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Operator-valued Camassa–Holm systems and their integrability

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Abstract

We study a Sturm–Liouville-type operator with operator-valued coefficients and its iso-spectral deformations, related to a new two-component Camassa–Holm-type completely integrable dynamical systems. Based on a specially devised gradient-holonomic scheme, generalizing the one before developed for studying a Sturm–Liouville-type spectral problem on a spatially multidimensional Hilbert–Schmidt operator-valued Hilbert space, we constructed the related two compatible Poisson structures and an infinite hierarchy of commuting to each other conservation laws of the derived two-component Camassa–Holm-type Hamiltonian system. The latter makes it possible to state under some additional constraints its complete integrability, and in particular, to develop the corresponding inverse spectral-type-based method for constructing its exact solutions.

Keywords Sturm–Liouville-type spectral problem · Monodromy matrix operator · Hamiltonian systems · Poisson brackets · Conservation laws · Integrability

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1 Introduction

We study a Sturm–Liouville-type spectral problem of second order, defined on a spatially multidimensional Hilbert–Schmidt operator-valued space, whose iso-spectral deformations describe two-component operator-valued Camassa–Holm-type [1–3] completely integrable Hamiltonian systems. The corresponding classical systems were widely investigated [5, 7] during past decades, where there have been demonstrated very interesting singular peakon-type properties [3, 10, 17] of their solutions. The presented results concern new operator-valued two-component Camassa–Holm-type completely integrable dynamical systems, which are based on studying analytical properties of the corresponding Sturm–Liouville-type spectral problem and its iso-spectral deformations and, respectively, constitute continuation of a previously developed analytical scheme in works [11–14, 16], devoted to studying a Sturm–Liouville-type spectral problem on a spatially multidimensional Hilbert–Schmidt operator-valued Hilbert space, generating generalized operator-valued Korteweg–de Vries-type nonlinear dynamical systems within the gradient-holonomic approach, initiated before by S.P. Novikov in his classical works [18, 19]. As the corresponding classical spectral problem generates, respectively, the two-component Camassa–Holm-type [3, 5, 8] hydrodynamic-type evolution systems, we succeeded in constructing both their operator-valued Hamiltonian generalizations with respect to suitably related compatible Poisson structures and an infinite hierarchy of commuting to each other conservation laws. The latter makes it possible to prove under some additional constraints the complete integrability of the generalized operator-valued two-component Camassa–Holm-type dynamical system, and in particular, to develop the corresponding inverse spectral-type-based method for constructing its exact solutions.

2 Bilocal periodic spectral problem

Consider the following Sturm–Liouville-type periodic spectral problem:

$$Lg(x) := -\partial^2 g(x)/\partial x^2 + (\mu I + v(x)\lambda + \rho(x)\lambda^2 I)g, \tag{1}$$

where $\mu \in \mathbb{R}$ is a fixed parameter, $\lambda \in \mathbb{C}$ is the corresponding spectral parameter, coefficients $v, \rho \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathcal{B})$ and a function $g \in C^2(\mathbb{R}; \mathcal{B})$, with $\mathcal{B}$ being a unital operator algebra of the Hilbert–Schmidt [20, 21] operators on a Hilbert space H. We also will assume that the coefficient $\rho \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathcal{B})$ belongs to the center of the algebra $\mathcal{B}_2$, that is $[\rho(x), \mathcal{B}] = 0$ for all $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$. The algebra $\mathcal{B}$ is endowed with a natural trace-norm $||\cdot||$, defined by the relationship $||A|| := (\mathrm{Tr}(A^*A))^{1/2}$ for an operator $A \in \mathcal{B}$. The spectrum $\sigma(L) \subset \mathbb{C}$ of the operation (1) is determined by the condition $\sup_{x\in\mathbb{R}} ||f(x)|| < \infty$. Its structure is in a general case very complicated [4, 18] even in the classical case of the algebra $\mathcal{B}_0 = \mathbb{C}$; nonetheless, we will analyze it, following the approach devised in [14, 16], having assumed that there holds the constraint $[\rho(x), \mathcal{B}] = 0$ for all $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$ and made use of general functional–operator relationships, generated by the related spectral problem (1).

