Letters in Mathematical Physics (2022) 112:22
https://doi.org/10.1007/s11005-022-01517-2

Remarks on the geodesically completeness and the smoothing effect on asymptotically Minkowski spacetimes

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Received: 19 August 2021 / Revised: 17 January 2022 / Accepted: 9 February 2022 /
Published online: 8 March 2022

Abstract

In this note, we study a geometric property of asymptotically Minkowski spacetimes and an analytic property of the wave operator. More precisely, our first main results show that asymptotically Minkowski spacetimes are geodesically complete under a null non-trapping condition. Secondly, we prove that Sobolev index of a real principal type estimate used in a previous work is optimal.

Keywords Geodesic completeness · Smoothing effects · Self-adjointness

Mathematics Subject Classification 35L05 · 53C22

1 Introduction

Let g_0 be the Minkowski metric on $\mathbb{R}^{n+1}$ and g_0^{-1} be its dual metric:

$$g_0 = -dx_1^2 + dx_2^2 + ... + dx_{n+1}^2 = (g_{0,jk})_{j,k=1}^{n+1},$$
$$g_0^{-1} = -\partial_{x_1}^2 + \partial_{x_2}^2 + ... + \partial_{x_{n+1}}^2 = (g_0^{jk})_{j,k=1}^{n+1},$$

where we denote

$$x = (t, y) \in \mathbb{R} \times \mathbb{R}^n,$$

that is, t is the time variable and y is the space variable in the spacetime. We denote $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and introduce the function space

$$S_k(\mathbb{R}^{n+1}) := \{a \in C^\infty(\mathbb{R}^{n+1}) \mid |\partial_x^\alpha a(x)| \leq C_\alpha \langle x \rangle^{k-|\alpha|}\}, \qquad k \in \mathbb{R}.$$

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Definition 1 A Lorentzian metric g on $\mathbb{R}^{n+1}$ is called asymptotically Minkowski if the inverse matrix $g(x)^{-1} = (g^{jk}(x))_{j,k=1}^{n+1}$ of $g(x)$ satisfies $g^{jk} - g_0^{jk} \in S_{-\mu}(\mathbb{R}^{n+1})$ for some $\mu > 0$ and for all $j, k = 1, \ldots, n+1$.

Remark 1.1 A Lorentzian metric $g(x) = (g_{jk}(x))_{j,k=1}^{n+1}$ is asymptotically Minkowski if and only if $g_{jk} - g_{0,jk} \in S_{-\mu}(\mathbb{R}^{n+1})$ for some $\mu > 0$ and for all $j, k = 1, \ldots, n+1$. This definition seems more natural than Definition 1, where we impose a condition on the inverse matrix $g(x)^{-1}$. However, in studies of PDE, Definition 1 is more useful since the principal symbol of the wave operator $-\Box_g$ explicitly depends on the inverse matrix $g(x)^{-1}$.

The Feynman propagator, which is an inverse of $-\Box_g$ and satisfies a certain wavefront condition, is a fundamental object in quantum field theory. In [5, 6, 9, 10] and [20], the Feynman propagator is constructed on various spacetimes including asymptotically Minkowski spacetimes. In [19], it is proved that on asymptotically Minkowski spacetimes, the (anti-)Feynman propagator constructed in [9] and [10] coincides with the outgoing resolvent of the wave operator. Such an identity should hold since, for the exact Minkowski spacetime, the Feynman propagator is defined by the outgoing resolvent in the physics literature. See also the review article [11].

In this short note, we give supplementary results on the geometry and the property of the wave operator on asymptotically Minkowski spacetimes: One is a result on the completeness of asymptotically Minkowski spacetimes and the other is a result on the optimality of the local smoothing estimate used in [19]. As is stated below, the both results are closely related to essential self-adjointnness of the wave operator $-\Box_g$ although they are results from different view points. These hopefully give a clue to solve the conjecture by Dereziński–Siemssen [7,§1.7], which states that wave operators on various spacetimes are essentially self-adjoint.

