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Hadamard property of the *in* and *out* states for Dirac fields on asymptotically static spacetimes

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Abstract
We consider massive Dirac equations on asymptotically static spacetimes with a Cauchy surface of bounded geometry. We prove that the associated quantized Dirac field admits *in* and *out* states, which are asymptotic vacuum states when some time coordinate tends to $\mp\infty$. We also show that the *in/out* states are Hadamard states.

Keywords Hadamard states · Microlocal spectrum condition · Pseudo-differential calculus · Scattering theory · Dirac equation · Curved spacetimes

Mathematics Subject Classification 81T20 · 35S05 · 35Q41

1 Introduction

1.1 In/out vacuum states

The construction of a *distinguished quantum state* for a quantized field on a curved background has been studied extensively in various contexts in Quantum Field Theory.

If the background spacetime has no global symmetries but only asymptotic ones, one can try to specify a distinguished quantum state by its asymptotic behavior, for example at early or late times.

An often studied situation arises when the background spacetime (M, g) has a product structure $M = \mathbb{R} \times \Sigma$ and the metric g becomes asymptotic to *static* metrics when $t \to \pm\infty$. One can then at least heuristically consider *asymptotic vacua*, the

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so-called *in* and *out* states, which look like vacuum states for the asymptotic static metrics when $t \to \mp\infty$.

Let us mention for example the wave and Klein–Gordon fields on Minkowski space, in external electromagnetic potentials [25, 36, 41], or on curved spacetimes with special asymptotic symmetries, [7–9, 44].

Besides the existence of the *in* and *out* states, an important question is to ensure that they satisfy the *Hadamard condition* [26].

Nowadays regarded as an indispensable ingredient in the perturbative construction of interacting fields (see, e.g., the recent reviews [10, 22]), this property accounts for the correct short-distance behavior of two-point functions. It can be conveniently formulated as a condition on the *wave front set* of the state's two-point functions [34].

The above questions were solved in [14] for *massive Klein–Gordon fields*, using a combination of scattering theory arguments and global pseudodifferential calculus.

In this paper we consider this problem for *massive Dirac fields*, using similar methods. Let us now describe in more details the results of this paper.

1.2 Free Dirac fields

To define the *in/out* vacuum states, we first briefly recall some background on free Dirac fields, see Sect. 2 for more details.

Let (M, g) an even-dimensional globally hyperbolic spacetime, equipped with a spin structure. We denote by $S(M)$ the bundle of spinors and by $D = \not{D} + m$ a Dirac operator on M.

The CAR $*$-algebra of free Dirac fields, denoted by $\mathrm{CAR}(D)$, is the unital $*$-algebra generated by symbols

$$\psi(u),\ \psi^*(u),\ \mathbb{1},\ \text{for } u \in C_0^\infty(M; S(M)),$$

with relations

$$\begin{aligned}
&(i)\ u \mapsto \psi^*(u) \text{ resp. } u \mapsto \psi(u) \text{ is } \mathbb{C} \text{ linear resp. anti-linear},\\
&(ii)\ \psi(u)^* = \psi^*(u),\\
&(iii)\ \phi(Du) = \phi^*(Du) = 0,\\
&(iv)\ [\psi(u), \psi(v)]_+ = [\psi^*(u), \psi^*(v)]_+ = 0,\ [\psi(u), \psi^*(v)]_+ = \mathrm{i}(u|Gv)_M \mathbb{1},
\end{aligned} \tag{1.1}$$

where $[\cdot,\cdot]_+$ is the anti-commutator, G is the causal propagator for D and $(\cdot|\cdot)_M$ is the canonical hermitian scalar product on spinors. Conditions *iii)* and *iv)* express the field equation and the CAR respectively.

$\mathrm{CAR}(D)$ is an example of the abstract CAR $*$-algebra $\mathrm{CAR}(\mathcal{Y}, \nu)$, see Sect. 2.3 for the pre-Hilbert space

$$(\mathcal{Y}, \nu) = \left(\frac{C_0^\infty(M; S(M))}{DC_0^\infty(M; S(M))}, \mathrm{i}(\cdot|G\cdot)_M \right).$$

The 'field operators' $\psi^{(*)}(u)$ for $u \in C_0^\infty(M; S(M))$ are traditionally called 'space-time fields'.

Other equivalent pre-Hilbert spaces are also convenient to discuss Dirac fields, see 2.3.3. One of them is the space $\mathrm{Sol}_{\mathrm{sc}}(D)$ of smooth, space compact solutions of $Du = 0$, with the Hilbertian scalar product recalled in (2.14), or equivalently the space $C_0^\infty(\Sigma; S(\Sigma))$ of Cauchy data, equipped with the Hilbertian scalar product ν_Σ recalled in (2.15).

The use of space-time fields is important to formulate the Hadamard condition for states on $\mathrm{CAR}(D)$, see 1.3.4 below.

1.2.1 Quasi-free states

We recall that a quasi-free state ω on $\mathrm{CAR}(\mathcal{Y}, \nu)$ is uniquely determined by its covariances $\lambda^\pm$, which have hermitian forms on $\mathcal{Y}$ satisfying

$$\lambda^\pm \geq 0,\ \lambda^+ + \lambda^- = \nu. \tag{1.2}$$

Their relationship with the state ω is given by

$$\omega(\psi(y_1)\psi^*(y_2)) = \overline{y}_1 \cdot \lambda^+ y_2,\ \omega(\psi^*(y_2)\psi(y_1)) = \overline{y}_1 \cdot \lambda^- y_1,\ \ y_1, y_2 \in \mathcal{Y}.$$

It is convenient to identify $\lambda^\pm$ with linear maps $c^\pm$ on $\mathcal{Y}$ by setting

$$\lambda^\pm =: \nu \circ c^\pm.$$

The conditions (1.2) become

$$c^\pm \geq 0 \text{ for } \nu,\ c^+ + c^- = \mathbb{1}. \tag{1.3}$$

Going back to Dirac fields, it follows that after fixing a Cauchy surface Σ, a pair of linear maps $c^\pm : C_0^\infty(\Sigma; S(\Sigma)) \to \mathcal{D}'(\Sigma; S(\Sigma))$ such that

$$c^+ + c^- = \mathbb{1},\ c^\pm \geq 0 \text{ for } \nu_\Sigma,$$

defines a unique quasi-free state ω on $\mathrm{CAR}(D)$.

1.3 Results

Let us now describe more in details the results of this paper. We first describe the class of spacetimes that we will consider.

1.3.1 Asymptotically static spacetimes

We will consider a spacetime of *even dimension* n of the form $M = \mathbb{R} \times \Sigma$, where Σ is a d-dimensional manifold, equipped with a metric

$$g = -c^2(x)\mathrm{d}t^2 + (\mathrm{dx}^i + b^i(x)\mathrm{d}t)h_{ij}(x)(\mathrm{dx}^j + b^j(x)\mathrm{d}t),$$

where $x = (t, \mathrm{x}) \in M$, $c \in C^\infty(M;\mathbb{R})$ is a strictly positive function, $b \in C^\infty(M; T\Sigma)$ and $h \in C^\infty(M; \otimes^2_{\mathrm{s}} T^*\Sigma)$ is a t-dependent Riemannian metric on Σ. We will assume that when $t \to \pm\infty$ the metric g converges to *static metrics*

$$g_{\mathrm{out/in}} = -c_{\mathrm{out/in}}(\mathrm{x})\mathrm{d}t^2 + h_{\mathrm{out/in}}(\mathrm{x})\mathrm{dx}^2.$$

The convergence of g to $g_{\mathrm{out/in}}$ is assumed to be uniform in the space variable x. More precisely, one assumes that there exists $\mu > 0$ such that

$$\begin{array}{ll} \partial_t^k \partial_{\mathrm{x}}^\alpha (h(x) - h_{\mathrm{out/in}}(\mathrm{x})) \in O(\langle t\rangle^{-\mu-k}), & \\ \partial_t^k \partial_{\mathrm{x}}^\alpha b(x) \in O(\langle t\rangle^{-1-\mu-k}), & k \in \mathbb{N}, \alpha \in \mathbb{N}^d, \\ \partial_t^k \partial_{\mathrm{x}}^\alpha (c(x) - c_{\mathrm{out/in}}(\mathrm{x})) \in O(\langle t\rangle^{-\mu-k}), & \end{array} \tag{1.4}$$

in an appropriate uniform sense in $\mathrm{x} \in \Sigma$, using the notion of Riemannian manifolds of *bounded geometry*, see hypotheses (H1), (H2) in 3.1.3 for the precise formulation.

We consider a Dirac operator

$$D = \not{D} + m$$

and assume that $m(t, \mathrm{x})$ converges to $m_{\mathrm{out/in}}(\mathrm{x})$ when $t \to \pm\infty$ in a similar uniform way, see hypothesis (H3) in 3.1.3 for the precise formulation.

It follows that D converges when $t \to \pm\infty$ to *asymptotic Dirac operators* $D_{\mathrm{out/in}}$, which are associated to the static metrics $g_{\mathrm{out/in}}$.

1.3.2 Vacua for the asymptotic Dirac operators

The vector field ∂_t is Killing for the static metrics $g_{\mathrm{out/in}}$, which implies that one can define the *vacuum states* $\omega^{\mathrm{vac}}_{\mathrm{out/in}}$ for $D_{\mathrm{out/in}}$, see Sect. 2.4, using the projections

$$c^{\pm\mathrm{vac}}_{\mathrm{out/in}} := \mathbb{1}_{\mathbb{R}^\pm}(H_{\mathrm{out/in}}),$$

where $H_{\mathrm{out/in}}$ are selfadjoint operators on $L^2(\Sigma; S(\Sigma))$, for the canonical Hilbertian scalar product on $S(\Sigma)$. The operators $H_{\mathrm{out/in}}$ are the *generators* of the unitary group induced by the spinorial Lie derivative $\mathcal{L}_{\partial_t}$ on solutions of $D_{\mathrm{out/in}}\psi = 0$, see Sect. 2.4.

To define the vacuum states $\omega^{\mathrm{vac}}_{\mathrm{out/in}}$ in an unambiguous way, one needs to assume that

$$\mathrm{Ker}\, H_{\mathrm{out/in}} = \{0\}, \tag{1.5}$$

i.e., the absence of *zero modes*. If (1.5) is violated, then in physics language one needs to decide if zero modes are considered as particles or as anti-particles.

In this paper, we strengthen (1.5) by requiring that

$$0 \notin \sigma(H_{\text{out/in}}),$$

see hypothesis (H4), ie that the asymptotic Dirac operators $D_{\text{out/in}}$ are *massive* in the terminology of 2.4.3.

1.3.3 The *in/out* vacuum states

Let us now explain the definition of the *in/out* states for D. We set $\Sigma_s = \{s\} \times \Sigma$, fix the reference time $t = 0$ and denote by $U(t, s) : C_0^\infty(\Sigma_s; S(\Sigma_s)) \to C_0^\infty(\Sigma_t; S(\Sigma_t))$ the Cauchy evolution for the Dirac operator D.

Let us set for $\pm T \gg 1$:

$$c_T^\pm = U(0, T)c_{\text{out/in}}^{\pm\text{vac}}U(T, 0), \text{ acting on } C_0^\infty(\Sigma_0; S(\Sigma_0)).$$

The maps $c_T^\pm$ correspond intuitively to the *vacuum state at (late or early) time T*.

One expects that the limits

$$c_{\text{out/in}}^\pm = \lim_{T\to\pm\infty} U(0, T)c_{\text{out/in}}^{\pm\text{vac}}U(T, 0) \tag{1.6}$$

exist in an appropriate sense (actually in some operator norm topology).

One can show that $c_{\text{out/in}}^\pm$ are supplementary projections acting on $C_0^\infty(\Sigma_0; S(\Sigma_0))$, which are selfadjoint for the Hilbertian scalar product ν_Σ.

Therefore, one can associate to $c_{\text{out/in}}^\pm$ quasi-free states $\omega_{\text{out/in}}$ on $\text{CAR}(D)$, which are called the *in/out vacuum states* for D.

If we go back to space-time fields, we obtain

$$\omega_{\text{out/in}}(\psi(u)\psi^*(u)) = (u|\Lambda_{\text{out/in}}^+ u)_M,$$
$$\omega_{\text{out/in}}(\psi^*(u)\psi(u)) = (u|\Lambda_{\text{out/in}}^- u)_M, \ u \in C_0^\infty(M; S(M)),$$

where

$$\Lambda_{\text{out/in}}^\pm : C_0^\infty(M; S(M)) \to C^\infty(M; S(M))$$

are defined by

$$\Lambda_{\text{out/in}}^\pm(t, s) = U(t, 0)\mathrm{i}\gamma(n)c_{\text{out/in}}^\pm U(0, s), \tag{1.7}$$

and we write $\Lambda_{\text{out/in}}^\pm$ as operator-valued Schwartz kernels in the time variable, ie we use the formal identity

$$Au(t) = \int_{\mathbb{R}} A(t, s)u(s)\mathrm{d}s,$$

to define the 'time kernel' of some operator A acting on M. In (1.7) n is the future directed unit normal to Σ_0 and γ are the 'gamma matrices' (or Clifford multiplications) obtained from the spin structure on (M, g).

The operators $\Lambda^{\pm}_{\rm out/in}$ satisfy the analog of (1.1):

$$\begin{array}{rl}\text{(i)} & \Lambda^{\pm}_{\rm out/in} \geq 0 \text{ for } (\cdot|\cdot)_M,\\ \text{(ii)} & \Lambda^{+}_{\rm out/in} + \Lambda^{-}_{\rm out/in} = {\rm i}G,\\ \text{(iii)} & D \circ \Lambda^{\pm}_{\rm out/in} = \Lambda^{\pm}_{\rm out/in} \circ D = 0.\end{array}$$

1.3.4 Hadamard property of the *in /out* states

As explained above, the Hadamard condition allows to select among the plethora of states the physically meaningful ones, which should resemble the Minkowski vacuum, at least in the vicinity of any point of M.

The microlocal definition of Hadamard states for Dirac fields was first introduced by Hollands [21], who also proved its equivalence with the older characterization by the short distance asymptotics of its two-point functions. Hadamard states for Dirac fields were further studied in [24, 29, 37, 39].

To our knowledge the first paper proving existence of Hadamard states for Dirac fields in the general case is the recent paper by Islam and Strohmaier [23], although the construction of Hadamard states by the deformation argument of Fulling, Narcowich and Wald [11] was quite probably known to experts.

Another construction of Hadamard states on spacetimes of bounded geometry was given in [17] using global pseudodifferential calculus on a Cauchy surface. The methods used in the present paper are to a large extend an adaptation of the strategy in [17] to a scattering situation.

Let us now state the main result of this work, referring the reader to Sect. 3.1 for hypotheses (H).

Theorem 1.1 *Assume hypotheses* (Hi), $1 \leq i \leq 4$. *Then*

(1) *The norm limits* (1.6) *exist and define by* (1.7) pure *quasi-free states* $\omega_{\rm out/in}$ *called the* in/out vacuum states.
(2) $\omega_{\rm out/in}$ *is a* Hadamard state, *ie*

$$\mathrm{WF}(\Lambda^{\pm}_{\rm out/in}) \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}$$

where $\mathcal{N}^{\pm}$ *are the two connected components of the characteristic set* $\mathcal{N} = \{(x, \xi) \in T^*M \setminus o : \xi \cdot g^{-1}(x)\xi = 0\}$ *of* D.

1.3.5 Relationship with asymptotic fields

The definition of the out/in vacua is often introduced using *asymptotic fields*. Let us now explain the relationship between these two methods, using the space of Cauchy data $C_0^\infty(\Sigma; S(\Sigma))$ as our pre-Hilbert space.

If g satisfies (1.4) *for* $\mu > 1$ (with a similar short-range condition for the convergence of $m(t, x)$ when $t \to \pm\infty$), then it is easy to construct *Möller operators*

$$\Omega_{\text{out/in}} = \lim_{t \to \pm\infty} U_{\text{out/in}}(0, t) U(t, 0),$$

where $U(t, s)$ resp. $U_{\text{out/in}}(t, s)$ are the Cauchy evolutions for D resp. $D_{\text{out/in}}$.

Since $\Omega_{\text{out/in}}$ are unitary for the scalar product ν_Σ, they induce automorphisms $\tau_{\text{out/in}}$ of $\text{CAR}(D)$ defined by

$$\tau_{\text{out/in}}(\psi^{(*)}(f)) = \psi^{(*)}(\Omega_{\text{out/in}} f) =: \psi^{(*)}_{\text{out/in}}(f), \ \ f \in C_0^\infty(\Sigma; S(\Sigma)).$$

The out/in vacuum states can then be equivalently defined by

$$\omega_{\text{out/in}} = \omega^{\text{vac}}_{\text{out/in}} \circ \tau_{\text{out/in}}. \tag{1.8}$$

Note that the existence of the Möller operators $\Omega_{\text{out/in}}$ requires the short-range condition $\mu > 1$, while the direct construction of $\omega_{\text{out/in}}$ that we use here requires only the weaker condition $\mu > 0$, as was also the case in [14] for Klein–Gordon fields.

1.4 Outline of the proof

Let us now briefly explain the main ingredients in the proof of Theorem 1.1, which follows the general strategy in [17]. The first step consists in reducing the metric to the simpler form

$$g = -\mathrm{d}t^2 + h(t, \mathrm{x})\mathrm{dx}^2,$$

where the time-dependent Riemannian metric $h(t, \mathrm{x})\mathrm{dx}^2$ on Σ converges to Riemannian metrics $h_{\text{out/in}}(\mathrm{x})\mathrm{dx}^2$ when $t \to \pm\infty$. This is done in the usual way, by combining a conformal transformation and the well-known argument using the flow of the vector field ∇t.

One can use the covariance of Dirac operators and two-point functions under conformal transformations, see Sects. 2.2 and 2.3.7, to reduce ourselves to this simple situation.

In a second step, one uses parallel transport with respect to the vector field ∂_t to identify the spinor bundles at different times, and to reduce the Dirac equation $D\psi = 0$ to a time-dependent Schroedinger equation:

$$\partial_t \psi - \mathrm{i} H(t) \psi = 0,$$

where $H(t) = H(t, \mathrm{x}, \partial_{\mathrm{x}})$ is a first-order elliptic differential operator on Σ.

The third step is analogous to [17], where Hadamard states for Dirac fields are constructed using pseudodifferential calculus, with the difference that in our case we need to control the behavior of various operators when $t \to \pm\infty$.

