Letters in Mathematical Physics (2023) 113:44
https://doi.org/10.1007/s11005-023-01666-y

# Explicit formulas of the logarithmic couplings of certain staggered Virasoro modules

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Received: 5 December 2022 / Revised: 17 March 2023 / Accepted: 30 March 2023 /
Published online: 12 April 2023


**Abstract**

We construct a family of staggered Virasoro modules which are isomorphic to those constructed by Cromer. Using these staggered Virasoro modules, we verify the limit formulas of the logarithmic couplings given by Vasseur, Jacobsen and Saleur. Furthermore by using the formula of the norm of logarithmic primary proved by Yanagida, we present explicit formulas for the logarithmic couplings of these staggered Virasoro modules.

**Keywords** Virasoro algebra · Conformal field theory

**Mathematics Subject Classification** 81R10

## 1 Introduction

Logarithmic conformal field theories have been actively studied in both physics and mathematics in recent years. Unlike the case of rational conformal field theories, logarithmic conformal field theories admit indecomposable modules on which the Virasoro zero-mode $L_0$ acts non-semisimply. These indecomposable modules are called staggered modules or logarithmic modules and appear in the field of statistical mechanics, such as critical percolation and dilute polymers [5, 19, 23], and in the theory of fusion rules, such as the Nahm–Gaberdiel–Kausch algorithm [3, 6, 9] (there are also studies on the Virasoro fusion rules from the Schramm–Loewner evolution processes [13]). For each staggered module, an important invariant called logarithmic coupling can be defined [15] and it has been found that the value of the logarithmic coupling fixes the staggered module up to isomorphism [14]. There are many studies related to the logarithmic couplings. Let us briefly review a few of them. In the case of the Virasoro

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minimal models M(2, $p$), Mathieu and Ridout show that the logarithmic couplings can be written by combinatorial product [16]. In the paper [22], Vasseur, Jacobsen and Saleur derive limit formulas for the logarithmic couplings by studying in detail certain two point functions with deformation parameters. In the paper [4], Cromer also derive certain combinatorial formulas of the logarithmic couplings by using free field realization techniques. Recently, Nivesvivat and Ribault derive explicit formulas for the logarithmic couplings from the direction of the Liouville theory [18].

Let $p_+$ and $p_-$ be coprime integers such that $p_- > p_+ \geq 2$, and let

$$c_{p_+,p_-} := 1 - 6\frac{(p_+ - p_-)^2}{p_+ p_-}$$

be the central charge of the minimal model $\mathrm{M}(p_+, p_-)$. In this paper we examine the logarithmic couplings of certain staggered modules whose central charge are $c_{p_+,p_-}$. In Sect. 3.1, we construct certain staggered modules $F(\tau)$ whose $L_0$ nilpotent rank two by gluing certain Fock modules, and define certain finite length submodules $P(\tau)$ as the quotients of $F(\tau)$. We see that these staggered modules $F(\tau)$ are isomorphic to those constructed by Cromer [4]. In Sect. 3.2, we rederive the limit formula of the logarithmic couplings given by Vasseur, Jacobsen and Saleur [22] using the staggered modules $F(\tau)$. Furthermore, by using the results in [24], we present explicit formulas for the logarithmic couplings of the staggered modules $P(\tau)$. These results are stated in Theorem 3.2. As an application of the proofs of these theorems, in Sect. 3.3 we consider the structure of a slightly more complex staggered modules whose $L_0$ nilpotent rank two.

## 2 The structure of Fock modules

Recall that the Virasoro algebra $\mathcal{L}$ is the Lie algebra over $\mathbb{C}$ generated by $L_n (n \in \mathbb{Z})$ and $C$ (the central charge) with the relation

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} C\delta_{m+n,0}, \qquad [L_n, C] = 0.$$

Fix two coprime integers $p_+$, $p_-$ such that $p_- > p_+ \geq 2$, and let

$$c_{p_+,p_-} := 1 - 6\frac{(p_+ - p_-)^2}{p_+ p_-}$$

be the central charge of the minimal model $\mathrm{M}(p_+, p_-)$. In this paper we consider Virasoro modules on which $C$ acts as $c_{p_+,p_-} \cdot \mathrm{id}$. In this section we briefly review the structure of Fock modules whose central charges are $c_{p_+,p_-}$ in accordance with [12] and [21] (see also [7]).

### 2.1 Fock modules

The Heisenberg Lie algebra

$$\mathcal{H}=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}a_n\oplus\mathbb{C}K_{\mathcal{H}}$$

is the Lie algebra whose commutation is given by

$$[a_m,a_n]=m\delta_{m+n,0}K_{\mathcal{H}},\qquad [K_{\mathcal{H}},\mathcal{H}]=0.$$

Let

$$\mathcal{H}^{\pm}=\bigoplus_{n>0}\mathbb{C}a_{\pm n},\quad \mathcal{H}^{0}=\mathbb{C}a_0\oplus\mathbb{C}K_{\mathcal{H}},\quad \mathcal{H}^{\geq}=\mathcal{H}^{+}\oplus\mathcal{H}^{0}.$$

For any $\alpha\in\mathbb{C}$, let $\mathbb{C}|\alpha\rangle$ be the one dimensional $\mathcal{H}^{\geq}$ module defined by

$$a_n|\alpha\rangle=\delta_{n,0}\alpha|\alpha\rangle\ (n\geq 0),\qquad K_{\mathcal{H}}|\alpha\rangle=|\alpha\rangle.$$

For any $\alpha\in\mathbb{C}$, the bosonic Fock module is defined by the following induced module

$$F_\alpha=\mathrm{Ind}_{\mathcal{H}^{\geq}}^{\mathcal{H}}\mathbb{C}|\alpha\rangle.$$

Let

$$a(z)=\sum_{n\in\mathbb{Z}}a_nz^{-n-1}$$

be the bosonic current. Then we have the following operator expansion

$$a(z)a(w)\sim\frac{1}{(z-w)^2}.$$

We define the energy-momentum tensor

$$T(z):=\frac{1}{2}:a(z)a(z):+\frac{\rho}{2}\partial a(z),\qquad \rho:=\sqrt{\frac{2p_-}{p_+}}-\sqrt{\frac{2p_+}{p_-}}.$$

where $:\ :$ is the normal order product. The Fourier modes of $T(z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}$ generate the Virasoro algebra whose central charge is $c_{p_+,p_-}$.