1.1 First result

It is a classical question how the essential self-adjointness of pseudodifferential operators is related to the completeness of the associated Hamilton flow ([17,Vol II], [2]). In the case of the Laplace operator P on a semi-Riemannian manifold M, this corresponds to a relation between the essential self-adjointness of P and the geodesic completeness of M. It is well known that on a geodesically complete Riemannian manifold, the Laplace operator is essentially self-adjoint on C_c^∞. In [2] and references therein, such a relation is studied for more general Schrödinger-type operators on complete Riemmanian manifolds.

In [3], it is shown that the completeness of the Hamilton flow is equivalent to the essential self-adjointness on generic closed Lorentzian surfaces. The authors in [3] conjecture that the completeness of the Hamilton flow implies the essential self-adjointness for non-elliptic operators on closed manifolds. In [19], this conjecture is solved for general real principal type operators on the one-dimensional torus. As a related result, in [13], the author gives an example of a Lorentzian manifold (M, g), which is geodesically complete and globally hyperbolic, but the wave operator associated with g is not essentially self-adjoint.

Recently, it was shown in [20] and [16] that the wave operator on an asymptotically Minkowski spacetime is essentially self-adjoint on C_c^∞ under a null non-trapping condition. However, on this manifold, a relation to the geodesic completeness was not revealed. In this note, we show that a null non-trapping condition implies the completeness on this manifold, including timelike and spacelike completeness.

We introduce a non-trapping condition, which is a bit weaker than the condition in [9, 10, 16] and [19] in the sense that the completeness of the null geodesics is not assumed here.

Definition 2 [1,Definition 11.17] Let (M, g) be a Lorentzian manifold and $T_0, T_1 \in [-\infty, \infty]$ with $T_0 < T_1$. We say that a maximally extended geodesic $\gamma : (T_0, T_1) \to M$ is forward (resp. backward) non-trapping if for each $t_0 \in (-T_0, T_1)$, $\gamma|_{[t_0,T_1)}$ (resp. $\gamma|_{(-T_0,t_0]}$) fails to have compact closure. We way that γ is non-trapping if γ is both forward and backward non-trapping. The Lorentzian manifold (M, g) is called null non-trapping (or null disprisoning) if every non-constant maximally extended null geodesic is non-trapping.

Theorem 1.2 *Suppose* $(\mathbb{R}^{n+1}, g)$ *is asymptotically Minkowski and null non-trapping in the sense of Definition 2. Then* $(\mathbb{R}^{n+1}, g)$ *is geodesically complete.*

1.2 Second result

Next, we consider the exact Minkowski spacetime $(\mathbb{R}^{n+1}, g_0)$ and the operator

$$P = \partial_t^2 - \Delta_y.$$

It is well known that P is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{n+1})$. We denote the unique self-adjoint extension of P by the same symbol. In [19,Proposition 3.2], it is proved that the resolvent $(P-i)^{-1}$ is a continuous map from $L^2(\mathbb{R}^{n+1})$ to $\langle x\rangle^{\frac{1}{2}+\varepsilon} H^{\frac{1}{2}}(\mathbb{R}^{n+1})$:

$$(P-i)^{-1} : L^2(\mathbb{R}^{n+1}) \to \langle x\rangle^{\frac{1}{2}+\varepsilon} H^{\frac{1}{2}}(\mathbb{R}^{n+1}), \quad \varepsilon > 0, \tag{1.1}$$

where the space $H^k(\mathbb{R}^{n+1})$ is the usual Sobolev space. This mapping property plays a crucial role for the proof of the limiting absorption principle in [19] on asymptotically Minkowski spacetimes. On the other hand, the radial estimate ([4,Propositions A.3, A.4], [19,Theorem A.3], [20,(2), (3)]) and the propagation of singularities imply that the resolvent $(P-i)^{-1}$ is a continuous map from : $\langle x\rangle^{-\frac{1}{2}-\varepsilon} L^2(\mathbb{R}^{n+1})$ to $\langle x\rangle^{\frac{1}{2}+\varepsilon} H^1(\mathbb{R}^{n+1})$

