



Monodromy free linear equations and many-body systems

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Abstract

We further develop the approach to many-body systems based on finding conditions of existence of meromorphic solutions to certain linear partial differential and difference equations which serve as auxiliary linear problems for nonlinear integrable equations such as KP, BKP, CKP and different versions of the Toda lattice. These conditions imply equations of the time evolution for poles of singular solutions to the nonlinear equations which are equations of motion for integrable many-body systems of Calogero–Moser and Ruijsenaars–Schneider type. A new many-body system is introduced, which governs dynamics of poles of elliptic solutions to the Toda lattice of type B.

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1 Introduction

Dynamics of poles of singular solutions to nonlinear integrable equations is a well-known subject in the theory of integrable systems. Investigations in this direction were initiated in the seminal paper [1]. In [2, 3], it was shown that poles x_i of rational solutions to the Kadomtsev–Petviashvili (KP) equation move as particles of the many-body Calogero–Moser system [4–6] with the pairwise potential $1/(x_i - x_j)^2$. This remarkable connection was further generalized to elliptic (double-periodic) solutions in [7]: Poles x_i of the elliptic solutions were shown to move according to the equations of motion of Calogero–Moser particles with the elliptic interaction potential $\wp(x_i - x_j)$, where \wp is the elliptic Weierstrass \wp -function. This many-body system of classical mechanics is known to be integrable. For a review of the models of the Calogero–Moser type, see [8].

The correspondence between singular solutions of nonlinear integrable equations and integrable many-body systems allows for generalizations in various directions. The extension to the matrix KP equation was discussed in [9]; in this case the poles and matrix residues at the poles move as particles of the spin generalization of the Calogero–Moser model known also as the Gibbons–Hermsen model [10]. In the paper [11] by the authors, the dynamics of poles of elliptic solutions to the (matrix) two-dimensional Toda lattice was shown to be isomorphic to the relativistic extension of the Calogero–Moser system known also as the Ruijsenaars–Schneider system [12, 13] and its version with spin degrees of freedom [11]. Another generalizations concern B- and C-versions of the KP equation (BKP and CKP). Equations of motion for poles of elliptic solutions to the BKP and CKP equations were recently obtained in [14] and [15], respectively. For a review, see [16].

The method suggested in [2, 7] and used in all subsequent works on the subject is based on the well-known fact that nonlinear integrable equations such as KP, BKP, CKP and the 2D Toda lattice can be represented as compatibility conditions for systems of certain overdetermined linear problems which are partial differential or difference equations in two variables (space and time). The suggested scheme of finding the dynamics of poles consists in substituting the elliptic solution not in the nonlinear equation itself but in the linear problems for it, using a suitable pole ansatz for the wave function depending on a spectral parameter.

In this paper, we systematically use another method suggested by one of the authors in [17] on the example of the KP/Calogero–Moser correspondence and further discussed in [18, 19]. The key point of this method is existence of meromorphic solutions to the linear partial differential or difference equations. We call them *monodromy free linear equations*. It turns out that the conditions of existence of meromorphic solu-

tions in the space variable are equivalent to equations of motion for the poles which are equations of motion for integrable many-body systems of Calogero–Moser and Ruijsenaars–Schneider type.

We also prove that existence of at least one meromorphic solution implies existence of a whole family of meromorphic wave solutions depending on a spectral parameter.

For completeness, we include in this paper the analysis of meromorphic solutions to the linear problems for the KP and Toda lattice equations leading to the Calogero–Moser and Ruijsenaars–Schneider systems, respectively. (This is contained in the earlier works [17, 18].) Conditions of existence of meromorphic solutions to the linear problems for the BKP and CKP equations as well as for the Toda lattice of type C were not discussed in the literature; in this paper we find them and show that they are equivalent to the equations of motion for poles of these equations obtained in [14, 15, 22].

The main new result of this paper is the equations of motion (3.59) for poles of elliptic solutions to the Toda lattice of type B recently introduced in [20]. They have the form

$$\ddot{x}_i + \sum_{k=1, \neq i}^N \dot{x}_i \dot{x}_k \left(\zeta(x_{ik} + \eta) + \zeta(x_{ik} - \eta) - 2\zeta(x_{ik}) \right) - U(x_{i1}, \dots, x_{iN}) = 0, \tag{1.1}$$

where dot means the time derivative, $x_{ik} \equiv x_i - x_k$,

$$U(x_{i1}, \dots, x_{iN}) = \sigma(2\eta) \left[\prod_{j \neq i} \frac{\sigma(x_{ij} + 2\eta)\sigma(x_{ij} - \eta)}{\sigma(x_{ij} + \eta)\sigma(x_{ij})} - \prod_{j \neq i} \frac{\sigma(x_{ij} - 2\eta)\sigma(x_{ij} + \eta)}{\sigma(x_{ij} - \eta)\sigma(x_{ij})} \right] \tag{1.2}$$

and $\sigma(x)$, $\zeta(x)$ are the standard Weierstrass functions (see Appendix A). These equations are obtained from the condition of existence of meromorphic solutions to the differential–difference auxiliary linear problem for the Toda lattice of type B. We also show that the same equations can be obtained by restriction of the Ruijsenaars–Schneider dynamics with respect to the time flow $\partial_{t_1} - \partial_{\bar{t}_1}$ of the system containing $2N$ particles to the half-dimensional subspace of the $4N$ -dimensional phase space corresponding to the configuration in which the particles stick together joining in pairs such that the distance between particles in each pair is equal to η . This configuration is destroyed by the flow $\partial_{t_1} + \partial_{\bar{t}_1}$ but is preserved by the flow $\partial_{t_1} - \partial_{\bar{t}_1}$ (and, hypothetically, by all higher flows $\partial_{t_k} - \partial_{\bar{t}_k}$), and the time evolution in $t = t_1 - \bar{t}_1$ of the pairs with coordinates x_i is given by equations (1.1), (1.2). This fact is not so surprising if we recall that the tau-function $\tau^{\text{Toda}}(x)$ of the Toda lattice (whose zeros move as Ruijsenaars–Schneider particles) is connected with the tau-function $\tau(x)$ of the Toda lattice of type B (whose zeros move according to equations (1.1)) by the relation

$$\tau^{\text{Toda}}(x) = \tau(x) \tau(x - \eta) \tag{1.3}$$

(see [20]), so zeros of $\tau^{\text{Toda}}(x)$ stick together in pairs.

To avoid a confusion, we should stress that what we mean by the Toda lattice of type B or C is very different from the systems introduced in [23] under similar names. Our equations are natural integrable discretizations of the BKP and CKP equations, that is why we found it appropriate to call them “Toda lattices of type B and C.”

The reader should be aware that the notation for coefficients of Laurent expansions below is valid only throughout each section and the same notation may mean different things in different sections. We hope that this will not lead to a misunderstanding since each section is devoted to its own linear equation and the contents do not intersect.

2 Differential equations

2.1 The KP case

We start with a warm-up exercise following [17].

Consider the linear equation

$$(\partial_t - \partial_x^2 - 2u)\psi = 0, \quad (2.1)$$

which is one of the auxiliary linear problems for the KP equation. It is easy to see that if $u(x)$ has a pole a in the complex x -plane, it must be a second-order pole. Expanding the left-hand side in a neighborhood of the pole, one can find a necessary condition of existence of a meromorphic solution in this neighborhood.

Proposition 2.1 *If equation (2.1) with*

$$u(x) = -\frac{1}{(x-a)^2} + u_0 + u_1(x-a) + \dots, \quad (2.2)$$

has a meromorphic in x solution, then the condition

$$\ddot{a} + 4u_1 = 0 \quad (2.3)$$

holds, where dot means the time derivative.

Proof Let the expansion of $\psi(x)$ around the point a be of the form

$$\psi(x) = \frac{\alpha}{x-a} + \beta + \gamma(x-a) + \delta(x-a)^2 + \dots \quad (2.4)$$

Substituting expansions (2.2), (2.4) in the left-hand side of (2.1), we see that the highest (third-order)-order poles cancel identically. Equating the coefficients in front of $(x-a)^{-2}$, $(x-a)^{-1}$ and $(x-a)^0$ to zero, we get the conditions

$$\begin{cases} \dot{\alpha} + 2\beta = 0, \\ \dot{\alpha} + 2\gamma - 2u_0\alpha = 0, \\ \dot{\beta} - \dot{\alpha}\gamma - 2u_0\beta - 2u_1\alpha = 0, \end{cases} \tag{2.5}$$

Taking t -derivative of the first equation, plugging $\dot{\alpha}$ and $\dot{\beta}$ from the second and the third ones and using the first one again, we obtain the necessary condition (2.3). \square

One can see that this condition encodes equations of motion for the (elliptic in general) Calogero–Moser system. Indeed, let $u(x)$ be the doubly-periodic meromorphic function

$$u(x) = - \sum_i \wp(x - x_i), \tag{2.6}$$

where $\wp(x)$ is the Weierstrass \wp -function, then the expansion (2.2) near the pole at $a = x_i$ holds true with

$$u_0 = - \sum_{j \neq i} \wp(x_i - x_j), \quad u_1 = - \sum_{j \neq i} \wp'(x_i - x_j),$$

so the conditions (2.3) for each x_i read

$$\ddot{x}_i = 4 \sum_{j \neq i} \wp'(x_i - x_j), \tag{2.7}$$

which are the equations of motion for the elliptic Calogero–Moser system.

Next, we show that (2.3) is simultaneously a sufficient condition for local existence of a meromorphic wave solution to equation (2.1), i.e., a solution depending on a spectral parameter k with the expansion of the form

$$\psi(x) = e^{kx+k^2t} \left(1 + \sum_{s \geq 1} \xi_s k^{-s} \right), \quad k \rightarrow \infty. \tag{2.8}$$

Proposition 2.2 *Suppose that condition (2.3) for the pole a of $u(x)$ holds. Then, all wave solutions of equation (2.1) of the form (2.8) are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular elsewhere in this neighborhood.*

Proof Substitution of the series (2.8) into the equation (2.1) gives the recurrence relation

$$2\xi'_{s+1} = \dot{\xi}_s - 2u\xi_s - \xi''_s, \quad s \geq 0, \quad \xi_0 \equiv 1. \tag{2.9}$$

In particular, at $s = 0$ we have

$$\xi'_1 = -u. \tag{2.10}$$

Let the Laurent expansion of ξ_s near the pole at $x = a$ be of the form

$$\xi_s = \frac{r_s}{x - a} + r_{s,0} + r_{s,1}(x - a) + \dots, \tag{2.11}$$

and the expansion of $u(x)$ be as in (2.2). The solution is meromorphic if the residue of the right-hand side of (2.9) vanishes:

$$2 \operatorname{res}_{x=a} \xi'_{s+1} = \dot{r}_s + 2r_{s,1} - 2u_0r_s = 0. \tag{2.12}$$