We construct time-dependent projections $P^{\pm}(t)$ such that

$$\begin{array}{l}(1)\ P^{\pm}(t) - \mathbb{1}_{\mathbb{R}^{\pm}}(H_{\text{out/in}}) \in O(t^{-\mu}) \text{ when } t \to \pm\infty,\\ (2)\ \partial_t U(0,t) P^{\pm}(t) U(t,0) \in O(t^{-1-\mu})\Psi^{-\infty},\end{array}$$

where $\Psi^{-\infty}$ is some ideal of smoothing operators on Σ. (1) implies that to prove the existence of the limits (1.6), it suffices to consider instead

$$\lim_{t\to\pm\infty} U(0,t)P^{\pm}(t)U(t,0)$$

which exists by (2) and the Cook argument. This prove the existence of the *in /out* states $\omega_{\text{out/in}}$ for D. Integrating (2) from 0 to $\pm\infty$, we also obtain that

$$c^{\pm}_{\text{out/in}} - P^{\pm}(0) \text{ are smoothing operators on } \Sigma.$$

It is shown in [17] that $P^{\pm}(0)$ are projections which generate a Hadamard state, which, since $c^{\pm}_{\text{out/in}} - P^{\pm}(0)$ are smoothing, proves the Hadamard property of $\omega_{\text{out/in}}$.

1.5 Plan of the paper

Let us now discuss the plan of this paper.

In Sect. 2 we recall the quantization of Dirac fields on curved spacetimes. In Sect. 3 we describe the geometric framework of asymptotically static spacetimes and the spin structures and Dirac operators on such spacetimes.

In Sect. 4 we give a brief overview of Shubin's pseudodifferential calculus on manifolds of bounded geometry and of its time-dependent version that we will use in this paper. Finally, Sect. 5 contains the proof of Theorem 1.1 and the various reduction procedures that are used.

1.6 Notation

1.6.1 Lorentzian manifolds

We use the mostly + signature convention for Lorentzian metrics. All Lorentzian manifolds considered in this paper will be *orientable* and connected.

1.6.2 Bundles

If $E \xrightarrow{\pi} M$ is a bundle, we denote by $C^\infty(M;E)$ resp. $C_0^\infty(M;E)$ the set of smooth resp. smooth and compactly supported sections of E.

If $E \xrightarrow{\pi} M$ is a vector bundle, we denote by $\mathcal{D}'(M;E)$ resp. $\mathcal{E}'(M;E)$ the space of distributional resp. compactly supported distributional sections of E.

1.6.3 Matrices

Since we will often use frames of vector bundles, we will denote by $\boldsymbol{M}$ a matrix in $M_n(\mathbb{R})$ or $M_N(\mathbb{C})$ and by M the associated endomorphism.

1.6.4 Frames and frame indices

We use the letters $0 \leq a \leq d$ for frame indices on TM or T^*M, and $1 \leq a \leq d$ for frame indices on $T\Sigma$ or $T^*\Sigma$, if $\Sigma \subset M$ is a space like hypersurface. If g is a metric on M and $(e_a)_{0\leq a\leq d}$ is a local frame of TM, we set $g_{ab} = e_a{\cdot}ge_b$ and $g^{ab} = e^a{\cdot}g^{-1}e^b$, where $(e^a)_{0\leq a\leq d}$ is the dual frame.

We use capital letters $1 \leq A \leq N$ for frame indices of the spinor bundle $S(M)$.

If $\mathcal{F}$ is, for example, a local frame of TM, we denote by $\mathcal{F}\boldsymbol{t}$ the frame obtained by the right action of $\boldsymbol{t} \in M_n(\mathbb{R})$ on $\mathcal{F}$.

We use capital letters $1 \leq A \leq N$ for frame indices of the spinor bundle $S(M)$.

1.6.5 Vector spaces

if $\mathcal{X}$ is a real or complex vector space, we denote by $\mathcal{X}'$ its dual. If $\mathcal{X}$ is a complex vector space, we denote by $\mathcal{X}^*$ its anti-dual, ie the space of anti-linear forms on $\mathcal{X}$ and by $\overline{\mathcal{X}}$ its conjugate, ie $\mathcal{X}$ equipped with the complex structure $-\mathrm{i}$.

A linear map $a \in L(\mathcal{X}, \mathcal{X}')$ is a bilinear form on $\mathcal{X}$, whose action on pairs of vectors is denoted by $x_1 \cdot ax_2$. Similarly a linear map $a \in L(\mathcal{X}, \mathcal{X}^*)$ is a sesquilinear form on $\mathcal{X}$, whose action is denoted by $\overline{x}_1 \cdot ax_2$. We denote by a', resp. a^* the transposed resp. adjoint of a. The space of symmetric resp. hermitian forms on $\mathcal{X}$ is denoted by $L_{\mathrm{s}}(\mathcal{X}, \mathcal{X}')$ resp. $L_{\mathrm{h}}(\mathcal{X}, \mathcal{X}^*)$.

1.6.6 Maps

We write $f : A \xrightarrow{\sim} B$ if $f : A \to B$ is a bijection. We use the same notation if A, B are topological spaces resp. smooth manifolds, replacing bijection by homeomorphism, resp. diffeomorphism.

2 Quantization of Dirac equations on curved spacetimes

In this section we recall well-known facts, see, e.g., [6, 21, 30, 43] about Dirac equations and Dirac quantum fields on curved spacetimes.

2.1 Dirac equations on curved spacetimes

Let us denote by $\mathrm{SO}^{\uparrow}(1, d)$ and $\mathrm{Spin}^{\uparrow}(1, d)$ the restricted Lorentz and Spin groups (ie the connected component of Id in $\mathrm{O}(1, d)$ and $\mathrm{Pin}(1, d)$) and $Ad : \mathrm{Spin}^{\uparrow}(1, d) \to \mathrm{SO}^{\uparrow}(1, d)$ the double sheeted covering.

We recall that a *spacetime* is an oriented and time oriented Lorentzian manifold.

2.1.1 Spin structures

Let (M, g) a spacetime of even dimension $n = 1 + d$ and let $P\mathrm{SO}^{\uparrow}(M, g)$ the $\mathrm{SO}^{\uparrow}(1, d)$-principal bundle of oriented and time oriented orthonormal frames of TM.

We recall that a *spin structure* on (M, g) is given by a $\mathrm{Spin}^{\uparrow}(1, d)$-principal bundle $P\mathrm{Spin}(M, g)$ with a bundle morphism $\chi : P\mathrm{Spin}(M, g) \to P\mathrm{SO}^{\uparrow}(M, g)$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
\mathrm{Spin}^{\uparrow}(1,d) & \longrightarrow & P\mathrm{Spin}(M,g) & & \\
\Big\downarrow{\scriptstyle Ad} & & \Big\downarrow{\scriptstyle \chi} & \searrow^{\pi'} & \\
 & & & & M. \\
 & & & \nearrow_{\pi} & \\
\mathrm{SO}^{\uparrow}(1,d) & \longrightarrow & P\mathrm{SO}^{\uparrow}(M,g) & &
\end{array}
\tag{2.1}
$$

We assume that (M, g) has a spin structure $P\mathrm{Spin}(M, g)$. Let us recall that a Lorentzian manifold admits a spin structure if and only if its second Stiefel–Whitney class $w_2(TM)$ is trivial, see [32, 33]. It admits a *unique* spin structure if in addition its first Stiefel–Whitney class $w_1(M)$ is trivial, which is equivalent to the fact that M is orientable, see, e.g., [33]. In our situation, M is orientable hence spin structures on (M, g) are unique if they exist. If $n = 4$ and (M, g) is globally hyperbolic, it admits a (unique) spin structure, see [18, 19].

We denote by $\mathrm{Cliff}(M, g)$, $S(M)$ the associated Clifford and spinor bundles.

The map $TM \to End(S(M))$ obtained from the embedding $TM \to \mathrm{Cliff}(M, g)$ and the canonical map $\mathrm{Cliff}(M, g) \to End(S(M))$ will be denoted by

$$
TM \ni u \mapsto \gamma(u) \in End(S(M)), \tag{2.2}
$$

and is often called the *Clifford multiplication*. The spin connection will be denoted by ∇^S.

It is well known, see, e.g., [43], [12, Sect. 17.6] that there exists a (unique up to multiplication by strictly positive constants) non-degenerate Hermitian form β acting on the fibers of $S(M)$ such that

$$
\begin{array}{l}
\gamma^*(u)\beta = -\beta\gamma(u), \ u \in TM, \\
\mathrm{i}\beta\gamma(e) > 0, \text{ for all } e \in TM \text{ time-like and future directed}, \\
u \cdot \overline{\psi} \cdot \beta\psi = \overline{\nabla^S_u \psi} \cdot \beta\psi + \overline{\psi} \cdot \beta\nabla^S_u \psi, \ \forall u \in C^\infty(M; TM), \psi \in C^\infty(M; S(M)).
\end{array}
\tag{2.3}
$$

For later use we summarize the properties of ∇^S, γ and β that we will need. We have:

$$
\begin{array}{l}
\nabla^S_u \gamma(v)\psi = \gamma(v)\nabla_u\psi + \gamma(\nabla_u v)\psi, \\
u \cdot \overline{\psi} \cdot \beta\psi = \overline{\nabla^S_u \psi} \cdot \beta\psi + \overline{\psi} \cdot \beta\nabla^S_u \psi, \\
u, v \in C^\infty(M; TM), \psi \in C^\infty(M; S(M)),
\end{array}
\tag{2.4}
$$

where ∇ is the metric connection on (M, g)

2.1.2 Dirac operators

Fixing a smooth section $m \in C^\infty(M; L(S(M)))$ with $m^*\beta = \beta m$, we consider a Dirac operator

$$D = \not{D} + m, \tag{2.5}$$

where $\not{D}$ is locally expressed (on an open set $U \subset M$ over which $S(M)$ and TM are trivialized) as

$$\not{D} = g^{ab}\gamma(e_a)\nabla^S_{e_b},$$

where $(e_a)_{0\leq a\leq d}$ is a local frame over U.

2.1.3 Selfadjointness

For $\psi_1, \psi_2 \in C^\infty(M; \mathcal{S}(M))$ one defines the 1-form $J(\psi_1, \psi_2) \in C^\infty(M; T^*M)$ by

$$J(\psi_1, \psi_2)\cdot u := \overline{\psi}_1\cdot\beta\gamma(u)\psi_2, \quad u \in C^\infty(M; TM),$$

and one deduces from (2.4) that

$$\nabla^\mu J_\mu(\psi_1, \psi_2) = -\overline{D\psi_1}\cdot\beta\psi_2 + \overline{\psi}_1\cdot\beta D\psi_2, \quad \psi_i \in C^\infty(M; S(M)).$$

Using then Stokes formula, this implies that the Dirac operator D is formally selfadjoint on $C_0^\infty(M; S(M))$ with respect to the indefinite Hermitian form

$$(\psi_1|\psi_2)_M := \int_M \overline{\psi}_1\cdot\beta\psi_2 \, dVol_g. \tag{2.6}$$

2.1.4 Characteristic manifold

The *principal symbol* $\sigma_{\rm pr}(D)$ equals

$$\sigma_{\rm pr}(D)(x, \xi) = \gamma(g^{-1}(x)\xi), \ (x, \xi) \in T^*M \setminus o,$$

where$o = X \times \{0\}$ is the zero section in T^*M.

The *characteristic manifold* of D is

$$\mathrm{Char}(D) := \{(x, \xi) \in T^*M \setminus o : \sigma_{\rm pr}(D)(x, \xi) \text{ not invertible}\},$$

equal to

$$\mathrm{Char}(D) = \{(x, \xi) \in T^*M \setminus o : \xi\cdot g^{-1}(x)\xi = 0\} =: \mathcal{N},$$

by the Clifford relations. We denote as usual by $\mathcal{N}^{\pm}$ the two connected components of $\mathcal{N}$, where

$$\mathcal{N}^{\pm}:=\{(x,\xi)\in\mathcal{N} : \pm\xi\cdot v>0 \text{ for } v\in T_xM \text{ future directed}\}. \tag{2.7}$$

2.1.5 Retarded/advanced inverses

Let us assume in addition that (M,g) is globally hyperbolic. Then, (see [6] for Dirac operators in 4 dimensions, or [31] for more general prenormally hyperbolic operators), D admits unique *retarded/advanced inverses* $G_{\rm ret/adv}: C_0^\infty(M;S(M))\to C_{\rm sc}^\infty(M;S(M))$ such that:

$$\begin{cases} DG_{\rm ret/adv}=G_{\rm ret/adv}D=\mathbb{1}, \\ \operatorname{supp} G_{\rm ret/adv}u\subset J_{\pm}(\operatorname{supp} u), \quad u\in C_0^\infty(M;S(M)), \end{cases}$$

where $J_{\pm}(K)$ are the future/past causal shadows of $K\Subset M$.

Using the fact that D is formally selfadjoint with respect to $(\cdot|\cdot)_M$ and the uniqueness of $G_{\rm ret/adv}$, we obtain that

$$G^*_{\rm ret/adv}=G_{\rm adv/ret},$$

where the adjoint is computed with respect to $(\cdot|\cdot)_M$. One defines then the *causal propagator*

$$G:=G_{\rm ret}-G_{\rm adv}$$

which satisfies

$$\begin{cases} DG=GD=0, \\ \operatorname{supp} Gu\subset J(\operatorname{supp} u), \quad u\in C_0^\infty(M;S(M)), \\ G^*=-G, \end{cases} \tag{2.8}$$

where $J(K)=J_-(K)\cup J_+(K)$ is the causal shadow of $K\Subset M$.

2.1.6 The Cauchy problem

Let $\Sigma\subset M$ be a smooth, space-like Cauchy surface and denote by n its future directed unit normal and by $S(\Sigma)$ the restriction of the spinor bundle $S(M)$ to Σ and

$$\varrho_\Sigma : C^\infty(M;S(M))\ni\psi\longmapsto\psi\restriction_\Sigma\in C^\infty(\Sigma;S(\Sigma))$$

the restriction to Σ. The Cauchy problem

$$\begin{cases} D\psi=0, \\ \varrho_\Sigma\psi=f, \quad f\in C_0^\infty(\Sigma;S(\Sigma)), \end{cases}$$

is globally well-posed, see, e.g., [31], the solution being denoted by $\psi = U_\Sigma f$. We have, see, e.g., [5, Thm. 19.63]:

$$U_\Sigma f(x) = -\int_\Sigma G(x, y)\gamma(n(y)) f(y)\mathrm{d}Vol_h, \tag{2.9}$$

where h is the Riemannian metric induced by g on Σ.

We equip $C_0^\infty(\Sigma; S(\Sigma))$ with the indefinite Hermitian form

$$(f_1|f_2)_\Sigma := \int_\Sigma \overline{f}_1 \cdot \beta f_2 \,\mathrm{d}Vol_h. \tag{2.10}$$

For $g \in \mathcal{E}'(\Sigma; S(\Sigma))$, we define $\varrho_\Sigma^* g \in \mathcal{D}'(M; S(M))$ by

$$\int_M \overline{\varrho_\Sigma^* g} \cdot \beta u \,\mathrm{d}dVol_g := \int_\Sigma \overline{g} \cdot \beta \varrho_\Sigma u \mathrm{d}Vol_h, \; u \in C^\infty(\Sigma; S(\Sigma)),$$

i.e., ϱ_Σ^* is the adjoint of ϱ_Σ with respect to the scalar products $(\cdot|\cdot)_M$ and $(\cdot|\cdot)_\Sigma$. We can rewrite (2.9) as

$$U_\Sigma f = (\varrho_\Sigma G)^* \gamma(n) f, \quad f \in C_0^\infty(\Sigma; S(\Sigma)). \tag{2.11}$$

2.1.7 Cauchy evolution

Let us assume that M is foliated by a family $(\Sigma_t)_{t\in\mathbb{R}}$ of space-like smooth Cauchy surfaces, for example the level sets of a *Cauchy time function*, see Sect. 3.1 for the definition.

Denoting the restriction of $S(M)$ to Σ_t by $S(\Sigma_t)$ and ϱ_{Σ_t} by ϱ_t, one can introduce the *Cauchy evolution*

$$U(t, s) : C_0^\infty(\Sigma_s; S(\Sigma_s)) \to C_0^\infty(\Sigma_t; S(\Sigma_t)), \; t, s \in \mathbb{R}$$

defined by

$$U(t, s) f = \varrho_t U_{\Sigma_s} f \;\; f \in C_0^\infty(\Sigma_s; S(\Sigma_s)).$$

2.2 Conformal transformations

We briefly discuss conformal transformations and refer to [17, 2.7.2] or [12, Sect. 17.13] for details.

Let $c \in C^\infty(M)$ with $c(x) > 0$ and $\tilde{g} = c^{-2}g$. Then the spin and spinor bundles for $(M, \tilde{g})$ are identical to those for (M, g). One has:

$$\begin{aligned}
&\tilde{\gamma}(X) = c^{-1}\gamma(X),\ \tilde{\beta} = c\beta,\\
&\tilde{\nabla}^S_C = \nabla^S_C - \tfrac{1}{2}c^{-1}\gamma(X)\gamma(\nabla c) + \tfrac{1}{2}c^{-1}X\cdot dc,\\
&\tilde{\not{D}} = c^{\frac{n+1}{2}}\not{D}c^{\frac{1-n}{2}},\\
&\tilde{D}:=\tilde{\not{D}} + \tilde{m} = c^{\frac{n+1}{2}}Dc^{\frac{1-n}{2}} \text{ for } \tilde{m} = cm.
\end{aligned} \tag{2.12}$$

2.3 Quantization of Dirac equation on curved spacetimes

We now recall the algebraic quantization of Dirac equations, due to Dimock [6].

2.3.1 *CAR* ∗-algebras

Let $(\mathcal{Y}, \nu)$ be a pre-Hilbert space. The *CAR ∗-algebra* over $(\mathcal{Y}, \nu)$, denoted by $\mathrm{CAR}(\mathcal{Y}, \nu)$, is the unital complex ∗-algebra generated by elements $\psi(y)$, $\psi^*(y)$, $y \in \mathcal{Y}$, with the relations

$$\begin{aligned}
&\psi(y_1 + \lambda y_2) = \psi(y_1) + \overline{\lambda}\psi(y_2),\\
&\psi^*(y_1 + \lambda y_2) = \psi(y_1) + \lambda\psi^*(y_2), \quad y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C},\\
&[\psi(y_1), \psi(y_2)]_+ = [\psi^*(y_1), \psi^*(y_2)]_+ = 0,\\
&[\psi(y_1), \psi^*(y_2)]_+ = \overline{y}_1 \cdot \nu y_2 \mathbb{1}, \quad y_1, y_2 \in \mathcal{Y},\\
&\psi(y)^* = \psi^*(y), \quad y \in \mathcal{Y},
\end{aligned} \tag{2.13}$$

where $[A, B]_+ = AB + BA$ is the anti-commutator.