$$(P-i)^{-1} : \langle x\rangle^{-\frac{1}{2}-\varepsilon} L^2(\mathbb{R}^{n+1}) \to \langle x\rangle^{\frac{1}{2}+\varepsilon} H^1(\mathbb{R}^{n+1}), \quad \varepsilon > 0. \tag{1.2}$$

Namely, (1.2) states that if f has an additional decay (compared with the case (1.1)), then its image $(P-i)^{-1}f$ gains a better regularity. So a natural question is whether the Sobolev index $\frac{1}{2}$ in (1.1) is optimal or not. In this note, we give an affirmative answer to this question.

Theorem 1.3 *For the Minkowski spacetime* $(\mathbb{R}^{n+1}, g_0)$*, for all* $\varepsilon > 0$*, there exists* $f \in L^2(\mathbb{R}^{n+1})$ *such that* $(P - i)^{-1} f \notin H^{\frac{1}{2}+\varepsilon}_{loc}(\mathbb{R}^{n+1})$.

Usual real principal type estimates are used for proving existence of a local solution ([12,Theorem 26.1.7]). Estimates like (1.1) and (1.2) might be regarded as global versions of the real principal type estimates. Theorem (1.3) states that a global analogue of the usual real principal type estimates with Sobolev index 1 does not hold if we impose $f \in L^2(\mathbb{R}^{n+1})$ only. We note that the Sobolev index $\frac{1}{2}$ in (1.1) is the same as in the local smoothing effects for time-dependent Schrödinger equations (see also [19,§1.2]).

Regularity estimates such as (1.1) and (1.2) are also related to essential self-adjoiness of differential operators. Indeed, the argument in [16,before Proposition 3.1] implies that in order to prove essential self-adjointness of the wave operator, it suffices to deduce that this operator has a kind of regularity. On the other hand, the results in [18,proof of Theorem 5.1] show that essential self-adjointness of a differential operator may break due to the existence of its eigenfunction, which is not smooth enough. Hence, it seems important to know the regularity of differential operators quantitatively. Theorem 1.3 shows that it strongly depends on weight functions of function spaces in non-compact spaces.

1.3 Organization and notation

This paper is organized as follows: In Sect. 2, the proof of the geodesic completeness is given. In Sect. 3, we show the optimality of the mapping property (1.1). Moreover, in Appendix, we give a short proof of (1.1) and (1.2) on the exact Minkowski spacetime.

We fix some notations. We use the Sobolev spaces: $H^k(\mathbb{R}^{n+1}) = \langle D\rangle^{-k} L^2(\mathbb{R}^{n+1})$ for $k \in \mathbb{R}$. We write the variable on $T^*\mathbb{R}^{n+1}$ by (x, ξ) with $x = (t, y) \in \mathbb{R} \times \mathbb{R}^n$ and $\xi = (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^n$. We denote the Fourier transform on $\mathbb{R}^m$ by $\mathcal{F} f(y) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} e^{-iy\cdot\eta} f(y) dy$. For a Banach space X, we denote the norm of X by $\|\cdot\|_X$. If X is a Hilbert space, we write the inner product of X by $(\cdot,\cdot)_X$, where $(\cdot,\cdot)_X$ is linear with respect to the right variable. For a Lorentzian manifold (M, g), we say that $v \in T_p M$ is a null vector if $g_p(v, v) = 0$.

2 Geodesic completeness of asymptotically Minkowski spacetimes

2.1 Completeness of trapped geodesics

In this subsection, we discuss the completeness of non-null trapped geodesics. The following lemma is possibly well known; however, the author could not find a suitable reference. Hence, we give a proof here.