At $s = 0$, we have $\operatorname{res}_{x=a} \xi'_1 = 0$ from (2.10). We are going to prove (2.12) by induction in s . Assume that (2.12) holds for some s , then it is easy to see that the condition (2.3) implies that it holds for $s + 1$. Indeed, substituting the expansion (2.11) into the equation and equating the coefficients of $(x - a)^{-2}$, $(x - a)^{-1}$ and $(x - a)^0$ to zero, we get the conditions

$$\begin{cases} 2r_{s+1} = -r_s\dot{a} - 2r_{s,0}, \\ \dot{r}_s + 2r_{s,1} - 2u_0r_s = 0, \\ 2r_{s+1,1} = \dot{r}_{s,0} - r_{s,1}\dot{a} - 2u_0r_{s,0} - 2u_1r_s. \end{cases} \tag{2.13}$$

Substituting them into the right-hand side of (2.12) at $s \rightarrow s + 1$, we have:

$$\begin{aligned} 2 \operatorname{res}_{x=a} \xi'_{s+2} &= \dot{r}_{s+1} + 2r_{s+1,1} - 2u_0r_{s+1} \\ &= \left(-\frac{1}{2}\dot{r}_s\dot{a} - \frac{1}{2}r_s\ddot{a} - \dot{r}_{s,0}\right) + \left(\dot{r}_{s,0} - r_{s,1}\dot{a} - 2u_0r_{s,0} - 2u_1r_s\right) + \left(u_0r_s\dot{a} + 2u_0r_{s,0}\right), \end{aligned}$$

where we have used the first and the third equations in (2.13). After cancellations, we get:

$$2 \operatorname{res}_{x=a} \xi'_{s+2} = -\frac{1}{2}r_s(\ddot{a} + 4u_1) - \frac{1}{2}\dot{a}(\dot{r}_s + 2r_{s,1} - 2u_0r_s) = 0.$$

(The second term vanishes by the induction assumption, and the first one is zero due to the condition (2.3).) □

2.2 The CKP case

Consider the linear equation

$$(\partial_t - \partial_x^3 - 6u\partial_x - 3u')\psi = 0, \tag{2.14}$$

which is one of the auxiliary linear problems for the CKP equation.

Proposition 2.3 *Suppose that $u(x)$ in (2.14) has a pole at $x = a$ and equation (2.14) has a meromorphic solution, then the condition*

$$\dot{a} + 6u_0 = 0 \tag{2.15}$$

holds, where u_0 is the coefficient in the Laurent expansion

$$u(x) = -\frac{1}{2(x-a)^2} + u_0 + u_1(x-a) + \dots \tag{2.16}$$

Proof We have the expansion near the pole at $x = a$:

$$\psi(x) = \frac{\alpha}{x-a} + \beta + \gamma(x-a) + \delta(x-a)^2 + \dots \tag{2.17}$$

Substituting the expansions in the left-hand side of (2.14), we see that the highest (fourth-order)-order poles cancel identically. The necessary condition of cancellation of the third-order poles is $\beta = 0$. Equating the coefficients in front of $(x-a)^{-2}$, $(x-a)^{-1}$ and $(x-a)^0$ to zero, we get the conditions

$$\begin{cases} \alpha\dot{a} + 6\alpha u_0 = 0, \\ \dot{\alpha} + 3\alpha u_1 + 3\delta = 0, \\ \gamma\dot{a} + 6\gamma u_0 = 0. \end{cases} \tag{2.18}$$

The first and the third equations are equivalent and lead to the necessary condition (2.15). □

Let $u(x)$ be the elliptic function

$$u(x) = -\frac{1}{2} \sum_i \wp(x - x_i), \tag{2.19}$$

then the expansion (2.16) near the pole at $x = x_i$ holds true with $u_0 = -\frac{1}{2} \sum_{j \neq i} \wp(x_i - x_j)$, so the conditions (2.15) for each x_i read

$$\dot{x}_i = 3 \sum_{j \neq i} \wp(x_i - x_j), \tag{2.20}$$

which are the equations of motion for poles of elliptic solutions to the CKP equation derived in [15]. As shown in [15], they are obtained by restriction of the third flow of the Calogero–Moser system to the submanifold of turning points.

Next, we show that (2.15) is simultaneously a sufficient condition for local existence of meromorphic wave solutions to equation (2.14) with the expansion of the form

$$\psi = e^{kx+k^3t} \left(1 + \sum_{s \geq 1} \xi_s k^{-s} \right), \quad k \rightarrow \infty. \tag{2.21}$$

Proposition 2.4 *Suppose that condition (2.15) for the pole a of $u(x)$ holds. Then, all wave solutions of equation (2.14) of the form (2.21) are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular elsewhere in this neighborhood.*

Proof Substitution of the series into equation (2.14) gives the recurrence relation

$$\begin{aligned} \dot{\xi}_s - 3\xi'_{s+2} - 3\xi''_{s+1} - \xi'''_s - 6u\xi_{s+1} - 6u\xi'_s - 3u'\xi_s &= 0, \\ s \geq -1, \quad \xi_{-1} \equiv 0, \quad \xi_0 \equiv 1. \end{aligned} \tag{2.22}$$

In particular, at $s = -1$ we have

$$\xi'_1 = -2u. \tag{2.23}$$

It is convenient to represent equation (2.22) in the form

$$f'_s = \dot{\xi}_s - 6u\xi_{s+1} - 3u\xi'_s, \quad f_s = 3\xi_{s+2} + 3\xi'_{s+1} + \xi''_s + 3u\xi_s. \tag{2.24}$$

Let the Laurent expansion of ξ_s near the pole at $x = a$ be

$$\xi_s = \frac{r_s}{x-a} + r_{s,0} + r_{s,1}(x-a) + r_{s,2}(x-a)^2 + \dots, \tag{2.25}$$

and the expansion of $u(x)$ be as in (2.16). The solution is meromorphic if the residue of the right-hand side of (2.24) vanishes:

$$\operatorname{res}_{x=a} f'_s = \dot{r}_s + 3r_{s+1,1} + 3r_{s,2} - 6u_0r_{s+1} + 3u_1r_s = 0. \tag{2.26}$$

At $s = -1$, we have $\operatorname{res}_{x=a} f'_{-1} = 0$ from (2.23). We are going to prove (2.26) by induction in s . Assume that (2.26) holds for some s , then it is easy to see that the condition (2.15) implies that it holds for $s + 1$. Indeed, substituting the expansion (2.25) into the equation and equating the coefficients of $(x-a)^{-3}$, $(x-a)^{-2}$, $(x-a)^{-1}$ and $(x-a)^0$ to zero, we get the conditions

$$\begin{cases} r_{s+1} + r_{s,0} = 0, \\ 3r_{s+2} + r_s \dot{a} + 3r_{s+1,0} + 6u_0 r_s = 0, \\ \dot{r}_s + 3r_{s+1,1} + 3r_{s,2} - 6u_0 r_{s+1} + 3u_1 r_s = 0, \\ \dot{r}_{s,0} - r_{s,1} \dot{a} - 3r_{s+2,1} - 3r_{s+1,2} - 6u_0 r_{s+1,0} - 6u_0 r_{s,1} - 3u_1 r_{s+1} = 0. \end{cases} \tag{2.27}$$

Substituting them into the right-hand side of (2.26) at $s \rightarrow s + 1$, we have:

$$\begin{aligned} \operatorname{res}_{x=a} f'_{s+1} &= \dot{r}_{s+1} + 3r_{s+2,1} + 3r_{s+1,2} - 6u_0 r_{s+2} + 3u_1 r_{s+1} \\ &= (2u_0 r_s - r_{s,1})(\dot{a} + 6u_0) - 2u_0(3r_{s+2} \\ &\quad + r_s \dot{a} + 3r_{s+1,0} + 6u_0 r_s) = 0. \end{aligned}$$

(The second term vanishes because of the second equation in (2.27), and the first one is zero due to the condition (2.15).) □

2.3 The BKP case

Consider the linear equation

$$(\partial_t - \partial_x^3 - 6u\partial_x)\psi = 0, \tag{2.28}$$

which is one of the auxiliary linear problems for the BKP equation.

Proposition 2.5 *Suppose that $u(x)$ in (2.28) has a pole at $x = a$ and equation (2.28) has a meromorphic solution, then the condition*

$$\ddot{a} + 6\dot{u}_0 - 12u_1(\dot{a} + 6u_0) - 36u_3 = 0 \tag{2.29}$$

holds, where u_0, u_1, u_3 are coefficients in the Laurent expansion

$$u(x) = -\frac{1}{(x-a)^2} + u_0 + u_1(x-a) + u_2(x-a)^2 + u_3(x-a)^3 + \dots \tag{2.30}$$

Proof We have the expansion near the pole at $x = a$:

$$\psi(x) = \frac{\alpha}{x-a} + \beta + \gamma(x-a) + \delta(x-a)^2 + \varepsilon(x-a)^3 + \mu(x-a)^4 + \dots \tag{2.31}$$

Substituting the expansions in the left-hand side of (2.28), we see that possible fourth- and third-order poles cancel identically. Equating the coefficients in front of $(x-a)^{-2}$, $(x-a)^{-1}$, $(x-a)^0$ and $(x-a)$ to zero, we get the conditions

$$\begin{cases} \alpha \dot{\alpha} + 6\alpha u_0 + 6\gamma = 0, \\ \dot{\alpha} + 6\alpha u_1 + 12\delta = 0, \\ \dot{\beta} - \gamma \dot{\alpha} - 6\gamma u_0 + 6\alpha u_2 + 12\varepsilon = 0, \\ \dot{\gamma} - 2\delta \dot{\alpha} - 12\delta u_0 - 6\gamma u_1 + 6\alpha u_3 = 0. \end{cases} \quad (2.32)$$

Note that the terms with the coefficient μ in (2.31) which might enter the last condition actually cancel. Taking t -derivative of the first equation, we have

$$\ddot{\alpha} + 6\dot{u}_0 + 6\frac{\dot{\gamma}}{\alpha} - 6\frac{\gamma\dot{\alpha}}{\alpha^2} = 0.$$

Plugging here $\dot{\alpha}$ from the second equation and $\dot{\gamma}$ from the fourth one, we obtain the necessary condition (2.29). \square

One can see that this condition encodes equations of motion for the many-body system obtained in [14] as a dynamical system for motion of poles of elliptic solutions to the BKP equation. Indeed, let $u(x)$ be the doubly-periodic meromorphic function

$$u(x) = - \sum_i \wp(x - x_i), \quad (2.33)$$

then the expansion (2.30) near the pole at $x = x_i$ holds true with

$$u_0 = - \sum_{j \neq i} \wp(x_i - x_j), \quad u_1 = - \sum_{j \neq i} \wp'(x_i - x_j), \quad u_3 = -\frac{1}{6} \sum_{j \neq i} \wp'''(x_i - x_j)$$

and $\dot{u}_0 = - \sum_{j \neq i} (\dot{x}_i - \dot{x}_j) \wp'(x_i - x_j)$. Therefore, the conditions (2.29) for each x_i give the equations of motion derived in [14]:

$$\ddot{x}_i + 6 \sum_{j \neq i} (\dot{x}_i + \dot{x}_j) \wp'(x_i - x_j) - 72 \sum_{j \neq k \neq i} \wp(x_i - x_j) \wp'(x_i - x_k) = 0. \quad (2.34)$$

(The identity $\wp'''(x) = 12\wp(x)\wp'(x)$ is used.) As shown in [21], the same equations can be obtained by restriction of the third flow of the elliptic Calogero–Moser system to the subspace of the phase space in which $2N$ particles stick together in pairs in such a way that the two particles in each pair are in one and the same point. This configuration is immediately destroyed by the second flow of the Calogero–Moser system but is preserved by the third one, and coordinates of the pairs are subject to equations (2.34).