Let us preliminarily represent the differential–operator expression (1) in the following differential matrix form:

$$df/dx = l(x;\lambda)f, \tag{2}$$

with $f := (f_1, f_2)^\intercal \in C^1(\mathbb{R}; \mathcal{B}^2)$ and

$$l(x;\lambda) := \begin{pmatrix} 0 & I \\ w(x;\lambda) & 0 \end{pmatrix}, \tag{3}$$

where we denoted by $w(x;\lambda) := \mu I + \lambda v(x) + \lambda^2 \rho(x) = \mu I + \langle w(x)|(\lambda, \lambda^2)^\intercal\rangle \in \mathcal{B}$, $w(x) := (v(x), \rho(x))^\intercal \in \mathcal{B}^2$, $x \in \mathbb{R}, \lambda \in \mathbb{C}$, as well as we also denoted, for brevity, by $\langle\cdot|\cdot\rangle$ the usual bilinear form on $\mathbb{C}^2$. Solutions to the matrix problem (2) are effectively described by means of its fundamental solution $F(x; s) \in C^2(\mathbb{R};\ \mathcal{B}^2 \otimes \mathcal{B}^{*,2})$ for any $x, s \in \mathbb{R}$, satisfying the following properties:

$$dF(x,s;\lambda)/dx = l(x;\lambda))F(x,s;\lambda), \quad F(x,s;\lambda)|_{x=s} = I \tag{4}$$

for all $\lambda \in \mathbb{C}$. Taking into account that the operator-valued function $w(x;\lambda) \in \mathcal{B}$ is 2π-periodic with respect to the variable $x \in \mathbb{R}$, based on the fundamental solution $F(x,s;\lambda) \in C^2(\mathbb{R};\ \mathcal{B}^2 \otimes \mathcal{B}^{*,2})$, $x, s \in \mathbb{R}$, one can construct [22] the corresponding monodromy matrix $S(x;\lambda) := F(x,x;\lambda) \in C^2(\mathbb{R};\ \mathcal{B}^2 \otimes \mathcal{B}^{*,2})$, $x \in \mathbb{R}$, satisfying [14, 19] the following differential–commutator Novikov–Lax relationship:

$$dS(x;\lambda)/dx = [l(x;\lambda), S(x;\lambda)] \tag{5}$$

and study the corresponding iso-spectral deformations of the Sturm–Liouville-type spectral problem (1). We will be mainly interested in the functional properties of the trace-functional

$$\gamma(\lambda) := \mathrm{Tr}\ (S(x;\lambda)) = \mathrm{reg} \int_{\mathbb{R}^n} \mathrm{tr}\ S(x;\lambda; y|y)dy), \tag{6}$$

where "tr" is the usual matrix trace operation and "reg (...)" means the usual linear regularization of the corresponding trace-functional, whose existence *a priori* follows [20] owing to the Hilbert–Schmidt structure of the operator algebra $\mathcal{B}$. Note here that owing to the commutator structure of (5) one immediately follows that $d\gamma(\lambda)/dx = 0$ for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, that is the expression (6) defines a smooth functional invariant $\Delta(\lambda) : \mathcal{M} \to \mathbb{C}$ on a suitably chosen operator–functional manifold $\mathcal{M} \subset \{v, \rho \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathcal{B})\}$.

3 Functional–operator properties of $\gamma(\lambda)$

It is easy to observe that the functional (6) is analytical with respect to the parameter $\lambda \in \mathbb{C}$. Assuming, in addition, that it is also smooth by Fréchet on the introduced above operator manifold $\mathcal{M}$, one can calculate the corresponding gradient $\varphi(x;\lambda) := \operatorname{grad}\Delta(x;\lambda) \in T(\mathcal{M}) \subset C^\infty(\mathbb{R};\ \mathcal{B}^2)$ as an element of the cotangent space $T(\mathcal{M})$ to the functional–operator manifold $\mathcal{M}$, which is defined by means of the following variational relationship:

$$\delta\gamma(\lambda)\ = (\varphi(x;\lambda)|\delta w(x)) := \int_0^{2\pi} dx \operatorname{Tr}\langle\varphi(x;\lambda)|\delta w(x)\rangle. \tag{7}$$

