binom{m}{n} \binom{2n}{n-j}. \end{aligned}$$

Now the duplication formula for the Gamma function entails:

$$\frac{(2n)!}{n!} = 4^n \frac{\Gamma(n + 1/2)}{\sqrt{\pi}}$$

so that

$$\begin{aligned} & \sum_{n=j}^m (-1)^n (k-1)^{n-j} \binom{m}{n} \binom{2n}{n-j} \\ &= \frac{m!}{\sqrt{\pi} k^{2j}} \sum_{n=0}^{m-j} \frac{(-4)^{n+j} \Gamma(n+j+1/2)}{(m-j-n)! n! \Gamma(n+2j+1)} \frac{(k-1)^n}{k^{2n}} \\ &= \frac{m! (-4)^j \Gamma(j+1/2)}{(m-j)! \sqrt{\pi} \Gamma(2j+1) k^{2j}} \sum_{n=0}^{m-j} \binom{m-j}{n} \frac{(-1)^n (j+1/2)_n}{(2j+1)_n} \left[\frac{4(k-1)}{k^2} \right]^n \\ &= \frac{(-1)^j}{k^{2j}} \binom{m}{j} \sum_{n=0}^{m-j} \frac{(j-m)_n (j+1/2)_n}{n! (2j+1)_n} \left[\frac{4(k-1)}{k^2} \right]^n \end{aligned}$$

where we used the Pochhammer symbol:

$$(x)_n := \frac{\Gamma(x+n)}{\Gamma(x)}, \quad x > 0, \quad (j-m)_n = (-1)^n \frac{(m-j)!}{(m-j-n)!}, \quad 0 \leq n \leq m-j.$$

Since $(j-m)_n = 0$ for any $n > m-j$, $1 \leq j \leq m$, then we recognize the Gauss hypergeometric (terminating) series:

$$\sum_{n=0}^{m-j} \frac{(j-m)_n (j+1/2)_n}{n! (2j+1)_n} [4(k-1)]^n = {}_2F_1 \left(j-m, j+\frac{1}{2}, 2j+1; \frac{4(k-1)}{k^2} \right)$$

and the lemma is proved. □

From this lemma, it follows that the asymptotic entanglement entropy may be written as:

$$\begin{aligned} -\tau \left(\tilde{W}_t^k \ln \left[\tilde{W}_t^k \right] \right) &= \frac{\ln(1 + (k-1)e^{-t}) - (k-1)e^{-t}}{1 + (k-1)e^{-t}} \\ &\quad - \frac{1}{1 + (k-1)e^{-t}} \tau \left(\tilde{W}_\infty^k \ln \left[\tilde{W}_\infty^k \right] \right) \\ &\quad + k \sum_{m \geq 2} \frac{1}{m(m-1)} \sum_{j=1}^m \frac{(-1)^j}{k^{2j}} \binom{m}{j}^2 \\ &\quad F_1 \left(j-m, j+\frac{1}{2}, 2j+1; \frac{4(k-1)}{k^2} \right) \tau [(T_k U_t T_k U_t^*)^j]. \end{aligned}$$

Equivalently,

$$\begin{aligned} & \frac{1}{1 + (k - 1)e^{-t}} \tau \left(\tilde{W}_\infty^k \ln \left[\tilde{W}_\infty^k \right] \right) - \tau \left(\tilde{W}_t^k \ln \left[\tilde{W}_t^k \right] \right) \\ &= \frac{\ln(1 + (k - 1)e^{-t}) - (k - 1)e^{-t}}{1 + (k - 1)e^{-t}} \\ &+ k \sum_{m \geq 2} \frac{1}{m(m - 1)} \sum_{j=1}^m \frac{(-1)^j}{k^{2j}} \binom{m}{j} {}_2F_1 \\ &\times \left(j - m, j + \frac{1}{2}, 2j + 1; \frac{4(k - 1)}{k^2} \right) \tau[(T_k U_t T_k U_t^*)^j]. \end{aligned}$$

In particular, when $k = 2$, the Gauss hypergeometric Theorem entails:

$$\begin{aligned} {}_2F_1 \left(j - m, j + \frac{1}{2}, 2j + 1; 1 \right) &= \frac{(j + 1/2)_{m-j}}{(2j + 1)_{m-j}} = \frac{\Gamma(m + 1/2)\Gamma(2j + 1)}{\Gamma(j + 1/2)\Gamma(m + j + 1)} \\ &= \frac{\Gamma(m + 1/2)4^j j!}{\sqrt{\pi}\Gamma(m + j + 1)}, \end{aligned}$$

whence

$$\begin{aligned} & \frac{1}{1 + e^{-t}} \tau \left(\tilde{W}_\infty^2 \ln \left[\tilde{W}_\infty^2 \right] \right) - \tau \left(\tilde{W}_t^2 \ln \left[\tilde{W}_t^2 \right] \right) = \frac{\ln(1 + e^{-t}) - e^{-t}}{1 + e^{-t}} \\ &+ 2 \sum_{m \geq 2} \frac{m!\Gamma(m + 1/2)}{\sqrt{\pi}m(m - 1)} \sum_{j=1}^m \frac{(-1)^j}{\Gamma(m + j + 1)(m - j)!} \tau[(T_2 U_t T_2 U_t^*)^j]. \end{aligned}$$

Moreover, $T_2 U_t T_2 U_t^*$ is unitary and is distributed as U_{2t} so that:

$$\tau[(T_2 U_t T_2 U_t^*)^j] = \frac{e^{-jt}}{j} L_{j-1}^{(1)}(2jt), \quad j \geq 1,$$

where $L_{j-1}^{(1)}$ is the Laguerre polynomial of index one.