By the energy-momentum tensor $T(z)$, each Fock module $F_\alpha$ has the structure of a Virasoro module whose central charge is $c_{p_+,p_-}$. The $L_0$ weight of the highest weight vector $|\alpha\rangle$ is given by

$$L_0|\alpha\rangle = \frac{1}{2}\alpha(\alpha - \rho)|\alpha\rangle.$$

Let us denote

$$h_\alpha := \frac{1}{2}\alpha(\alpha - \rho).$$

For any $\alpha \in \mathbb{C}$, the Fock module $F_\alpha$ has the following $L_0$ weight decomposition

$$F_\alpha = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} F_\alpha[n], \qquad F_\alpha[n] := \{v \in F_\alpha \setminus \{0\} \mid L_0 v = (h_\alpha + n)v\},$$

where each weight space $F_\alpha[n]$ has a basis

$$\{a_{-\lambda}|\alpha\rangle \mid \lambda \vdash n\}$$

with $a_{-\lambda} = a_{-\lambda_k} \cdots a_{-\lambda_1}$ for a partition $\lambda = (\lambda_1, \dots, \lambda_k)$.

We define the following conformal vector in $F_0$

$$T = \frac{1}{2}\left(a_{-1}^2 + \rho a_{-2}\right)|0\rangle.$$

**Definition 2.1** The Fock module $F_0$ carries the structure of a $\mathbb{Z}_{\geq 0}$-graded vertex operator algebra, with

$$Y(|0\rangle, z) = \mathrm{id}, \qquad Y(a_{-1}|0\rangle, z) = a(z), \qquad Y(T, z) = T(z).$$

We denote this vertex operator algebra by $\mathcal{F}_\rho$.

### 2.2 The structure of Fock modules

We set $\alpha_\pm = \pm\sqrt{2p_\mp/p_\pm}$. For $r, s, n \in \mathbb{Z}$ we introduce the following symbols

$$\alpha_{r,s;n} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_- + \frac{\sqrt{2p_+p_-}}{2}n$$

For $r, s, n \in \mathbb{Z}$, we set

$$F_{r,s;n} = F_{\alpha_{r,s;n}}, \qquad h_{r,s;n} := \frac{1}{2}\alpha_{r,s;n}(\alpha_{r,s;n} - \rho).$$

For any $h \in \mathbb{C}$, let $L(h)$ be the irreducible Virasoro module whose highest weight is $h$ and the central charge $C = c_{p_+,p_-} \cdot \mathrm{id}$. For $r, s, n \in \mathbb{Z}$, let $L(h_{r,s;n}) = L_{r,s;n}$. For

$1 \leq r \leq p_+ - 1$, $1 \leq s \leq p_- - 1$, we set

$$r^\vee := p_+ - r, \qquad s^\vee := p_- - s.$$

Before describing the structure of Fock modules, let us introduce the notion of socle series.

**Definition 2.2** Let $M$ be a finite length Virasoro module. Let $\mathrm{Soc}(M)$ be the socle of $M$, that is $\mathrm{Soc}(M)$ is the maximal semisimple submodule of $M$. Since $M$ is finite length, we have the sequence of the submodule

$$0 \leq \mathrm{Soc}_1(M) \leq \mathrm{Soc}_2(M) \leq \cdots \leq \mathrm{Soc}_n(M) = M$$

such that $\mathrm{Soc}_1(M) = \mathrm{Soc}(M)$ and $\mathrm{Soc}_{i+1}(M)/\mathrm{Soc}_i(M) = \mathrm{Soc}(M/\mathrm{Soc}_i(M))$. We call such a sequence of the submodules of $M$ the socle series of $M$.

The following proposition is due to Feigin and Fuchs [7].

**Proposition 2.1** *As the Virasoro module, there are four cases of socle series for the Fock modules $F_{r,s;n} \in \mathcal{F}_\rho - \mathrm{mod}$:*

1. *For each $1 \leq r \leq p_+ - 1,\ 1 \leq s \leq p_- - 1,\ n \in \mathbb{Z}$, we have*

$$0 \leq \mathrm{Soc}_1(F_{r,s;n}) \leq \mathrm{Soc}_2(F_{r,s;n}) \leq \mathrm{Soc}_3(F_{r,s;n}) = F_{r,s;n}$$

*with*

$$\mathrm{Soc}_1(F_{r,s;n}) = \mathrm{Soc}(F_{r,s;n}) = \bigoplus_{k\geq 0} L_{r,s^\vee;|n|+2k+1},$$

$$\mathrm{Soc}_2(F_{r,s;n})/\mathrm{Soc}_1(F_{r,s;n}) = \bigoplus_{k\geq a} L_{r,s;|n|+2k} \oplus \bigoplus_{k\geq 1-a} L_{r^\vee,s^\vee;|n|+2k},$$

$$F_{r,s;n}/\mathrm{Soc}_2(F_{r,s;n}) = \bigoplus_{k\geq 0} L_{r^\vee,s;|n|+2k+1},$$

*where $a = 0$ if $n \geq 0$ and $a = 1$ if $n < 0$.*

2. *For each $1 \leq s \leq p_- - 1,\ n \in \mathbb{Z}$, we have*

$$0 \leq \mathrm{Soc}_1(F_{p_+,s;n}) \leq \mathrm{Soc}_2(F_{p_+,s;n}) = F_{p_+,s;n}$$

*with*

$$\mathrm{Soc}_1(F_{p_+,s;n}) = \mathrm{Soc}(F_{p_+,s;n}) = \bigoplus_{k\geq 0} L_{p_+,s^\vee;|n|+2k+1},$$

$$\mathrm{Soc}_2(F_{p_+,s;n})/\mathrm{Soc}_1(F_{p_+,s;n}) = \bigoplus_{k\geq a} L_{p_+,s;|n|+2k}$$

*where $a = 0$ if $n \geq 1$ and $a = 1$ if $n < 1$.*

3. *For each* $1 \leq r \leq p_+ - 1,\ n \in \mathbb{Z}$*, we have*

$$0 \leq \mathrm{Soc}_1\left(F_{r,p_-;n}\right) \leq \mathrm{Soc}_2\left(F_{r,p_-;n}\right) = F_{r,p_-;n}$$

*with*

$$\mathrm{Soc}_1\left(F_{r,p_-;n}\right) = \mathrm{Soc}\left(F_{r,p_-;n}\right) = \bigoplus_{k\geq 0} L_{r,p_-;|n|+2k},$$
$$\mathrm{Soc}_2\left(F_{r,p_-;n}\right)/\mathrm{Soc}_1\left(F_{r,p_-;n}\right) = \bigoplus_{k\geq a} L_{r^\vee,p_-;|n|+2k-1}$$

*where* $a = 1$ *if* $n \geq 0$ *and* $a = 0$ *if* $n < 0$.

4. *For each* $n \in \mathbb{Z}$*, the Fock module* $F_{p_+,p_-;n}$ *is semi-simple as a Virasoro module*

$$\mathrm{Soc}\left(F_{p_+,p_-;n}\right) = F_{p_+,p_-;n} = \bigoplus_{k\geq 0} L_{p_+,p_-;|n|+2k}.$$

For the four groups of Fock modules in Proposition 2.1, we call the first group of Fock modules “braided-type", the second and third groups of Fock modules “chain-type" and the last group of Fock modules “semisimple-type".