Next, we will show that (2.29) is simultaneously a sufficient condition for local existence of meromorphic wave solutions to equation (2.28) of the form

$$\psi(x) = e^{kx+k^3t} \left(1 + \sum_{s \geq 1} \xi_s k^{-s} \right), \quad k \rightarrow \infty. \tag{2.35}$$

Proposition 2.6 *Suppose that condition (2.29) for the pole of $u(x)$ holds. Then, all wave solutions of equation (2.28) of the form (2.35) are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular elsewhere in this neighborhood.*

Proof Substitution of the series into the equation (2.28) gives the recurrence relation

$$\dot{\xi}_s - 3\xi'_{s+2} - 3\xi''_{s+1} - \xi'''_s - 6u\xi_{s+1} - 6u\xi'_s = 0, \quad s \geq -1, \quad \xi_{-1} \equiv 0, \quad \xi_0 \equiv 1. \tag{2.36}$$

In particular, at $s = -1$ we have

$$\xi'_1 = -2u. \tag{2.37}$$

It is convenient to represent equation (2.36) in the form

$$g'_s = \dot{\xi}_s - 6u\xi_{s+1} - 6u\xi'_s, \quad g_s = 3\xi_{s+2} + 3\xi'_{s+1} + \xi''_s. \tag{2.38}$$

Let the Laurent expansion of ξ_s near the pole at $x = a$ be

$$\xi_s = \frac{r_s}{x-a} + r_{s,0} + r_{s,1}(x-a) + r_{s,2}(x-a)^2 + r_{s,3}(x-a)^3 + \dots, \tag{2.39}$$

and the expansion of $u(x)$ be as in (2.30). The solution is meromorphic if the residue of the right-hand side of (2.38) vanishes:

$$\operatorname{res}_{x=a} g'_s = \dot{r}_s + 6r_{s+1,1} + 12r_{s,2} - 6u_0r_{s+1} + 6u_1r_s = 0. \tag{2.40}$$

At $s = -1$, we have $\operatorname{res}_{x=a} g'_{-1} = 0$ from (2.37). As before, we will prove (2.40) by induction in s . Assume that (2.40) holds for some s , then it is easy to see that the condition (2.29) implies that it holds for $s + 1$. Indeed, substituting the expansion (2.39) into the equation and equating the coefficients of $(x-a)^{-2}$, $(x-a)^{-1}$, $(x-a)^0$ and $(x-a)$ to zero, we get the conditions

$$\begin{cases} 3r_{s+2} + r_s \dot{a} + 6r_{s+1,0} + 6r_{s,1} + 6u_0 r_s = 0, \\ \dot{r}_s + 6r_{s+1,1} + 12r_{s,2} - 6u_0 r_{s+1} + 6u_1 r_s = 0, \\ \dot{r}_{s,0} - r_{s,1} \dot{a} - 3r_{s+2,1} + 12r_{s,3} - 6u_0 r_{s+1,0} - 6u_0 r_{s,1} - 6u_1 r_{s+1} + 6u_2 r_s = 0, \\ \dot{r}_{s,1} - 2r_{s,2} \dot{a} - 6r_{s+2,2} - 12r_{s+1,3} - 6u_0 r_{s+1,1} - 12u_0 r_{s,2} \\ - 6u_1 r_{s+1,0} - 6u_1 r_{s,1} - 6u_2 r_{s+1} + 6u_3 r_s = 0. \end{cases} \tag{2.41}$$

Substituting them into the right-hand side of (2.40) at $s \rightarrow s + 1$, we have:

$$\begin{aligned} 3 \operatorname{res}_{x=a} g'_{s+1} &= 3\dot{r}_{s+1} + 18r_{s+2,1} + 36r_{s+1,2} - 18u_0 r_{s+2} + 18u_1 r_{s+1} \\ &= -r_{s-1}(\ddot{a} + 6\dot{u}_0 - 12u_1(\dot{a} + 6u_0) - 36u_3) \\ &\quad - \dot{a}(\dot{r}_{s-1} + 12r_{s-1,2} + 6r_{s,1} - 6u_0 r_s + 6u_1 r_{s-1}) \\ &\quad - 6u_0(r_s \dot{a} + 3r_{s+2} + 6r_{s+1,0} + 6r_{s,1} + 6u_0 r_s) - 6u_1(r_{s-1} \dot{a} + 3r_{s+1} \\ &\quad + 6r_{s,0} + 6r_{s-1,1} + 6u_0 r_{s-1}) = 0. \end{aligned}$$

(The last two terms vanish because of the first equation in (2.41), the second term vanishes due to the induction assumption, and the first one is zero due to the condition (2.29).) □

3 Differential–difference equations

3.1 The Toda lattice case

The 2D Toda lattice equation is the compatibility condition for the linear differential–difference equations

$$\partial_{t_1} \psi(x) = \psi(x + \eta) + b(x)\psi(x), \tag{3.1}$$

$$\partial_{\bar{t}_1} \psi(x) = v(x)\psi(x - \eta), \tag{3.2}$$

where η is a parameter (the lattice spacing). Suppose that $b(x)$ and $v(x)$ are meromorphic functions; let us investigate when these equations have meromorphic solutions in x .

3.1.1 The linear problem with respect to t_1

First we consider the linear problem (3.1):

$$\partial_t \psi(x) = \psi(x + \eta) + b(x)\psi(x). \tag{3.3}$$

Let $b(x)$ have first-order poles at $x = a$ and $x = a - \eta$:

$$b(x) = \begin{cases} \frac{\nu}{x - a} + \mu_0 + O(x - a), & x \rightarrow a \\ -\frac{\nu}{x - a + \eta} + \mu_1 + O(x - a + \eta), & x \rightarrow a - \eta. \end{cases} \tag{3.4}$$

Proposition 3.1 *Suppose that $b(x)$ in (3.3) has poles at $x = a$ and $x = a - \eta$ with expansions near the poles of the form (3.4). If equation (3.3) has a meromorphic solution with a pole at the point a regular at $a \pm \eta$, then the condition*

$$\ddot{a} - \dot{a}(\mu_0 + \mu_1) = 0 \tag{3.5}$$

holds.

Proof Let the expansion of $\psi(x)$ around the point a be of the form

$$\psi(x) = \frac{\alpha}{x - a} + \beta + O(x - a), \tag{3.6}$$

then

$$\partial_t \psi(x) = \frac{\alpha \dot{a}}{(x - a)^2} + \frac{\dot{\alpha}}{x - a} + O(1).$$

Substituting the expansions around the point a in the equation, we write:

$$\frac{\alpha \dot{a}}{(x - a)^2} + \frac{\dot{\alpha}}{x - a} + O(1) = \left(\frac{\nu}{x - a} + \mu_0 + \dots \right) \left(\frac{\alpha}{x - a} + \beta + \dots \right).$$

Equating the coefficients in front of the poles, we obtain the conditions

$$\begin{cases} \nu = \dot{a}, \\ \dot{\alpha} = \nu\beta + \mu_0\alpha. \end{cases} \tag{3.7}$$

Around the point $a - \eta$, we have:

$$= \frac{\alpha}{x - a + \eta} + \beta - \left(\frac{\nu}{x - a + \eta} + \mu_1 + \dots \right) \left(\psi(a - \eta) + (x - a + \eta)\psi'(a - \eta) + \dots \right).$$

Equating the coefficients in front of the terms of order $(x - a + \eta)^{-1}$ and $(x - a + \eta)^0$, we obtain the conditions

$$\begin{cases} \alpha = v\psi(a - \eta), \\ \partial_t \psi(a - \eta) = \beta - \mu_1 \psi(a - \eta) - v\psi'(a - \eta). \end{cases} \tag{3.8}$$

Taking the time derivative of the first equation in (3.8) and using (3.7), we get

$$\dot{\alpha} = \ddot{a}\psi(a - \eta) + \dot{a}\dot{\psi}(a - \eta), \tag{3.9}$$

where

$$\dot{\psi}(a - \eta) = \partial_t \psi(a - \eta) + \dot{a}\psi'(a - \eta) \tag{3.10}$$

is the full time derivative of $\psi(a - \eta)$. Now, combining equations (3.7)–(3.10), we arrive at the condition (3.5). \square

Suppose now that $b(x)$ is an elliptic function of x having $2N$ first-order poles in the fundamental domain at some points x_j and at the points $x_j - \eta$, then it must have the form

$$b(x) = \sum_j \dot{x}_j \left(\zeta(x - x_j) - \zeta(x - x_j + \eta) \right). \tag{3.11}$$

Let $a = x_i$ for some i , then the coefficients μ_0, μ_1 in (3.4) are

$$\begin{aligned} \mu_0 &= \sum_{k \neq i} \dot{x}_k \zeta(x_i - x_k) - \sum_k \dot{x}_k \zeta(x_i - x_k + \eta), \\ \mu_1 &= \sum_{k \neq i} \dot{x}_k \zeta(x_i - x_k) - \sum_k \dot{x}_k \zeta(x_i - x_k - \eta) \end{aligned} \tag{3.12}$$

and (3.5) is equivalent to the equation of motion

$$\ddot{x}_i + \sum_{k \neq i} \dot{x}_i \dot{x}_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right) = 0 \tag{3.13}$$

of the Ruijsenaars–Schneider model.