2.3.2 Quasi-free states

As usual a *state* on $\mathrm{CAR}(\mathcal{Y}, \nu)$ is a linear map $\omega : \mathrm{CAR}(\mathcal{Y}, \nu) \to \mathbb{C}$ which is positive and normalized, ie

$$\omega(A^*A) \geq 0,\ \omega(\mathbb{1}) = 1,\ A \in \mathrm{CAR}(\mathcal{Y}, \nu).$$

– a state ω is *quasi-free* if:

$$\begin{aligned}
&\omega(\textstyle\prod_{i=1}^n \psi^*(y_i) \prod_{j=1}^m \psi(y'_j)) = 0, \quad \text{if } n \neq m,\\
&\omega(\textstyle\prod_{i=1}^n \psi^*(y_i) \prod_{j=1}^n \psi(y'_j)) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n \omega(\psi^*(y_i\psi(y_{\sigma(i)})),
\end{aligned}$$

where S_n is the set of permutations of $\{1, \ldots, n\}$.

– a quasi-free state is uniquely determined by its *covariances* $\lambda^\pm \in L_\mathrm{h}(\mathcal{Y}, \mathcal{Y}^*)$, defined by

$$\omega(\psi(y_1)\psi^*(y_2))=:\overline{y}_1\cdot\lambda^+ y_2, \quad \omega(\psi^*(y_2)\psi(y_1))=:\overline{y}_1\cdot\lambda^- y_2, \quad y_1, y_2 \in \mathcal{Y}.$$

The following two results are well-known, see, e.g., [5, Sect. 17.2.2].

Proposition 2.1 *Let* $\lambda^{\pm} \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$. *Then the following statements are equivalent :*

(1) $\lambda^{\pm}$ *are the covariances of a gauge invariant quasi-free state on* $\mathrm{CAR}(\mathcal{Y}, \nu)$*;*
(2) $\lambda^{\pm} \geq 0$ *and* $\lambda^{+} + \lambda^{-} = \nu$.

Proposition 2.2 *A quasi-free state* ω *on* $\mathrm{CAR}(\mathcal{Y}, \nu)$ *is pure if and only if there exist projections* $c^{\pm} \in L(\mathcal{Y})$ *such that*

$$\lambda^{\pm} = \nu \circ c^{\pm}, c^{+} + c^{-} = \mathbb{1}.$$

2.3.3 Pre-Hilbert spaces

We now recall several equivalent pre-Hilbert spaces appearing in the quantization of the Dirac equation.

Let us denote by $\mathrm{Sol}_{\mathrm{sc}}(D)$ the space of smooth, space compact solutions of the Dirac equation

$$D\psi = 0.$$

For $\psi_1, \psi_2 \in \mathrm{Sol}_{\mathrm{sc}}(D)$ we set

$$\overline{\psi}_1 \cdot \nu \psi_2 := \int_{\Sigma} \mathrm{i} J_{\mu}(\psi_1, \psi_2) n^{\mu} \mathrm{d}Vol_h = (\varrho_{\Sigma} \psi_1 | \mathrm{i}\gamma(n) \varrho_{\Sigma} \psi_2)_{\Sigma}, \tag{2.14}$$

where Σ is a smooth space-like Cauchy surface.

Using that $\nabla^{\mu} J_{\mu}(\psi_1, \psi_2) = 0$, the rhs in (2.14) is independent on the choice of Σ. Moreover, by (2.3) ν is a positive definite scalar product on $\mathrm{Sol}_{\mathrm{sc}}(D)$.

Setting:

$$\overline{f}_1 \cdot \nu_{\Sigma} f_2 := \mathrm{i} \int_{\Sigma} \overline{f}_1 \cdot \beta \gamma(n) f_2 \mathrm{d}Vol_h, \tag{2.15}$$

we obtain that

$$\varrho_{\Sigma} : (\mathrm{Sol}_{\mathrm{sc}}(D), \nu) \to (C_0^{\infty}(\Sigma; S(\Sigma)), \nu_{\Sigma})$$

is unitary, with inverse U_{Σ}.

It is also well-known, see, e.g., [6], that $G : C_0^{\infty}(M; S(M)) \to \mathrm{Sol}_{\mathrm{sc}}(D)$ is surjective with kernel $DC_0^{\infty}(M; S(M))$ and that

$$G : \left(\frac{C_0^{\infty}(M; S(M))}{DC_0^{\infty}(M; S(M))}, \mathrm{i}(\cdot | G \cdot)_M \right) \to (\mathrm{Sol}_{\mathrm{sc}}(D), \nu)$$

is unitary. Summarizing, the maps

$$\left(\tfrac{C_0^{\infty}(M; S(M))}{DC_0^{\infty}(M; S(M))}, \mathrm{i}(\cdot | G \cdot)_M\right) \xrightarrow{G} (\mathrm{Sol}_{\mathrm{sc}}(D), \nu) \xrightarrow{\varrho_{\Sigma}} (C_0^{\infty}(\Sigma; S(\Sigma)), \nu_{\Sigma}) \tag{2.16}$$

are unitary maps between pre-Hilbert spaces.

2.3.4 CAR∗-algebra for Dirac fields

We denote by $\mathrm{CAR}(D)$ the $*$-algebra $\mathrm{CAR}(\mathcal{Y}, \nu)$ for $(\mathcal{Y}, \nu)$ one of the equivalent pre-Hilbert spaces in (2.16).

We use the Hermitian form $(\cdot|\cdot)_M$ in (2.6) to pair $C_0^\infty(M; S(M))$ with $\mathcal{D}'(M; S(M))$ and to identify continuous sesquilinear forms on $C_0^\infty(M; S(M))$ with continuous linear maps from $C_0^\infty(M; S(M))$ to $\mathcal{D}'(M; S(M))$.

We use the Hermitian form $(\cdot|\cdot)_\Sigma$ in (2.10) in the same way on the Cauchy surface Σ.

It is natural to require a weak continuity of the *spacetime covariances* $\Lambda^\pm$ of a state ω on $\mathrm{CAR}(D)$ defined by:

$$(u|\Lambda^+ u)_M{:=}\omega(\psi(u)\psi^*(u)),\ (u|\Lambda^- u){:=}\omega(\psi^*(u)\psi(u)),\ u \in C_0^\infty(M; S(M)).$$

Therefore, one considers states on $\mathrm{CAR}(D)$ whose spacetime covariances satisfy:

$$\begin{array}{rl} \text{(i)} & \Lambda^\pm : C_0^\infty(M; S(M)) \to \mathcal{D}'(M; S(M)) \text{ are linear continuous,} \\ \text{(ii)} & \Lambda^\pm \geq 0 \text{ with respect to } (\cdot|\cdot)_M, \\ \text{(iii)} & \Lambda^+ + \Lambda^- = \mathrm{i}G, \\ \text{(iv)} & D \circ \Lambda^\pm = \Lambda^\pm \circ D = 0. \end{array} \tag{2.17}$$

Alternatively, one can define ω by its *Cauchy surface covariances* $\lambda_\Sigma^\pm$, which satisfy

$$\begin{array}{rl} \text{(i)} & \lambda_\Sigma^\pm : C_0^\infty(\Sigma; S(\Sigma)) \to \mathcal{D}'(\Sigma; S(\Sigma)) \text{ are linear continuous,} \\ \text{(ii)} & \lambda_\Sigma^\pm \geq 0 \text{ for } (\cdot|\cdot)_\Sigma, \\ \text{(iii)} & \lambda_\Sigma^+ + \lambda_\Sigma^- = \mathrm{i}\gamma(n). \end{array} \tag{2.18}$$

Using (2.11) one can show by the same arguments as for Klein–Gordon fields, see [16, Prop. 7.5] that

$$\begin{array}{l} \Lambda^\pm = (\varrho_\Sigma G)^* \lambda_\Sigma^\pm (\varrho_\Sigma G), \\ \lambda_\Sigma^\pm = (\varrho_\Sigma^* \gamma(n))^* \Lambda^\pm (\varrho_\Sigma^* \gamma(n)). \end{array} \tag{2.19}$$

We recall that if $E_i \xrightarrow{\pi_i} M_i$, $i = 1, 2$ are two vector bundles with typical fibers V_i, one can define the vector bundle $E_1 \boxtimes E_2 \xrightarrow{\pi} M_1 \times M_2$ with typical fiber $V_1 \otimes V_2$. If $\{U_{i,j_i}\}_{j_i \in I_i}$ and t_{i,j_i,k_i} are coverings and transition maps for $E_i \xrightarrow{\pi_i} M_i$, then one takes $\{U_{1,j_1} \times U_{2,j_2}\}_{(j_1,j_2)\in I_1 \times I_2}$ as covering of $M_1 \times M_2$ and $t_{1,j_1,k_1} \otimes t_{2,j_2,k_2} \in L(V_1 \otimes V_2)$ as transition maps of $E_1 \boxtimes E_2 \xrightarrow{\pi} M_1 \times M_2$.

By the Schwartz kernel theorem, we can identify $\Lambda^\pm$ with distributional sections in $\mathcal{D}'(M \times M; S(M) \boxtimes S(M))$, still denoted by $\Lambda^\pm$.

2.3.5 The role of the Cauchy evolution

Recall from 2.1.7 that we denoted by $U(t, s)$ the Cauchy evolution associated to a foliation by the Cauchy surfaces $(\Sigma_t)_{t\in\mathbb{R}}$.

If ω is a quasi-free state on CAR(D), then denoting by $\lambda^{\pm}(t)$ its Cauchy surface covariances on Σ_t one has obviously

$$\lambda^{\pm}(t) = U(s,t)^{*}\lambda^{\pm}(s)U(s,t),\ \ t,s \in \mathbb{R}. \tag{2.20}$$

2.3.6 Hadamard states

The *wavefront set* of $A \in \mathcal{D}'(M \times M; S(M) \boxtimes S(M))$ is defined in the natural way: introducing local trivializations of $S(M)$ one can assume that $A \in \mathcal{D}'(M\times M; M_N(\mathbb{C}))$ where $N = \mathrm{rank} S(M)$ and the wavefront set of a matrix valued distribution is simply the union of the wavefront sets of its entries.

We will identify $T^*(M \times M)$ with $T^*M \times T^*M$. If $\Gamma \subset T^*M \times T^*M$ then one sets

$$\Gamma' {:=} \{((x,\xi),(x',\xi')) : ((x,\xi),(x',-\xi')) \in \Gamma\}.$$

For example $\mathrm{WF}(\delta(x-x')) = \Delta$, where $\Delta \subset T^*M \times T^*M$ is the diagonal.

We recall that $\mathcal{N}^{\pm}$ are the two connected components of $\mathcal{N}$, see (2.7).

The following definition of Hadamard states is due to Hollands [21].

Definition 2.3 ω is a *Hadamard state* if

$$\mathrm{WF}(\Lambda^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}.$$

The following proposition, see [17, Prop. 3.8] gives a sufficient condition for the Cauchy surface covariances $\lambda^{\pm}_{\Sigma}$ to generate a Hadamard state.

Proposition 2.4 *Let*

$$\lambda^{\pm}_{\Sigma} {=:} \mathrm{i}\gamma(n)c^{\pm}$$

be the Cauchy surface covariances of a quasi-free state ω*. Assume that* $c^{\pm}$ *are continuous from* $C_0^{\infty}(\Sigma; \mathcal{S}_{\Sigma})$ *to* $C^{\infty}(\Sigma; \mathcal{S}_{\Sigma})$ *and from* $\mathcal{E}'(\Sigma; \mathcal{S}_{\Sigma})$ *to* $\mathcal{D}'(\Sigma; \mathcal{S}_{\Sigma})$*, and that for some neighborhood* U *of* Σ *in* M *we have*

$$\mathrm{WF}(U_{\Sigma} \circ c^{\pm})' \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^*\Sigma,\ \ \textit{over } U \times \Sigma, \tag{2.21}$$

where $\mathcal{F} \subset T^*M$ *is a conic set with* $\mathcal{F} \cap \mathcal{N} = \emptyset$*. Then* ω *is a Hadamard state.*

2.3.7 Action of conformal transformations

Let us now study the action of the conformal transformations recalled in Sect. 2.2. If $\tilde{D}$ is the Dirac operator for $\tilde{g}$, its causal propagator is

$$\tilde{G} = c^{\frac{n-1}{2}} G c^{-\frac{n+1}{2}}.$$

If we set

$$\begin{aligned} W\tilde{\psi} &= c^{\frac{1-n}{2}}\tilde{\psi},\ \tilde{\psi} \in C_0^\infty(M; S(M)),\\ W^*\psi &= c^{\frac{n+1}{2}}\psi,\ \psi \in C_0^\infty(M, ; S(M)),\\ Uf &= c^{\frac{n-1}{2}} f, f \in C_0^\infty(\Sigma; S(\Sigma)), \end{aligned}$$

then a routine computation gives the following proposition.

Proposition 2.5 *The following diagram is commutative, with all arrows unitary:*

$$\begin{array}{ccccc} \left(\frac{C_0^\infty(M;S(M))}{DC_0^\infty(M;S(M))}, (\cdot|\mathrm{i}G\cdot)_M\right) & \xrightarrow{G} & (\mathrm{Sol}_{\mathrm{sc}}(D), \nu) & \xrightarrow{\varrho_\Sigma} & (C_0^\infty(\Sigma; S(\Sigma)), \nu_\Sigma)\\ \downarrow W^* & & \downarrow W^{-1} & & \downarrow U\\ \left(\frac{C_0^\infty(\tilde{M};S(M))}{\tilde{D}C_0^\infty(\tilde{M};S(M))}, (\cdot|\mathrm{i}\tilde{G}\cdot)_{\tilde{M}}\right) & \xrightarrow{\tilde{G}} & (\mathrm{Sol}_{\mathrm{sc}}(\tilde{D}), \tilde{\nu}) & \xrightarrow{\tilde{\varrho}_\Sigma} & (C_0^\infty(\Sigma; S(\Sigma)), \tilde{\nu}_\Sigma) \end{array}$$

Let us now consider the action of conformal transformations on quasi-free states. Let $\Lambda^\pm$ be the spacetime covariances of a quasi-free state ω for D. Then

$$\tilde{\Lambda}^\pm = c^{\frac{n-1}{2}}\Lambda^\pm c^{-\frac{n+1}{2}} \tag{2.22}$$

are the spacetime covariances of a quasi-free state $\tilde{\omega}$ for $\tilde{D}$, and

$$\tilde{\lambda}_\Sigma^\pm = (U^*)^{-1}\lambda_\Sigma^\pm U^{-1} = c^{\frac{n-1}{2}}\lambda_\Sigma^\pm c^{\frac{1-n}{2}},$$

if $\lambda_\Sigma^\pm$, resp. $\tilde{\lambda}_\Sigma^\pm$ are the Cauchy surface covariances of ω, resp. $\tilde{\omega}$.

Clearly ω is a Hadamard state iff $\tilde{\omega}$ is.

2.4 The vacuum state for Dirac fields on static spacetimes

The basic example of a state for Dirac fields is the *vacuum state* on static spacetimes. Let us recall its definition, following [4].

2.4.1 Vacuum state associated to a Killing field

Let (M, g) a globally hyperbolic spacetime with a spin structure. The *Lie derivative* of a spinor field is defined as (see [28]):

$$\begin{aligned} \mathcal{L}_X\psi &= \nabla_X^S\psi + \tfrac{1}{8}((\nabla_a X)_b - (\nabla_b X)_a)\gamma^a\gamma^b\psi,\\ \psi &\in C^\infty(M; S(M)),\ X \in C^\infty(M; TM). \end{aligned} \tag{2.23}$$

If X is a complete Killing vector field, and the mass m in (2.5) satisfies $X{\cdot}dm = 0$, then $[D, \mathcal{L}_X] = 0$, see, e.g., [13, Appendix A]. It follows that the flow ϕ_s generated by $\mathcal{L}_X$ preserves $\mathrm{Sol}_{\mathrm{sc}}(D)$ and one can easily show, using (2.4) and (2.23) that it preserves the Hilbertian scalar product ν.

It hence defines a unique strongly continuous unitary group $(\mathrm{e}^{\mathrm{i}sH})_{s\in\mathbb{R}}$ on the completion of $(\mathrm{Sol}_{\mathrm{sc}}(D), \nu)$, whose generator H is, by Nelson's invariant domain theorem, the closure of $\mathrm{i}^{-1}\mathcal{L}_X$ on $\mathrm{Sol}_{\mathrm{sc}}(D)$.

If Σ is a smooth space-like Cauchy surface, we denote by H_Σ the corresponding generator on the completion of $(C_0^\infty(\Sigma; S(\Sigma)), \nu_\Sigma)$.

The following definition is taken from [4].

Definition 2.6 Assume that

$$\mathrm{Ker}\, H_\Sigma = \{0\}. \tag{2.24}$$

The *vacuum state* ω^{vac} associated to the complete Killing field X is the quasi-free state defined by the Cauchy surface covariances:

$$\lambda^{\pm\mathrm{vac}} := \mathrm{i}\gamma(n)\mathbb{1}_{\mathbb{R}^\pm}(H_\Sigma).$$

Unlike the bosonic case, X does not need to be time-like in order to be able to define the associated vacuum state.

2.4.2 Vacuum state on static spacetimes

We now discuss the vacuum state on static spacetimes. We will assume that $M = \mathbb{R}\times\Sigma$ is equipped with the static metric $g = -c^2(\mathrm{x})\mathrm{d}t^2 + h(\mathrm{x})\mathrm{dx}^2$, where $c \in C^\infty(\Sigma; \mathbb{R})$ with $c(\mathrm{x}) > 0$ and h is a Riemannian metric on Σ. We set

$$\tilde{g} = c^{-2}g = -\mathrm{d}t^2 + \tilde{h}(\mathrm{x})\mathrm{dx}^2,$$

which is ultra-static. The restriction of $S(M)$ to Σ_t is independent on t and denoted by $S(\Sigma)$, see [17, Subsect. 7.1].

We consider a *static Dirac operator*

$$D = \not{D} + m,$$

where $m \in C^\infty(\Sigma, ; \mathbb{R})$ is *independent on* t.

The corresponding Dirac operator on $(M, \tilde{g})$ is

$$\tilde{D} = \tilde{\not{D}} + \tilde{m},\ \ \tilde{m} = cm.$$

If $(\tilde{e}_j)_{1\leq j\leq d}$ is a local orthonormal frame for $\tilde{h}$ and $\tilde{e}_0 = \partial_t$, we have setting $\tilde{\gamma}_0 = \tilde{\gamma}(\tilde{e}_0)$:

$$\tilde{D} = -\tilde{\gamma}_0(\partial_t - \mathrm{i}\tilde{H}_\Sigma)$$

for

$$\tilde{H}_\Sigma = \mathrm{i}\tilde{\gamma}_0(\tilde{\gamma}(\tilde{e}_j)\tilde{\nabla}^S_{\tilde{e}_j} + \tilde{m}) =: \tilde{H}_{0\Sigma} + \mathrm{i}\tilde{\gamma}_0\tilde{m}. \tag{2.25}$$

From (2.23), we obtain that $\mathcal{L}_{\tilde{e}_0} = \tilde{\nabla}^S_{\tilde{e}_0} = \partial_t$ and hence the generator of the Lie derivative w.r.t. the Killing vector field ∂_t equals $\tilde{H}_\Sigma$ on $C_0^\infty(\Sigma; S(\Sigma))$. We still denote by $\tilde{H}_\Sigma$ its closure for the Hilbertian scalar product $\tilde{\nu}_\Sigma$.