### 2.3 Screening operators and Felder complex

As detailed in [20], we can define the non-trivial screening currents

$$Q_+^{[r]}(z) \in \mathrm{Hom}_{\mathbb{C}}(F_{r,k;l}, F_{-r,k;l})[[z, z^{-1}]], \quad r \geq 1, k, l \in \mathbb{Z},$$
$$Q_-^{[s]}(z) \in \mathrm{Hom}_{\mathbb{C}}(F_{k,s;l}, F_{k,-s;l})[[z, z^{-1}]], \quad s \geq 1, k, l \in \mathbb{Z}.$$

These fields satisfy the following operator product expansion

$$T(z)Q_\pm^{[\bullet]}(w) = \frac{Q_\pm^{[\bullet]}(w)}{(z-w)^2} + \frac{\partial_w Q_\pm^{[\bullet]}(w)}{z-w} + \cdots$$

In particular zero modes

$$\mathrm{Res}_{z=0} Q_+^{[r]}(z)\mathrm{d}z = Q_+^{[r]} \in \mathrm{Hom}_{\mathbb{C}}(F_{r,k}, F_{-r,k}), \quad r \geq 1, k \in \mathbb{Z},$$
$$\mathrm{Res}_{z=0} Q_-^{[s]}(z)\mathrm{d}z = Q_-^{[s]} \in \mathrm{Hom}_{\mathbb{C}}(F_{k,s}, F_{k,-s}), \quad s \geq 1, k \in \mathbb{Z}$$

commute with every Virasoro mode. These zero modes are called screening operators.

For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$ and $n \in \mathbb{Z}$, we define the following Virasoro modules

1. For $1 \leq r < p_+,\ 1 \leq s \leq p_-,\ n \in \mathbb{Z}$

$$K_{r,s;n;+} = \ker\left(Q_+^{[r]} : F_{r,s;n} \to F_{r^\vee,s;n+1}\right),$$
$$X_{r^\vee,s;n+1;+} = \mathrm{im}\left(Q_+^{[r]} : F_{r,s;n} \to F_{r^\vee,s;n+1}\right).$$

2. For $1 \leq r \leq p_+,\ 1 \leq s < p_-,\ n \in \mathbb{Z}$

$$K_{r,s;n;-} = \ker\left(Q_-^{[s]} : F_{r,s;n} \to F_{r,s^\vee;n-1}\right),$$
$$X_{r,s^\vee;n-1;-} = \mathrm{im}\left(Q_-^{[s]} : F_{r,s;n} \to F_{r,s^\vee;n-1}\right).$$

The following propositions are due to Felder [8].

**Proposition 2.2** *The socle series of $K_{r,s;n;\pm}$ and $X_{r,s;n;\pm}$ are given by:*

*1. For $1 \leq r \leq p_+ - 1,\ 1 \leq s \leq p_- - 1$ and $n \in \mathbb{Z}$, we have*

$$0 \leq S^K_{1;\pm} := \mathrm{Soc}(K_{r,s;n;\pm}) \leq S^K_{2;\pm} := K_{r,s;n;\pm}$$
$$0 \leq S^X_{1;\pm} := \mathrm{Soc}(X_{r,s;n;\pm}) \leq S^X_{2;\pm} := X_{r,s;n;\pm}$$

*such that*

$n \geq 0$

$$S^K_{1;+} = \bigoplus_{k\geq 1} L_{r,s^\vee;n+2k-1},\quad \overline{S^K_{2;+}} = \bigoplus_{k\geq 1} L_{r,s;n+2(k-1)},\quad S^X_{1;+} = \bigoplus_{k\geq 1} L_{r,s^\vee;n+2k},\quad \overline{S^X_{2;+}} = \bigoplus_{k\geq 1} L_{r,s;n+2k-1},$$

$n \leq -1$

$$S^K_{1;+} = \bigoplus_{k\geq 1} L_{r,s^\vee;-n+2k-1},\quad \overline{S^K_{2;+}} = \bigoplus_{k\geq 1} L_{r,s;-n+2k},\quad S^X_{1} = \bigoplus_{k\geq 1} L_{r,s^\vee;-n+2(k-1)},\quad \overline{S^X_{2;+}} = \bigoplus_{k\geq 1} L_{r,s;-n+2k-1},$$

$n \geq 1$

$$S^K_{1;-} = \bigoplus_{k\geq 1} L_{r,s^\vee;n+2k-1},\quad \overline{S^K_{2;-}} = \bigoplus_{k\geq 1} L_{r,s;n+2(k-1)},\quad S^X_{1;-} = \bigoplus_{k\geq 1} L_{r,s^\vee;n+2(k-1)},\quad \overline{S^X_{2;-}} = \bigoplus_{k\geq 1} L_{r^\vee,s^\vee;n+2k-1},$$

$n \leq 0$

$$S^K_{1;-} = \bigoplus_{k\geq 1} L_{r,s^\vee;-n+2k-1},\quad \overline{S^K_{2;-}} = \bigoplus_{k\geq 1} L_{r,s;-n+2k},\quad S^X_{1;-} = \bigoplus_{k\geq 1} L_{r,s^\vee;-n+2k},\quad \overline{S^X_{2;-}} = \bigoplus_{k\geq 1} L_{r^\vee,s^\vee;-n+2k-1},$$

*where $\overline{S^K_{2;\pm}} = S^K_{2;\pm}/S^K_{1;\pm}$ and $\overline{S^X_{2;\pm}} = S^X_{2;\pm}/S^X_{1;\pm}$.*

2. *For* $1 \le r \le p_+ - 1,\ s = p_-,\ n \in \mathbb{Z}$, *we have*

$$X_{r,p_-;n} = \mathrm{Soc}(F_{r,p_-;n}).$$

3. *For* $r = p_+,\ 1 \le s \le p_- - 1,\ n \in \mathbb{Z}$, *we have*

$$X_{p_+,s;n} = \mathrm{Soc}(F_{p_+,s;n}).$$

**Proposition 2.3** *1. For* $1 \le r < p_+,\ 1 \le s < p_-$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_+^{[r]}$ *and* $Q_+^{[r^\vee]}$ *define the* Felder complex

$$\cdots \to F_{r^\vee,s;n-1} \xrightarrow{Q_+^{[r^\vee]}} F_{r,s;n} \xrightarrow{Q_+^{[r]}} F_{r^\vee,s;n+1} \to \cdots .$$