Lemma 2.1 *Suppose that a Lorentzian manifold* (M, g) *is null non-trapping and satisfies the second axiom of countability. Let* $\gamma : (a, b) \to M$ *be a maximally extended* *forward (resp. backward) trapped geodesic, satisfying*

$$g(\gamma'(t), \gamma'(t)) \neq 0 \quad \text{for all} \quad t \in (a, b),$$

that is, γ is not null. Then, γ is forward complete (resp. backward complete) in the sense that $b = \infty$ (resp. $a = -\infty$).

Proof We consider the forward case only. Since M satisfies the second axiom of countability, there exists a complete Riemannian metric on M (see [15]). We fix a complete Riemannian metric h on M. Let

$$SM = \{(x, v) \in TM \mid h(v, v) = 1\}$$

t be the sphere bundle associated with h. Since γ is trapped, there exists $b \in \mathbb{R}$ such that $\gamma([0, b)) \subset K$ for a compact set $K \subset M$.

Set

$$v(t) = \frac{1}{h(\gamma'(t), \gamma'(t))^{\frac{1}{2}}}\gamma'(t) \in S_{\gamma(t)}M.$$

Let φ_t be the geodesic flow on TM and $\pi : TM \to M$ be the natural projection. Since $(\gamma(t), v(t)) \in SM|_K := SM \cap \pi^{-1}(K)$ and $SM|_K$ is compact, there exist a sequence $\{t_i\}_i$ and $(q, v) \in SM|_K$ such that $t_i \to b$ and $(\gamma(t_i), v(t_i)) \to (q, v) \in SM$ as $i \to \infty$. We shall show that v is not null. We suppose v is a null vector. Then, the null non-trapping condition implies the existence of $T > 0$ such that $\varphi_t(q, v)$ exists for $0 \leq t \leq T$ and $\varphi_T(q, v) \notin SM|_K$. By the continuous dependence of initial data of ODE, we have $\varphi_T(\gamma(t_i), v(t_i)) \to \varphi_T(q, v)$ as $i \to \infty$. In particular, $\varphi_T(\gamma(t_i), v(t_i)) \notin SM|_K$. This contradicts

$$\varphi_T(\gamma(t_i), v(t_i)) = \varphi_{\frac{T}{h(\gamma'(t),\gamma'(t))^{\frac{1}{2}}}}(\gamma(t_i), \gamma'(t_i)) \in SM|_K.$$

Thus, it follows that v is a not null vector.

Suppose γ is not forward complete. Then, we have $b < \infty$ and

$$0 \neq g(v, v) = \lim_{i\to\infty} g(v(t_i), v(t_i)) = \lim_{i\to\infty} \frac{g(\gamma'(t_i), \gamma'(t_i))}{h(\gamma'(t_i), \gamma'(t_i))}.$$

Notice that $g(\gamma'(t_i), \gamma'(t_i))$ is nonzero constant. Thus, we have

$$\lim_{i\to\infty} h(\gamma'(t_i), \gamma'(t_i)) = \frac{g(\gamma'(t_i), \gamma'(t_i))}{g(v, v)} < \infty$$

Since $\gamma([0, b)) \subset K$ and the Riemannian metric h is complete, it contradicts $b < \infty$ and [8,Proposition 2.1]. ◻

2.2 Completeness of non-trapping orbits

In the rest of this section, we assume g is an asymptotically Minkowski metric on $\mathbb{R}^{n+1}$. We set $p(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} g^{ij}(x)\xi_i\xi_j$ for $(x,\xi) \in T^*\mathbb{R}^{n+1}$. Let $(y(t,x,\xi), \eta(t,x,\xi))$ denote the solution to the Hamilton equations:

$$\begin{cases} \frac{d}{dt}y(t,x,\xi) = \partial_\xi p(y(t,x,\xi), \eta(t,x,\xi)), \\ \frac{d}{dt}\eta(t,x,\xi) = -\partial_x p(y(t,x,\xi), \eta(t,x,\xi)), \end{cases} \qquad \begin{cases} y(0,x,\xi) = x, \\ \eta(0,x,\xi) = \xi. \end{cases} \tag{2.1}$$