Let us now consider the original Dirac operator D. Using (2.12), one checks that

$$D = -c^{-1}\gamma(e_0)(\partial_t - \mathrm{i}H_\Sigma), \tag{2.26}$$

$$H_\Sigma := c^{\frac{1-n}{2}}\tilde{H}_\Sigma c^{\frac{n-1}{2}}, \tag{2.27}$$

where $e_a = c^{-1}\tilde{e}_a$. By Proposition 2.5 we know that H_Σ with domain $c^{\frac{1-n}{2}}\,\mathrm{Dom}\,\tilde{H}_\Sigma$ is selfadjoint for the scalar product ν_Σ. It equals the generator of the unitary group associated to $\mathcal{L}_{\partial_t}$ considered in 2.4.1.

Applying the discussion in 2.3.7, we can define:

Definition 2.7 Assume that $\mathrm{Ker}\, H_\Sigma = \{0\}$. Then the *vacuum state* ω^{vac} for D is the quasi-free state with Cauchy surface covariances

$$\lambda^{\pm\mathrm{vac}} = \mathrm{i}\gamma(e_0)\mathbb{1}_{\mathbb{R}^\pm}(H_\Sigma).$$

2.4.3 Massive Dirac operators

Definition 2.8 The static Dirac operator D is called *massive* if

$$0 \notin \sigma(H_\Sigma). \tag{2.28}$$

It is a standard fact that if (2.28) holds, then ω^{vac} is a Hadamard state, see, e.g., [38, Thm. 5.1]. Another proof is given in [17, Subsect. 7.1]. If $0 \in \sigma(H_\Sigma)$ but $\mathrm{Ker}\, H_\Sigma = \{0\}$, then one can encounter infrared problems.

Let us give a simple sufficient condition for (2.28). Using the Clifford relations and (2.4), we obtain that

$$\tilde{H}^2_\Sigma = \tilde{H}^2_{0\Sigma} + \tilde{\gamma}(\tilde{h}^{-1}d\tilde{m}) + \tilde{m}^2.$$

Since $A = \tilde{\gamma}(\tilde{h}^{-1}\mathrm{d}\tilde{m})$ is selfadjoint for $\tilde{\nu}_\Sigma$ with $A^2 = \mathrm{d}\tilde{m}\cdot\tilde{h}^{-1}d\tilde{m}$, we obtain that if

$$\inf_\Sigma \tilde{m}^2 - \mathrm{d}\tilde{m}\cdot\tilde{h}^{-1}\mathrm{d}\tilde{m} > 0 \tag{2.29}$$

Then $0 \notin \sigma(\tilde{H})$. In terms of c, m (2.29) becomes:

$$\inf_\Sigma (c^2m^2 - d(cm)\cdot h^{-1}d(cm)) > 0, \tag{2.30}$$

Note that (2.30) holds if $c \equiv 1$ and $m(\mathrm{x}) \equiv m_0 \neq 0$.

3 Dirac operators on asymptotically static spacetimes

3.1 Asymptotically static spacetimes

We fix an orientable $d-$dimensional manifold Σ equipped with a reference Riemannian metric k such that (Σ, k) is of bounded geometry, and consider $M = \mathbb{R}_t \times \Sigma_{\rm x}$, setting $x = (t, {\rm x})$, $n = 1 + d$ is even.

3.1.1 Bounded geometry

Roughly speaking a Riemannian manifold (Σ, k) is of bounded geometry if its radius of injectivity is strictly positive and if the metric and all its derivatives, expressed in normal coordinates at a point x, satisfy estimates which are *uniform* with respect to the point x.

The two basic examples are compact Riemannian manifolds and $\mathbb{R}^d$ with the flat metric, but many other non-compact Riemannian manifolds are of bounded geometry, like for example asymptotically hyperbolic Riemannian manifolds.

After fixing a background Riemannian metric, one can define in a canonical way various global spaces, like spaces of bounded tensors, Sobolev spaces, bounded differential operators.

Roughly speaking an object is bounded, if, when expressed in normal coordinates at a base point x, the object and all its derivatives satisfy estimates which are uniform with respect to x.

The main interest for us is that on a Riemannian manifold of bounded geometry one can define a global pseudodifferential calculus, the *Shubin calculus*, which shares several important properties with the pseudodifferential calculus on compact manifolds or the uniform pseudodifferential calculus on $\mathbb{R}^d$.

3.1.2 Lorentzian metric

We equip M with a Lorentzian metric g of the form

$$g = -c^2(x){\rm d}t^2 + ({\rm dx}^i + b^i(x){\rm d}t)h_{ij}(x)({\rm dx}^j + b^j(x){\rm d}t), \qquad (3.1)$$

where $c \in C^\infty(M;\mathbb{R})$, $c(x) > 0$, $b \in C^\infty(M; T\Sigma)$ and $h \in C^\infty(M; \otimes_{\rm s}^2 T^*\Sigma)$ is a t-dependent Riemannian metric on Σ.

We recall that $\tilde{t} \in C^\infty(M;\mathbb{R})$ is called a *time function* if $\nabla\tilde{t}$ is a time-like vector field. It is called a *Cauchy time function* if in addition its level sets are Cauchy hypersurfaces.

By [2, Thm. 2.1] we know that (M, g) is globally hyperbolic and t is a Cauchy time function.

3.1.3 Asymptotically static spacetimes

We consider also two static metrics on M:

$$g_{\rm out/in} = -c^2_{\rm out/in}({\rm x}){\rm d}t^2 + h_{\rm out/in}({\rm x}){\rm dx}^2,$$

where $h_{\mathrm{out/in}}$, resp. $c_{\mathrm{out/in}}$ are two Riemannian metrics, resp. smooth functions on Σ such that:

$$h_{\mathrm{out/in}} \in BT^0_2(\Sigma, k),\ h^{-1}_{\mathrm{out/in}} \in BT^2_0(\Sigma, k),\ c_{\pm\infty}, c^{-1}_{\pm\infty} \in BT^0_0(\Sigma, k). \tag{H1}$$

Concerning the asymptotic behavior of g when $t \to \pm\infty$, we assume that

$$\begin{aligned} &h(x) - h_{\mathrm{out/in}}(\mathrm{x}) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; BT^0_2(\Sigma, k)),\\ &b(x) \in \mathcal{S}^{-1-\mu}(\mathbb{R}; BT^1_0(\Sigma, k)),\\ &c(x) - c_{\mathrm{out/in}}(\mathrm{x}) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; BT^0_0(\Sigma, k)), \end{aligned} \tag{H2}$$

for some $\mu > 0$, where $BT^p_q(\Sigma, k)$ is the Fréchet space of bounded q, p-tensors, see, e.g., [42] or [17, Subsect. 4.1], and the space $\mathcal{S}^{\delta}(\mathbb{R}; \mathcal{F})$ for $\mathcal{F}$ a Fréchet space is defined in Sect. 4.2.

In other words the metric g is asymptotic to the static metrics $g_{\mathrm{out/in}}$ when $t \to \pm\infty$.

For later use we also fix $m \in C^{\infty}(M; \mathbb{R})$, representing a variable mass and $m_{\pm\infty} \in C^{\infty}(\Sigma; \mathbb{R})$ such that

$$m(x) - m_{\pm\infty}(\mathrm{x}) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; BT^0_0(\Sigma, k)). \tag{H3}$$

3.1.4 Orthogonal decomposition

We recall now the well-known orthogonal decomposition of g associated to the Cauchy time function t. We set

$$v := \frac{g^{-1}\mathrm{d}t}{\mathrm{d}t \cdot g^{-1}\mathrm{d}t} = \partial_t + b^i \partial_{\mathrm{x}^i},$$

which using (H1), (H2) is a complete vector field on M. Denoting its flow by ϕ_t, we have:

$$\phi_t(0, \mathrm{y}) = (t, \mathrm{x}(t, 0, \mathrm{y})),\ t \in \mathbb{R},\ \mathrm{y} \in \Sigma,$$

where $\mathrm{x}(t, s, \cdot)$ is the flow of the time-dependent vector field b on Σ. We also set

$$\chi : \mathbb{R} \times \Sigma \ni (t, \mathrm{y}) \mapsto (t, \mathrm{x}(t, 0, \mathrm{y})) \in \mathbb{R} \times \Sigma. \tag{3.2}$$

The following lemma is proved in [14, Appendix A.4]. Bounded diffeomorphisms on a manifold of bounded geometry are defined for example in [14, Def. 3.3].

Lemma 3.1 *Assume* (H1), (H2). *Then*

$$\hat{g} := \chi^* g = -\hat{c}^2(t, \mathrm{y})\mathrm{d}t^2 + \hat{h}(t, \mathrm{y})\mathrm{d}\mathrm{y}^2, \tag{3.3}$$

for $\hat{c} \in C^\infty(\mathbb{R} \times M)$, $\hat{h} \in C^\infty(\mathbb{R}; T_2^0(\Sigma))$. *Moreover, there exist bounded diffeomorphisms* $\mathrm{x}_{\mathrm{out/in}}$ *of* (Σ, k) *such that if:*

$$\begin{aligned}\hat{h}_{\mathrm{out/in}} &:= \mathrm{x}^*_{\mathrm{out/in}} h_{\mathrm{out/in}},\\ \hat{c}_{\mathrm{out/in}} &:= \mathrm{x}^*_{\mathrm{out/in}} c_{\mathrm{out/in}},\end{aligned}$$

then:

$$\begin{aligned}&\hat{h}_{\mathrm{out/in}} \in BT_2^0(\Sigma, k),\ \hat{h}^{-1}_{\mathrm{out/in}} BT_0^2(\Sigma, k),\\ &\hat{c}_{\mathrm{out/in}}, \hat{c}^{-1}_{\mathrm{out/in}} \in BT_0^0(\Sigma, k),\end{aligned}$$

and furthermore,

$$\begin{aligned}\hat{h} - \hat{h}_{\mathrm{out/in}} &\in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_2^0(\Sigma, k)),\\ \hat{c} - \hat{c}_{\mathrm{out/in}} &\in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_0^0(\Sigma, k)),\\ \chi^* m - m_{\pm\infty} &\in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_0^0(\Sigma, k)).\end{aligned}$$

After applying the isometry $\chi : (M, \hat{g}) \xrightarrow{\sim} (M, g)$ in Lemma 3.1, removing the hats to simplify notation and denoting y again by x, we can assume that

$$g = -c^2(t, \mathrm{x})\mathrm{d}t^2 + h(t, \mathrm{x})\mathrm{d}\mathrm{x}^2,$$

with

$$\begin{aligned}&h - h_{\mathrm{out/in}} \in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_2^0(\Sigma, k)),\\ &c - c_{\mathrm{out/in}} \in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_0^0(\Sigma, k)),\\ &m - m_{\pm\infty} \in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_0^0(\Sigma, k)),\\ &h_{\mathrm{out/in}} \in BT_2^0(\Sigma, k),\ h^{-1}_{\mathrm{out/in}} BT_0^2(\Sigma, k),\\ &c_{\mathrm{out/in}}, c^{-1}_{\mathrm{out/in}} \in BT_0^0(\Sigma, k).\end{aligned} \tag{3.4}$$

3.1.5 Conformal transformation

We set

$$\tilde{g} := c^{-2} g = -\mathrm{d}t^2 + \tilde{h}(t, \mathrm{x})\mathrm{d}\mathrm{x}^2$$

and obtain that

$$\begin{aligned}&\tilde{h} - \tilde{h}_{\mathrm{out/in}} \in \mathcal{S}^{-\mu}(\mathbb{R}^\pm, BT_2^0(\Sigma, k)),\ \text{with}\\ &\tilde{h}_{\mathrm{out/in}} = c^{-2}_{\mathrm{out/in}} h_{\mathrm{out/in}} \in BT_2^0(\Sigma, k),\ \tilde{h}^{-1}_{\mathrm{out/in}} BT_0^2(\Sigma, k).\end{aligned} \tag{3.5}$$

3.2 Spin structures

Let us assume that (M, g) admits a spin structure $P\mathrm{Spin}(M, g)$. We denote by $\mathrm{Cliff}(M, g)$, $S(M)$ the Clifford and spinor bundles over (M, g).

By well-known results on conformal transformations of spin structures, see, e.g., [30, Lemma 5.27], [20] [17, 2.7.2] $(M, \tilde{g})$ also admits a spin structure and the spinor bundle for $(M, \tilde{g})$ is equal to $S(M)$.

Before further discussing the spin structure on (M, g) or $(M, \tilde{g})$, we prove a lemma. We set $\tilde{h}_t = \tilde{h}(t, \cdot)$.

Lemma 3.2 *Let us fix a bounded atlas $(V_i, \psi_i)_{i\in\mathbb{N}}$ for $(\Sigma, \tilde{h}_0)$.*

Let $\mathcal{F}_i = (e_{i,j})_{1\leq j\leq d}$ oriented orthonormal frames for $\tilde{h}_0$ over V_i such that $e_{i,j}$ for $i \in \mathbb{N}$, $1 \leq j \leq d$ are a bounded family in $BT^1_0(V_i, k)$. Let $\mathcal{F}_i(t) = (e_{i,j}(t))_{1\leq j\leq d}$ the oriented orthonormal frames for $\tilde{h}_t$ over V_i obtained by parallel transport with respect to ∂_t of $\mathcal{F}_i$ for the metric $\tilde{g}$. Then:

(1) *$e_{i,j}(\pm\infty) = \lim_{t\to\pm\infty} e_{i,j}(t)$ exist and the family $e_{i,j}(\pm\infty)$ for $i \in \mathbb{N}$, $1 \leq j \leq d$ is bounded in $BT^1_0(V_i, k)$.*
(2) *$\mathbb{R}^{\pm} \ni t \mapsto e_{i,j}(t) - e_{i,j}(\pm\infty)$ form a bounded family in $\mathcal{S}^{-\mu}(\mathbb{R}, BT^0_0(V_i, k))$.*

Proof Let us forget the index i for the moment. Let x^α, $1 \leq \alpha \leq d$ be local coordinates on V obtained from $\psi : V \to B_d(0, 1)$ and let $x^0 = t$. Denoting by $\Gamma^\mu_{\varrho\nu}$ the Christoffel symbols for g in the local coordinates $(x^\mu)_{0\leq\mu\leq d}$ over $U = \mathbb{R} \times V$, we have $\Gamma^\mu_{0\nu} = \frac{1}{2} h^{\mu\varrho}\partial_t h_{\varrho\nu}$.

Putting back the index i we see from (3.5) that $\mathbb{R} \in t \mapsto \Gamma^\mu_{i,0\nu}(t)$ form a bounded family in $\mathcal{S}^{-1-\mu}(\mathbb{R}, BT^0_0(V_i))$. Denoting $e_{i,j}(t)$ simply by $u(t)$ and setting $u = u^\alpha\partial_{x^\alpha}$ over V, we obtain that $u(t)$ solves:

$$\begin{cases} \partial_t u^\alpha(t) + \Gamma^\alpha_{0\beta}(t) u^\beta(t) = 0, \\ u^\alpha(0) = e^\alpha_{i,j}. \end{cases}$$

From the above estimates on $\Gamma^\mu_{i,0\nu}(t)$ and standard estimates on solutions of linear differential equations, we obtain (1). It follows that $u(t)$ also solves

$$\begin{cases} \partial_t u^\alpha(t) + \Gamma^\alpha_{0\beta}(t) u^\beta(t) = 0, \\ \lim_{t\to\pm\infty} u^\alpha(t) = u^\alpha(\pm\infty). \end{cases}$$

Again the same estimates (integrating now from $t = \pm\infty$ instead of from $t = 0$) prove (2) and complete the proof of the lemma. □

3.2.1 Spin structures

Since M is a Cartesian product and from the form of $\tilde{g}$, further simplifications occur, see, e.g., [1] or [17, Subsect. 2.6].

Let us set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and set $\tilde{h}_{\mathrm{in/out}} = \tilde{h}_{\mp\infty}$ for coherence of notation.

We can use the local frames $\mathcal{F}_i(t)$ over V_i to obtain local trivializations of $P\mathrm{SO}(\Sigma, \tilde{h}_t)$ for $t \in \overline{\mathbb{R}}$. The associated transition functions are independent on t.

By the arguments in [17, Subsect. 2.6], we obtain unique spin structures on $(\Sigma, \tilde{h}_t)$ for $t \in \overline{\mathbb{R}}$.

The transition functions of $P\mathrm{Spin}(M, \tilde{g})$ are independent on t and induce a spin structure on $(\Sigma, \tilde{h}_t)$ whose transition functions are also independent on t. If $S_t(\Sigma)$ denotes the restriction of $S(M)$ to Σ_t, then $S_t(\Sigma)$ is independent on t and denoted by $S(\Sigma)$.

Conversely the spin structure on $(\Sigma, \tilde{h}_{\pm\infty})$ induces a spin structure on $(\Sigma, \tilde{g}_{\pm\infty})$ for $\tilde{g}_{\pm\infty} = -\mathrm{d}t^2 + \tilde{h}_{\pm\infty}(\mathrm{x})\mathrm{dx}^2$ and by conformal invariance a spin structure on $(M, g_{\pm\infty})$. The associated spinor bundle is again equal to $S(M)$.

3.3 Dirac operators

We consider the Dirac operator locally given by

$$D := \not{D} + m, \ \ \not{D} = g^{ab}\gamma(e_a)\nabla^S_{e_b} \tag{3.6}$$

where $(e_a)_{0\leq a\leq d}$ is some local frame of TM and ∇^S is the spin connection.

3.3.1 Conformal transformation

By Sect. 2.2 we obtain that

$$D = c^{-\frac{n+1}{2}}\tilde{D}c^{\frac{n-1}{2}} \text{ for } \tilde{D} = \tilde{\not{D}} + \tilde{m}, \ \ \tilde{m} = cm, \tag{3.7}$$

with

$$\begin{aligned} &\tilde{m} - \tilde{m}_{\mathrm{out/in}} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}, BT^0_0(\Sigma, k)),\\ &\tilde{m}_{\mathrm{out/in}} = c_{\mathrm{out/in}}m_{\mathrm{out/in}}, \ \tilde{m}_{\mathrm{out/in}}, \tilde{m}^{-1}_{\mathrm{out/in}} \in BT^0_0(\Sigma, k). \end{aligned} \tag{3.8}$$

3.3.2 Asymptotic Dirac operators

Let

$$D_{\mathrm{out/in}} = \not{D}_{\mathrm{out/in}} + m_{\mathrm{out/in}}$$

the asymptotic Dirac operators obtained from the spin structures $P\mathrm{Spin}(M, g_{\mathrm{out/in}})$.

We will assume

$$D_{\mathrm{out/in}} \text{ are } \textit{massive} \text{ ie } 0 \notin \sigma(H_{\mathrm{out/in}}), \tag{H4}$$

see 2.4.3. A sufficient condition for (H4) is given in (2.30).