*This complex is exact everywhere except in* $F_{r,s} = F_{r,s;0}$ *where the cohomology is given by*

$$\ker Q_+^{[r]} / \mathrm{im}\, Q_+^{[r^\vee]} \simeq L_{r,s;0}.$$

2. *For* $1 \le r < p_+,\ 1 \le s < p_-$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_-^{[s]}$ *and* $Q_-^{[s^\vee]}$ *define the* Felder complex

$$\cdots \to F_{r,s^\vee;n+1} \xrightarrow{Q_-^{[s^\vee]}} F_{r,s;n} \xrightarrow{Q_-^{[s]}} F_{r,s^\vee;n-1} \to \cdots .$$

*This complex is exact everywhere except in* $F_{r,s} = F_{r,s;0}$ *where the cohomology is given by*

$$\ker Q_-^{[s]} / \mathrm{im}\, Q_-^{[s^\vee]} \simeq L_{r,s;0}.$$

3. *For* $1 \le r < p_+$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_+^{[r]}$ *and* $Q_+^{[r^\vee]}$ *define the* Felder complex

$$\cdots \to F_{r^\vee,p_-;n-1} \xrightarrow{Q_+^{[r^\vee]}} F_{r,p_-;n} \xrightarrow{Q_+^{[r]}} F_{r^\vee,p_-;n+1} \to \cdots$$

*and this complex is exact.*

4. *For* $1 \le s < p_-$ *and* $n \in \mathbb{Z}$ *the screening operators* $Q_-^{[s]}$ *and* $Q_-^{[s^\vee]}$ *define the* Felder complex

$$\cdots \to F_{p_+,s^\vee;n+1} \xrightarrow{Q_-^{[s^\vee]}} F_{p_+,s;n} \xrightarrow{Q_-^{[s]}} F_{p_+,s^\vee;n-1} \to \cdots .$$

*and this complex is exact.*

## 3 Staggered Virasoro modules

In this section, we construct certain staggered modules by gluing bosonic Fock modules and, by using these staggered modules, we rederive the limit formulas of the logarithmic couplings given by Vasseur, Jacobsen and Saleur [22] (see also [11, 18]). Furthermore, by using the formula of the norm of logarithmic primary proved by Yanagida [24], we give explicit formulas for the logarithmic couplings. We see that the staggered Virasoro module which will be constructed in this section are isomorphic to those constructed by Cromer [4]. In this section, we identify any Virasoro modules that are isomorphic among each other.

### 3.1 Construction of certain staggered Virasoro modules

We set

$$A_{p_+,p_-} := \{ \alpha_{r,s;n} \mid r, s, n \in \mathbb{Z} \}.$$

Let $U(\mathcal{L})$ be the universal enveloping algebra of the Virasoro algebra.

**Definition 3.1** 1. We define $\mathcal{T}^{\searrow}_{p_+,p_-}$ to be the subset of $A^2_{p_+,p_-}$ such that every element $\tau = (\alpha_1, \alpha_2) \in A^2_{p_+,p_-}$ satisfies the following conditions:

- $h_{\alpha_1} < h_{\alpha_2}$.
- The two Fock modules $F_{\alpha_1}$ and $F_{\alpha_2}$ are contained in the same Felder complex given in Proposition 2.3 and adjacent to each other as

$$\cdots \to F_{\alpha_1} \xrightarrow{Q_\tau} F_{\alpha_2} \to \cdots,$$

where we denote the screening operator from $F_{\alpha_1}$ to $F_{\alpha_2}$ by $Q_\tau$.

2. We define $\mathcal{T}^{\nearrow}_{p_+,p_-}$ to be the subset of $A^2_{p_+,p_-}$ such that every element $\tau = (\alpha_1, \alpha_2) \in A^2_{p_+,p_-}$ satisfies the following conditions:

- $h_{\alpha_1} > h_{\alpha_2}$.
- The two Fock modules $F_{\alpha_1}$ and $F_{\alpha_2}$ are contained in the same Felder complex given in Proposition 2.3 and adjacent to each other as

$$\cdots \to F_{\alpha_1} \xrightarrow{Q_\tau} F_{\alpha_2} \to \cdots,$$

where we denote the screening operator from $F_{\alpha_1}$ to $F_{\alpha_2}$ by $Q_\tau$.

3. We define $\mathcal{T}_{p_+,p_-} = \mathcal{T}^{\searrow}_{p_+,p_-} \sqcup \mathcal{T}^{\nearrow}_{p_+,p_-}$

Let $\hat{a}$ be the dual of the zero mode $a_0$ defined by

$$[a_m, \hat{a}] = \delta_{m,0}\mathrm{id}. \tag{3.1}$$

For any $\alpha, \beta \in \mathbb{C}$, let us identify $e^{\beta \hat{a}}|\alpha\rangle = |\alpha + \beta\rangle$.

Fix any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$. Let $v$ be any $L_0$-homogeneous vector of $F_{\alpha_1}$ and let $A \in U(\mathcal{L})$ be any $L_0$-homogeneous element. Let $h_{\alpha_1} + n_1$ and $n_2$ be the $L_0$-weight of $v$ and $A$, respectively. For any $\epsilon \in \mathbb{C}^\times$, let us consider the following operator

$$[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}] = Q_\tau e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}} - e^{-t\hat{a}} A e^{\epsilon \hat{a}} Q_\tau$$

on $F_{\alpha_1}$, where $F_\alpha \in \mathcal{F}_\rho - \mathrm{mod}$ for all $\alpha \in \mathbb{C}$. Note that $[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}]v \in F_{\alpha_2}[n_1 + n_2 + h_{\alpha_1} - h_{\alpha_2}]$. Let us write $[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}]v$ as

$$[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}]v = \sum_{\lambda \vdash N_{1,2}} f_\lambda(\epsilon) a_{-\lambda} |\alpha_2\rangle,$$

where $f_\lambda(\epsilon)$ are some polynomials of $\epsilon$ and $N_{1,2} = n_1 + n_2 + h_{\alpha_1} - h_{\alpha_2}$. Since $[Q_\tau, A] = 0$, we can see that every $f_\lambda(\epsilon)$ is divisible by $\epsilon$. Then we define

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \neq 0}} \frac{1}{\epsilon} \left[ Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}} \right] v := \sum_{\lambda \vdash N_{1,2}} (\epsilon^{-1} f_\lambda(\epsilon)|_{\epsilon=0}) a_{-\lambda} |\alpha_2\rangle.$$

We introduce the following $\mathbb{C}$-linear operators.

**Definition 3.2** For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$, we define the following $\mathbb{C}$-linear operator

$$\Lambda_\tau(A) = \lim_{\substack{\epsilon \to 0 \\ \epsilon \neq 0}} \frac{1}{\epsilon} \left[ Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}} \right], \quad \text{for } A \in U(\mathcal{L}),$$

where $F_{\alpha_1+\epsilon}, F_{\alpha_2+\epsilon} \in \mathcal{F}_\rho - \mathrm{mod}$ for all $\epsilon \in \mathbb{C}$.