It is well known that $t \mapsto y(t,x,\xi)$ is a geodesic on the Lorentzian manifold (M,g) with the initial value x and the initial velocity $g(x)^{-1}\xi$. Moreover, we denote the Hamiltonian vector field H_p of the function $p \in C^\infty(T^*\mathbb{R}^{n+1}; \mathbb{R})$ by

$$H_p = \sum_{j=1}^{n+1} \left(\partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}\right)$$

The following lemma follows from a direct calculation. See [16,Lemma A.1] for a proof.

Lemma 2.2 *There exist $M > 0$ and $R_0 > 1$ such that*

$$H_p^2(|x|^2) \geq M|\xi|^2$$

for any $(x,\xi) \in \{(y,\eta) \in T^*\mathbb{R}^n \mid |y| > R_0,\ |\eta| \neq 0\}$.

We also mention a result on an extension of solutions to the Hamilton equation.

Lemma 2.3 *Let $(x,\xi) \in T^*\mathbb{R}^{n+1}$. Suppose that the solution $(y(t,x,\xi), \eta(t,x,\xi))$ to (2.1) exists for a time interval (T_0, T_1) and that there exists $C > 0$ such that*

$$|\eta(t,x,\xi)| \leq C \quad \textit{for} \quad t \in (T_0, T_1).$$

Then, the solution $(y(t,x,\xi), \eta(t,x,\xi))$ can be extended to a time interval beyond (T_0, T_1).

Proof We note $|\partial_\xi p(x,\xi)| \leq C'|\xi|$ with a constant $C' > 0$. By the assumption and the Hamilton equation, we have $|y(t,x,\xi)| \leq |x| + CC'|t|$. Thus, $(y(t,x,\xi), \eta(t,x,\xi))$ stays in a fixed compact set for $t \in (T_0, T_1)$. The standard theory of ODE gives our conclusion. □

Now we show that non-trapping orbits are complete on the asymptotically Minkowski spacetimes. The following proposition is proved by a slight modification of the proof in [16], which follows the strategy in [14].

Proposition 2.4 *Fix $(x_0, \xi_0) \in T^*\mathbb{R}^n$ with $\xi_0 \neq 0$ and suppose that (x_0, ξ_0) is forward (resp. backward) non-trapping in the sense that there exists $T \in (0, \infty]$ (resp. $[-\infty.0)$) such that*

$$\lim_{t\to T, t<T} |y(t, x_0, \xi_0)| = \infty \quad (\textit{resp.} \quad \lim_{t\to T, t>T} |y(t, x_0, \xi_0)| = \infty)$$

Then, there exist $C_1, C_2 > 0$ such that

$$C_1 \leq |\eta(t, x_0, \xi_0)| \leq C_2 \quad \textit{for} \quad 0 \leq t < T \quad (\textit{resp.} -T < t \leq 0).$$

Moreover, it follows that $|T| = \infty$ and that the orbit $(y(t, x_0, \xi_0), \eta(t, x_0, \xi_0))$ is forward (resp. backward) complete.

Proof We consider the forward case only. We write $y(t) = y(t, x_0, \xi_0)$ and $\eta(t) = \eta(t, x_0, \xi_0)$. Let R_0 be as in Lemma 2.2, and we let $R_1 \geq R_0$ which is determined later. We first note that by the forward non-trapping condition and Lemma 2.2, there exits $0 \leq t_0 < T$ such that for $t_0 \leq t < T$, we have

$$|y(t)| \geq R_1, \quad \frac{d}{dt}|y(t)|^2 \geq 0. \tag{2.2}$$

Indeed, it is easy to see that there are $0 < s_0 < t_0 < T$ such that $\frac{d^2}{dt^2}|y(t)|^2 > 0$ for $t \geq s_0$, and $\frac{d}{dt}|y(t)|^2|_{t=t_0} > 0$. Then for all $t_0 \leq t < T$, the condition (2.2) is satisfied.