4 Pseudodifferential calculus

In this section we will recall Shubin's global pseudodifferential calculus on manifolds of bounded geometry and its time-dependent versions. We refer the reader to [27, 42] for the original exposition and to [16] for a more recent one. We are interested in pseudodifferential operators acting on sections of spinor bundles, which are considered in [17].

4.1 Notations

Let (Σ, k) a Riemannian manifold of bounded geometry see [3, 35] or [16, Thm. 2.2] for an equivalent definition. We refer the reader to [17, Subsect. 4.1] for the definitions below.

We denote by $BT^p_q(\Sigma, k)$ the space of bounded (q, p) tensors on Σ. Let also $E \xrightarrow{\pi} \Sigma$ a vector bundle of bounded geometry.

We denote by $S^m_{\rm ph}(T^*\Sigma; L(E))$ the space of $L(E)$-valued poly-homogenous symbols of order m on Σ, see, e.g., [17, Sect. 4.1].

The ideal of smoothing operators is denoted by $\mathcal{W}^{-\infty}(\Sigma; L(E))$, and one sets

$$\Psi^m(\Sigma; L(E)) = {\rm Op}(S^m_{\rm ph}(T^*\Sigma; L(E))) + \mathcal{W}^{-\infty}(\Sigma; L(E)),$$

for some quantization map Op obtained from a bounded atlas and bounded partition of unity of (Σ, k).

4.2 Time-dependent pseudodifferential operators

We will also consider time-dependent pseudodifferential operators, adapted to the geometric situation considered in Sect. 3.2.

We first introduce some notation.

Let $\mathcal{F}$ a Fréchet space whose topology is defined by the seminorms $\|\cdot\|_p$, $p \in \mathbb{N}$ and $\delta \in \mathbb{R}$. We denote by $\mathcal{S}^\delta(\mathbb{R}; \mathcal{F})$ the space of smooth functions $f : \mathbb{R} \to \mathcal{F}$ such that $\sup_{\mathbb{R}} \langle t\rangle^{k-\delta} \|\partial_t^k f(t)\|_p < \infty$ for all $k, p \in \mathbb{N}$. Equipped with the obvious seminorms, it is itself a Fréchet space.

Note that $\mathcal{S}^\delta(\mathbb{R}; \mathcal{F}) = \langle t\rangle^\delta \mathcal{S}^0(\mathbb{R}; \mathcal{F})$ so we can always reduce ourselves to $\delta = 0$.

Similarly we denote by $C^\infty_{\rm b}(\mathbb{R}; \mathcal{F})$ the space of smooth functions $f : \mathbb{R} \to \mathcal{F}$ such that $\sup_{\mathbb{R}} \|\partial_t^k f(t)\|_p < \infty$ for all $k, p \in \mathbb{N}$, with the analogous Fréchet space topology.

We use this notation to define the spaces $\mathcal{S}^\delta(\mathbb{R}; S^m_{\rm ph}(T^*\Sigma; L(E)))$, $\mathcal{S}^\delta(\mathbb{R}, \mathcal{W}^{-\infty}(\Sigma; L(E)))$ and $\mathcal{S}^\delta(\mathbb{R}; \Psi^m(\Sigma; L(E)))$.

For example, if (Σ, k) equals $\mathbb{R}^n$ equipped with the flat metric, then $\mathcal{S}^\delta(\mathbb{R}; S^m_{\rm ph}(T^*\mathbb{R}^n))$ is the space of smooth functions $a : \mathbb{R} \times T^*\mathbb{R}^n \to \mathbb{C}$ such that there exist for $j \in \mathbb{N}$ functions $a_{m-j} : \mathbb{R} \times T^*(\mathbb{R}^n) \to \mathbb{C}$, homogeneous of degree

$m-j$ in ξ with

$$\sup_{\mathbb{R}\times T^*\mathbb{R}^n\setminus o} \langle t\rangle^{-\delta+k}\langle\xi\rangle^{-m+j|\beta|}|\partial_t^k\partial_x^\alpha\partial_\xi^\beta a_{m-j}(t,x,\xi)|<\infty,\ k\in\mathbb{N},\alpha,\beta\in\mathbb{N}^n,$$

and for any $N\in\mathbb{N}$

$$\sup_{\mathbb{R}\times T^*\mathbb{R}^n\setminus o} \langle t\rangle^{-\delta+k}\langle\xi\rangle^{-m+N+1+|\beta|}|\partial_t^k\partial_x^\alpha\partial_\xi^\beta (a-\sum_{j=0}^N a_{m-j}(t,x,\xi))|$$
$$<\infty,\ k\in\mathbb{N},\alpha,\beta\in\mathbb{N}^n.$$

Similarly $\mathcal{S}^\delta(\mathbb{R},\mathcal{W}^{-\infty}(\mathbb{R}^n))$ is the space of smooth functions $a:\mathbb{R}\to B(L^2(\mathbb{R}^n))$ such that

$$\sup_{\mathbb{R}}\langle t\rangle^{-\delta+k}\|\partial_t^k a(t)\|_{B(H^{-m}(\mathbb{R}^n),H^m(\mathbb{R}^n))}<\infty,\ k,m\in\mathbb{N},$$

where $H^m(\mathbb{R}^n)$ are the usual Sobolev spaces.

For simplicity of notation $\mathcal{S}^\delta(\mathbb{R};S^m_{\rm ph}(T^*\Sigma;L(E)))$ or $\mathcal{S}^\delta(\mathbb{R};\Psi^m(\Sigma;L(E)))$ will often simply be denoted by $\mathcal{S}^{\delta,m}$, $\Psi^{\delta,m}$.

4.2.1 Principal symbol

If $A(t)=\mathrm{Op}(a(t))+R_{-\infty}(t)\in\mathcal{S}^\delta(\mathbb{R};\Psi^m(\Sigma;L(E)))$, its principal symbol is

$$\sigma_{\rm pr}(A)(t):=[a](t)\in\mathcal{S}^\delta(\mathbb{R};S^m_{\rm ph}(T^*\Sigma;L(E)))/\mathcal{S}^\delta(\mathbb{R};S^{m-1}_{\rm ph}(T^*\Sigma;L(E))).$$

$\sigma_{\rm pr}(A)(t)$ is independent on the decomposition of $A(t)$ as $\mathrm{Op}(a)(t)+R_{-\infty}(t)$ and on the choice of the good quantization map Op. As usual we choose a representative of $\sigma_{\rm pr}(A)(t)$ which is homogeneous of order m on the fibers of $T^*\Sigma$.

4.2.2 Ellipticity

An operator $A(t)\in\mathcal{S}^\delta(\mathbb{R};\Psi^m(\Sigma;L(E)))$ is *elliptic* if $\sigma_{\rm pr}(A)(t,x,\xi)$ is invertible for all $t\in\mathbb{R}$ and

$$\sup_{t\in\mathbb{R},(x,\xi)\in T^*\Sigma,|\xi|=1}\|\sigma_{\rm pr}(A)^{-1}(t,x,\xi)\|<\infty.$$

To define the norm above, one chooses a bounded Hilbert space structure on the fibers of E, the definition being independent on its choice.

Proposition 4.1 *Let* $A(t)\in\mathcal{S}^\epsilon(\mathbb{R};\Psi^m(\Sigma;L(E)))$*,* $\epsilon\in\mathbb{R}, m\geq 0$ *elliptic. Then the following holds:*

(1) $A(t)$ *is closeable on* $C_0^\infty(\Sigma;E)$ *with* Dom $A^{\rm cl}(t)=H^m(\Sigma;L(E))$.

(2) *if there exists* $\delta > 0$ *such that* $[-\delta, \delta] \cap \sigma(A^{\rm cl}(t)) = \emptyset$ *for* $t \in \mathbb{R}$*, then* $A^{-1}(t) \in \mathcal{S}^{-\epsilon}(\mathbb{R}; \Psi^{-m}(\Sigma; L(E)))$ *and*

$$\sigma_{\rm pr}(A^{-1})(t) = (\sigma_{\rm pr}(A))^{-1}(t).$$

Proof the same result is proved in [17, Prop. 5.8], with $\mathcal{S}^{\delta}(\mathbb{R}; \Psi^m)$ replaced by $C_{\rm b}^{\infty}(\mathbb{R}; \Psi^m)$, where $C_{\rm b}^{\infty}(\mathbb{R}; \mathcal{F})$ is defined at the beginning of Sect. 4.2. Note that $a(t) \in \mathcal{S}^{\delta}(\mathbb{R}; \mathcal{F})$ iff $\langle t\rangle^{-\delta-n}\partial_t^n a(t) \in C_{\rm b}^{\infty}(\mathbb{R}; \mathcal{F})$ for all $n \in \mathbb{N}$. Using that $\partial_t A^{-1}(t) = -A^{-1}(t)\partial_t A(t) A^{-1}(t)$ and similar identities for higher derivatives of $A^{-1}(t)$ combined with the above remark, we obtain the proposition. □

4.3 Functional calculus

4.3.1 Elliptic selfadjoint operators

Let us fix a bounded Hilbertian structure $(\cdot|\cdot)_E$ on the fibers of E and define the scalar product

$$(u|v) = \int_{\Sigma} (u(x)|v(x))_E {\rm d}Vol_g, \ \ u, v \in C_0^{\infty}(\Sigma; E).$$

Let $H(t) \in \mathcal{S}^{\delta}(\mathbb{R}; \Psi^m(\Sigma; L(E)))$ be elliptic, symmetric on $C_0^{\infty}(\Sigma; E)$. Using Proposition 4.1, one easily shows that its closure is selfadjoint with domain $H^m(\Sigma; E)$. Note also that its principal symbol $\sigma_{\rm pr}(H)(t, x, \xi)$ is selfadjoint for the Hilbertian scalar product on E_x.

4.3.2 Functional calculus

We now extend some results in [17] on functional calculus for selfadjoint pseudodifferential operators to our situation. We first recall some definitions from [17, Subsect. 5.3] about pseudodifferential operators with parameters.

One denotes by $\tilde{S}^m(\Sigma; L(E))$ the space of symbols $b \in C^{\infty}(\mathbb{R}_{\lambda} \times T^*\Sigma; L(E))$ such that if $b(\lambda) = b(\lambda, \cdot) \in C^{\infty}(T^*\Sigma; L(E))$ and $T_i b(\lambda)$ are the push-forwards of $b(\lambda)$ associated to a covering $\{U_i\}_{i\in\mathbb{N}}$ of Σ, we have:

$$\partial_{\lambda}^{\gamma}\partial_x^{\alpha}\partial_{\xi}^{\beta} b_i(\lambda, x, \xi) \in O(\langle\xi\rangle + \langle\lambda\rangle)^{m-|\beta|-\gamma}, \ (\lambda, x, \xi) \in \mathbb{R} \times T^*B(0, 1)$$

uniformly with respect to $i \in \mathbb{N}$. One denotes by $\widetilde{S}^m_{\rm h}(T^*\Sigma; L(E))$ the subspace of such symbols which are homogeneous w.r.t. (λ, ξ) and by $\widetilde{S}^m_{\rm ph}(T^*\Sigma; L(E))$ the subspace of poly-homogeneous symbols.

One also defines the ideal $\widetilde{\mathcal{W}}^{-\infty}(\Sigma; L(E))$ as the set of smooth functions $b : \mathbb{R} \ni \lambda \mapsto b(\lambda) \in \mathcal{W}^{-\infty}(\Sigma; L(E))$ such that

$$\|\partial_{\lambda}^{\gamma} b(\lambda)\|_{B(H^{-m}(\Sigma), H^m(\Sigma))} \in O(\langle\lambda\rangle^{-n}), \ \forall, m, n, \gamma \in \mathbb{N},$$

and set

$$\widetilde{\Psi}^m(\Sigma; L(E)) := \mathrm{Op}(\widetilde{S}^m_{\mathrm{ph}}(T^*\Sigma; L(E))) + \widetilde{\mathcal{W}}^{-\infty}(\Sigma; L(E)).$$

As usual one defines the time-dependent versions of the above spaces:

$$\mathcal{S}^\delta(\mathbb{R}; \widetilde{S}^m_{\mathrm{ph}}(T^*\Sigma; L(E))),\ \mathcal{S}^\delta(\mathbb{R}; \widetilde{\mathcal{W}}^{-\infty}(\Sigma; L(E))), \mathcal{S}^\delta(\mathbb{R}; \widetilde{\Psi}^m(\Sigma; L(E))).$$

We define the *principal symbol* of $A(t) \in \mathcal{S}^\delta(\mathbb{R}; \widetilde{\Psi}^m(\Sigma; L(E)))$ as in 4.2.1, using the poly-homogeneity.

Proposition 4.2 *Let $H(t) \in \mathcal{S}^\delta(\mathbb{R}; \Psi^1(\Sigma; L(E)))$ elliptic and formally selfadjoint. Let us still denote by $H(t)$ its closure, which is selfadjoint on $H^1(\Sigma; E)$ by Proposition 4.1. Assume that there exists $\delta > 0$ such that $[-\delta, \delta] \cap \sigma(H(t)) = \emptyset$ for $t \in I$.*

Then $\lambda \mapsto (H(t) + \mathrm{i}\lambda)^{-1}$ belongs to $\mathcal{S}^\delta(\mathbb{R}; \widetilde{\Psi}^{-1}(\Sigma; L(E)))$ with principal symbol $(\sigma_{\mathrm{pr}}(H(t)) + \mathrm{i}\lambda)^{-1}$.

Proof The C^∞_{b} version of the proposition is proved in [17, Prop. 5.9]. We use the same remark as in the proof of Proposition 4.1 to extend it to the $\mathcal{S}^\delta$ case. Details are left to the reader. □

Proposition 4.3 *Let $H(t) \in \mathcal{S}^0(\mathbb{R}; \Psi^1(\Sigma; L(E)))$ be elliptic, symmetric on $C^\infty_0(\Sigma; E)$, and let us denote still by $H(t)$ its selfadjoint closure. Assume that there exists $\delta > 0$ such that $[-\delta, \delta] \cap \sigma(H(t)) = \emptyset$ for $t \in \mathbb{R}$.*

Assume in addition that there exist $H_\infty \in \Psi^1(\Sigma; L(E))$, elliptic symmetric on $C^\infty_0(\Sigma; E)$ with $0 \notin \sigma(H_\infty)$ such that

$$H(t) - H_\infty \in \mathcal{S}^{-\mu}(\mathbb{R}; \Psi^1(\Sigma; L(E))).$$

Then

(1) *the spectral projections $\mathbb{1}_{\mathbb{R}^\pm}(H(t))$ belong to $\mathcal{S}^0(\mathbb{R}; \Psi^0(\Sigma; L(E)))$ and*

$$\sigma_{\mathrm{pr}}(\mathbb{1}_{\mathbb{R}^\pm}(H(t))) = \mathbb{1}_{\mathbb{R}^\pm}(\sigma_{\mathrm{pr}}(H(t))).$$

Moreover, $\mathbb{1}_{\mathbb{R}^\pm}(H(t)) - \mathbb{1}_{\mathbb{R}^\pm}(H_\infty)$ belongs to $\mathcal{S}^{-\mu}(\mathbb{R}; \Psi^0(\Sigma; L(E)))$.

(2) *$S(t) = (H^2(t) + 1)^{\frac{1}{2}}$ belongs to $\mathcal{S}^0(\mathbb{R}; \Psi^1(\Sigma; L(E)))$ and*

$$\sigma_{\mathrm{pr}}(S(t)) = |\sigma_{\mathrm{pr}}(H(t))|.$$

Moreover, $S(t) - S_\infty$ belongs to $\mathcal{S}^{-\mu}(\mathbb{R}; \Psi^1(\Sigma; L(E)))$ for $S_\infty = (H^2_\infty + 1)^{\frac{1}{2}}$.

Proof By Proposition 4.2 we have

$$(\mathrm{i}\lambda - H(t))^{-1} = \mathrm{Op}(a(t, \lambda)) + R_{-\infty}(t, \lambda), \tag{4.1}$$

where $a(t) \in \mathcal{S}^0(\mathbb{R}; \widetilde{S}^{-1}(T^*\Sigma; L(E)))$ and $R_{-\infty}(t) \in \mathcal{S}^0(\mathbb{R}; \widetilde{\mathcal{W}}^{-\infty}(\Sigma; L(E))$ satisfies:

$$\langle t\rangle^p \|\partial_\lambda^n \partial_t^p R_{-\infty}(t,\lambda)\|_{B(H^{-m}(\Sigma),H^m(\Sigma))} \in O(\langle\lambda\rangle)^{-m}, \ \forall p,m,n \in \mathbb{N},$$

uniformly for $t \in \mathbb{R}$.

The principal symbol of $a(t)$ is $(\mathrm{i}\lambda - \sigma_{\mathrm{pr}}(H))^{-1}$, which means that

$$\mathrm{Op}(a(t,\lambda)) - \mathrm{Op}((\mathrm{i}\lambda - \sigma_{\mathrm{pr}}(H)(t))^{-1} \in \mathcal{S}^0(\mathbb{R}; \widetilde{\Psi}^{-2}(\Sigma; L(E))). \tag{4.2}$$

For $a \neq 0$ we have

$$|a|^{-1} = \frac{2}{\pi}\int_0^{+\infty}(a+\mathrm{i}\lambda)^{-1}(a-\mathrm{i}\lambda)^{-1}\mathrm{d}\lambda, \tag{4.3}$$

hence

$$|H(t)|^{-1} = \frac{2}{\pi}\int_0^{+\infty}(H(t)+\mathrm{i}\lambda)^{-1}(H(t)-\mathrm{i}\lambda)^{-1}\mathrm{d}\lambda. \tag{4.4}$$

From Proposition 4.2 we obtain that $|H(t)|^{-1} \in \mathcal{S}^0(\mathbb{R}; \Psi^{-1}(\Sigma; L(E)))$. We also deduce from (4.4) using the second resolvent formula that $|H(t)|^{-1} - |H_\infty|^{-1} \in \mathcal{S}^{-\mu}(\mathbb{R}; \Psi^{-1}(\Sigma; L(E)))$. This implies that $\mathrm{sgn}(H(t)) \in \mathcal{S}^0(\mathbb{R}; \Psi^{-0}(\Sigma; L(E)))$ and $\mathrm{sgn}(H(t)) - \mathrm{sgn}(H_\infty) \in \mathcal{S}^{-\mu}(\mathbb{R}; \Psi^0(\Sigma; L(E)))$.

Moreover, since the principal symbol of $(H(t)+\mathrm{i}\lambda)^{-1}$ equals $(\sigma_{\mathrm{pr}}(H(t))+\mathrm{i}\lambda)^{-1}$, applying once more (4.3) we obtain that $\sigma_{\mathrm{pr}}(\mathrm{sgn}(H(t)))$ equals $\mathrm{sgn}(\sigma_{\mathrm{pr}}(H(t)))$.