**Proposition 3.1** *For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$, the operator $\Lambda_\tau$ satisfies the following the properties of the derivative*

$$\Lambda_\tau(AB) = \Lambda_\tau(A)B + A\Lambda_\tau(B), \quad A, B \in U(\mathcal{L}).$$

***Proof*** For any $A, B \in U(\mathcal{L})$, we have

$$\begin{aligned}
&\left[Q_\tau, e^{-\epsilon \hat{a}} AB e^{\epsilon \hat{a}}\right] \\
&= \left[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}} \cdot e^{-\epsilon \hat{a}} B e^{\epsilon \hat{a}}\right] \\
&= \left[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}\right] e^{-\epsilon \hat{a}} B e^{\epsilon \hat{a}} + e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}} \left[Q_\tau, e^{-\epsilon \hat{a}} B e^{\epsilon \hat{a}}\right] \\
&= \left[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}\right] B + A[Q_\tau, e^{-\epsilon \hat{a}} B e^{\epsilon \hat{a}}] \\
&\quad + \left[Q_\tau, e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}}\right](e^{-\epsilon \hat{a}} B e^{\epsilon \hat{a}} - B) + (e^{-\epsilon \hat{a}} A e^{\epsilon \hat{a}} - A)\left[Q_\tau, e^{-\epsilon \hat{a}} B e^{\epsilon \hat{a}}\right].
\end{aligned}$$

Dividing both sides by $\epsilon$ and taking the limit, we have the property of the derivative. □

For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$, set

$$F(\tau) = F_{\alpha_1} \oplus F_{\alpha_2}.$$

**Definition 3.3** Fix any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$. For $A \in U(\mathcal{L})$, we define the following operator $J_\tau(A)$ on $F(\tau)$:

$$J_\tau(A) = \begin{cases} A + \Lambda_\tau(A) & \text{on } F_{\alpha_1}, \\ A & \text{on } F_{\alpha_2}. \end{cases}$$

By Proposition 3.1, we obtain the following proposition.

**Proposition 3.2** *For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$, we have*

$$J_\tau(AB) = J_\tau(A)J_\tau(B), \quad \text{for any } A, B \in U(\mathcal{L}).$$

By this proposition, we see that $J_\tau$ defines a structure of Virasoro module on $F(\tau)$. We denote this Virasoro module by $(F(\tau), J_\tau)$. Let us compute the $J_\tau(L_n)$ action of the Virasoro module $(F(\tau), J_\tau)$.

**Proposition 3.3** *For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$ and $n \in \mathbb{Z}$, the $J_\tau(L_n)$ action on the vector subspace $F_{\alpha_1} \subset F(\tau)$ is given by*

$$J_\tau(L_n) = L_n + [Q_\tau, a_n].$$

***Proof*** Note that the ordinary action of $L_n$ on the Fock modules in $\mathcal{F}_\rho$ − mod is given by

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_m a_{n-m} : -\frac{1}{2}\rho(n+1)a_n. \tag{3.2}$$

Let $v$ be any nonzero vector of $F_{\alpha_1}$. Then, by (3.1), (3.2) and $[Q_\tau, L_n] = 0$, we have

$$\begin{aligned}(J_\tau(L_n) - L_n)v &= \Lambda_\tau(L_n)v \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}Q_\tau e^{-\epsilon\hat{a}}L_n e^{\epsilon\hat{a}}v - \lim_{\epsilon\to 0}\frac{1}{\epsilon}e^{-\epsilon\hat{a}}L_n e^{\epsilon\hat{a}}Q_\tau v \\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}Q_\tau\epsilon a_n v - \lim_{t\to 0}\frac{1}{\epsilon}\epsilon a_n Q_\tau v \\ &= [Q_\tau, a_n]v.\end{aligned}$$

□

***Remark 3.1*** Fix any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\nearrow}_{p_+,p_-}$. In [4], Cromer define the operator

$$V_n = \frac{1}{\alpha_2 - \alpha_1}[a_n, Q_\tau]$$

and define the structure of a Virasoro module on $F_{\alpha_1} \oplus F_{\alpha_2}$ as

$$J'_\tau(L_n) = \begin{cases} L_n + V_n & \text{on } F_{\alpha_1}, \\ L_n & \text{on } F_{\alpha_2}. \end{cases}$$

From Proposition 3.3, we see that the Virasoro modules $(F(\tau), J_\tau)$ and $(F(\tau), J'_\tau)$ are isomorphic.

By Proposition 3.3, we see that $(F(\tau), J_\tau)$ has the structure of a staggered Virasoro module whose $L_0$-nilpotent rank two. In the following, we define a finite length submodule of the staggered module $(F(\tau), J_\tau)$.

Noting Proposition 2.1, we define the following symbols.

**Definition 3.4** 1. For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\searrow}_{p_+,p_-}$, let $S_\tau$ be the Shapovalov element

$$S_\tau = L_{-1}^{h_{\alpha_2} - h_{\alpha_1}} + \cdots \in U(\mathcal{L}) \setminus \{0\}$$

satisfying $S_\tau|\alpha_1\rangle = 0$.

2. For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\nearrow}_{p_+,p_-}$, let $S_\tau$ be the Shapovalov element

$$S_\tau = L_{-1}^{h_{\alpha_1} - h_{\alpha_2}} + \cdots \in U(\mathcal{L}) \setminus \{0\},$$

which gives the singular vector in $F_{\alpha_2}[h_{\alpha_1} - h_{\alpha_2}]$.

For $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\searrow}_{p_+,p_-}$ and $\epsilon \in \mathbb{C}^\times$, let us consider the vector $e^{-\epsilon\hat{a}} S_\tau|\alpha_1 + \epsilon\rangle \in F_{\alpha_1}$, where $F_{\alpha_1+\epsilon} \in \mathcal{F}_\rho - \text{mod}$. Let us write

$$e^{-\epsilon\hat{a}} S_\tau|\alpha_1 + \epsilon\rangle = \sum_{\lambda \vdash h_{\alpha_2} - h_{\alpha_1}} f_\lambda(\epsilon) a_{-\lambda}|\alpha_1\rangle,$$

where $f_\lambda(\epsilon)$ are some polynomials of $\epsilon$. Since $S_\tau|\alpha_1\rangle = 0$, we can see that every $f_\lambda(\epsilon)$ is divisible by $\epsilon$. Then we define

$$\lim_{\epsilon \to 0} \epsilon^{-1} e^{-\epsilon\hat{a}} S_\tau|\alpha_1 + \epsilon\rangle := \sum_{\lambda \vdash h_{\alpha_2} - h_{\alpha_1}} (\epsilon^{-1} f_\lambda(\epsilon)\ |_{\epsilon=0}) a_{-\lambda}|\alpha_1\rangle \in F_{\alpha_1}.$$

By the Jantzen filtration of the Fock module $F_{\alpha_1}$ (cf. [7, 12]), we can see that this vector is nonzero and becomes a cosingular vector in $F_{\alpha_1}[h_{\alpha_2} - h_{\alpha_1}]$. Thus, by Proposition 2.2, we have $Q_\tau(v_\tau) \in \mathbb{C}^\times|\alpha_2\rangle$ (cf. [8]).