The latter can be represented by means of simple calculations in the following compact and useful form:

$$\delta\gamma(\lambda) = (\ \operatorname{grad}\gamma(x;\lambda)|\delta w(x)) = \int_0^{2\pi} dx \operatorname{Tr}\left(S(x;\lambda)\delta l(x;\lambda)\right), \tag{8}$$

resulting in the following gradient covector expression:

$$\operatorname{grad}\gamma(x;\lambda) = (\lambda s_{12}(x;\lambda), \lambda^2 s_{12}(x;\lambda))^{\mathsf{T}} \tag{9}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Moreover, as it follows from the matrix relationship (5), the gradient covector (9) satisfies the characteristic Magri-type [23] relationship:

$$\lambda\vartheta\ \operatorname{grad}\Delta(x;\lambda) = \eta\ \operatorname{grad}\Delta(x;\lambda), \tag{10}$$

where operators $\vartheta, \eta\ :\ T^*(\mathcal{M}) \to T(\mathcal{M})$ are Poisson, compatible [14, 23, 24] operators on the functional–operator manifold $\mathcal{M}$ and equal to the following skew-symmetric differential–integral operator–matrix expressions:

$$\vartheta = \begin{pmatrix} 1/2\left(\partial_x v^+ + v^+\partial_x\right) & \partial_x\rho + \rho\partial_x - 1/2v^-\partial_x^{-1}v^- \\ \partial_x\rho + \rho\partial_x - 1/2v^-\partial_x^{-1}v^- & 0 \end{pmatrix} \tag{11}$$

and

$$\eta = \begin{pmatrix} I(\partial_x^3/2 - 2\mu\partial_x) & 0 \\ 0 & \partial_x\rho + \rho\partial_x - 1/2v^-\partial_x^{-1}v^- \end{pmatrix}, \tag{12}$$

where we denoted by $v^\pm : \mathcal{B} \to \mathcal{B}$, respectively, the anti-commutator/commutator $v^\pm(\cdot) := [v, (\cdot)]_\pm$ and took into account that $\rho^-\mathcal{B} := [\rho, \mathcal{B}] = 0$ owing to the assumption imposed on the coefficient $\rho \in C^\infty(\mathbb{R}/\{2\pi\mathbb{Z}\};\mathcal{B})$. Thus, we have proved the following proposition.

Proposition 1 *The differential–integral matrix operator expressions (11) and (12) determine a compatible pair of Poisson structures on the operator–functional manifold $\mathcal{M}$.*

The Poisson operators (11) and (12) make it possible to construct the corresponding symmetry recursion operator $\Lambda := \vartheta^{-1}\eta : T^*(\mathcal{M}) \to T^*(\mathcal{M})$, generating a countable hierarchy of commuting to each other functionals $\gamma_j : \mathcal{M} \to \mathbb{R}$, $j \in \mathbb{Z}_+$, via the relationships

$$\Lambda \text{ grad}\gamma_j = \text{ grad}\gamma_{j+1}, \tag{13}$$

where ϑ grad$\gamma_0 := 0$, that is

$$\{\gamma_j, \gamma_k\}_\vartheta = 0 = \{\gamma_j, \gamma_k\}_\eta \tag{14}$$

for all $j, k \in \mathbb{Z}_+$ with respect to the following two compatible Poisson brackets:

$$\{\alpha, \beta\}_\vartheta := (\text{ grad}\alpha|\vartheta \text{ grad}\beta), \quad \{\alpha, \beta\}_\eta := (\text{ grad}\alpha|\eta\text{grad}\beta), \tag{15}$$

defined for arbitrary smooth functionals $\alpha, \beta : \mathcal{M} \to \mathbb{R}$.