Take a constant $C_0 > 0$ such that

$$|\partial_x p(x, \xi)| \leq C_0|x|^{-1-\mu}|\xi|^2 \quad \text{for} \quad |x| \geq 1.$$

We write $\eta_0 = |\eta(t_0)|$ and $T_1 := \sup\{s \in [t_0, T) \mid \frac{1}{2}\eta_0 \leq |\eta(t)|\}$. By Lemma 2.2 and (2.2), we have

$$|y(t)|^2 \geq R_1^2 + \frac{M\eta_0^2}{8}(t - t_0)^2 \quad \text{for} \quad t_0 \leq t < T_1.$$

Since $R_1 \geq 1$, the Hamilton equation gives $|\eta'(t)| \leq C_0|y(t)|^{-1-\mu}|\eta(t)|^2$ for $t_0 \leq t < T_1$ and hence

$$|\frac{d}{dt}|\eta(t)|^{-1}| \leq C_0 \left(R_1^2 + \frac{M\eta_0^2}{8}(t - t_0)^2\right)^{-\frac{1+\mu}{2}} \quad \text{for} \quad t_0 \leq t < T_1.$$

Thus, for $t_0 \leq t < T_1$, we obtain

$$|\eta_0^{-1} - |\eta(t)|^{-1}| \leq C_0 \int_{t_0}^{t} \left(R_1^2 + \frac{M\eta_0^2}{8}(s - t_0)^2\right)^{-\frac{1+\mu}{2}} ds \leq \frac{2\sqrt{2}C_0C_\mu}{R_1^\mu M^{\frac{1}{2}}}\eta_0^{-1}$$

where $C_\mu = \int_0^\infty (1+s^2)^{-\frac{1+\mu}{2}} ds$. Taking R_1 large enough, we have $\frac{2}{3}\eta_0 \le |\eta(t)| \le \frac{4}{3}\eta_0$ for $t_0 \le t < T_1$. By the definition of T_1, we obtain $T_1 = T$. Moreover, by Lemma 2.3 and the inequality $\frac{2}{3}\eta_0 \le |\eta(t)| \le \frac{4}{3}\eta_0$ for $t_0 \le t < T = T_1$, we conclude $T = \infty$.

□

2.3 Proof of the first main result

Now we prove our first main result.

Proof *(Proof of Theorem 1.2)* Let $\gamma : (a,b) \to \mathbb{R}^{n+1}$ be a geodesic on an asymptotically Minkowski spacetime $(\mathbb{R}^{n+1}, g)$ satisfying the null non-trapping condition in the sense of Definition 2. When the geodesic γ satisfies the non-trapping condition, then the convexity near infinity ([19]) implies that $|\gamma(t)| \to \infty$ as $t \to a$ and $t \to b$. Hence, the geodesic γ is complete due to Proposition 2.4. Thus, we only consider the case when the geodesic γ is trapped.

Suppose that the geodesic γ is trapped. Then, it is not null by the null non-trapping condition. Now completeness of the geodesic γ follows from Lemma 2.1. □

3 Optimality of the local smoothing estimate

In this section, we assume that $g = g_0$ is the Minkowski metric on $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_y$.