**Definition 3.5** For any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$, we define the following vector:

$$v_\tau := \begin{cases} \lim_{\epsilon\to 0}\epsilon^{-1}e^{-\epsilon\hat{a}}S_\tau|\alpha_1+\epsilon\rangle \in F_{\alpha_1} & \tau \in \mathcal{T}^{\searrow}_{p_+,p_-}, \\ |\alpha_1\rangle & \tau \in \mathcal{T}^{\nearrow}_{p_+,p_-}. \end{cases} \tag{3.3}$$

**Definition 3.6** For $\tau \in \mathcal{T}_{p_+,p_-}$, we define the following finite length submodule of $(F(\tau), J_\tau)$

$$P(\tau) := J_\tau(U(\mathcal{L})).v_\tau.$$

Since $Q_\tau(v_\tau) \neq 0$, we see that every $P(\tau)$ has $L_0$ nilpotent rank two, and

$$J_\tau(S_\tau\sigma(S_\tau))v_\tau \neq 0,$$

where $\sigma$ is an anti-involution of $U(\mathcal{L})$ defined by $\sigma(L_n) = L_{-n} (n \in \mathbb{Z})$. Thus every $P(\tau)$ is an extension between certain two highest weight Virasoro modules.

### 3.2 The logarithmic couplings of $P(\tau)$

We review the definition of the logarithmic couplings (see [3, 14] for general cases). Fix any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\searrow}_{p_+,p_-}$. Let $V(h_{\alpha_1})$ and $V(h_{\alpha_2})$ be any highest weight modules whose highest weights are $h_{\alpha_1}$ and $h_{\alpha_2}$, respectively. Assume that there exists a non-trivial staggered module satisfying

$$0 \to V(h_{\alpha_1}) \xrightarrow{\iota} E_\tau \xrightarrow{p} V(h_{\alpha_2}) \to 0.$$

Let $x_0$ be the highest weight vector of $V(h_{\alpha_1})$ such that $\langle\iota(x_0), \iota(x_0)\rangle = 1$, and let $y_0$ be the highest weight vector of $V(h_{\alpha_2})$. Let $x = \iota(x_0)$ and fix any $L_0$-homogeneous vector $y \in E_\tau$ such that $p(y) = y_0$. Then we have

$$(L_0 - h_{\alpha_2})y = cS_\tau x, \qquad \sigma(S_\tau)y = \beta' x,$$

where $c(\neq 0)$ and $\beta'$ are some constants. We then define $\beta(E_\tau)$ by

$$\beta(E_\tau) = \frac{\beta'}{c}. \tag{3.4}$$

One can check that this $\beta(E_\tau)$ is a unique constant independent of the choice of $y$. The $\beta(E_\tau)$ is called the logarithmic coupling of $E_\tau$ or the indecomposability parameter of $E_\tau$.

In the following, let us determine the logarithmic coupling of $P(\tau)$ for any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$. Before that, let us introduce the formula of the norm of logarithmic primary proved by Yanagida [24] as follows.

Let $M(h, c_{p_+,p_-})$ be the Verma module of the Virasoro algebra whose highest weight and the central charge are $h \in \mathbb{C}$ and $C = c_{p_+,p_-} \cdot \mathrm{id}$. Let $|h\rangle$ be the highest weight vector of $M(h, c_{p_+,p_-})$. Note that, for $r, s \geq 1$, $M(h_{r,s;0}, c_{p_+,p_-})$ has the singular vector whose $L_0$-weight is $h_{r,s;0} + rs$. Let $S_{r,s} \in U(\mathcal{L})$ be the Shapovalov element corresponding to this singular vector, normalized as

$$S_{r,s}|h_{r,s;0}\rangle = (L_{-1}^{rs} + \cdots)|h_{r,s;0}\rangle.$$

For $r, s \geq 1$ and $h \in \mathbb{C}$, let us consider the value $\langle h|\sigma(S_{r,s})S_{r,s}|h\rangle$, where we choose a norm of the highest weight vector $|h\rangle \in M(h, c_{p_+,p_-})$ as $\langle h|h\rangle = 1$. We can see that this value is a polynomial of $h$ and is divisible by $(h - h_{r,s;0})$. A more detailed value is given by the following theorem.

**Theorem 3.1** ([24]) *For $r, s \geq 1$ and $h \in \mathbb{C}$,*

$$\langle h|\sigma(S_{r,s})S_{r,s}|h\rangle = R_{r,s}(h - h_{r,s;0}) + O((h - h_{r,s;0})^2),$$

*where $R_{r,s}$ is given by*

$$R_{r,s} = 2 \prod_{\substack{(k,l)\in\mathbb{Z}^2,\\ 1-r\leq k\leq r, 1-s\leq l\leq s,\\ (k,l)\neq(0,0),(r,s)}} \left(k\left(\frac{p_+}{p_-}\right)^{-\frac{1}{2}} + l\left(\frac{p_+}{p_-}\right)^{\frac{1}{2}}\right).$$

We obtain the following limit and combinatorial formulas of the logarithmic coupling $\beta(P(\tau))$.

**Theorem 3.2** *Fix any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}_{p_+,p_-}$ and let $(r, s)$ be the element in $\mathbb{Z}^2_{\geq 1}$ such that $S_\tau = S_{r,s}$. Then the logarithmic coupling $\beta(P(\tau))$ is given by the following formula:*

1. *In the case of $\tau \in \mathcal{T}^{\searrow}_{p_+,p_-}$, the logarithmic coupling of $P(\tau)$ is given by*

$$\begin{aligned} \beta(P(\tau)) &= -\frac{\frac{d}{d\epsilon}\langle\alpha_1 + \epsilon|\sigma(S_\tau)S_\tau|\alpha_1 + \epsilon\rangle|_{\epsilon=0}}{\frac{d}{d\epsilon}(h_{\alpha_2+\epsilon} - h_{\alpha_1+\epsilon})|_{\epsilon=0}} \\ &= \frac{1}{2}\frac{2\alpha_1 - \rho}{\alpha_1 - \alpha_2}R_{r,s}, \end{aligned} \tag{3.5}$$