4 Camassa–Holm-type Hamiltonian systems and their reduction

Consider the constructed above invariant functionals $\gamma_j : \mathcal{M} \to \mathbb{R}$, $j \in \mathbb{Z}_+$, on the Poisson manifold $\mathcal{M}$ and define the following infinite hierarchy of Hamiltonian systems:

$$\frac{\partial}{\partial t_j}(v, \rho)^\intercal := -\eta \text{ grad}\gamma_j[v, \rho], \tag{16}$$

where $t_j \in \mathbb{R}$, $j \in \mathbb{Z}_+$, are the related evolution parameters. Having assumed now that $t_0 := x \in \mathbb{R}$ at $\gamma_0 := H_0$, one can easily obtain that

$$\begin{aligned} v_x &= -(\partial_x^3/2 - 2\mu\partial_x) \text{ grad}_v H_0, \\ \rho_x &= -\left[(\partial_x\rho + \rho\partial_x) - 1/2v^-\partial_x^{-1}v^-\right] \text{ grad}_\rho H_0. \end{aligned} \tag{17}$$

The latter allows to define a new operator variable $u \in \mathcal{B}$ via the substitution $v := \left(-\partial_x^2 + 4\mu\right)u + k \in \mathcal{B}$ for some constant operator $k \in \mathcal{B}$, subject to which there hold the following relationships:

$$\text{grad}_v H_0 = 2u = -2\left(\partial_x^2 - 4\mu\right)^{-1}(v - k), \quad \text{grad}_\rho H_0 = -I, \tag{18}$$

generated by the following Hamiltonian functional:

$$H_0 = \int_0^{2\pi} dx \text{Tr}\,(uv - uk - \rho)\,. \tag{19}$$

Making use of the Hamiltonian functional (19) and the first Poisson operator (11), one can construct a new evolution operator flow

$$\frac{\partial}{\partial t}(v, \rho)^\intercal := -\vartheta \text{ grad}H_0[v, \rho], \tag{20}$$

on the operator manifold $\mathcal{M}$ with respect to the temporal evolution parameter $t \in \mathbb{R}$, or, equivalently,

$$
\begin{aligned}
-u_{xxt} + 4\mu u_t &= -2v^+ u_x - v_x^+ u + \rho_x - 1/2 v^- \partial_x^{-1} v^- \\
\rho_t &= -2\rho_x u - 4\rho u_x + v^- \partial_x^{-1} \left(v^- u \right),
\end{aligned}
\tag{21}
$$

being exactly a two-component Camassa–Holm-type operator dynamical system, naturally generalizing the one before obtained in [5]. We formulate the obtained results as the following theorem.

Theorem 2 *The two-component Camassa–Holm-type system (21) possesses an infinite hierarchy of commuting conservation laws and represents a completely integrable bi-Hamiltonian operator flow on the operator–functional manifold* $\mathcal{M}$.

It is here worth to observe, similarly to that in the work [25], that the whole construction, presented above, remains unchanged, if we replace the scalar parameter $\mu \in \mathbb{R}$ by a constant linear operator $\tilde{\mu} \in \mathcal{B}$, where for any $a(x) \in \mathcal{B}$, $x \in \mathbb{R}$, the kernel of the operator $\tilde{\mu} a \in \mathcal{B}$ equals $(\tilde{\mu} a)(x; y|z) = \int_{\mathbb{R}^n} ds \tilde{\mu}(y|s) a(x; s|z) dx$ for $y, z \in \mathbb{R}^n$. Then, one easily constructs the corresponding skew-symmetric differential–integral matrix operators

$$
\tilde{\vartheta} = \begin{pmatrix} 1/2 \left(\partial_x v^+ + v^+ \partial_x - \tilde{\mu}^- \partial_x^{-1} v^- - v^- \partial_x^{-1} \tilde{\mu}^- \right) & \partial_x \rho + \rho \partial_x - 1/2 v^- \partial_x^{-1} v^- \\ \partial_x \rho + \rho \partial_x - 1/2 v^- \partial_x^{-1} v^- & 0 \end{pmatrix}
\tag{22}
$$

and

$$
\tilde{\eta} = \begin{pmatrix} I \partial_x^3 /2 - \ \tilde{\mu}^+ \partial_x & 0 \\ 0 & \partial_x \rho + \rho \partial_x - 1/2 v^- \partial_x^{-1} v^- \end{pmatrix},
\tag{23}
$$