3.1 An explicit formula for the resolvent $(P-i)^{-1}$

We recall $P = \partial_t^2 - \Delta_y$ on $\mathbb{R}^{n+1}$. First, we calculate an explicit formula for the resolvent $(P-i)^{-1}$. A similar formula also appears in the proof of [4,Theorem C.1]. To do this, at first, we shall calculate the imaginary part of the symbol of $\sqrt{-\Delta_y - i}$. Set

$$A := \sqrt{-\Delta_y + i} = A_1 - iA_2 \quad A_1, A_2 : \text{self-adjoint}$$

with $A_2 \ge 0$. First, we compute the symbols of A_1, A_2.

Lemma 3.1 *For $\eta \in \mathbb{R}^n$, set $a = \sqrt{|\eta|^2 - i} = re^{i\theta}$ with $\operatorname{Im} a \le 0$. Then we have*

$$r = (|\eta|^4 + 1)^{\frac{1}{4}}, \quad \operatorname{Im} a = -\frac{1}{2^{\frac{1}{2}}(\sqrt{|\eta|^4+1} + |\eta|^2)^{\frac{1}{2}}}.$$

In particular, $C^{-1}\langle\eta\rangle \le |a| \le C\langle\eta\rangle$ and $C^{-1}\langle\eta\rangle^{-1} \le |\operatorname{Im} a| \le C\langle\eta\rangle^{-1}$ with a constant $C > 0$.

Proof Since $a^2 = |\eta|^2 - i$, we have $r^2 = |a|^2 = |a^2| = ||\eta|^2 - i| = (|\eta|^4 + 1)^{\frac{1}{2}}$, which implies $r = (|\eta|^4+1)^{\frac{1}{4}}$. Moreover, it follows that $r^2\cos 2\theta + ir^2 \sin 2\theta = r^2 e^{2i\theta} = |\eta|^2 - i$ and

$$\cos 2\theta = \frac{|\eta|^2}{(|\eta|^4+1)^{\frac{1}{2}}}, \quad \sin 2\theta = \frac{-1}{(|\eta|^4+1)^{\frac{1}{2}}},$$
$$\sin^2\theta = \frac{1}{2(|\eta|^4+1)^{\frac{1}{2}}((|\eta|^4+1)^{\frac{1}{2}}+|\eta|^2)}.$$

This calculation with $\mathrm{Im}\, a \le 0$ implies

$$\mathrm{Im}\, a = r\sin\theta = -\frac{1}{2^{\frac{1}{2}}((|\eta|^4+1)^{\frac{1}{2}}+|\eta|^2)^{\frac{1}{2}}}.$$

□

We write

$$a_1 = \mathrm{Re}\, a, \quad a_2 = -\mathrm{Im}\, a, \quad A_1 = a_1(D_\eta), \quad A_2 = a_2(D_\eta).$$

Since $C^{-1}\langle\eta\rangle^{-1} \le |\mathrm{Im}\, a| \le C\langle\eta\rangle^{-1}$, we have

$$C^{-1}\|u\|_{H^k(\mathbb{R}^n_y)} \le \|A_2^{-k}u\|_{L^2(\mathbb{R}^n_y)} \le C\|u\|_{H^k(\mathbb{R}^n_y)} \tag{3.1}$$

for all $k \in \mathbb{R}$ with a constant $C > 0$.

Now we calculate an expression for the resolvent $(P-i)^{-1}$.

Proposition 3.2 *For $f \in L^2(\mathbb{R}^{n+1})$, we have*

$$\begin{aligned}(P-i)^{-1}f(t,y) =& \frac{i}{2}A^{-1}\int_{-\infty}^{t} e^{-i(t-s)A}f(s,y)ds - \frac{i}{2}A^{-1}\int_{+\infty}^{t} e^{i(t-s)A}f(s,y)ds \\ =& \frac{i}{2}A^{-1}\int_{-\infty}^{\infty} e^{-i|t-s|A}f(s,y)ds. \end{aligned}\tag{3.2}$$