Let us show that (3.5) is a sufficient condition for existence of a meromorphic wave solution to equation (3.3) depending on a spectral parameter k with a pole at $x = a$ and regular at the points $x = a \pm \eta$. This solution can be found in the form

$$\psi(x) = k^{x/\eta} e^{kt} \left(1 + \sum_{s \geq 1} \xi_s(x) k^{-s} \right), \quad k \rightarrow \infty. \tag{3.14}$$

Proposition 3.2 *Suppose that $b(x)$ in (3.3) has poles at $x = a$ and $x = a - \eta$ with expansions near the poles of the form (3.4) and condition (3.5) for the pole of $b(x)$ holds. Then, all wave solutions of equation (3.3) of the form (3.14) are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular at $x = a \pm \eta$.*

Proof Substituting the expansions into the equation, we obtain the recurrence relation for the coefficients ξ_s :

$$\xi_{s+1}(x) - \xi_{s+1}(x + \eta) = b(x)\xi_s(x) - \dot{\xi}_s(x), \quad s \geq 0 \tag{3.15}$$

(It is convenient to put $\xi_0 = 1$.) In particular,

$$\xi_1(x) - \xi_1(x + \eta) = b(x). \tag{3.16}$$

Let the Laurent expansion of ξ_s near the point a be

$$\xi_s(x) = \frac{r_s}{x - a} + r_{s,0} + r_{s,1}(x - a) + \dots, \quad x \rightarrow a. \tag{3.17}$$

Substituting this into (3.15) with $x \rightarrow a$ and equating the coefficients at the poles, we get the recurrence relation

$$r_{s+1} = \dot{a}r_{s,0} + \mu_0 r_s - \dot{r}_s. \tag{3.18}$$

Expanding (3.15) near the point $x \rightarrow a - \eta$ and equating the coefficients in front of $(x - a + \eta)^{-1}$ and $(x - a + \eta)^0$, we get

$$\begin{cases} r_{s+1} = \dot{a}\xi_s(a - \eta), \\ r_{s+1,0} - \xi_{s+1}(a - \eta) = \mu_1 \xi_s(a - \eta) + \dot{a}\xi'_s(a - \eta) + \partial_t \xi_s(a - \eta). \end{cases} \tag{3.19}$$

The meromorphic wave solution with the required properties exists if the sum of residues of the right-hand side of (3.15) at the points $x = a$ and $x = a - \eta$ is equal to zero for all $s \geq 0$. This sum of residues is given by

$$R_s = \dot{a}r_{s,0} + \mu_0 r_s - \dot{r}_s - \dot{a}\xi_s(a - \eta).$$

We are going to prove that $R_s = 0$ for all $s \geq 0$ by induction. This is obviously true for $s = 0$ due to equation (3.16). Assume that $R_s = 0$ for some s , this is our induction assumption. Using (3.19), we find:

$$\begin{aligned} R_{s+1} &= \dot{a}(r_{s+1,0} - \xi_{s+1}(a - \eta)) + \mu_0 r_{s+1} - \dot{r}_{s+1} \\ &= (\dot{a}(\mu_0 + \mu_1) - \ddot{a})\xi_s(a - \eta) - \partial_t R_s = 0 \end{aligned}$$

because the first term vanishes due to the condition (3.5) and the second one vanishes due to the induction assumption. □

3.1.2 The linear problem with respect to \bar{t}_1

Consider now the linear problem (3.2):

$$\partial_t \psi(x) = v(x)\psi(x - \eta). \quad (3.20)$$

Suppose that the function $v(x)$ has a second-order pole at $x = a$ with the expansion

$$v(x) = \frac{\nu}{(x - a)^2} + \frac{\mu}{x - a} + O(1), \quad x \rightarrow a. \quad (3.21)$$

We also assume that $v(x)$ has a zero at $x = a - \eta$: $v(a - \eta) = 0$.

Proposition 3.3 *Suppose that $v(x)$ in (3.20) has a second-order pole at $x = a$ with the expansion (3.21) and $v(a - \eta) = 0$. If equation (3.20) has a meromorphic solution with a simple pole at the point a regular at $a \pm \eta$, then the condition*

$$v\dot{a} + \mu\dot{a}^2 - \dot{v}a = 0 \quad (3.22)$$

holds.

Proof For $\psi(x)$ near the point a , we have

$$\psi(x) = \frac{\alpha}{x - a} + O(1), \quad x \rightarrow a. \quad (3.23)$$

Substituting the expansions around the point a in the equation (3.20), we write:

$$\frac{\alpha\dot{a}}{(x - a)^2} + \frac{\dot{\alpha}}{x - a} = \left(\frac{\nu}{(x - a)^2} + \frac{\mu}{x - a} + O(1) \right) (\psi(a - \eta) + (x - a)\psi'(a - \eta) + \dots).$$

Equating the coefficients in front of the poles, we obtain the conditions

$$\begin{cases} \alpha\dot{a} = \nu\psi(a - \eta), \\ \dot{\alpha} = \nu\psi'(a - \eta) + \mu\psi(a - \eta). \end{cases} \quad (3.24)$$

At $x = a - \eta$, equation (3.20) gives $\partial_t \psi(a - \eta) = 0$ due to the fact that $v(a - \eta) = 0$, so the full time derivative of $\psi(a - \eta)$ is

$$\dot{\psi}(a - \eta) = \dot{a}\psi'(a - \eta). \quad (3.25)$$

Combining the time derivative of the first equation in (3.24) with the second one and using (3.25), we obtain the condition (3.22). \square

Suppose that $v(x)$ is an elliptic function of the form

$$v(x) = \prod_{j=1}^N \frac{\sigma(x - x_j + \eta)\sigma(x - x_j - \eta)}{\sigma^2(x - x_j)}, \tag{3.26}$$

then, setting $a = x_i$, we have:

$$v = -\sigma^2(\eta) \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)\sigma(x_i - x_j - \eta)}{\sigma^2(x_i - x_j)}, \tag{3.27}$$

$$\mu = v \sum_{k \neq i} \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right), \tag{3.28}$$

and the condition (3.22) is equivalent to the equation of motion which is the same as (3.13).

Similarly to the previous case, equation (3.22) is a sufficient condition for local existence of a meromorphic wave solution to equation (3.20) depending on a spectral parameter k with a pole at $x = a$ and regular at the points $x = a \pm \eta$. However, the arguments need some modifications.

Proposition 3.4 *Suppose that $v(x)$ in (3.20) has a second-order pole at $x = a$ with expansion near the pole of the form (3.21) and $v(a \pm \eta) = 0$. Let $v(x)$ be of the form*

$$v(x) = e^{\varphi(x) - \varphi(x - \eta)}, \tag{3.29}$$

where $\varphi(x)$ has logarithmic singularities at $x = a$ and $x = a - \eta$, so that

$$\dot{\varphi}(x) = \begin{cases} \frac{\dot{a}}{x - a} + \varphi_0 + O(x - a), & x \rightarrow a, \\ -\frac{\dot{a}}{x - a + \eta} + \varphi_1 + O(x - a + \eta), & x \rightarrow a - \eta. \end{cases} \tag{3.30}$$

Suppose that condition (3.22) for the pole of $v(x)$ holds. Then, all wave solutions of equation (3.20) of the form

$$\psi(x) = k^{-x/\eta} e^{kt + \varphi(x)} \left(1 + \sum_{s \geq 1} \xi_s(x) k^{-s} \right), \quad k \rightarrow \infty \tag{3.31}$$

are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular at $x = a \pm \eta$.

Proof Expanding the equality $\partial_t \log v(x) = \dot{\varphi}(x) - \dot{\varphi}(x - \eta)$ near the point $x = a$ with the help of (3.21) and comparing the coefficients, we get:

$$\varphi_0 - \varphi_1 = \frac{\dot{v}}{v} - \frac{\mu}{v} \dot{a}. \tag{3.32}$$

Substituting the expansions into the equation, we obtain the recurrence relation for the coefficients ξ_s :

$$\xi_{s+1}(x - \eta) - \xi_{s+1}(x) = \dot{\varphi}(x)\xi_s(x) + \dot{\xi}_s(x), \quad s \geq 0. \tag{3.33}$$

In particular,

$$\xi_1(x - \eta) - \xi_1(x) = \dot{\varphi}(x). \tag{3.34}$$

The factor $e^{\varphi(x)}$ in (3.31) has a pole at $x = a$ and a zero at $x = a - \eta$. This means that the coefficients $\xi_s(x)$ are regular at $x = a$ and may have a pole at $x = a - \eta$. Let the Laurent expansion of ξ_s near the point $a - \eta$ be of the form

$$\xi_s(x) = \frac{r_s}{x - a + \eta} + r_{s,0} + r_{s,1}(x - a + \eta) + \dots, \quad x \rightarrow a - \eta. \tag{3.35}$$

The meromorphic wave solution with the required properties exists if the sum of residues of the right-hand side of (3.33) at the points $x = a$ and $x = a - \eta$ is equal to zero for all $s \geq 0$. This can be proved by induction in the same way as for the linear problem (3.3). □

3.2 The case of the Toda lattice of type C

The Toda lattice of type C was introduced in the paper [22] by the authors. The auxiliary linear problem has the form

$$\partial_t \psi(x) = \psi(x + \eta) + \frac{1}{2} \dot{\varphi}(x)\psi(x) + v(x)\psi(x - \eta), \tag{3.36}$$

where $v(x) = e^{\varphi(x) - \varphi(x - \eta)}$. We assume that $\dot{\varphi}(x)$ is expanded near the points $x = a$ and $x = a - \eta$ as in (3.30) and

$$v(x) = \frac{\nu}{(x - a)^2} + \frac{\mu}{x - a} + O(1), \quad x \rightarrow a. \tag{3.37}$$

We also note that the relation (3.32) holds.

Proposition 3.5 *Suppose that $v(x) = e^{\varphi(x) - \varphi(x - \eta)}$ in (3.36) has a second-order pole at $x = a$ with the expansion (3.37), $v(a - \eta) = 0$ and $\dot{\varphi}(x)$ is expanded near the points $x = a$ and $x = a - \eta$ as in (3.30). If equation (3.36) has a meromorphic solution with a simple pole at the point a regular at $a \pm \eta$, then the condition*

$$\dot{a}^2 = -4\nu \tag{3.38}$$

holds.

Proof Let the function $\psi(x)$ have a pole at $x = a$ with the expansion near this point of the form (3.6). Substituting the expansions near $x = a$ into the equation and equating the coefficients in front of the highest poles, we get the condition

$$\alpha \dot{a} + 2v\psi(a - \eta) = 0. \tag{3.39}$$

The same procedure at the pole at $x = a - \eta$ leads to the condition

$$2\alpha = \dot{a}\psi(a - \eta). \tag{3.40}$$

Combining (3.39) and (3.40), we obtain the condition (3.38). □

Suppose now that $v(x)$ is the elliptic function (3.26), then, expanding it near the point $a = x_i$, we see that v is given by (3.27). Equations (3.38) for any i are then equivalent to the equations of motion for poles of elliptic solutions to the Toda lattice of type C:

$$\dot{x}_i = 2\sigma(\eta) \prod_{j \neq i} \frac{(\sigma(x_i - x_j + \eta)\sigma(x_i - x_j - \eta))^{1/2}}{\sigma(x_i - x_j)}. \tag{3.41}$$

These equations were obtained in [22]. The properly taken limit $\eta \rightarrow 0$ leads to equations (2.20).

Let us show that (3.38) is a sufficient condition for existence of a meromorphic wave solution to equation (3.36) depending on a spectral parameter k with a pole at $x = a$ and regular at the points $x = a \pm \eta$.

Proposition 3.6 *Suppose that $v(x) = e^{\varphi(x) - \varphi(x - \eta)}$ in (3.36) has a second-order pole at $x = a$ with expansion near the pole of the form (3.37) and $v(a - \eta) = 0$. Assume also that the function $\dot{\varphi}(x)$ has expansion of the form (3.30). If condition (3.22) for the pole of $v(x)$ holds, then all wave solutions of equation (3.36) of the form*

$$\psi(x) = k^{x/\eta} e^{kt} \left(1 + \sum_{s \geq 1} \xi_s(x) k^{-s} \right), \quad k \rightarrow \infty \tag{3.42}$$

are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular at $x = a \pm \eta$.