which prove to be Poisson and compatible on the operator manifold $\mathcal{M}$, satisfying the related gradient relationships

$$
\tilde{\vartheta} \ \mathrm{grad} \tilde{\gamma}_{j+1} = \tilde{\eta} \ \mathrm{grad} \tilde{\gamma}_j
\tag{24}
$$

for an infinite hierarchy of commuting to each other smooth functionals $\tilde{\gamma}_j : \mathcal{M} \to \mathbb{R}$, $j \in \mathbb{Z}_+$, that is

$$
\{\tilde{\gamma}_j, \tilde{\gamma}_k\}_{\tilde{\vartheta}} = 0 = \{\tilde{\gamma}_j, \tilde{\gamma}_k\}_\eta
\tag{25}
$$

with respect to the corresponding two compatible Poisson brackets:

$$
\{\alpha, \beta\}_{\tilde{\vartheta}} := (\ \mathrm{grad}\alpha | \tilde{\vartheta} \mathrm{grad}\beta), \quad \{\alpha, \beta\}_\eta := (\ \mathrm{grad}\alpha | \tilde{\eta}\ \mathrm{grad}\beta),
\tag{26}
$$

defined for arbitrary smooth functionals $\alpha, \beta : \mathcal{M} \to \mathbb{R}$. The obtained result we can formulate as the following proposition.

Proposition 3 *The skew-symmetric differential–integral matrix operator expressions (22) and (23) determine on the operator–functional manifold* $\mathcal{M}$ *a compatible pair of Poisson structures.*

The hierarchy of smooth functionals $\tilde{\gamma}_j : \mathcal{M} \to \mathbb{R}$, $j \in \mathbb{Z}_+$, is naturally generated by a Hamiltonian functional $\tilde{H}_0 : \mathcal{M} \to \mathbb{R}$ via the recursion expressions

$$\operatorname{grad}\tilde{\gamma}_j := \left(\tilde{\vartheta}^{-1}\tilde{\eta}\right)^j \operatorname{grad}\tilde{H}_{0,} \tag{27}$$

and satisfies the determining condition

$$\begin{aligned} v_x &= -(\partial_x^3/2 - \tilde{\mu}^+\partial_x) \operatorname{grad}_v \tilde{H}_0, \\ \rho_x &= -\left[(\partial_x \rho + \rho\partial_x) - 1/2v^-\partial_x^{-1}v^-\right] \operatorname{grad}_\rho \tilde{H}_0. \end{aligned} \tag{28}$$

Upon the operator substitution $(\partial_x^2 - 2\tilde{\mu}^+)u := v - \tilde{k} \in \mathcal{B}$ for a constant operator element $\tilde{k} \in \mathcal{B}$, one easily obtains the functional expression,

$$\tilde{H}_0 = \int_0^{2\pi} dx \operatorname{Tr}(uv - u\tilde{k} - \rho), \tag{29}$$

simultaneously generating a new two-component Camassa–Holm-type Hamiltonian flow

$$\frac{\partial}{\partial t}(v, \rho)^{\mathsf{T}} := -\tilde{\vartheta} \operatorname{grad}\tilde{H}_0[v, \rho] \tag{30}$$

with respect to the temporal evolution parameter $t \in \mathbb{R}$, equivalent to the following new two-component Camassa–Holm-type integrable system

$$\begin{aligned} &u_{xxt} - 4\mu^+u_t = -2v^+u_x - v_x^+u + \rho_x + \tilde{\mu}^-\partial_x^{-1}\left(v^-u\right) + v^-\partial_x^{-1}\left(\tilde{\mu}^-u\right) - 1/2v^-\partial_x^{-1}v^-, \\ &\rho_t = -2\rho_x u - 4\rho u_x + v^-\partial_x^{-1}\left(v^-u\right), \end{aligned} \tag{31}$$

on the operator manifold $\mathcal{M}$. The obtained result we can formulate as the next theorem.