Writing $\mathbb{1}_{\mathbb{R}^\pm}(\lambda) = \frac{1}{2}(1 \pm \mathrm{sgn}(\lambda))$ this implies (1). To prove (2) we deduce from (4.3) that

$$\begin{aligned}(a+1)^{-\frac{1}{2}} &= \frac{2}{\pi}\int_0^{+\infty}(a+s^2+1)^{-1}\mathrm{d}s \\ &= \frac{2}{\pi}\int_1^{+\infty}(a+\lambda^2)^{-1}\lambda(\lambda^2-1)^{-\frac{1}{2}}\mathrm{d}\lambda,\end{aligned} \tag{4.5}$$

hence

$$(H^2(t)+1)^{-\frac{1}{2}} = \frac{2}{\pi}\int_1^{+\infty}(H(t)+\mathrm{i}\lambda)^{-1}(H(t)-\mathrm{i}\lambda)^{-1}\lambda(\lambda^2-1)^{-\frac{1}{2}}\mathrm{d}\lambda. \tag{4.6}$$

we obtain that $(H^2(t)+1)^{-\frac{1}{2}} \in \mathcal{S}^0(\mathbb{R}; \Psi^{-1}(\Sigma; L(E)))$. We also deduce from (4.6) that $(H^2(t)+1)^{-\frac{1}{2}} - (H_\infty^2+1)^{-\frac{1}{2}} \in \mathcal{S}^{-\mu}(\mathbb{R}; \Psi^{-1}(\Sigma; L(E)))$. We write then $(H^2(t)+1)^{\frac{1}{2}} = (H^2+1)(H^2(t)+1)^{-\frac{1}{2}}$ and obtain (2). □

5 The in/out vacuum states

In this section we prove Theorem 1.1.

5.1 Reduction of the Dirac operator

In this subsection we consider the Dirac operator $\tilde{D}$ obtained from D by conformal transformation, see Sects. 2.2 and 3.1.5, 3.3.1. We recall that the spinor bundle for $(M, \tilde{g})$ is identical to the one for (M, g) and hence denoted by $S(M)$, and that the restriction $S_t(\Sigma)$ of the spinor bundle $S(M)$ to Σ_t is independent of t, and denoted by $S(\Sigma)$.

We recall also from Sect. 2.2 that $S(M)$ is equipped with the time positive Hermitian form $\tilde{\beta}$ see (2.12) and we denote by $\tilde{\beta}_t$ its restriction to $S(\Sigma)$. Also we denote by $\tilde{\gamma}_t : T_{\Sigma_t}M \to L(S_t(\Sigma))$ the restrictions of $\tilde{\gamma}$ defined in (2.12) to $S(\Sigma_t)$.

We will denote by (x, k) local coordinates on $T^*\Sigma$ and by (t, x, τ, k) local coordinates on T^*M.

The first step consists in reducing the Dirac equation $\tilde{D}\psi = 0$ to a time-dependent Schroedinger equation

$$\partial_t\psi - \mathrm{i}\mathsf{H}(t)\psi = 0,$$

where $\mathsf{H}(t)$ is some time-dependent selfadjoint operator. To this end it is necessary to identify the elements of spinor bundles at different times by parallel transport. We recall that $e_0 = \partial_t$ and $(e_j)_{1\leq j\leq d}$ are the local frames constructed in Lemma 3.2. We start by an easy proposition.

Proposition 5.1

$$\begin{aligned}
(i)\ & \tilde{\gamma}_t(e_0) - \tilde{\gamma}_{\pm\infty}(e_0) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C^\infty_{\mathrm{b}}(V; L(S(\Sigma)))),\\
(ii)\ & \tilde{\gamma}_t(e_j(t)) - \tilde{\gamma}_{\pm\infty}(e_j(\pm\infty)) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C^\infty_{\mathrm{b}}(V; L(S(\Sigma)))),\\
(iii)\ & \beta_t - \beta_{\pm\infty} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C^\infty_{\mathrm{b}}(V; L(S(\Sigma), S(\Sigma)^*))),\\
(iv)\ & \nabla^S_{e_0} - \nabla^{S_{\pm\infty}}_{e_0} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^1(\Sigma; S(\Sigma))),\\
(v)\ & \nabla^S_{e_j(t)} - \nabla^{S_{\pm\infty}}_{e_j(\pm\infty)} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^1(\Sigma; S(\Sigma))).
\end{aligned}$$

Proof We fix a bounded atlas $(V_i, \psi_i)_{i\in\mathbb{N}}$ of (Σ, h_0) and set $U_i = \mathbb{R}\times V_i$. We fix a bounded family $(\mathcal{F}_i)_{i\in\mathbb{N}}$ of oriented orthonormal frames for h_0 over V_i and denote by $\mathcal{F}_i(t) = (e_{i,j}(t))_{1\leq j\leq d}$ the orthonormal frames obtained by parallel transport as in Lemma 3.2. Since $e_0 = \partial_t$, $\mathcal{E}_i = (e_{i,a})_{0\leq a\leq d}$ are then oriented, time-oriented orthonormal frames for g over $U_i = \mathbb{R}\times V_i$.

We use the spin frames $\mathcal{B}_i(t) = (E_{i,A}(t))_{1\leq A\leq N}$ of $S(\Sigma)$ associated to the frames $\mathcal{E}_i(t) = (e_{i,a}(t))_{0\leq a\leq d}$ over $\{t\}\times V_i$. From the estimates in Lemma 3.2, we obtain that $\mathcal{B}_i(\pm\infty) = \lim_{t\to\pm\infty}\mathcal{B}_i(t)$ exist and that

$$\begin{aligned}
& E_{i,A}(\pm\infty) \in C^\infty_{\mathrm{b}}(V_i, S(\Sigma)),\\
& E_{i,A}(t) - E_{i,A}(\pm\infty) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C^\infty_{\mathrm{b}}(V_i, S(\Sigma))) \text{ uniformly w.r.t. } i\in\mathbb{N}.
\end{aligned} \tag{5.1}$$

We recall from 3.2.1 that from the transition functions $\boldsymbol{o}_{ij}(\pm\infty) : V_{ij} \to \mathrm{SO}(d)$ one obtains the spin structures $P\mathrm{Spin}(\Sigma; h_{\pm\infty})$ introduced above. The frames $\mathcal{B}_i(\pm\infty)$ are the frames associated to the $\mathcal{E}_i(\pm\infty)$ for this spin structure.

Let us now forget the index i and denote by $(\boldsymbol{\psi}^A)_{1\leq A\leq N} \in \mathbb{C}^N$ the components of ψ in the frame $\mathcal{B}$. The dual frames are as usual denoted by $(e^a)_{0\leq a\leq d}$, $(E^A)_{1\leq A\leq N}$ so for example $\boldsymbol{\psi}^A = \psi \cdot E^A$.

Denoting by $\boldsymbol{\gamma}_t(u)$ the matrix of $\tilde{\gamma}_t(u)$ in the frame $\mathcal{B}(t)$, we have also

$$\boldsymbol{\gamma}_t(u) = \boldsymbol{\gamma}_a u^a(t),\ u^a(t) := u \cdot e^a(t). \tag{5.2}$$

where $\boldsymbol{\gamma}_a \in M_N(\mathbb{C})$ for $0 \leq a \leq d$ are the usual gamma matrices. Using Lemma 3.2 and (5.1), we obtain that $\lim_{t\to\pm\infty} \tilde{\gamma}_t(e_a(t)) \in L(S(\Sigma))$ exist and that

$$\tilde{\gamma}_t(e_a(t)) - \lim_{t\to\pm\infty} \tilde{\gamma}_t(e_a(t)) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C_{\mathrm{b}}^{\infty}(V; L(S(\Sigma)))). \tag{5.3}$$

If we reintroduce the index i and set $V = V_i$, then the seminorms in (5.3) are uniform with respect to $i \in \mathbb{N}$. Because of (5.2), the limits $\lim_{t\to\pm\infty} \tilde{\gamma}_t(e_a(t))$ are equal to $\tilde{\gamma}_{\pm\infty}(e_a(\pm\infty))$. This proves *ii). i)* is proved similarly.

Let us now denote by $\boldsymbol{\beta}_t$ the matrix of $\tilde{\beta}_t$ in the frame $\mathcal{B}(t)$. We have

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}$$

where $\boldsymbol{\beta} \in M_N(\mathbb{C})$ is a Hermitian matrix such that

$$\boldsymbol{\beta}\boldsymbol{\gamma}_a = -\boldsymbol{\gamma}_a^*\boldsymbol{\beta},\ \mathrm{i}\boldsymbol{\beta}\boldsymbol{\gamma}_0 > 0.$$

This implies as above that $\lim_{t\to\pm\infty} \tilde{\beta}_t$ exist and

$$\tilde{\beta}_t - \lim_{t\to\pm\infty} \tilde{\beta}_t \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C_{\mathrm{b}}^{\infty}(V; L(S(\Sigma), S(\Sigma)^*))). \tag{5.4}$$

Again because of (5.2) the limits $\lim_{t\to\pm\infty} \tilde{\beta}_t$ are equal to $\tilde{\beta}_{\pm\infty}$, which proves *iii)*.

Finally, see, e.g., [17, 2.5.6] we have:

$$\tilde{\nabla}^S_{e_a}\boldsymbol{\psi}^A = \partial_a\boldsymbol{\psi}^A + \sigma^A_{aC}\boldsymbol{\psi}^C, \tag{5.5}$$

where

$$\partial_a f = e_a \cdot df,\ \sigma^A_{aC} = E^A \cdot \sigma_a E_C,\ \sigma_a = \frac{1}{4}\Gamma^c_{ab}\tilde{\gamma}(e_c)g^{bd}\tilde{\gamma}(e_d),\ \Gamma^c_{ab} = \nabla_{e_a}e_b \cdot e^c.$$

Using (3.5) and the properties of $(e_j(t))_{1\leq j\leq d}$ in Lemma 3.2, we obtain by a routine computation that

$$\begin{aligned}&\Gamma^0_{0b}(t) = \Gamma^c_{00}(t) = 0,\ \Gamma^0_{ab}(t),\ \Gamma^a_{0b}(t) \in \mathcal{S}^{-1-\mu}(\mathbb{R}; C_{\mathrm{b}}^{\infty}(V)),\\ &\Gamma^c_{ab}(t) - \Gamma^c_{ab}(\pm\infty) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C_{\mathrm{b}}^{\infty}(V)) \text{ if } a, b, c \neq 0.\end{aligned} \tag{5.6}$$

If we reintroduce the index i and set $V = V_i$, then the seminorms in (5.6) are uniform with respect to $i \in \mathbb{N}$. Therefore, the limits $\lim_{t\to\pm\infty} \tilde{\nabla}^S_{e_j(t)}$ exist and

$$\tilde{\nabla}^S_{e_j(t)} - \lim_{t\to\pm\infty} \tilde{\nabla}^S{}_{e_j(t)} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^1(\Sigma; S(\Sigma))).$$

Using (5.5) we obtain also that $\lim_{t\to\pm\infty} \tilde{\nabla}^S_{e_j(t)} = \tilde{\nabla}^{S_{\pm\infty}}(e_j(\pm\infty))$, which proves *v)*. The proof of *iv)* is similar. □

5.1.1 Identification by parallel transport

For $f \in C^\infty(\Sigma_s; S(\Sigma_s))$ we denote by $\mathcal{T}(s)f = \psi$ the solution of

$$\begin{cases} \tilde{\nabla}^S_{\partial_t}\psi = 0 \text{ in } \mathbb{R}\times\Sigma, \\ \psi_{|\Sigma_s} = f, \end{cases} \tag{5.7}$$

and set

$$\mathcal{T}(t,s)f = \mathcal{T}(s)f\restriction_{\Sigma_t},$$

$$\begin{aligned} &\mathcal{T} : C^\infty(\mathbb{R}; C^\infty(\Sigma, S(\Sigma))) \to C^\infty(M; S(M)) \\ &\psi(t) \mapsto (\mathcal{T}\psi)(t) = |\tilde{h}_t|^{-\frac14}|\tilde{h}_0|^{\frac14}\mathcal{T}(t,0)\psi(t), \end{aligned} \tag{5.8}$$

We denote by $\tilde{\nu}_0$ the Hilbertian scalar product

$$\overline{f}\cdot\tilde{\nu}_0 f := \mathrm{i}\int_\Sigma \overline{f}\cdot\tilde{\beta}_0\tilde{\gamma}_0(e_0)f|\tilde{h}_0|^{\frac12}\mathrm{dx},\;\; f \in C_0^\infty(\Sigma, S(\Sigma)).$$

Using (2.4) we obtain the following lemma, see [17, Lemma 6.1].

Lemma 5.2 *One has*

(1) $\mathcal{T}(s,t)\tilde{\gamma}_t(e_0)\mathcal{T}(t,s) = \tilde{\gamma}_s(e_0)$, $t, s \in I$,
(2) $\mathcal{T}(s,t)\tilde{\gamma}_t(e_j(t))\mathcal{T}(t,s) = \tilde{\gamma}_s(e_j(s))$, $t, s \in I$,
(3) $\mathcal{T}(t,s)^*\tilde{\beta}_t\mathcal{T}(t,s) = \tilde{\beta}_s$, $t, s \in I$.

5.1.2 Reduction of the Dirac operator

Proposition 5.3 *Let*

$$D := \mathcal{T}^{-1}\tilde{D}\mathcal{T}.$$

Then

(1) *the map*

$$\mathcal{T} : (\mathrm{Sol}_{\mathrm{sc}}(D), \tilde{\nu}_0) \xrightarrow{\sim} (\mathrm{Sol}_{\mathrm{sc}}(\tilde{D}), \tilde{\nu}_0)$$

is unitary.

(2) *We have*

$$D = -\tilde{\gamma}_0(e_0)\partial_t + \mathrm{i}\tilde{\gamma}_0(e_0)H(t),$$

where $H(t) \in \mathcal{S}^0(\mathbb{R}, \Psi^1(\Sigma, S(\Sigma)))$ *has the following properties:*

(2i) $\sigma_{\mathrm{pr}}(H(t))(\mathrm{x}, k) = -\tilde{\gamma}_0(e_0)\tilde{\gamma}_0(\tilde{h}_t(\mathrm{x})^{-1}k)$.

(2ii) *there exist* $H_{\pm\infty} \in \Psi^1(\Sigma; L(S(\Sigma)))$ *elliptic, formally selfadjoint for* ν_0 *such that*

$$H(t) - H_{\pm\infty} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^1(\Sigma; L(S(\Sigma)))).$$

$H_{\pm\infty}$ *is selfadjoint with domain* $H^1(\Sigma; S(\Sigma))$ *and* $0 \notin \sigma(H_{\pm\infty})$.

(2iii) $H(t)$ *is formally selfadjoint for* $\tilde{\nu}_0$ *and selfadjoint with domain* $H^1(\Sigma; S(\Sigma))$.

Proof (1) is obvious since $\mathcal{T}(0,0) = \mathbb{1}$. We have $\mathcal{T}^{-1}\tilde{\gamma}(e_0)\mathcal{T} = \tilde{\gamma}_0(e_0)$ by Lemma 5.2 and

$$\mathcal{T}\partial_t\mathcal{T}^{-1} = \tilde{\nabla}^S_{e_0} - \frac{1}{4}\partial_t|\tilde{h}_t||\tilde{h}_t|^{-1}. \tag{5.9}$$

If we fix over some open set $U = \mathbb{R}\times V$, a local oriented and time oriented orthonormal frame $(e_a)_{0\leq a\leq d}$ as in Lemma 3.2, we have

$$\begin{aligned} &\mathcal{T}^{-1}D\mathcal{T} = -\tilde{\gamma}_0(e_0)\partial_t + \mathrm{i}\tilde{\gamma}_0(e_0)\mathsf{H}(t),\\ &\mathsf{H}(t) = \mathcal{T}^{-1}H(t)\mathcal{T} + \tfrac{1}{4}\partial_t|\tilde{h}_t||\tilde{h}_t|^{-1}, \text{ for}\\ &H(t){:=}\mathrm{i}\tilde{\gamma}_t(e_0)\tilde{\gamma}_t(e_j(t))\tilde{\nabla}^S_{e_j(t)} + \mathrm{i}\tilde{\gamma}_t(e_0)\tilde{m}, \end{aligned} \tag{5.10}$$

where in the second line we sum only over $1 \leq j \leq d$.

Let us now prove the properties of $\mathsf{H}(t)$ stated in (2). By Proposition 5.1 we obtain that

$$H(t) - H_{\pm\infty} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^1(\Sigma; L(S(\Sigma)))), \tag{5.11}$$

for

$$H_{\mathrm{out/in}} = \mathrm{i}\tilde{\gamma}_\infty(e_0)(\tilde{\gamma}_\infty(e_j(\pm\infty))\tilde{\nabla}^{\mathrm{out/in}}_{e_j(\pm\infty)} + \tilde{m}_{\mathrm{out/in}}). \tag{5.12}$$

Let us now consider the maps $\mathcal{T}(t,s)$. We have

$$\tilde{\nabla}^S_{e_0}\psi = \partial_t\psi + \frac{1}{4}\Gamma^a_{0b}\boldsymbol{\gamma}_a\boldsymbol{\gamma}^b\psi = \partial_t\psi,$$

since $\nabla_{e_0} e_a = 0$. It follows that the matrix of $\mathcal{T}(t,s)$ in the bases $\mathcal{B}(s)$ and $\mathcal{B}(t)$ equals the identity matrix. Using then (5.1) we obtain that the limits $\mathcal{T}(\pm\infty, 0) = \lim_{t\to\pm\infty} \mathcal{T}(t,0) \in C_{\rm b}^\infty(\Sigma; L(S(\Sigma)))$ exist and that

$$\mathcal{T}(t,0) - \mathcal{T}(\pm\infty, 0) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; C_{\rm b}^\infty(\Sigma; L(S(\Sigma)))). \tag{5.13}$$

Combining (5.11) and (5.13), we obtain that

$$\mathsf{H}_{\rm out/in} := \mathcal{T}(0, \pm\infty)|\tilde{h}_{\rm out/in}|^{\frac{1}{4}}|\tilde{h}_0|^{-\frac{1}{4}} H_{\pm\infty}|\tilde{h}_0|^{\frac{1}{4}}|\tilde{h}_{\rm out/in}|^{-\frac{1}{4}}\mathcal{T}(\pm\infty, 0) \tag{5.14}$$

belongs to $\Psi^1(\Sigma; L(S(\Sigma)))$ and

$$\mathsf{H}(t) - \mathsf{H}_{\rm out/in} \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^1(\Sigma; L(S(\Sigma)))). \tag{5.15}$$

The principal symbol of $\mathsf{H}(t)$ is clearly equal to $-\tilde{\gamma}_0(e_0)\tilde{\gamma}_0(\tilde{h}_t(\mathrm{x})^{-1}k)$, which proves (2i).