Proof Substituting the expansions into the equation, we obtain the recurrence relation for the coefficients ξ_s in (3.42):

$$\xi_{s+1}(x) - \xi_{s+1}(x + \eta) = \frac{1}{2} \dot{\varphi}(x) \xi_s(x) - \dot{\xi}_s(x) - v(x) \xi_{s-1}(x - \eta), \quad s \geq 0, \tag{3.43}$$

where we set $\xi_{-1} = 0, \xi_0 = 1$. In particular,

$$\xi_1(x) - \xi_1(x + \eta) = \frac{1}{2} \dot{\varphi}(x). \tag{3.44}$$

Assuming the expansion of $\xi_s(x)$ near the point a of the form (3.17), we get from (3.43) expanded near this point:

$$\begin{cases} \frac{1}{2} r_s \dot{a} + v \xi_{s-1}(a - \eta) = 0, \\ r_{s+1} = \frac{1}{2} \dot{a} r_{s,0} + \frac{1}{2} \varphi_0 r_s - \dot{r}_s - v \xi'_{s-1}(a - \eta) - \mu \xi_{s-1}(a - \eta). \end{cases} \tag{3.45}$$

The expansion near the point $x = a - \eta$ gives:

$$\begin{cases} r_{s+1} = \frac{1}{2} \dot{a} \xi_s(a - \eta), \\ \xi_{s+1}(a - \eta) - r_{s+1,0} = \frac{1}{2} \varphi_1 \xi_s(a - \eta) - \frac{1}{2} \dot{a} \xi'_s(a - \eta) - \partial_t \xi_s(a - \eta). \end{cases} \tag{3.46}$$

Let

$$R_s = \frac{1}{2} \dot{a}(r_{s,0} - \xi_s(a - \eta)) + \frac{1}{2} \varphi_0 r_s - \dot{r}_s - v \xi'_{s-1}(a - \eta) - \mu \xi_{s-1}(a - \eta)$$

be sum of the residues at the points $x = a$ and $x = a - \eta$ which must be zero. At $s = 0$, this is true due to (3.44). Our induction assumption is that $R_s = 0$ for some s ; let us show that this implies that $R_{s+1} = 0$. We have:

$$R_{s+1} = \frac{1}{2} \dot{a}(r_{s+1,0} - \xi_{s+1}(a - \eta)) + \frac{1}{2} \varphi_0 r_{s+1} - \dot{r}_{s+1} - v \xi'_s(a - \eta) - \mu \xi_s(a - \eta).$$

Substituting here the recurrence relations (3.45), (3.46), we obtain, after some calculations and cancellations:

$$R_{s+1} = \frac{1}{4} \left((\varphi_0 - \varphi_1) \dot{a} - 4\mu - 2\ddot{a} \right) \xi_s(a - \eta) - \frac{1}{4} (\dot{a}^2 + 4v) \xi'_s(a - \eta) - \partial_t R_s.$$

The last two terms are equal to zero by virtue of the induction assumption and the condition (3.38). As for the first term, we have:

$$\begin{aligned} (\varphi_0 - \varphi_1) \dot{a} - 4\mu - 2\ddot{a} &= \left(\frac{\dot{v}}{v} - \frac{\mu}{v} \dot{a} \right) \dot{a} - 4\mu - 2\ddot{a} \\ &= (\dot{a}^2 + 4v) \frac{\dot{v} - \mu \dot{a}}{v \dot{a}} - \frac{1}{\dot{a}} \partial_t (\dot{a}^2 + 4v) = 0 \end{aligned}$$

by virtue of the condition (3.38). Therefore, $R_{s+1} = 0$ and we have proved the existence of a meromorphic wave solution. □

3.3 The case of the Toda lattice of type B

3.3.1 Existence of a meromorphic solution

The Toda lattice of type B was recently introduced by the authors in [20]. The linear equation for the first time flow has the form

$$\partial_t \psi(x) = v(x)(\psi(x + \eta) - \psi(x - \eta)). \tag{3.47}$$

Assume that the function $v(x)$ has a second-order pole at $x = a$ with the expansion

$$v(x) = \frac{\nu}{(x - a)^2} + \frac{\mu}{x - a} + O(1), \quad x \rightarrow a. \tag{3.48}$$

We also assume that $v(x)$ has zeros at $x = a - \eta$ and $x = a + \eta$:

$$v(x) = \begin{cases} (x - a - \eta)V^+(a) + O((x - a - \eta)^2), & x \rightarrow a + \eta, \\ (x - a + \eta)V^-(a) + O((x - a + \eta)^2), & x \rightarrow a - \eta. \end{cases} \tag{3.49}$$

Proposition 3.7 *Suppose that $v(x)$ in (3.47) has a second-order pole at $x = a$ and zeros at $x = a \pm \eta$ with the expansions (3.48), (3.49). If equation (3.47) has a meromorphic solution with a simple pole at the point a regular at $a \pm \eta$, then the condition*

$$v\ddot{a} + \mu\dot{a}^2 - \dot{a} + v^2(V^+(a) + V^-(a)) = 0 \tag{3.50}$$

holds.

Proof The principal part of the Laurent expansion of $\psi(x)$ near the point $x = a$ is

$$\psi(x) = \frac{\alpha}{x - a} + O(1), \quad x \rightarrow a. \tag{3.51}$$

As $x \rightarrow a$, we have from equation (3.47):

$$\frac{\alpha\dot{a}}{(x - a)^2} + \frac{\dot{\alpha}}{x - a} = \left(\frac{\nu}{(x - a)^2} + \frac{\mu}{x - a} + O(1) \right) (\psi(a + \eta) - \psi(a - \eta) + (x - a)(\psi'(a + \eta) - \psi'(a - \eta)) + \dots).$$

Equating the coefficients in front of the poles, we obtain the conditions

$$\begin{cases} \alpha\dot{a} = \nu(\psi(a + \eta) - \psi(a - \eta)), \\ \dot{\alpha} = \mu(\psi(a + \eta) - \psi(a - \eta)) + \nu(\psi'(a + \eta) - \psi'(a - \eta)). \end{cases} \tag{3.52}$$

At $x = a \pm \eta$, equation (3.47) gives:

$$\partial_t \psi(a \pm \eta) = \mp \alpha V^\pm(a). \tag{3.53}$$

Therefore,

$$\dot{\psi}(a \pm \eta) = \mp \alpha V^\pm(a) + \dot{a} \psi'(a \pm \eta). \tag{3.54}$$

Taking the time derivative of the first equation in (3.52) and combining it with the other equations, we obtain the condition (3.50). \square

3.3.2 Dynamics of poles of elliptic solutions

Suppose that $v(x)$ is an elliptic function of the form

$$v(x) = \prod_{j=1}^N \frac{\sigma(x - x_j + \eta)\sigma(x - x_j - \eta)}{\sigma^2(x - x_j)}, \tag{3.55}$$

then, setting $a = x_i$, we have:

$$v = -\sigma^2(\eta) \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)\sigma(x_i - x_j - \eta)}{\sigma^2(x_i - x_j)}, \tag{3.56}$$

$$\mu = v \sum_{k \neq i} \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right), \tag{3.57}$$

$$V^\pm(x_i) = \pm \frac{\sigma(2\eta)}{\sigma^2(\eta)} \prod_{j \neq i} \frac{\sigma(x_i - x_j \pm 2\eta)\sigma(x_i - x_j)}{\sigma^2(x_i - x_j \pm \eta)}, \tag{3.58}$$

and the condition (3.50) is equivalent to the equation of motion

$$\begin{aligned} & \ddot{x}_i + \sum_{k \neq i} \dot{x}_i \dot{x}_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right) \\ & - \sigma(2\eta) \left[\prod_{j \neq i} \frac{\sigma(x_i - x_j + 2\eta)\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j + \eta)\sigma(x_i - x_j)} - \prod_{j \neq i} \frac{\sigma(x_i - x_j - 2\eta)\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j - \eta)\sigma(x_i - x_j)} \right] = 0. \end{aligned} \tag{3.59}$$

As shown below, the properly taken limit $\eta \rightarrow 0$ of this equation coincides with equation (2.34). In the rational limit, one should substitute $\zeta(x) \rightarrow 1/x$, $\sigma(x) \rightarrow x$.

In Appendix C, we show that the same equations can be obtained by restriction of the Ruijsenaars–Schneider dynamics with respect to the time flow $\partial_{t_1} - \partial_{\bar{t}_1}$ of the system containing $2N$ particles to the half-dimensional subspace of the $4N$ -dimensional phase space corresponding to the configuration in which the particles stick together in pairs

such that the distance between particles in each pair is equal to η . This configuration is destroyed by the flow $\partial_{t_1} + \partial_{\bar{t}_1}$ but is preserved by the flow $\partial_{t_1} - \partial_{\bar{t}_1}$. We conjecture that it is preserved also by all higher flows $\partial_{t_k} - \partial_{\bar{t}_k}$. The time evolution in $t = t_1 - \bar{t}_1$ of the pairs with coordinates x_i is given by equations (3.59).

3.3.3 The limit $\eta \rightarrow 0$

The “non-relativistic limit” $\eta \rightarrow 0$ in (3.55) yields

$$v(x) = 1 + \eta^2 u(x) + O(\eta^4), \tag{3.60}$$

where $u(x)$ is given by (2.33). Then, the limit of the difference operator $v(x)(e^{\eta\partial_x} - e^{-\eta\partial_x})$ is

$$v(x)(e^{\eta\partial_x} - e^{-\eta\partial_x}) = 2\eta\partial_x + \frac{\eta^3}{3}(\partial_x^3 + 6u(x)\partial_x) + O(\eta^5), \tag{3.61}$$

i.e., in the next-to-leading order the differential operator $\partial_x^3 + 6u\partial_x$ participating in the linear problem for the BKP equation arises (see equation (2.28)). Let us pass to the variables

$$X = x + 2\eta t, \quad T = \frac{\eta^3}{3} t, \tag{3.62}$$

then $\partial_X = \partial_x$, $\partial_T = \frac{3}{\eta^3}(\partial_t - 2\eta\partial_x)$ and in the limit $\eta \rightarrow 0$ the linear problem (3.47) becomes $\partial_T\psi = (\partial_X^3 - 6u\partial_X)\psi$ which is the linear problem (2.28) for the BKP equation.

Taking the change of variables (3.62) into account, let us find the $\eta \rightarrow 0$ limit of equation (3.59). We have:

$$\sigma(x - x_i) = \sigma\left(X - \frac{6}{\eta^2} T - x_i\right) = \sigma(X - X_i),$$

whence $X_i = \frac{6}{\eta^2} T + x_i$ and $\dot{x}_i = \partial_t x_i = -2\eta + \frac{\eta^3}{3} \partial_T X_i$. Expanding equation (3.59) in powers of η , we obtain:

$$\begin{aligned} & \frac{\eta^6}{9} \partial_T^2 X_i - 4\eta^2 \sum_{k \neq i} \left(1 - \frac{\eta^2}{6} \partial_T X_i\right) \\ & \left(1 - \frac{\eta^2}{6} \partial_T X_k\right) \left(\eta^2 \wp'(X_{ik}) + \frac{\eta^4}{12} \wp'''(X_{ik}) + O(\eta^6)\right) - U_i = 0, \end{aligned}$$

where $X_{ik} \equiv X_i - X_k$ and

$$\begin{aligned}
 U_i &= \sigma(2\eta) \left[\prod_{j \neq i} \frac{\sigma(X_{ij} + 2\eta)\sigma(X_{ij} - \eta)}{\sigma(X_{ij} + \eta)\sigma(X_{ij})} - \prod_{j \neq i} \frac{\sigma(X_{ij} - 2\eta)\sigma(X_{ij} + \eta)}{\sigma(X_{ij} - \eta)\sigma(X_{ij})} \right] \\
 &= -4\eta^4 \sum_{j \neq i} \wp'(X_{ij}) + 8\eta^6 \sum_{j \neq i} \wp(X_{ij}) \sum_{l \neq i} \wp'(X_{il}) - \eta^6 \sum_{j \neq i} \wp'''(X_{ij}) + O(\eta^8).
 \end{aligned}$$

It is easy to see that the terms of order η^4 cancel in the equation and in the leading-order η^6 equation (2.34) arises.