Theorem 4 *The two-component Camassa–Holm-type system (31) possesses an infinite hierarchy of commuting conservation laws and represents a completely integrable bi-Hamiltonian operator flow on the operator–functional manifold $\mathcal{M}$.*

It can be easily checked that the above-obtained operator-valued two-component Camassa–Holm-type Hamiltonian systems (21) and (31) on the operator manifold $\mathcal{M}$ reduce in the case, when the algebra $\mathcal{B} \to \mathbb{C}$, to the classical two-component Camassa–Holm-type Hamiltonian system. What is important to mention subject to the spectral problem (1) that it generates nontrivial integrable operator Hamiltonian systems only for the case when the coefficient constraint $[\rho(x), \mathcal{B}] = 0$ is imposed for all $x \in \mathbb{R}/\{2\pi\mathbb{Z}\}$. Exactly such a special matrix operator case was already constructed in [6] for the well-known Kontsevich [9] dynamical system)

$$\left.\begin{aligned} du/dx &:= \{h, u\}_\natural = uv - uv^{-1} - v^{-1} \\ dv/dx &:= \{h, v\}_\natural = -vu + vu^{-1} + u^{-1} \end{aligned}\right\} \tag{32}$$

on the operator space $\mathcal{B} = \mathbb{C}\langle u^{\pm 1}, v^{\pm}\rangle$, which is Hamiltonian with respect to the following Hamiltonian function $h = u + v + u^{-1} + v^{-1} + u^{-1} + v^{-1} \in \mathcal{B}$ and the Poisson bracket

$$\{u, v\}_{\natural} = -uv, \quad \{u, u\}_{\natural} = 0_{\natural} = \{v, v\}_{\natural} \tag{33}$$

for generating elements $u, v \in \mathcal{B}$. This system possesses the Lax-type representation

$$dS(u, v; \lambda)/dx = [l(u, v; \lambda), S(u, v; \lambda)], \tag{34}$$

where operator matrices $l, S \in \mathrm{End}(\mathcal{B}^2)$ are equal to such expressions:

$$S(u, v; \lambda) = \begin{pmatrix} \lambda\left(v^{-1} + u\right) & \lambda^2 v + \lambda\left(v^{-1}u^{-1} + u^{-1} + 1\right) \\ \lambda v^{-1} + u & 1 + \lambda\left(v + v^{-1}u^{-1} + u^{-1}\right) \end{pmatrix}, \tag{35}$$

$$l(u, v; \lambda) = \begin{pmatrix} v^{-1} - v + u & \lambda v \\ v^{-1} & u \end{pmatrix}, \tag{36}$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter, $S(u, v; \lambda) := \lambda^2 \operatorname{grad}\gamma(\lambda)|_+ \in \mathrm{End}(\mathcal{B}^2)[\lambda]$ owing to (8), and was recently analyzed in [15] by means of the Lie-algebraic approach. A more detailed analysis of the related spectral problem

$$df(x; \lambda)/dx = l(u, v; \lambda) f(x; \lambda) \tag{37}$$

for generating elements $u, v \in C^{\infty}(\mathbb{R}/\{2\pi\mathbb{Z}\}; \mathcal{B})$ on the space of functions $f \in L_{\infty}(\mathbb{R}; \mathcal{B})$ is under investigation and planned to be presented in a forthcoming work.

5 Conclusions

We have analyzed a generalized periodic Sturm–Liouville-type spectral problem with coefficients from some operator ring and studied its invariant iso-spectral deformations by means of a previously developed gradient-holonomic analytic scheme. Based on specially constructed compatible Poisson structures, we succeeded in deriving an infinite hierarchy of commuting to each other smooth functionals on our operator manifold and, respectively, related commuting Hamiltonian systems, among which we presented a new two-component operator Camassa–Holm-type integrable Hamiltonian system and some its modification.

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Author Contributions All authors contributed equally.

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Declarations

Conflict of interest The authors report no potential conflict of interest.