   *where we choose a norm of the highest weight vector $|\alpha\rangle$ as $\langle\alpha|\alpha\rangle = 1$ for any Fock modules $F_\alpha$.*
2. *In the case of $\tau \in \mathcal{T}^{\nearrow}_{p_+,p_-}$, the logarithmic coupling of $P(\tau)$ is given by*

$$\begin{aligned} \beta(P(\tau)) &= -\frac{\frac{d}{d\epsilon}\langle\alpha_2 + \epsilon|\sigma(S_\tau)S_\tau|\alpha_2 + \epsilon\rangle|_{\epsilon=0}}{\frac{d}{d\epsilon}(h_{\alpha_1+\epsilon} - h_{\alpha_2+\epsilon})|_{\epsilon=0}} \\ &= \frac{1}{2}\frac{2\alpha_2 - \rho}{\alpha_2 - \alpha_1}R_{r,s}. \end{aligned} \tag{3.6}$$

***Proof*** We only prove (3.5). (3.6) can be proved in the same way. Fix any $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\searrow}_{p_+,p_-}$. Note that $Q_\tau|\alpha_1\rangle = 0$. Then, from Proposition 3.3 and (3.3), we have

$$\begin{aligned}(J_\tau(L_0) - h_{\alpha_2})v_\tau &= (\alpha_1 - \alpha_2)Q_\tau(v_\tau),\\ J_\tau(\sigma(S_\tau))v_\tau &= \sigma(S_\tau)v_\tau,\\ J_\tau(S_\tau)|\alpha_1\rangle &= Q_\tau(v_\tau).\end{aligned} \tag{3.7}$$

Thus, by (3.7) and the definition of the logarithmic couplings (3.4), we obtain

$$\begin{aligned}\beta(P(\tau)) &= -(\alpha_2 - \alpha_1)^{-1}\frac{d}{d\epsilon}\langle\alpha_1|\sigma(S_\tau)e^{-\epsilon\hat{a}}S_\tau e^{\epsilon\hat{a}}|\alpha_1\rangle|_{\epsilon=0}\\ &= -(\alpha_2 - \alpha_1)^{-1}\frac{d}{d\epsilon}\langle\alpha_1|e^{-\epsilon\hat{a}}\sigma(S_\tau)S_\tau e^{\epsilon\hat{a}}|\alpha_1\rangle|_{\epsilon=0}.\end{aligned} \tag{3.8}$$

Note that

$$\frac{d}{d\epsilon}h_{\alpha+\epsilon}|_{\epsilon=0} = \frac{2\alpha - \rho}{2} \tag{3.9}$$

for any $\alpha \in \mathbb{C}$. Thus, by (3.8), we obtain the limit formula

$$\beta(P(\tau)) = -\frac{\frac{d}{d\epsilon}\langle\alpha_1 + \epsilon|\sigma(S_\tau)S_\tau|\alpha_1 + \epsilon\rangle|_{\epsilon=0}}{\frac{d}{d\epsilon}(h_{\alpha_2+\epsilon} - h_{\alpha_1+\epsilon})|_{\epsilon=0}}. \tag{3.10}$$

Let $(r, s)$ be the element in $\mathbb{Z}^2_{\geq 1}$ such that $S_\tau = S_{r,s}$. Then, by Theorem 3.1 and (3.9), we have

$$\frac{d}{d\epsilon}\langle\alpha_1 + \epsilon|\sigma(S_\tau)S_\tau|\alpha_1 + \epsilon\rangle|_{\epsilon=0} = \frac{1}{2}(2\alpha_1 - \rho)R_{r,s}.$$

Therefore, by (3.10), we obtain

$$\beta(P(\tau)) = \frac{1}{2}\frac{2\alpha_1 - \rho}{\alpha_1 - \alpha_2}R_{r,s}.$$

□

**Corollary 3.1** *For any* $\tau = (\alpha_1, \alpha_2) \in \mathcal{T}^{\searrow}_{p_+,p_-}$, *let* $\tau^\vee = (\rho - \alpha_2, \rho - \alpha_1) \in \mathcal{T}^{\nearrow}_{p_+,p_-}$. *Then we have* $\beta(P(\tau)) = \beta(P(\tau^\vee))$.

***Remark 3.2*** In [4], Cromer derive certain summation formulas of the logarithmic couplings $\beta(P(\tau))$. It would be an interesting problem to see if our formula can be derived directly from theirs.

For example, noting Proposition 2.1, let us compute some values of $\beta(P(\tau))$ in the case of $(p_+, p_-) = (2, 3)$:

1. For $\tau = (\alpha_{1,1;0}, \alpha_{1,2;-1})$,

$$\beta(P(\tau)) = \frac{1}{2}\frac{2\alpha_{1,1;0}-\rho}{\alpha_{1,1;0}-\alpha_{1,2;-1}}R_{1,1} = -\frac{1}{2}.$$

2. For $\tau = (\alpha_{1,2;0}, \alpha_{1,1;-1})$,

$$\beta(P(\tau)) = \frac{1}{2}\frac{2\alpha_{1,2;0}-\rho}{\alpha_{1,2;0}-\alpha_{1,1;-1}}R_{1,2} = -\frac{5}{18}.$$

3. For $\tau = (\alpha_{1,1;-1}, \alpha_{1,2;-2})$,

$$\beta(P(\tau)) = \frac{1}{2}\frac{2\alpha_{1,1;-1}-\rho}{\alpha_{1,1;-1}-\alpha_{1,2;-2}}R_{3,1} = -420.$$

4. For $\tau = (\alpha_{1,2;-1}, \alpha_{1,1;-2})$,

$$\beta(P(\tau)) = \frac{1}{2}\frac{2\alpha_{1,2;-1}-\rho}{\alpha_{1,2;-1}-\alpha_{1,1;-2}}R_{3,2} = -\frac{10780000}{243}.$$

5. For $\tau = (\alpha_{1,2;0}, \alpha_{1,2;1})$,

$$\beta(P(\tau)) = \frac{1}{2}\frac{2\alpha_{1,2;0}-\rho}{\alpha_{1,2;0}-\alpha_{1,2;1}}R_{1,2} = \frac{10}{27}.$$

These values coincide with the logarithmic couplings $\beta_{1,4}$, $\beta_{1,5}$, $\beta_{1,7}$, $\beta_{1,8}$ and $\beta_{3,1}$ in [3], respectively (see [16, 22] for other values of logarithmic couplings, but note that the normalization of the Shapovalov elements $S_{r,s}$ is different from our case).

### 3.3 Other rank two staggered modules

In the following we present a slightly more complex staggered modules whose $L_0$ nilpotent rank two.