3.3.4 Existence of meromorphic wave solutions

Similarly to the previous cases, equation (3.50) is a sufficient condition for local existence of a meromorphic wave solution to equation (3.47) depending on a spectral parameter k with a pole at $x = a$ and regular at the points $x = a \pm \eta$. However, the proof requires more sophisticated calculations than in the Toda lattice case. To proceed, we represent $v(x)$ in the form

$$v(x) = \frac{\tau(x + \eta)\tau(x - \eta)}{\tau^2(x)} \tag{3.63}$$

which is motivated by the result of [20]. ($\tau(x)$ is the tau-function of the Toda lattice of type B.) We assume that $\tau(x)$ has a simple zero at the point a and that it is regular and nonzero in some neighborhood of this point including the points $a \pm \eta$:

$$\tau(x) = (x - a)\rho(x - a), \tag{3.64}$$

where the function $\rho(x)$ is regular and nonzero at $x = 0$. It depends also on the time t . Then, the coefficients in (3.48), (3.49) are expressed as

$$v = -\eta^2 \frac{\rho(\eta)\rho(-\eta)}{\rho^2(0)}, \quad \mu = v \left(\frac{\rho'(\eta)}{\rho(\eta)} + \frac{\rho'(-\eta)}{\rho(-\eta)} - 2 \frac{\rho'(0)}{\rho(0)} \right), \tag{3.65}$$

$$V^\pm(a) = \pm \frac{2}{\eta} \frac{\rho(0)\rho(\pm 2\eta)}{\rho^2(\pm \eta)}. \tag{3.66}$$

It is also convenient to introduce the function

$$\varphi_+(x) = \log \frac{\tau(x - \eta)}{\tau(x)}. \tag{3.67}$$

The function $\dot{\varphi}_+(x)$ has simple poles at the points $x = a$ and $x = a + \eta$ with the expansions

$$\dot{\varphi}_+(x) = \begin{cases} \frac{\dot{a}}{x - a} + \varphi_0 + \dots, & x \rightarrow a, \\ -\frac{\dot{a}}{x - a - \eta} + \varphi_1 + \dots, & x \rightarrow a + \eta, \end{cases} \tag{3.68}$$

where

$$\begin{aligned} \varphi_0 &= \dot{a} \left(\frac{1}{\eta} - \frac{\rho'(-\eta)}{\rho(-\eta)} + \frac{\rho'(0)}{\rho(0)} \right) + \frac{\dot{\rho}(-\eta)}{\rho(-\eta)} - \frac{\dot{\rho}(0)}{\rho(0)}, \\ \varphi_1 &= \dot{a} \left(\frac{1}{\eta} + \frac{\rho'(\eta)}{\rho(\eta)} - \frac{\rho'(0)}{\rho(0)} \right) - \frac{\dot{\rho}(\eta)}{\rho(\eta)} + \frac{\dot{\rho}(0)}{\rho(0)}. \end{aligned} \tag{3.69}$$

It is easy to check that

$$\varphi_0 - \varphi_1 = \frac{\dot{v}}{v} - \frac{\mu}{v} \dot{a}.$$

Proposition 3.8 *Assume that $v(x)$ in (3.47) is represented in the form (3.63), $\tau(x)$ has a simple zero at a point a and $\tau(a + \eta)\tau(a - \eta) \neq 0$. If condition (3.50) holds, then all wave solutions of equation (3.47) of the form*

$$\psi(x) = k^{x/\eta} e^{kt + \varphi_+(x)} \left(1 + \sum_{s \geq 1} \xi_s(x) k^{-s} \right), \quad k \rightarrow \infty, \tag{3.70}$$

where $\varphi_+(x)$ is given by (3.67), are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular at $x = a \pm \eta$.

Proof Substituting the series (3.70) into the equation (3.47), we obtain the recurrence relation for the coefficients ξ_s :

$$\xi_{s+1}(x + \eta) - \xi_{s+1}(x) = \dot{\varphi}_+(x)\xi_s(x) + \dot{\xi}_s(x) + v(x)v(x - \eta)\xi_{s-1}(x - \eta), \quad s \geq 0, \tag{3.71}$$

where we set $\xi_{-1} = 0, \xi_0 = 1$. The factor $e^{\varphi_+(x)}$ in (3.70) has a pole at $x = a$ and a zero at $x = a + \eta$. This means that the coefficients $\xi_s(x)$ are regular at $x = a$ and may have a pole at $x = a + \eta$. Let the Laurent expansion of ξ_s near the point $a + \eta$ be

$$\xi_s(x) = \frac{r_s}{x - a - \eta} + r_{s,0} + O(x - a - \eta), \quad x \rightarrow a + \eta. \tag{3.72}$$

The meromorphic wave solution with the required properties exists if the sum of residues of the right-hand side of (3.71) at the points $x = a$ and $x = a + \eta$ is equal to

zero for all $s \geq 0$. This can be proved by induction in a similar way as before but the calculations are more involved.

To proceed, we need some properties of the coefficient function $v(x)v(x - \eta)$ in (3.71). We have:

$$v(x)v(x - \eta) = \frac{\tau(x - 2\eta)\tau(x + \eta)}{\tau(x - \eta)\tau(x)},$$

whence

$$v(x)v(x - \eta) = \frac{\nu V^+(a)}{x - a - \eta} + O(1), \quad x \rightarrow a + \eta, \quad v(x)v(x - \eta)\Big|_{x=a-\eta} = 0, \tag{3.73}$$

$$v(x)v(x - \eta) = \frac{\nu V^-(a)}{x - a} + \Omega \nu V^-(a) + O(x - a), \quad x \rightarrow a, \tag{3.74}$$

where

$$\Omega = \frac{3}{2\eta} + \frac{\rho'(-2\eta)}{\rho(-2\eta)} - \frac{\rho'(-\eta)}{\rho(-\eta)} + \frac{\rho'(\eta)}{\rho(\eta)} - \frac{\rho'(0)}{\rho(0)}. \tag{3.75}$$

Expanding equation (3.71) near the point $x = a + \eta$, we get, equating the coefficients in front of the poles:

$$r_{s+1} = \dot{a}r_{s,0} - \varphi_1 r_s - \dot{r}_s - \nu V^+(a)\xi_{s-1}(a). \tag{3.76}$$

Similarly, expanding equation (3.71) near the point $x = a$, we get, equating the coefficients in front of $(x - a)^{-1}$ and $(x - a)^0$:

$$\begin{cases} r_{s+1} = \dot{a}\xi_s(a) + \nu V^-(a)\xi_{s-1}(a - \eta), \\ r_{s+1,0} - \xi_{s+1}(a) = \dot{\xi}_s(a) + \varphi_0 \xi_s(a) + \nu V^-(a)\xi'_{s-1}(a - \eta) + \nu V^-(a)\Omega \xi_{s-1}(a - \eta), \end{cases} \tag{3.77}$$

where $\dot{\xi}_s(a) = \dot{a}\xi'_s(a) + \partial_t \xi_s(a)$ is the full time derivative. At the point $x = a - \eta$, all terms in equation (3.71) are regular and the equation gives:

$$\xi_{s+1}(a) - \xi_{s+1}(a - \eta) = \partial_t \varphi_+(a - \eta)\xi_s(a - \eta) + \partial_t \xi_s(a - \eta) \tag{3.78}$$

. (The last term in (3.71) vanishes at this point.)

Let

$$R_s = \dot{a}(r_{s,0} - \xi_s(a)) - \varphi_1 r_s - \dot{r}_s - \nu V^+(a)\xi_{s-1}(a) - \nu V^-(a)\xi_{s-1}(a - \eta)$$

be sum of the residues at the left-hand side of (3.71) at the points $x = a$ and $x = a + \eta$ which must be zero. It is seen from (3.71) that $R_1 = 0$. Our induction assumption is that $R_s = 0$ for some s ; let us show that this implies that $R_{s+1} = 0$. We have:

$$R_{s+1} = \dot{a}(r_{s+1,0} - \xi_{s+1}(a)) - \varphi_1 r_{s+1} - \dot{r}_{s+1} - \nu V^+(a)\xi_s(a) - \nu V^-(a)\xi_s(a - \eta).$$

A straightforward calculation which uses recurrence relations (3.76), (3.77) and (3.78) yields:

$$R_{s+1} = \left[\dot{a} \left(\frac{\dot{\nu}}{\nu} - \frac{\mu}{\nu} \dot{a} \right) - \ddot{a} - \nu(V^+(a) + V^-(a)) \right] - \partial_t R_s + \nu V^-(a)\xi_{s-1}(a - \eta) \left[\dot{a}\Omega - \varphi_1 - \partial_t \log(\nu V^-(a)) + \partial_t \varphi_+(a - \eta) \right].$$

The first two terms vanish by virtue of the condition (3.50) and the induction assumption. Using equations (3.65), (3.66), (3.69) and (3.75), one can show that the third term also vanishes. Therefore, we have proved that from $R_s = 0$ it follows that $R_{s+1} = 0$ which implies the existence of a meromorphic wave solution. \square

4 Fully difference equation

The case of fully difference equation was considered by one of the authors in [18] but we find it appropriate to include it here for completeness.

Let us consider the difference equation

$$\psi_{t+1}(x) = \psi_t(x + \eta) + u_t(x)\psi_t(x), \quad u_t(x) = \frac{\tau_t(x)\tau_{t+1}(x + \eta)}{\tau_t(x + \eta)\tau_{t+1}(x)} \tag{4.1}$$

which serves as the auxiliary linear problem for the Hirota bilinear difference equation for the tau-function $\tau_t(x)$ [24, 25].