5 Concluding remarks

In this paper, we introduced a dynamical random density matrix built out of $k \geq 2$ independent unitary Brownian motions whose large size limit has, up to a normalization, the same moments as those of the free Jacobi process $PU_t PU_t^* P$ (in the compressed algebra) subject to $\tau(P) = 1/k$. Motivated by our previous results proved in [12] valid for $k = 2$, we derived for any $k \geq 2$ a binomial-type expansion for these moments where the non normal operator T_k (except when $k = 2$) plays a key role and extends the orthogonal symmetry $S = 2P - \mathbf{1}$ associated with P . We also provided expressions

for the average purity and for the entanglement entropy of the large-size limiting density matrix at fixed time $t > 0$. Our expression for this entropy involves the moments $\tau[(T_k U_t T_k U_t^*)^j]$, $j \geq 0$, which do not seem to admit a simple form unless $k = 2$, as suggested by the analysis of the characteristic curves of the pde satisfied by their generating function.

On the other hand, the Lebesgue decomposition of the spectral distribution of J_t for arbitrary orthogonal projections (P, Q) was obtained in [14]. Specializing Theorem 1.1. there to $P = Q$ with $\tau(P) = 1/k$, we deduce that the corresponding distribution admits a density (with respect to Lebesgue measure) of the form:

$$\mu_t^k(x) = \frac{1}{2\pi} \frac{g_t^k(2 \arccos(\sqrt{x}))}{\sqrt{x - x^2}},$$

where g_t^k is the density of the spectral distribution of the unitary operator $SU_t SU_t^*$, $\tau(S) = 2\tau(P) - 1 = (2 - k)/k$. Similarly, g_t^k admits a very complicated expression unless $k = 2$.

Finally, it was recently proved in [11] (see eq. (3) there) that

$$\tau[(PQP)^n] - \tau[((1 - P)(1 - Q)(1 - P))^n] = \tau(P) + \tau(Q) - 1,$$

for any self-adjoint projections (P, Q) with arbitrary traces $\tau(P), \tau(Q), \in (0, 1)$. In particular, if $Q = U_t P U_t^*$ and $\tau(P) = 1/k$ then we readily deduce:

$$\tau[((1 - P)U_t(1 - P)U_t^*(1 - P))^n] = \tau[(P U_t P U_t^* P)^n] + 1 - \frac{2}{k}.$$

Consequently, the moments of the free Jacobi process associated with $1 - P$ may be deduced from those of the free Jacobi process associated with P :

$$\frac{k}{k - 1} \tau[((1 - P)U_t(1 - P)U_t^*(1 - P))^n] = \frac{k}{k - 1} \tau[(P U_t P U_t^* P)^n] + \frac{k - 2}{k - 1}.$$

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A. Moments of stationary distribution

In this appendix, we derive another expression of

$$m_n(\infty) = \int x^n \tilde{\mu}_\infty^k(dx) = \frac{1}{2\pi} \int x^{n-1/2} \frac{\sqrt{4(k - 1) - k^2 x}}{(1 - x)} \mathbf{1}_{[0, 4(k-1)/k^2]}(x) dx.$$

To the best of our best knowledge, formula (34) below never appeared in literature. Compared to (21), it has the merit to separate the case $k = 2$ corresponding to the arcsine distribution from other values $k \geq 3$. Our main ingredients are two properties satisfied by the Gauss hypergeometric function.

To proceed, perform the variable change $x = 4(k - 1)y/k^2$ to write:

$$\begin{aligned}
 m_n(\infty) &= \frac{4^{n+1}(k-1)^{n+1}}{2\pi k^{2n+1}} \int y^{n-1/2} \frac{\sqrt{1-y}}{1-4(k-1)y/k^2} \mathbf{1}_{[0,1]}(y) dy \\
 &= \frac{4^n(k-1)^{n+1}}{\sqrt{\pi}k^{2n+1}} \frac{\Gamma(n+1/2)}{\Gamma(n+2)} {}_2F_1\left(1, n + \frac{1}{2}, n + 2; \frac{4(k-1)}{k^2}\right) \\
 &= \frac{4^n(k-1)^{n+1}}{n!k^{2n+1}} \left\{ (-1)^n \sqrt{1-z} \frac{d^n}{dz^n} \left[(1-z)^{n-1/2} {}_2F_1\left(\frac{1}{2}, 1, 2; z\right) \right] \right\}_{z=4(k-1)/k^2} \\
 &= 2 \frac{4^n(k-1)^{n+1}}{n!k^{2n+1}} \left\{ \sqrt{1-z} (-1)^n \frac{d^n}{dz^n} \left[\frac{(1-z)^{n-1/2}}{1+\sqrt{1-z}} \right] \right\}_{z=4(k-1)/k^2} \\
 &= 2 \frac{4^n(k-1)^{n+1}}{n!k^{2n+1}} \left\{ \sqrt{z} \frac{d^n}{dz^n} \left[\frac{z^{n-1/2}}{1+\sqrt{z}} \right] \right\}_{z=(k-2)/k^2} \\
 &= 2 \frac{4^n(k-1)^{n+1}}{n!k^{2n+1}} \left\{ \sqrt{z} \frac{d^n}{dz^n} \left[z^{n-1/2} - \frac{z^n}{1+\sqrt{z}} \right] \right\}_{z=(k-2)/k^2} \\
 &= \frac{2(k-1)^{n+1}}{k^{2n+1}} \left\{ \binom{2n}{n} - \frac{4^n}{n!} \sqrt{z} \frac{d^n}{dz^n} \left[\frac{z^n}{1+\sqrt{z}} \right] \right\}_{z=(k-2)/k^2}.
 \end{aligned}$$

Here, the second equality follows from the Euler integral representation of the Gauss hypergeometric function, the third and fourth ones follow from the variational formula (25), p.102 and formula (6), p.101 in [13]. Using direct computations, we readily see that

$$\frac{d^n}{dz^n} \left[\frac{z^n}{1+\sqrt{z}} \right] = \frac{\mathcal{P}_n(\sqrt{z})}{2^n(1+\sqrt{z})^{n+1}}$$

for some polynomial of degree n . For instance

$$\begin{aligned}
 \mathcal{P}_0(x) &= 1, \mathcal{P}_1(x) = x + 2, \mathcal{P}_2(x) = 3x^2 + 9x + 8, \\
 \mathcal{P}_3(x) &= 15x^3 + 60x^2 + 87x + 48.
 \end{aligned}$$

Consequently,

$$m_n(\infty) = \frac{2(k-1)^{n+1}}{k^{2n+1}} \left\{ \binom{2n}{n} - \frac{(k-2)}{2n!(k-1)^{n+1}} k^n \mathcal{P}_n\left(\frac{k-2}{k}\right) \right\}, \tag{34}$$

which reduces for $k = 2$ to the known formula:

$$m_n(\infty) = \frac{(2n)!}{2^{2n}(n!)^2}.$$

Appendix B. Free cumulants of an orthogonal projection

The first part of the proof of (29) is a routine computation in free probability theory and we refer the reader to [18] for further details on this machinery. Start with the Cauchy transform of P :

$$\tau[(z - P)^{-1}] = \frac{\alpha}{z - 1} + \frac{1 - \alpha}{z} = \frac{z + \alpha - 1}{z(z - 1)}, \quad z \notin \{0, 1\}.$$

Next, consider the equation

$$yz^2 - z(y + 1) + 1 - \alpha = 0,$$

for y lying in a neighborhood of zero. Then the K -transform of P reads:

$$K(y) = \frac{y + 1 + \sqrt{y^2 + 1 - 2y(1 - 2\alpha)}}{2y},$$

and in turn, its R -transform is given by

$$R(y) = K(y) - \frac{1}{y} = \frac{1}{2} \left[1 + \frac{\sqrt{y^2 + 1 - 2y(1 - 2\alpha)} - 1}{y} \right].$$

It remains to write down the Taylor expansion of the function:

$$f_\alpha : y \mapsto \frac{\sqrt{y^2 + 1 - 2y(1 - 2\alpha)} - 1}{y}.$$

To this end, we appeal to the generating series of Legendre polynomials:

$$\sum_{n=0}^{\infty} P_n(x)y^n = \frac{1}{\sqrt{1 + y^2 - 2xy}}, \quad |y| < 1.$$

Indeed, setting $\beta := 1 - 2\alpha \in [-1, 1]$, one has

$$[yf_\alpha(y)]' = (y - \beta) \sum_{n \geq 0} P_n(\beta)y^n$$

so that

$$\begin{aligned} f_\alpha(y) &= \sum_{n \geq 1} \frac{y^n}{n + 1} [P_{n-1}(\beta) - \beta P_n(\beta)] - \beta \\ &= \sum_{n \geq 1} \frac{y^n}{2n + 1} [P_{n-1}(\beta) - P_{n+1}(\beta)] - \beta \end{aligned}$$

where the last equality follows from the recurrence relation:

$$(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x).$$

Extracting the Taylor coefficients of f_α and recalling the definition

$$R(y) = \sum_{n \geq 0} \kappa_{n+1}(P)y^n$$

we get (29).

Note that since Legendre polynomials are orthogonal with respect to the uniform distribution in $[-1, 1]$, they are parity preserving. In particular,

$$P_{2n+1}(0) = 0, \quad P_{2n}(0) = (-1)^n \frac{(1/2)_n}{n!},$$

so that one recovers (28) after some computations.

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