Note that, unlike the chain type Fock modules, every braided type Fock module is included in two different Felder complex given in Proposition 2.3. Let $\mathcal{T}^{\mathrm{br}}_{p_+,p_-}$ be the subset of $\mathcal{T}_{p_+,p_-}$ consisting any element $\tau = (\alpha_1, \alpha_2)$ such that $F_{\alpha_1}$ is braided type. We set

$$\Upsilon_{p_+,p_-} := \left\{((\alpha_1,\alpha_2),(\alpha_1',\alpha_2')) \in \mathcal{T}^{\mathrm{br}}_{p_+,p_-} \times \mathcal{T}^{\mathrm{br}}_{p_+,p_-} \mid \alpha_1 = \alpha_1' \wedge \alpha_2 \neq \alpha_2'\right\}.$$

Fix any $(\tau, \tau') = ((\alpha_1, \alpha_2), (\alpha_1, \alpha_2')) \in \Upsilon_{p_+,p_-}$. We set

$$F(\tau, \tau') = F_{\alpha_1} \oplus F_{\alpha_2} \oplus F_{\alpha_2'}.$$

For $A \in U(\mathcal{L})$, we define the following operator $J_{\tau,\tau'}(A)$ on $F(\tau,\tau')$:

$$J_{\tau,\tau'}(A) = \begin{cases} A + \Lambda_\tau(A) + \Lambda_{\tau'}(A) & \text{on } F_{\alpha_1}, \\ A & \text{on } F_{\alpha_2} \oplus F_{\alpha_2'}. \end{cases}$$

Then, by Proposition 3.1, we have

$$J_{\tau,\tau'}(AB) = J_{\tau,\tau'}(A) J_{\tau,\tau'}(B), \quad \text{for any } A, B \in U(\mathcal{L}).$$

Thus we see that $J_{\tau,\tau'}$ defines the structure of a Virasoro module on $F(\tau,\tau')$. Similar to Proposition 3.3, we have the following proposition.

**Proposition 3.4** *For any $(\tau,\tau') = ((\alpha_1,\alpha_2),(\alpha_1,\alpha_2')) \in \Upsilon_{p_+,p_-}$ and $n \in \mathbb{Z}$, the $J_{\tau,\tau'}(L_n)$ action on the vector subspace $F_{\alpha_1} \subset F(\tau,\tau')$ is given by*

$$J_{\tau,\tau'}(L_n) = L_n + [Q_\tau, a_n] + [Q_{\tau'}, a_n].$$

**Definition 3.7** For $(\tau,\tau') \in \Upsilon_{p_+,p_-}$, we define the following finite length submodule of the staggered module $(F(\tau,\tau'), J_{\tau,\tau'})$

$$P(\tau,\tau') := J_{\tau,\tau'}(U(\mathcal{L})).v_\tau,$$

where $v_\tau$ is the vector of $F_{\alpha_1}$ defined by (3.3).

We define the subset $\Upsilon^{\text{Min}}_{p_+,p_-} \subset \Upsilon_{p_+,p_-}$ as

$$\Upsilon^{\text{Min}}_{p_+,p_-} = \{((\alpha_{r,s;0}, \alpha_{r^\vee,s;1}), (\alpha_{r,s;0}, \alpha_{r,s^\vee;-1})) \mid 1 \le r < p_+, 1 \le s < p_-\}.$$

**Theorem 3.3** *For any $(\tau,\tau') \in \Upsilon^{\text{Min}}_{p_+,p_-}$, $P(\tau,\tau')$ has two subquotients whose logarithmic couplings are the same as those of $P(\tau)$ and $P(\tau')$, respectively.*

***Proof*** Fix any element

$$(\tau,\tau') = ((\alpha_{r,s;0}, \alpha_{r^\vee,s;1}), (\alpha_{r,s;0}, \alpha_{r,s^\vee;-1})) \in \Upsilon^{\text{Min}}_{p_+,p_-}.$$

Let us consider the staggered module $P(\tau,\tau')$. By Proposition 3.4, we have

$$(J_{\tau,\tau'}(L_0) - h_{r^\vee,s;1}) v_\tau = -r\alpha_+ Q_+^{[r]}(v_\tau) - s\alpha_- Q_-^{[s]}(v_\tau), \tag{3.11}$$

where $v_\tau \in F_{r,s;0}$ is defined by (3.3). By using Theorem 3.1 we have

$$J_{\tau,\tau'}(S_{r,s}\sigma(S_{r,s})) v_\tau = \frac{1}{2}(2\alpha_{r,s;0} - \rho) R_{r,s} (Q_+^{[r]}(v_\tau) + Q_-^{[s]}(v_\tau)). \tag{3.12}$$

By (3.11) and (3.12), we see that $P(\tau, \tau')$ has two submodules $J_{\tau,\tau'}(U(\mathcal{L})).Q_+^{[r]}(v_\tau)$ and $J_{\tau,\tau'}(U(\mathcal{L})).Q_-^{[s]}(v_\tau)$, and the logarithmic couplings of the quotient modules

$$P(\tau, \tau')/J_{\tau,\tau'}(U(\mathcal{L})).Q_-^{[s]}(v_\tau), \qquad P(\tau, \tau')/J_{\tau,\tau'}(U(\mathcal{L})).Q_+^{[r]}(v_\tau)$$

are the same as those of $P(\tau)$ and $P(\tau')$, respectively. □

## 4 Future works

In this paper, we have constructed the infinite length staggered Virasoro modules $(F(\tau), J_\tau)$ and $(F(\tau, \tau'), J_{\tau,\tau'})$ using certain limit operations. We have not examined the detailed subquotient structure of these staggered modules. If we try to investigate the structure of these staggered modules directly from the definitions, we will have to calculate the actions of any modes of the screening currents $Q_\pm^{[\bullet]}(z)$. The results of classification for isomorphism classes of staggered modules by [14] are considered important to avoid the difficulties of direct calculation.

It is known that there is a constant difference between the logarithmic coupling $\beta(\lambda)$ associated with a given rank two staggered module $\lambda$ and a certain constant $\widetilde{\beta}(\lambda)$ characterizing the two point function $\langle \psi_\lambda(z)\psi_\lambda(w)\rangle$ [3, 10], where $\psi_\lambda(z)$ is the field which generates $\lambda$. As shown in [10, 22], this constant $\widetilde{\beta}(\lambda)$ also has a limit formula similar to $\beta(\lambda)$. It is an interesting problem to examine the explicit formula of $\widetilde{\beta}(\lambda)$ in the case of $\lambda = F(\tau)$. We believe that the value of the image of the screening operators [4, 17, 21] is important to investigate this problem.

We believe that our limit method is valid for other models as well. For example, in the case of staggered modules of $N = 1$ superconformal minimal models [2], we can construct certain staggered Neveu–Schwarz modules, by gluing Neveu–Schwarz Fock modules. The theory of admissible Jack polynomials [1] is considered to be important to investigate the logarithmic couplings of these staggered Neveu–Schwarz modules.

**Acknowledgements** We would like to thank Akihiro Tsuchiya, Koji Hasegawa, Gen Kuroki and Masaru Sugawara for useful discussions.

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