Proposition 4.1 *Let $\tau_t(x)$ have a simple zero at some point a_t : $\tau_t(a_t) = 0$, $\tau'_t(a_t) \neq 0$ and $\tau_t(a_t - \eta)\tau_{t+1}(a_t) \neq 0$. Then, the necessary condition that equation (4.1) has a meromorphic solution with a simple pole at $x = a_t$ regular at $x = a_t \pm \eta$, $x = a_{t+1}$ is*

$$\frac{\tau_{t+1}(a_t)\tau_t(a_t - \eta)\tau_{t-1}(a_t + \eta)}{\tau_{t+1}(a_t - \eta)\tau_t(a_t + \eta)\tau_{t-1}(a_t)} = -1. \tag{4.2}$$

Proof Tending x to $a_{t+1}, a_t - \eta$ and $a_{t+1} - \eta$ in equation (4.1), we obtain the relations

$$\begin{cases} \alpha_{t+1} = \frac{\tau_t(a_{t+1})\tau_{t+1}(a_{t+1} + \eta)}{\tau'_{t+1}(a_{t+1})\tau_t(a_{t+1} + \eta)} \psi_t(a_{t+1}), \\ \alpha_t = -\frac{\tau_t(a_t - \eta)\tau_{t+1}(a_t)}{\tau'_t(a_t)\tau_{t+1}(a_t - \eta)} \psi_t(a_t - \eta), \\ \psi_{t+1}(a_{t+1} - \eta) = \psi_t(a_{t+1}). \end{cases} \tag{4.3}$$

Combining them, we arrive at the condition (4.2). □

Let $\tau_t(x)$ be an “elliptic polynomial” of the form

$$\tau_t(x) = \prod_{j=1}^N \sigma(x - x_j^t) \tag{4.4}$$

with N simple roots x_j^t in the fundamental domain, then the conditions (4.2) with $a_t = x_i^t$ for each $i = 1, \dots, N$ yield the equations

$$\prod_{j=1}^N \frac{\sigma(x_i^t - x_j^{t+1})\sigma(x_i^t - x_j^t - \eta)\sigma(x_i^t - x_j^{t-1} + \eta)}{\sigma(x_i^t - x_j^{t+1} - \eta)\sigma(x_i^t - x_j^t + \eta)\sigma(x_i^t - x_j^{t-1})} = -1. \tag{4.5}$$

These are equations of motion for the discrete time version of the Ruijsenaars–Schneider system first obtained in [26].

Finally, we will show that (4.2) is a sufficient condition for local existence of a meromorphic wave solution to equation (4.1) of the form

$$\psi_t(x) = k^{x/\eta} k^t \left(1 + \sum_{s \geq 1} \xi_s^t(x) k^{-s} \right) \tag{4.6}$$

having a simple pole at $x = a_t$ and regular at $x = a_t \pm \eta, x = a_{t+1}$.

Proposition 4.2 *Assume that $\tau_t(x)$ in (4.1) has a zero at some point a_t such that $\tau'_t(a_t) \neq 0, \tau_t(a_t - \eta)\tau_{t+1}(a_t) \neq 0$ and condition (4.2) holds. Then, all wave solutions to equation (4.1) of the form (4.6) are meromorphic in a neighborhood of the point a with a simple pole at $x = a$ and regular at $x = a \pm \eta$.*

Proof Substituting the series (4.6) into the equation, we obtain the recurrence relation

$$\xi_{s+1}^{t+1}(x) - \xi_{s+1}^t(x + \eta) = u_t(x)\xi_s^t(x). \tag{4.7}$$

Let the expansion of the function $\psi_t(x)$ near the pole be of the form

$$\psi_t(x) = \frac{r_s^t}{x - a_t} + r_{s,0}^t + O(x - a_t). \tag{4.8}$$

Tending x to a_{t+1} , $a_t - \eta$ and $a_{t+1} - \eta$ in equation (4.7), we obtain the equalities

$$\begin{cases} r_{s+1}^{t+1} = \frac{\tau_t(a_{t+1})\tau_{t+1}(a_{t+1} + \eta)}{\tau'_{t+1}(a_{t+1})\tau_t(a_{t+1} + \eta)} \xi_s^t(a_{t+1}), \\ r_{s+1}^t = -\frac{\tau_t(a_t - \eta)\tau_{t+1}(a_t)}{\tau'_t(a_t)\tau_{t+1}(a_t - \eta)} \xi_s^t(a_t - \eta), \\ \xi_s^{t+1}(a_{t+1} - \eta) = \xi_s^t(a_{t+1}), \end{cases} \tag{4.9}$$

where we shifted $s \rightarrow s - 1$ in the last equation. Combining them and taking into account the condition (4.2), we see that the two expressions for r_{s+1}^t that follow from (4.7) actually coincide whence we conclude that the solution of the form (4.6) exists. \square

5 Concluding remarks

In this paper, we have further developed the approach to integrable many-body systems based on monodromy free linear equations, i.e., on finding conditions of existence of meromorphic solutions to linear partial differential and difference equations for all cases which arise as linear problems for integrable nonlinear equations. Some of them were previously discussed by one of the authors in [17–19]. We have also shown that these conditions are simultaneously sufficient conditions for local existence of meromorphic wave solutions depending on a spectral parameter. These conditions straightforwardly lead to equations of motion for poles of elliptic solutions to nonlinear integrable equations. It turns out that the dynamics of poles is isomorphic to many-body systems of Calogero–Moser type which include the Calogero–Moser system itself, its relativistic extension (the Ruijsenaars–Schneider system) and a rather exotic system with three-body interaction found in [14]. We note that the approach of this paper is the shortest way to obtain the equations of motion. Presumably, all these systems are integrable as they arise as certain finite-dimensional reductions of integrable nonlinear systems with infinitely many degrees of freedom. Independent proofs of integrability exist for the Calogero–Moser and Ruijsenaars–Schneider systems, while for the system introduced in [14] this is still a hypothesis.

The main new result of this paper is the many-body system with equations of motion (1.1), (1.2) representing dynamics of poles of elliptic solutions to the Toda lattice of type B recently introduced in [20]. This system can be regarded as a kind of relativistic extension of the system found in [14] with equations of motion (2.34) in the sense that the former is related to the latter in the same way as the Ruijsenaars–Schneider system is related to the Calogero–Moser system.

There are some interesting open problems. First, it is not clear whether the system (1.1), (1.2) is Hamiltonian. Second, a commutation representation for it is not yet known. Presumably, such commutation representation is of the form of the Manakov’s triple as it is the case for the system (2.34) which arises from (1.1), (1.2) in the $\eta \rightarrow 0$ limit. Last but not least, there is the problem of proving integrability of the system

(1.1), (1.2). At the moment, we know only one integral of motion which is $I = \sum_i \dot{x}_i$.

Its conservation ($\dot{I} = 0$) can be seen directly from the equations of motion taking into account that

$$\sum_{i=1}^N U(x_{i1}, \dots, x_{iN}) = 0$$

which follows from the fact that the left-hand side is sum of the residues at the $2N$ poles of the elliptic function

$$f(x) = \prod_{j=1}^N \frac{\sigma(x - x_j + 2\eta)\sigma(x - x_j - \eta)}{\sigma(x - x_j + \eta)\sigma(x - x_j)}$$

at the points $x = x_i, x = x_i - \eta, i = 1, \dots, N$.

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Appendix A: The Weierstrass functions

Throughout the main text, we use the standard Weierstrass functions: the σ -function, the ζ -function and the \wp -function.

The Weierstrass σ -function with quasi-periods $2\omega, 2\omega'$ such that $\text{Im}(\omega'/\omega) > 0$ is defined by the infinite product

$$\sigma(x) = \sigma(x | \omega, \omega') = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s} + \frac{x^2}{2s^2}}, \quad s = 2\omega m + 2\omega' m' \quad \text{with integer } m, m'. \quad (A1)$$

It is an odd entire quasiperiodic function in the complex plane. The Weierstrass ζ -function is defined as

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)} = \frac{1}{x} + \sum_{s \neq 0} \left(\frac{1}{x-s} + \frac{1}{s} + \frac{x}{s^2}\right). \quad (A2)$$

It is an odd function with first-order poles at the points of the lattice $s = 2\omega m + 2\omega' m'$ with integer m, m' . The definition of the Weierstrass \wp -function is

$$\wp(x) = -\zeta'(x) = \frac{1}{x^2} + \sum_{s \neq 0} \left(\frac{1}{(x-s)^2} - \frac{1}{s^2}\right). \quad (A3)$$

It is an even double-periodic function with periods $2\omega, 2\omega'$ and with second-order poles at the points of the lattice $s = 2\omega m + 2\omega' m'$ with integer m, m' .

The monodromy properties of the σ -function under shifts by the quasi-periods are as follows:

$$\begin{aligned} \sigma(x + 2\omega) &= -e^{2\zeta(\omega)(x+\omega)}\sigma(x), \\ \sigma(x + 2\omega') &= -e^{2\zeta(\omega')(x+\omega')}\sigma(x). \end{aligned} \tag{A4}$$

The ζ -function acquires an additive constant when the argument is shifted by any quasi-period:

$$\begin{aligned} \zeta(x + 2\omega) &= \zeta(x) + \zeta(\omega), \\ \zeta(x + 2\omega') &= \zeta(x) + \zeta(\omega'). \end{aligned} \tag{A5}$$

These constants are related by the identity $2\omega'\zeta(\omega) - 2\omega\zeta(\omega') = \pi i$.

Appendix B: The Ruijsenaars–Schneider model

Here, we collect some facts on the elliptic Ruijsenaars–Schneider system [12] following the paper [13]. The N -particle elliptic Ruijsenaars–Schneider system is a completely integrable model. The canonical Poisson brackets between coordinates and momenta are $\{x_i, p_j\} = \delta_{ij}$. The integrals of motion in involution have the form

$$I_k = \sum_{I \subset \{1, \dots, N\}, |I|=k} \exp\left(\sum_{i \in I} p_i\right) \prod_{i \in I, j \notin I} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \quad k = 1, \dots, N. \tag{B1}$$

It is convenient to put $I_0 = 1$. Important particular cases of (B1) are

$$I_1 = H_1 = \sum_i e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)} \tag{B2}$$

which is the Hamiltonian H_1 of the chiral Ruijsenaars–Schneider model and

$$I_N = \exp\left(\sum_{i=1}^N p_i\right). \tag{B3}$$

Let us denote the time variable of the Hamiltonian flow with the Hamiltonian H_1 by t_1 . The velocities of the particles are

$$\dot{x}_i = \frac{\partial H_1}{\partial p_i} = e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)}, \tag{B4}$$

where dot means the t_1 -derivative. The Hamiltonian equations $\dot{x}_i = \partial H_1 / \partial p_i$, $\dot{p}_i = -\partial H_1 / \partial x_i$ are equivalent to the following equations of motion:

$$\begin{aligned} \ddot{x}_i &= - \sum_{k \neq i} \dot{x}_i \dot{x}_k \left(\zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right) \\ &= \sum_{k \neq i} \dot{x}_i \dot{x}_k \frac{\wp'(x_i - x_k)}{\wp(\eta) - \wp(x_i - x_k)}. \end{aligned} \tag{B5}$$

The properly taken limit $\eta \rightarrow 0$ (the “non-relativistic limit”) gives equations of motion of the elliptic Calogero–Moser system.

One can also introduce integrals of motion I_{-k} as

$$I_{-k} = I_N^{-1} I_{N-k} = \sum_{I \subset \{1, \dots, N\}, |I|=k} \exp\left(-\sum_{i \in I} p_i\right) \prod_{i \in I, j \notin I} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}. \tag{B6}$$

In particular,

$$I_{-1} = \sum_i e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}. \tag{B7}$$

It can be easily verified that equations of motion in the time \bar{t}_1 corresponding to the Hamiltonian $\bar{H}_1 = \sigma^2(\eta) I_{-1}$ are the same as (B5).