References

1. Camassa, R., Holm, D.D.: Phys. Rev. Lett. **71**, 1661 (1993)
2. Camassa, R., Holm, D.D., Hyman, J.M.: A new integrable shallow water equation. Adv. Appl. Mech. **31**, 1–33 (1994)
3. Chen, M., Liu, S.-Q., Zhang, Y.: A two-component generalization of the Camassa-Holm equation and its solutions. Lett. Math. Phys. **75**, 1–15 (2006)
4. Dunford, N., Schwartz, J.T.: Linear Operators. Wiley, Spectral Operators (2013)
5. Aratyn, H., Gomes, J.F., Zimerman, A.H.: On negative flows of the AKNS hierarchy and a class of deformations of bi-Hamiltonian structure of hydrodynamic type. J. Phys. A: Math. General (2005). https://doi.org/10.1088/0305-4470/39/5/006
6. Efimovskaya, O., Wolf, T.: On Integrability of the Kontsevich Non-Abelian ODE system. Lett. Math. Phys. **100**(2), 161–170 (2012)
7. Ivanov, R.I., Lyons, T.: Dark solitons of the Qiao's hierarchy. J. Math. Phys. **53**, 123701–8 (2012). https://doi.org/10.1063/1.4764859
8. Novikov, V.: Generalisations of the Camassa-Holm equation, arXiv:0905.2219v1 [nlin.SI] 13 May (2009)
9. Kontsevich, M.: Noncommutative Identities. arXiv:1109.2469, (2011)
10. Freire, I.L.: A look on some results about Camassa-Holm type equations. Commun. Math. **29**, 115–130 (2021)
11. Bogolyubov, N.N., Jr., Prikarpatskii, A.K.: A bilocal periodic problem for the Sturm-Liouville and Dirac operators and some applications to the theory of nonlinear dynamical systems. I. Ukrainian Math. J. **42**(6), 702–707 (1990)
12. Prikarpatskyj, A.K., Bogolyubov, N.N.: A bilocal periodic problem for Sturm-Liouville and Dirac differential operators, and some applications in the theory of nonlinear dynamical systems. Dokl. Akad. Nauk SSSR. Dokl. Math. **310**(1), 29–32 (1990)
13. Prykarpatskyj, A.K., Samoilenko, V., Hr., Andrushkiw, R.I.: Algebraic structure of the gradient-holonomic algorithm for Lax integrable nonlinear dynamical systems. II The reduction via Dirac and canonical quantization procedure. J. Math. Phys. **35**, 4088–4116 (1994). https://doi.org/10.1063/1.530844
14. Blackmore, D., Prykarpatsky, A., Samoylenko, V.: Nonlinear dynamical systems of mathematical physics: spectral and differential-geometric integrability analysis. World Scientific, NY (2011)
15. Hentosh, O.E., Balinsky, A.A., Prykarpatski, A.K.: Poisson structures on (non) associative noncommutative algebras and integrable Kontsevich type Hamiltonian systems. Ann. Math. Phys. **3**(1), 001–006 (2020). https://doi.org/10.17352/amp.000010
16. Mitropolsky Yu. A., Bogolubov N. N. (Jr.), Prykarpatsky A. K., Samoylenko V. H.: Integrable dynamical systems: differential-geometric and spectral aspects, Kiev, Naukova Dumka (1987)
17. Constantin, A., Strauss, W.A.: Stability of peakons. Comm. Pure Appl. Math. **53**(5), 603–610 (2000)
18. Novikov, S., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E.: Theory of Solitons. The Inverse Scattering Method. Monographs in Contemporary Mathematics, Springer, (1984)
19. Novikov, S.P.: Funct. Anal. Appl. **8**(3), 236–246 (1974)
20. Moren, K.: Methods of Hilbert Spaces, 2nd, Revised Hafner Publisher (1973)

21. Reed, M., Simon, B.: Functional analysis: methods of modern mathematical physics, vol. 1. Academic Press (1980)
22. Titchmarsh, E.C.: Eigenfunction expansions associated with second order differential equations, Part 2. Oxford University Press, Oxford (1958)
23. Magri, F.: A Simple model of the integrable Hamiltonian equation. J. Math. Phys. **19**, 1156–1162 (1978)
24. Błaszak, M.: Multi-Hamiltonian theory of dynamical systems. Springer-Verlag, Berlin Heidelberg (1998)
25. Christov, O.: Geometric Integrability of Some Generalizations of the Camassa-Holm Equation, International Journal of Differential Equations, (2011), (1687-9643); https://doi.org/10.1155/2011/738509