Let us now prove the remaining parts of (2ii). From (2.25) we obtain that $\mathsf{H}_{\pm\infty}$ is the spatial part of the Dirac operator for the static metric $g_{\pm\infty}$. Using hypothesis (H4) (and remembering that we removed the tildes), we obtain that $0 \notin \sigma(H_{\pm\infty})$. The selfadjointness of $\mathsf{H}_{\rm out/in}$ on $H^1(\Sigma; S(\Sigma))$ follows by the usual ellipticity argument.

Finally, we know that if $\tilde{D}\psi = 0$ then

$$\int_\Sigma \overline{\psi}(t,\cdot)\cdot\tilde{\beta}_t\tilde{\gamma}_t(e_0)\psi(t,\cdot)|\tilde{h}_t|^{\frac{1}{2}}\mathrm{dx}$$

is independent on t; hence, if $\tilde{\psi} = \mathcal{T}^{-1}\psi$ we obtain using

$$\mathcal{T}\partial_t\mathcal{T}^{-1} = \tilde{\nabla}^S_{e_0} - \frac{1}{4}\partial_t|h_t||h_t|^{-1}$$

that

$$\int_\Sigma \overline{\tilde{\psi}}(t,\cdot)\cdot\tilde{\beta}_0\tilde{\gamma}_0(e_0)\tilde{\psi}(t,\cdot)|\tilde{h}_0|^{\frac{1}{2}}\mathrm{dx}$$

is independent on t. Since $\partial_t\tilde{\psi} = \mathrm{i}\mathsf{H}(t)\tilde{\psi}$, this implies that $\mathsf{H}(t) = \mathsf{H}^*(t)$ on $C_0^\infty(\Sigma; S(\Sigma))$ for ν_0. The fact that (the closure of) $\mathsf{H}(t)$ is then selfadjoint on $H^1(\Sigma; S(\Sigma))$ follows from the standard argument, using the ellipticity of $\mathsf{H}(t)$. □

5.2 Some preparations

The space $C_0^\infty(\mathbb{R}\times\Sigma; S(\Sigma))$ is equipped with the Hilbertian scalar product

$$\overline{\psi}\cdot\tilde{\nu}\psi = \int_{\mathbb{R}\times\Sigma} \overline{\psi}\cdot\tilde{\beta}_0\tilde{\gamma}_0(e_0)\psi\,\mathrm{d}t|\tilde{h}_0|^{\frac{1}{2}}\mathrm{d}x,$$

while $C_0^\infty(\Sigma; S(\Sigma))$ is equipped with

$$\overline{f}\cdot\tilde{\nu}_0 f = \int_\Sigma \overline{f}\cdot\tilde{\beta}_0\tilde{\gamma}_0(e_0) f|\tilde{h}_0|^{\frac{1}{2}}\mathrm{d}x. \tag{5.16}$$

Adjoints of operators will always be computed with respect to these scalar products. Our reference Hilbert space is

$$\mathcal{H} = L^2(\Sigma; S(\Sigma)),$$

equal to the completion of $C_0^\infty(\Sigma; S(\Sigma))$ for $\tilde{\nu}_0$.

The following lemma is the analog of [17, Lemma 6.3].

Lemma 5.4 *There exists $R_{-\infty} \in C_0^\infty(\mathbb{R}; \mathcal{W}^{-\infty}(\Sigma; S(\Sigma)))$ with $R_{-\infty}(t) = R_{-\infty}(t)^*$ and $\delta > 0$ such that*

$$\sigma(H(t) + R_{-\infty}(t)) \cap [-\delta, \delta] = \emptyset.$$

Proof we follow the proof in [17, Lemma 6.3]. By Proposition 5.3 (2), we know that there exists $\delta > 0$ such that $\sigma(\mathsf{H}(t)) \cap [-\delta, \delta] = \emptyset$ for $|t| \gg 1$, so the modification $R_{-\infty}(t)$ can be taken compactly supported in t. □

5.2.1 Unitary group

Let us denote by $\mathsf{U}(t, s)$, $s, t \in I$ the unitary evolution generated by $\mathsf{H}(t)$, ie the solution of

$$\begin{cases} \partial_t \mathsf{U}(t, s) = \mathrm{i}\mathsf{H}(t)\mathsf{U}(t, s), \\ \partial_s \mathsf{U}(t, s) = -\mathrm{i}\mathsf{U}(t, s)\mathsf{H}(s), \\ \mathsf{U}(s, s) = \mathbb{1}. \end{cases}$$

The properties of $\mathsf{H}(t)$ imply that $\mathsf{U}(t, s)$ is well-defined by a classical result of Kato, see, e.g., [40].

Lemma 5.5 *$U(t, s)$ are uniformly bounded in $B(H^m(\Sigma; S(\Sigma)))$ for $t, s \in \mathbb{R}$, $m \in \mathbb{R}$.*

Proof Let us set

$$S(t) := (\mathsf{H}^2(t) + 1)^{\frac{1}{2}}, \ S_{\mathrm{out/in}} := (\mathsf{H}^2_{\mathrm{out/in}} + 1)^{\frac{1}{2}}.$$

By Proposition 4.3 we obtain that

$$\begin{aligned} &S(t) \in \mathcal{S}^0(\mathbb{R}; \Psi^1(\Sigma; S(\Sigma))), \\ &S(t) - S_{\mathrm{out/in}} \in \mathcal{S}^{-\mu}(\mathbb{R}^\pm; \Psi^1(\Sigma; S(\Sigma))), \end{aligned} \tag{5.17}$$

and $\sigma_{\mathrm{pr}}(S(t))(x, k) = (k\cdot\tilde{h}_t^{-1}(\mathrm{x})k)^{\frac{1}{2}}$. This implies that

$$C_m^{-1}\|S^m(t)u\|_0 \leq \|u\|_m \leq C_m\|S^m(t)u\|_0, \ t \in \mathbb{R},$$

where we denote by $\|\cdot\|_m$ the norm in $H^m(\Sigma; S(\Sigma))$.

For $u \in S^m(s)C_0^\infty(\Sigma; S(\Sigma))$ we set

$$f(t) = \|\mathsf{U}(s,t)S^m(t)\mathsf{U}(t,s)S^{-m}(s)u\|_0,$$

which is finite since $\mathsf{U}(t,s)$ preserves $C_0^\infty(\Sigma; S(\Sigma))$. We have

$$\begin{aligned}|f'(t)| &\leq \|\mathsf{U}(s,t)\partial_t S^m(t)\mathsf{U}(t,s)S^{-m}(s)u\|_0\\ &= \|\mathsf{U}(s,t)\partial_t S^m(t)S^{-m}(t)\mathsf{U}(t,s)\mathsf{U}(s,t)S^m(t)\mathsf{U}(t,s)S^{-m}(s)u\|_0\\ &\leq \|\mathsf{U}(s,t)\partial_t S^m(t)S^{-m}(t)\mathsf{U}(t,s)\|_{B(\mathcal{H})}f(t) \leq Ct^{-1-\mu}f(t),\end{aligned}$$

where we use (5.17) in the last inequality. By Gronwall's inequality we obtain that $f(t) \leq Cf(s)$ for $t, s \in \mathbb{R}$ hence

$$\begin{aligned}&\|S^m(t)\mathsf{U}(t,s)S^{-m}(s)u\|\\ &= \|\mathsf{U}(s,t)S^m(t)\mathsf{U}(t,s)S^{-m}(s)u\| \leq C\|u\|,\ u \in S^m(s)C_0^\infty(\Sigma; S(\Sigma)),\end{aligned}$$

which proves the lemma since $S^m(s)C_0^\infty(\Sigma; S(\Sigma))$ is dense in $L^2(\Sigma; S(\Sigma))$. □

5.2.2 Some preparations

We next introduce some classes of maps between pseudodifferential operators. These classes are similar to the ones considered in [17, Subsect. 6.3], with the behavior for large times taken into account.

We will use the short hand notation introduced in Sect. 4.2 and denote $\mathcal{S}^\delta(\mathbb{R}; \Psi^m(\Sigma; L(S(\Sigma))))$ simply by $\mathcal{S}^{\delta,m}$. We set $\mathcal{S}^{\infty,\infty} = \bigcup_{\delta,m\in\mathbb{R}} \mathcal{S}^{\delta,m}$.

Definition 5.6 Let $\boldsymbol{\delta} : \mathbb{R} \to \mathbb{R}$ and $p \in \mathbb{R}$. We denote by $\mathcal{F}_{-\boldsymbol{\delta},-p}$ the set of maps $F : \mathcal{S}^{0,0} \to \mathcal{S}^{\infty,\infty}$ such that

$$F : \mathcal{S}^{-\mu,-1} \to \mathcal{S}^{-\mu-\boldsymbol{\delta}(\mu),-p},\ \forall \mu > 0,$$

and:

$$R_1 - R_2 \in \mathcal{S}^{-\mu-\epsilon,-1-j} \Rightarrow F(R_1) - F(R_2) \in \mathcal{S}^{-\mu-\boldsymbol{\delta}(\mu)-\epsilon,-p-j},\ \forall \epsilon > 0,\ j \in \mathbb{N}.$$

An element of $\mathcal{F}_{-\boldsymbol{\delta},-p}$ will be denoted by $F_{-\boldsymbol{\delta},-p}$. The following proposition is the analog of [15, Lemma A.1], [17, Prop. 6.6]. It is an abstract formulation of an ubiquitous argument in pseudodifferential calculus, consisting in solving recursive equations to determine successive terms in the symbolic expansion of a pseudodifferential operators.

Proposition 5.7 *Let $A \in \mathcal{S}^{-\mu_1,-1}$, $\mu_1 > 0$ and $F_{0,-2} \in \mathcal{F}_{0,-2}$. Then there exists a solution $R \in \mathcal{S}^{-\mu_1,-1}$, unique modulo $\mathcal{S}^{-\mu_1,-\infty}$ of the equation:*

$$R = A + F_{0,-2}(R) \ mod\ \mathcal{S}^{-\mu_1,-\infty}.$$

Proof Let us denote $F_{0,-2}$ simply by F. We set $S_0 = A$, $S_n = A + F(S_{n-1})$ for $n \geq 1$. We have $S_1 - S_0 = F(A)$ and $S_n - S_{n-1} = F(S_{n-1}) - F(S_{n-2})$. Since $F \in \mathcal{F}_{0,-2}$ we obtain by induction that $S_n - S_{n-1} \in \mathcal{S}^{-\mu_1,-(n+1)}$. We take $R \in \mathcal{S}^{-\mu_1,-1}$ such that $R \sim S_0 + \sum_0^\infty S_n - S_{n-1}$ which solves the equation modulo $\mathcal{S}^{-\mu_1,-\infty}$. If R_1, R_2 are two solutions, then $R_1 - F_2 = F(F_1) - F(R_2)$ modulo $\mathcal{S}^{-\mu_1,-\infty}$; hence, using that $F \in \mathcal{F}_{0,-2}$ we obtain by induction on n that $R_1 - R_2 \in \mathcal{S}^{-\mu_1,-n}$ for all $n \in \mathbb{N}$ which proves uniqueness modulo $\mathcal{S}^{-\mu_1,-\infty}$. □

We now collect some useful properties of the sets $\mathcal{F}_{-\boldsymbol{\delta},-p}$.

Lemma 5.8 (1) *If $A \in \mathcal{S}^{-\varrho,k}$ and $F_{-\boldsymbol{\delta},-p} \in \mathcal{F}_{-\boldsymbol{\delta},-p}$, then the maps*

$$\begin{aligned} AF_{-\boldsymbol{\delta},-p} &: R \mapsto AF_{-\boldsymbol{\delta},-p}(R), \\ F_{-\boldsymbol{\delta},-p}A &: R \mapsto F_{-\boldsymbol{\delta},-p}(R)A \end{aligned}$$

belong to $\mathcal{F}_{-\boldsymbol{\delta}-\varrho,-p+k}$ for $k \leq p$.

(2) *If $F_{-\boldsymbol{\delta}_i,-p_i} \in \mathcal{F}_{-\boldsymbol{\delta}_i,-p_i}$, then the map*

$$F_{-\boldsymbol{\delta_1},-p_1}F_{-\boldsymbol{\delta}_2,-p_2} : R \mapsto F_{-\boldsymbol{\delta_1},-p_1}(R)F_{-\boldsymbol{\delta_2}-p_2}(R)$$

belongs to $\mathcal{F}_{-\boldsymbol{\delta_1}-\boldsymbol{\delta_2}-\boldsymbol{\mu},-p_1-p_2}$, where $\boldsymbol{\mu}$ is the map $\mu \mapsto \mu$.

(3) *the map $R \to R^p$ belongs to $\mathcal{F}_{-(p-1)\boldsymbol{\mu},-p}$ for $p \in \mathbb{N}^*$.*

(4) *the map $R \mapsto \mathrm{e}^R$ belongs to $\mathcal{F}_{0,0}$.*

(5) *one has $\mathrm{e}^R = 1 + R + F_{-\boldsymbol{\mu},-2}(R)$, where $F_{-\boldsymbol{\mu},-2} \in \mathcal{F}_{-\boldsymbol{\mu},-2}$.*

Proof (1) and (2) are easy. We check that $R \mapsto R$ belongs to $\mathcal{F}_{0,-1}$ and use then (2) to obtain (3). To prove (4) we write $\mathrm{e}^R = \sum_{n\geq 0} \frac{1}{n!} R^n$ and obtain that $\mathrm{e}^R \in \mathcal{S}^{0,0}$ if $R \in \mathcal{S}^{-\mu,-1}$. We have

$$\mathrm{e}^{R_1} - \mathrm{e}^{R_2} = \int_0^1 \mathrm{e}^{\theta R_1}(R_1 - R_2)\mathrm{e}^{(1-\theta)R_2} \mathrm{d}\theta$$

and obtain that $\mathrm{e}^{R_1} - \mathrm{e}^{R_2} \in \mathcal{S}^{-\mu-\epsilon,-1-j}$ if $R_1 - R_2 \in \mathcal{S}^{-\mu-\epsilon,-1-j}$, which completes the proof of (4). To prove (5) we write

$$\mathrm{e}^R = \mathbb{1} + R + \int_0^1 (1-\theta)R^2 \mathrm{e}^{\theta R} \mathrm{d}\theta =: 1 + R + F(R)$$

and obtain by (2), (3) and (4) that $F \in \mathcal{F}_{\boldsymbol{\mu},-2}$. □

5.3 Construction of some projections

We now follow the constructions in [17, Subsect. 6.4], adapting the results to our framework.

Proposition 5.9 *There exist time-dependent projections*

$$P^\pm(t) \in \mathcal{S}^0(\mathbb{R}; \Psi^0(\Sigma; L(S(\Sigma))))$$

and time-dependent operators

$$R(t) \in \mathcal{S}^{-1-\mu}(\mathbb{R}; \Psi^{-1}(\Sigma; L(S(\Sigma))))$$

such that

(1) $P^{\pm}(t) = P^{\pm}(t)^*$, $P^+(t) + P^-(t) = \mathbb{1}$;
(2) $P^{\pm}(t) - \mathbb{1}_{\mathbb{R}^{\pm}}(H_{\rm out/in}) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}, \Psi^0(\Sigma; L(S(\Sigma))))$;
(3) $R(t) = R^*(t)$;
(4) $\partial_t P^{\pm}(t) + [P^{\pm}(t), {\rm i}\tilde{H}(t)] \in \mathcal{S}^{-1-\mu}(\mathbb{R}; \Psi^{-\infty}(\Sigma; L(S(\Sigma))))$ *for*

$$\tilde{H}(t) = {\rm e}^{{\rm i}R(t)} H(t) {\rm e}^{-{\rm i}R(t)} + {\rm i}^{-1}\partial_t {\rm e}^{{\rm i}R(t)} {\rm e}^{-{\rm i}R(t)};$$

(5)

$$WF(U(\cdot, 0){\rm e}^{-{\rm i}R(0)} P^{\pm}(0){\rm e}^{{\rm i}R(0)})' \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^*\Sigma,$$

for $\mathcal{F} = \{k = 0\} \subset T^*M$.

Proof we follow the proof of [17, Prop.6.8], taking into account the time decay of the various operators.

Step1. In Step 1 we replace $\mathsf{H}(t)$ by $\hat{\mathsf{H}}(t) = \mathsf{H}(t) + R_{-\infty}(t)$ as in Lemma 5.4. Let $\hat{\mathsf{U}}(t, s)$ the unitary group with generator $\hat{\mathsf{H}}(t)$. From Lemma 5.5 and Duhamel's formula, we obtain that $\mathsf{U}(t, s) - \hat{\mathsf{U}}(t, s) \in C^{\infty}_{\rm b}(\mathbb{R}^2; \mathcal{W}^{-\infty}(\Sigma; L(S(\Sigma))))$ so we can replace $\mathsf{H}(t)$ by $\hat{\mathsf{H}}(t)$. Denoting $\hat{\mathsf{H}}(t)$ again by $\mathsf{H}(t)$ we can assume without loss of generality that $[-\delta, \delta] \cap \sigma(\mathsf{H}(t)) = \emptyset$ for $t \in \mathbb{R}$.

By Proposition 4.3 the projections

$$P^{\pm}(t) = \mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}(t))$$

are well defined, selfadjoint with

$$\begin{aligned} &P^{\pm}(t) \in \mathcal{S}^0(\mathbb{R}; \Psi^0(\Sigma; L(S(\Sigma)))), \\ &P^{\pm}(t) - \mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in}) \in \mathcal{S}^{-\mu}(\mathbb{R}^{\pm}; \Psi^0(\Sigma; L(S(\Sigma)))), \end{aligned} \tag{5.18}$$

so properties (1) and (2) are satisfied. We have also

$$\sigma_{\rm pr}(P^{\pm})(t, {\rm x}, k) = \mathbb{1}_{\mathbb{R}^{\pm}}(\sigma_{\rm pr}(H)(t, {\rm x}, k)). \tag{5.19}$$

Since $\sigma_{\rm pr}(\mathsf{H}(t, {\rm x}, k)) = -\tilde{\gamma}_0\tilde{\gamma}(\tilde{h}_t^{-1}({\rm x})k)$, we obtain using the Clifford relations that:

$$\sigma_{\rm pr}(P^{\pm})(t, {\rm x}, k)\sigma_{\rm pr}(\mathsf{H})(t, {\rm x}, k) = \pm\epsilon(t, {\rm x}, k)\sigma_{\rm pr}(P^{\pm})(t, {\rm x}, k),$$

for $\epsilon(t, x, k) = (k \cdot \tilde{h}_t^{-1}({\rm x})k)^{\frac{1}{2}}$. By symbolic calculus this implies that

$$P^{\pm}(t)\mathsf{H}(t) = \pm\epsilon(t, \mathrm{x}, D_x)P^{\pm}(t) + R_0^{\pm}(t), \tag{5.20}$$

where $R_0^{\pm}(t) \in \mathcal{S}^0(\mathbb{R}; \Psi^0(\Sigma; S(\Sigma)))$.