The “physical” Hamiltonian of the Ruijsenaars–Schneider model is $H_+ = H_1 + \bar{H}_1$. Below in Appendix C we consider the Hamiltonian flow corresponding to the Hamiltonian $H_- = H_1 - \bar{H}_1$.

Appendix C: How Ruijsenaars–Schneider particles stick together

In this appendix, we show how to restrict the Ruijsenaars–Schneider dynamics of the $N = 2n$ -particle system to the subspace in which the particles stick together in n pairs such that

$$x_{2i} - x_{2i-1} = \eta, \quad i = 1, \dots, n. \tag{C1}$$

We introduce the variables

$$X_i = x_{2i-1}, \quad i = 1, \dots, n \tag{C2}$$

which are coordinates of the pairs. Such configuration is destroyed by the H_+ -Hamiltonian flow $\partial_{t_1} + \partial_{\bar{t}_1}$ but is preserved by the H_- -flow $\partial_t = \partial_{t_1} - \partial_{\bar{t}_1}$, as we shall see below.

The Hamiltonian H_- reads

$$H_- = \sum_i e^{p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j + \eta)}{\sigma(x_i - x_j)} - \sigma^2(\eta) \sum_i e^{-p_i} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \eta)}{\sigma(x_i - x_j)}.$$

For the velocities $\dot{x}_i = \partial H_- / \partial p_i$, we have:

$$\begin{aligned} \dot{x}_{2i-1} &= e^{p_{2i-1}} \prod_{j=1, \neq 2i-1}^{2n} \frac{\sigma(x_{2i-1, j} + \eta)}{\sigma(x_{2i-1, j})} + \sigma^2(\eta) e^{-p_{2i-1}} \prod_{j=1, \neq 2i-1}^{2n} \frac{\sigma(x_{2i-1, j} - \eta)}{\sigma(x_{2i-1, j})}, \\ \dot{x}_{2i} &= e^{p_{2i}} \prod_{j=1, \neq 2i}^{2n} \frac{\sigma(x_{2i, j} + \eta)}{\sigma(x_{2i, j})} + \sigma^2(\eta) e^{-p_{2i}} \prod_{j=1, \neq 2i}^{2n} \frac{\sigma(x_{2i, j} - \eta)}{\sigma(x_{2i, j})}, \end{aligned}$$

where $x_{ik} \equiv x_i - x_k$. Suppose that the momenta remain finite under the restriction to (C1). Then in terms of coordinates X_i of the pairs, we have from these formulas:

$$\begin{aligned} \dot{x}_{2i-1} &= \sigma(\eta) \sigma(2\eta) e^{-p_{2i-1}} \prod_{j=1, \neq i}^n \frac{\sigma(X_{ij} - 2\eta)}{\sigma(X_{ij})}, \\ \dot{x}_{2i} &= \frac{\sigma(2\eta)}{\sigma(\eta)} e^{p_{2i}} \prod_{j=1, \neq i}^n \frac{\sigma(X_{ij} + 2\eta)}{\sigma(X_{ij})}. \end{aligned} \tag{C3}$$

The dynamics preserves the configuration (C1) if $\dot{x}_{2i-1} = \dot{x}_{2i}$, whence we should require

$$e^{p_{2i-1} + p_{2i}} = \sigma^2(\eta) \prod_{j \neq i} \frac{\sigma(X_{ij} - 2\eta)}{\sigma(X_{ij} + 2\eta)}.$$

Resolving this constraint, we can put

$$p_{2i-1} = \alpha_i + P_i, \quad p_{2i} = \alpha_i - P_i, \quad i = 1, \dots, n, \tag{C4}$$

where

$$\alpha_i = \log \sigma(\eta) + \frac{1}{2} \sum_{j \neq i} \log \frac{\sigma(X_{ij} - 2\eta)}{\sigma(X_{ij} + 2\eta)} \tag{C5}$$

and P_i are arbitrary. We have thus restricted the original $4n$ -dimensional phase space \mathcal{F} with coordinates $\{p_j, x_j\}, j = 1, \dots, 2n$ to the $2n$ -dimensional subspace \mathcal{P} with coordinates $\{P_i, X_i\}, i = 1, \dots, n$ corresponding to joining the particles into pairs of

the form (C1). Equations (C3) are then equivalent to

$$\dot{X}_i = \sigma(2\eta)e^{-P_i} \prod_{j \neq i} \frac{(\sigma(X_{ij} - 2\eta)\sigma(X_{ij} + 2\eta))^{1/2}}{\sigma(X_{ij})}. \tag{C6}$$

Let us now pass to the second set of the Hamiltonian equations, $\dot{p}_i = -\partial H_- / \partial x_i$:

$$\begin{aligned} \dot{p}_i &= \sigma^2(\eta)e^{-p_i} \prod_{k=1, \neq i}^{2n} \frac{\sigma(x_{ik} - \eta)}{\sigma(x_{ik})} \sum_{j=1, \neq i}^{2n} \left(\zeta(x_{ij} - \eta) - \zeta(x_{ij}) \right) \\ &\quad - e^{p_i} \prod_{k=1, \neq i}^{2n} \frac{\sigma(x_{ik} + \eta)}{\sigma(x_{ik})} \sum_{j=1, \neq i}^{2n} \left(\zeta(x_{ij} + \eta) - \zeta(x_{ij}) \right) \\ &\quad + \sigma^2(\eta) \sum_{l=1, \neq i}^{2n} e^{-p_l} \prod_{k=1, \neq l}^{2n} \frac{\sigma(x_{lk} - \eta)}{\sigma(x_{lk})} \left(\zeta(x_{il} + \eta) - \zeta(x_{il}) \right) \\ &\quad - \sum_{l=1, \neq i}^{2n} e^{p_l} \prod_{k=1, \neq l}^{2n} \frac{\sigma(x_{lk} + \eta)}{\sigma(x_{lk})} \left(\zeta(x_{il} - \eta) - \zeta(x_{il}) \right). \end{aligned} \tag{C7}$$

Restricting to the subspace \mathcal{P} , we have:

$$\begin{aligned} \dot{p}_{2i-1} &= \sigma(\eta)\sigma(2\eta)e^{-\alpha_i - P_i} \prod_{k=1, \neq i}^n \frac{\sigma(X_{ik} - 2\eta)}{\sigma(X_{ik})} \\ &\quad \left[\sum_{j=1, \neq i}^n \left(\zeta(X_{ij} - 2\eta) - \zeta(X_{ij}) \right) + \zeta(\eta) - \zeta(2\eta) \right] \\ &\quad + \sigma(\eta)\sigma(2\eta) \sum_{l=1, \neq i}^n e^{-\alpha_l - P_l} \prod_{k=1, \neq l}^n \frac{\sigma(X_{lk} - 2\eta)}{\sigma(X_{lk})} \left(\zeta(X_{il} + \eta) - \zeta(X_{il}) \right) \\ &\quad - \frac{\sigma(2\eta)}{\sigma(\eta)} \sum_{l=1}^n e^{\alpha_l - P_l} \prod_{k=1, \neq l}^n \frac{\sigma(X_{lk} + 2\eta)}{\sigma(X_{lk})} \left(\zeta(X_{il} - 2\eta) - \zeta(X_{il} - \eta) \right) \\ &\quad + \sigma^{-1}(\eta)e^{\alpha_i + P_i} \prod_{k=1, \neq i}^n \frac{\sigma(X_{ik} + \eta)}{\sigma(X_{ik} - \eta)} - \sigma(\eta)e^{-\alpha_i + P_i} \prod_{k=1, \neq i}^n \frac{\sigma(X_{ik} - \eta)}{\sigma(X_{ik} + \eta)}. \end{aligned} \tag{C8}$$

When passing from (C7) to (C8) with the constraint (C1), one encounters expressions like $\sigma(x_{2i} - x_{2i-1} - \eta)\zeta(x_{2i} - x_{2i-1} - \eta)$ which is an indeterminacy of the form 0/0. To resolve it, one should put $x_{2i} - x_{2i-1} = \eta + \varepsilon$ and tend $\varepsilon \rightarrow 0$.

Taking the time derivative of (C6), we obtain:

$$\ddot{X}_i = -\sigma(2\eta)\dot{P}_i e^{-P_i} \prod_{j \neq i} \frac{(\sigma(X_{ij} - 2\eta)\sigma(X_{ij} + 2\eta))^{1/2}}{\sigma(X_{ij})} + \frac{1}{2} \sum_{j \neq i} \dot{X}_i (\dot{X}_i - \dot{X}_j) (\zeta(X_{ij} - 2\eta) + \zeta(X_{ij} + 2\eta) - 2\zeta(X_{ij})),$$

where we should substitute $\dot{P}_i = -\dot{\alpha}_i + \dot{p}_{2i-1}$ from (C8) taking into account (C6):

$$\begin{aligned} \dot{P}_i &= -\dot{\alpha}_i + \dot{X}_i \left[\sum_{j \neq i} (\zeta(X_{ij} - 2\eta) - \zeta(X_{ij})) + \zeta(\eta) - \zeta(2\eta) \right] \\ &+ \sum_{l \neq i} \dot{X}_l (\zeta(X_{il} + \eta) - \zeta(X_{il})) - \sum_l \dot{X}_l (\zeta(X_{il} - 2\eta) - \zeta(X_{il} - \eta)) \\ &+ e^{P_i} \prod_{k \neq i} \frac{\sigma^{1/2}(X_{ik} - 2\eta)\sigma(X_{ik} + \eta)}{\sigma^{1/2}(X_{ik} + 2\eta)\sigma(X_{ik} - \eta)} - e^{P_i} \prod_{k \neq i} \frac{\sigma^{1/2}(X_{ik} + 2\eta)\sigma(X_{ik} - \eta)}{\sigma^{1/2}(X_{ik} - 2\eta)\sigma(X_{ik} + \eta)}. \end{aligned}$$

Substituting here

$$\dot{\alpha}_i = \frac{1}{2} \sum_{j \neq i} (\dot{X}_i - \dot{X}_j) (\zeta(X_{ij} - 2\eta) - \zeta(X_{ij} + 2\eta)),$$

we obtain, after cancellations:

$$\begin{aligned} \ddot{X}_i &= - \sum_{j \neq i} \dot{X}_i \dot{X}_j (\zeta(X_{ij} + \eta) + \zeta(X_{ij} - \eta) - 2\zeta(X_{ij})) \\ &+ \sigma(2\eta) \left[\prod_{j \neq i} \frac{\sigma(X_{ij} + 2\eta)\sigma(X_{ij} - \eta)}{\sigma(X_{ij} + \eta)\sigma(X_{ij})} - \prod_{j \neq i} \frac{\sigma(X_{ij} - 2\eta)\sigma(X_{ij} + \eta)}{\sigma(X_{ij} - \eta)\sigma(X_{ij})} \right]. \end{aligned} \tag{C9}$$

These are equations (1.1), (1.2).

A similar calculation for \dot{p}_{2i} leads to the same result. This means that the restriction to the subspace \mathcal{P} is consistent.

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