Step 2. In Step 2 we find $R(t)$ such that (4) is satisfied. For ease of notation we denote simply by A a time-dependent pseudodifferential operator $A(t)$. By Lemma 5.8 we obtain easily that for $\tilde{\mathsf{H}}(t)$ defined in (4) we have:

$$\tilde{\mathsf{H}} = \mathsf{H} + [R, \mathrm{i}\mathsf{H}] + F_{-\boldsymbol{\mu},-1}(R). \tag{5.21}$$

We will look for R of the form

$$R = T(S) = P^+SP^+ + P^-S^*P^-, \ \ S \in \mathcal{S}^{0,-1}. \tag{5.22}$$

Note that if $F_{-\boldsymbol{\delta},-p} \in \mathcal{F}_{-\boldsymbol{\delta},-p}$ then the map $S \mapsto F_{-\boldsymbol{\delta},-p}(T(S))$ belongs also to $\mathcal{F}_{-\boldsymbol{\delta},-p}$ (note that the map $S \mapsto S^*$ belongs to $\mathcal{F}_{0,-1}$).

Since $P^{\pm}$ are projections, we have

$$\begin{aligned}&\partial_t P^{\pm} + [P^{\pm}, \mathrm{i}\tilde{\mathsf{H}}]\\ &\quad = P^+(\partial_t P^{\pm} + [P^{\pm}, \mathrm{i}\tilde{\mathsf{H}}])P^- + P^-(\partial_t P^{\pm} + [P^{\pm}, \mathrm{i}\tilde{\mathsf{H}}])P^+.\end{aligned}$$

Since the second term in the rhs above is the adjoint of the first, it suffices to find S such that

$$P^+(\partial_t P^+ + [P^+, \mathrm{i}\tilde{\mathsf{H}}])P^- \in \mathcal{S}^{-1-\mu,-\infty}. \tag{5.23}$$

Using (5.21), we obtain since $[P^{\pm}, \mathsf{H}] = 0$:

$$\begin{aligned}&P^+\left(\partial_t P^+ + [P^+, \mathrm{i}\tilde{\mathsf{H}}]\right)P^-\\ &\quad = P^+\left(\partial_t P^+ + P^+\mathsf{H}P^+S - SP^-\mathsf{H}P^- + F_{-\boldsymbol{\mu},-1}(S)\right)P^-\end{aligned}$$

We use now (5.20) denoting the scalar operator $\epsilon(t, \mathrm{x}, D_x) + m^2$ for $m \gg 1$ simply by ϵ and obtain:

$$\begin{aligned}&P^+\mathsf{H}P^+S - SP^-\mathsf{H}P^- = \epsilon S + S\epsilon + R_0^+S - SR_0^-\\ &\quad = 2\epsilon S + [S, \epsilon] + R_0^+S - SR_0^-.\end{aligned}$$

The maps $S \mapsto R_0^+S$, $S \mapsto SR_0^-$ belong to $\mathcal{F}_{0,-1}$ by Lemma 5.8, as the map $S \mapsto [\epsilon, S]$, since ϵ is scalar.

Therefore, Eq. (5.23) can be rewritten as

$$\partial_t P^+ + 2\epsilon S + F_{0,-1}(S) \in \mathcal{S}^{-1-\mu,-\infty},$$

or equivalently as

$$S + (2\epsilon)^{-1}\partial_t P^+ + -F_{0,-2}(S) \in \mathcal{S}^{-1-\mu,-\infty}. \tag{5.24}$$

where $F_{0,-2} : S \mapsto -(2\epsilon)^{-1} F_{0,-1}(S)$ belongs to $\mathcal{F}_{0,-2}$. We apply Proposition 5.7 to solve (5.24). We note that $-(2\epsilon)^{-1}\partial_t P^+ \in \mathcal{S}^{-1-\mu,-1}$ and we find $S \in \mathcal{S}^{-1-\mu,-1}$ such that

$$\partial_t P^+ + 2\epsilon S + F_{0,-1}(S) \in \mathcal{S}^{-1-\mu,-\infty}$$

and hence

$$\partial_t P^+ + [P^+, \mathrm{i}\tilde{\mathsf{H}}] = R_{-\infty} \in \mathcal{S}^{-1-\mu,-\infty}.$$

We have hence proved (4). Finally, (5) is proved exactly as in [17, Prop. 6.8]. □

5.4 The in/out vacua for *D*

In this subsection we construct the in/out vacua and prove their Hadamard property for D, $\tilde{D}$ and finally for the original Dirac operator D.

We recall from 2.1.7 that $U(t,s)$ is the Cauchy evolution for D associated to the foliation $(\Sigma_t)_{t\in\mathbb{R}}$. We denote by $L^2(\Sigma_t; S(\Sigma))$ the completion of $C_0^\infty(\Sigma_t; S(\Sigma))$ for the scalar product

$$\overline{f}\cdot\nu_t f = \mathrm{i}\int_{\Sigma_t} \overline{f}\cdot\beta\gamma(n)|h_t|^{\frac{1}{2}}\mathrm{d}x.$$

By the facts recalled in 2.3.3 $U(t,s) : L^2(\Sigma_s; S(\Sigma)) \to L^2(\Sigma_t; S(\Sigma))$ is unitary.

We denote by $\tilde{U}(t,s)$ the analogous Cauchy evolution for $\tilde{D}$. By (2.12) we have:

$$U(t,s) = c^{\frac{1-n}{2}}\tilde{U}(t,s)c^{\frac{n-1}{2}}. \tag{5.25}$$

From the definition (5.12) of $\tilde{H}_{\pm\infty}$ we obtain that the asymptotic Dirac operator $\tilde{D}_{\mathrm{out/in}}$ associated to the ultra-static metric $\tilde{g}_{\mathrm{out/in}}$ equals

$$\tilde{D}_{\mathrm{out/in}} = -\tilde{\gamma}_{\mathrm{out/in}}(\tilde{e}_0)(\partial_t - \mathrm{i}\tilde{H}_{\mathrm{out/in}}).$$

Recalling that $D_{\mathrm{out/in}}$ are the asymptotic Dirac operators associated to $g_{\mathrm{out/in}}$, we have as in Sect. 2.4:

$$D_{\mathrm{out/in}} = -c_{\mathrm{out/in}}^{-1}\gamma_{\pm\infty}(e_0)(\partial_t - \mathrm{i}H_{\mathrm{out/in}}),$$

and

$$H_{\mathrm{out/in}} = c_{\mathrm{out/in}}^{\frac{1-n}{2}}\tilde{H}_{\mathrm{out/in}}c_{\mathrm{out/in}}^{\frac{n-1}{2}}. \tag{5.26}$$

We first consider the operator D in 5.1.2.

Proposition 5.10 *Assume hypotheses* (Hi), $1 \le i \le 4$. *Then:*

(1) *the norm limits:*

$$P^{\pm}_{\rm out/in} = \lim_{t\to\pm\infty} U(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\rm out/in})U(t,0) \textit{ exist}.$$

(2) $P^{\pm}_{\rm out/in}$ *are selfadjoint projections for* $\tilde{\nu}_0$ *with* $P^{+}_{\rm out/in} + P^{-}_{\rm out/in} = \mathbb{1}$.

(3)

$$\mathrm{WF}(U(\cdot,0)P^{\pm}_{\rm out/in})' \subset (\mathcal{N}^{\pm}\cup\mathcal{F})\times T^*\Sigma$$

for $\mathcal{F} = \{k=0\}\subset T^*M$.

Proof Let $P^{\pm}(t)$, $R(t)$ be the operators constructed in Proposition 5.9. Setting

$$\tilde{P}^{\pm}(t) := \mathrm{e}^{-\mathrm{i}R(t)}P^{\pm}(t)\mathrm{e}^{\mathrm{i}R(t)},\ \tilde{\mathsf{U}}(t,s) := \mathrm{e}^{\mathrm{i}R(t)}\mathsf{U}(t,s)\mathrm{e}^{-\mathrm{i}R(s)},$$

we see that $\tilde{\mathsf{U}}(t,s)$ is a strongly continuous unitary group with generator

$$\tilde{\mathsf{H}}(t) = \mathrm{e}^{\mathrm{i}R(t)}\mathsf{H}(t)\mathrm{e}^{-\mathrm{i}R(t)} + \mathrm{i}^{-1}\partial_t\mathrm{e}^{\mathrm{i}R(t)}\mathrm{e}^{-\mathrm{i}R(t)}.$$

Since $P^{\pm}(t) - \mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in})$ and $R(t)$ are $O(t^{-\mu})$ in norm, we have $\tilde{P}^{\pm}(t) - \mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in}) \in O(t^{-\mu})$ and hence

$$\mathsf{U}(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in})\mathsf{U}(t,0) = \mathsf{U}(0,t)\tilde{P}^{\pm}(t)\mathsf{U}(t,0) + O(t^{-\mu}). \tag{5.27}$$

Next

$$\begin{aligned}\mathsf{U}(0,t)\tilde{P}^{\pm}(t)\mathsf{U}(t,0) &= \mathrm{e}^{-\mathrm{i}R(0)}\tilde{\mathsf{U}}(0,t)\mathrm{e}^{\mathrm{i}R(t)}\tilde{P}^{\pm}(t)\mathrm{e}^{-\mathrm{i}R(t)}\tilde{\mathsf{U}}(t,0)\mathrm{e}^{\mathrm{i}R(0)}\\ &= \mathrm{e}^{-\mathrm{i}R(0)}\tilde{\mathsf{U}}(0,t)P^{\pm}(t)\tilde{\mathsf{U}}(t,0)\mathrm{e}^{\mathrm{i}R(0)},\end{aligned}$$

and

$$\begin{aligned}&\partial_t\left(\tilde{\mathsf{U}}(0,t)P^{\pm}(t)\tilde{\mathsf{U}}(t,0)\right)\\ &= \tilde{\mathsf{U}}(0,t)\left(\partial_t P^{\pm}(t) + [P^{\pm}(t),\mathrm{i}\tilde{\mathsf{H}}(t)]\right)\tilde{\mathsf{U}}(t,0) = \tilde{\mathsf{U}}(0,t)R_{-\infty}(t)\tilde{\mathsf{U}}(t,0),\end{aligned} \tag{5.28}$$

where $R_{-\infty}(t)\in\mathcal{S}^{-1-\mu,-\infty}$, by Proposition 5.9. Therefore, by (5.27), (5.28) the limit

$$\mathsf{P}^{\pm}_{\rm out/in} = \lim_{t\to\pm\infty}\mathsf{U}(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in})\mathsf{U}(t,0) \text{ exists}$$

and

$$\mathsf{P}^{\pm}_{\rm out/in} = \tilde{P}^{\pm}(0) \pm \int_0^{\pm\infty}\tilde{\mathsf{U}}(0,t)R_{-\infty}(t)\tilde{\mathsf{U}}(t,0)\mathrm{d}t = \tilde{P}^{\pm}(0) + R_{\pm\infty},$$

where $R_{\pm\infty} \in \Psi^{-\infty}$, using the uniform estimates in Lemma 5.5 . $\mathsf{P}^{\pm}_{\rm out/in}$ are clearly selfadjoint projections for $\tilde{\nu}_0$ with $\mathsf{P}^{+}_{\rm out/in}(0) + \mathsf{P}^{-}_{\rm out/in}(0) = \mathbb{1}$.

From Proposition 5.9 (5) and the fact that $\mathsf{P}^{\pm}_{\rm out/in} - \tilde{P}^{+}(0)$ is a smoothing operator we obtain that $\mathrm{WF}(\mathsf{U}(\cdot, 0)\mathsf{P}^{\pm}_{\rm out/in})' \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^*\Sigma$. □

Next we consider the Dirac operator $\tilde{D}$.

Proposition 5.11 *Assume hypotheses* (Hi), $1 \leq i \leq 4$. *Then*

(1) *the norm limits*

$$\tilde{P}^{\pm}_{\rm out/in} = \lim_{t\to\pm\infty} \tilde{U}(0, t)\mathbb{1}_{\mathbb{R}^{\pm}}(\tilde{H}_{\pm\infty})\tilde{U}(t, 0) \text{ exist.}$$

(2) $\tilde{P}^{\pm}_{\rm out/in}$ *are selfadjoint projections for the scalar product* $\tilde{\nu}_0$ *with* $\tilde{P}^{+}_{\rm out/in} + \tilde{P}^{+}_{\rm out/in} = \mathbb{1}$.

(3)

$$\mathrm{WF}(\tilde{U}(\cdot, 0)\tilde{P}^{\pm}_{\rm out/in})' \subset (\mathcal{N}^{\pm} \cup \mathcal{F}) \times T^*\Sigma.$$

Proof We obtain easily from (5.8) that

$$\mathsf{U}(t, s) = \mathcal{T}(0, t)|\tilde{h}|_0^{-\frac{1}{4}}|\tilde{h}|_t^{\frac{1}{4}}\tilde{U}(t, s)|\tilde{h}|_0^{\frac{1}{4}}|\tilde{h}|_s^{-\frac{1}{4}}\mathcal{T}(s, 0). \tag{5.29}$$

This implies that

$$\begin{aligned} &\mathsf{U}(0, t)\mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in})\mathsf{U}(t, 0) \\ &= \tilde{U}(0, t)|\tilde{h}|_0^{\frac{1}{4}}|\tilde{h}|_t^{-\frac{1}{4}}\mathcal{T}(t, 0)\mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in})\mathcal{T}(0, t)|\tilde{h}|_0^{-\frac{1}{4}}|\tilde{h}|_t^{\frac{1}{4}}\tilde{U}(t, 0). \end{aligned}$$

By (5.13) $\mathcal{T}(0, t) - \mathcal{T}(0, \pm\infty) \in O(t^{-\mu})$ in norm and $|h|_t - |h|_{\rm out/in} \in O(t^{-\mu})$, hence

$$\begin{aligned} &\mathsf{U}(0, t)\mathbb{1}_{\mathbb{R}^{\pm}}(\mathsf{H}_{\rm out/in})\mathsf{U}(t, 0) \\ &= \tilde{U}(0, t)|\tilde{h}|_0^{\frac{1}{4}}|\tilde{h}|_{\rm out/in}^{-\frac{1}{4}}\mathcal{T}(\pm\infty, 0)\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\rm out/in})\mathcal{T}(0, \pm\infty)|\tilde{h}|_0^{-\frac{1}{4}}|\tilde{h}|_{\rm out/in}^{\frac{1}{4}} \\ &\tilde{U}(t, 0) + O(t^{-\mu}) = \tilde{U}(0, t)\mathbb{1}_{\mathbb{R}^{\pm}}(\tilde{H}_{\rm out/in})\tilde{U}(t, 0) + O(t^{-\mu}). \end{aligned}$$

where in the last line we use (5.14) and the fact that $\mathcal{T}(0, \pm\infty)|\tilde{h}|_0^{-\frac{1}{4}}|\tilde{h}|_{\rm out/in}^{\frac{1}{4}}$ is unitary for the scalar products $\tilde{\nu}_{\rm out/in}$ and $\tilde{\nu}_0$.

Therefore, the norm limit in (1) exist and equal the projections $\mathsf{P}^{\pm}_{\rm out/in}$ in Proposition 5.10. This also implies (2). (3) follows from (5.29) and the analogous statement in Proposition 5.10. □

Finally, we prove the main result of this paper.

Theorem 5.12 *Assume hypotheses* (Hi), $1 \leq i \leq 4$. *Then:*

(1) *the norm limits*

$$P^{\pm}_{\mathrm{out/in}} = \lim_{t\to\pm\infty} U(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\mathrm{out/in}})U(t,0) \textit{ exist.}$$

(2) *if*

$$\lambda^{\pm}_{\mathrm{out/in}} = \mathrm{i}\gamma(n)P^{\pm}_{\mathrm{out/in}}$$

$\lambda^{\pm}_{\mathrm{out/in}}$ *are the Cauchy surface covariances of a pure Hadamard state for* D $\omega_{\mathrm{out/in}}$ *called the* out/in vacuum state.

Proof Let us denote by c_t the restriction of the conformal factor c to Σ_t. From (2.12) we obtain that

$$U(t,s) = c_t^{\frac{1-n}{2}}\tilde{U}(t,s)c_s^{\frac{n-1}{2}},\ t,s\in\mathbb{R},$$

and hence

$$\begin{aligned}
&U(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\mathrm{out/in}})U(t,0)\\
&= c_0^{\frac{1-n}{2}}\tilde{U}(0,t)c_t^{\frac{n-1}{2}}\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\mathrm{out/in}})c_t^{\frac{1-n}{2}}\tilde{U}(t,0)c_0^{\frac{n-1}{2}}\\
&= c_0^{\frac{1-n}{2}}\tilde{U}(0,t)c_{\mathrm{out/in}}^{\frac{n-1}{2}}\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\mathrm{out/in}})c_{\mathrm{out/in}}^{\frac{1-n}{2}}\tilde{U}(t,0)c_0^{\frac{n-1}{2}} + O(t^{-\mu})\\
&= c_0^{\frac{1-n}{2}}\tilde{U}(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(\tilde{H}_{\mathrm{out/in}})\tilde{U}(t,0)c_0^{\frac{n-1}{2}} + O(t^{-\mu})
\end{aligned}$$

since $H_{\mathrm{out/in}} = c_{\mathrm{out/in}}^{\frac{1-n}{2}}\tilde{H}_{\mathrm{out/in}}c_{\mathrm{out/in}}^{\frac{n-1}{2}}$. By Proposition 5.11 we obtain that

$$P^{\pm}_{\mathrm{out/in}} = \lim_{t\to\pm\infty} U(0,t)\mathbb{1}_{\mathbb{R}^{\pm}}(H_{\mathrm{out/in}})U(t,0) = c_0^{\frac{1-n}{2}}\tilde{P}^{\pm}_{\mathrm{out/in}}c_0^{\frac{n-1}{2}}. \tag{5.30}$$

By Proposition 2.5 we obtain that $(c_0^{\frac{n-1}{2}})^*\tilde{\nu}_0 c_0^{\frac{n-1}{2}} = \nu_0$. By Proposition 5.11 $P^{\pm}_{\mathrm{out/in}}$ are hence selfadjoint projections for ν_0 with $P^{+}_{\mathrm{out/in}} + P^{-}_{\mathrm{out/in}} = \mathbb{1}$. Therefore, $\lambda^{\pm}_{\mathrm{out/in}}$ are the Cauchy surface covariances of pure Hadamard states $\omega_{\mathrm{out/in}}$ for D.

Finally, (5.30) and Proposition 5.11 (3) imply that $\mathrm{WF}(U(\cdot,0)P^{\pm}_{\mathrm{out/in}})' \subset (\mathcal{N}^{\pm}\cup\mathcal{F})\times T^*\Sigma$. Since $\mathcal{F}\cap\mathcal{N} = \emptyset$, we obtain by Proposition 2.4 that $\omega_{\mathrm{out/in}}$ are Hadamard states. □

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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