Proof Set $Rf(t,y) = \frac{i}{2}A^{-1}\int_{-\infty}^{\infty} e^{-i|t-s|A}f(s,y)ds$. Since P is essentially self-adjoint on $C_c^\infty(\mathbb{R}^{n+1})$, it suffices to prove that $(P-i)Rf = f$ for $f \in C_c^\infty(\mathbb{R}^{n+1})$ and that R is a bounded operator on $L^2(\mathbb{R}^{n+1})$. In fact, the essential self-adjointness of $P|_{C_c^\infty(\mathbb{R}^{n+1})}$ ensures that the domain $D(P)$ of P is written as $D(P) = \{u \in L^2(\mathbb{R}^{n+1}) \mid Pu \in L^2(\mathbb{R}^{n+1})\}$. Then, the relation $(P-i)Rf = f$ for $f \in C_c^\infty(\mathbb{R}^{n+1})$ and the boundedness of R on $L^2(\mathbb{R}^{n+1})$ show $Rf \in D(P)$ for $f \in C_c^\infty(\mathbb{R}^{n+1})$. Using the relation $(P-i)^{-1}(P-i)u = u$ for $u \in D(P)$ with $u = Rf$, we have $R|_{C_c^\infty(\mathbb{R}^{n+1})} = (P-i)^{-1}|_{C_c^\infty(\mathbb{R}^{n+1})}$. Hence, a density argument gives $R = (P-i)^{-1}$ on $L^2(\mathbb{R}^{n+1})$.

The identity $(P-i)Rf = f$ follows from a direct calculation. Hence, we shall prove that R is bounded on $L^2(\mathbb{R}^{n+1})$. Set $a_2(\eta) = -\mathrm{Im}\, a$. Using the Fourier transform in the y-variable and a scaling, we have

$$\begin{aligned}\|Rf\|_{L^2(\mathbb{R}^{n+1}_{t,y})} =&\|\mathcal{F}_{y\to\eta}Rf\|_{L^2(\mathbb{R}^{n+1}_{t,\eta})} \leq \frac{1}{2}\|a^{-1}\int_{\mathbb{R}} e^{-|t-s|a_2}|\mathcal{F}_{y\to\eta}f(s,\eta)|ds\|_{L^2(\mathbb{R}^{n+1}_{t,\eta})}\\ \leq&\frac{1}{2}\|a_2^{-\frac{3}{2}}a^{-1}\int_{\mathbb{R}} e^{-|t-s|}|\mathcal{F}_{y\to\eta}f(\frac{s}{a_2},\eta)|ds\|_{L^2(\mathbb{R}^{n+1}_{t,\eta})}\\ \leq&\frac{1}{2}\|e^{-|t|}\|_{L^1(\mathbb{R}_t)}\|a_2^{-\frac{3}{2}}a^{-1}\mathcal{F}_{y\to\eta}f(\frac{t}{a_2},\eta)\|_{L^2(\mathbb{R}^{n+1}_{t,\eta})}\\ =&\frac{1}{2}\|e^{-|t|}\|_{L^1(\mathbb{R}_t)}\|a_2^{-1}a^{-1}\mathcal{F}_{y\to\eta}f(t,\eta)\|_{L^2(\mathbb{R}^{n+1}_{t,\eta})} \leq C\|f\|_{L^2(\mathbb{R}^{n+1})}.\end{aligned}$$

Here we use the Young inequality in the third line and set

$$C = \frac{\|e^{-|t|}\|_{L^1(\mathbb{R}_t)}}{2}\sup_{\eta\in\mathbb{R}^n}|a_2(\eta)^{-1}a(\eta)^{-1}|,$$

which is finite by virtue of Lemma 3.1. This completes the proof. □

3.2 Proof of the second result

In this subsection, we shall prove Theorem 1.3. To do this, it suffices to find $f \in L^2(\mathbb{R}^{n+1})$ such that

$$\langle D_y\rangle^{\frac{1}{2}+\varepsilon}u \notin L^2_{loc}(\mathbb{R}^{n+1}),$$

where we set $u := (P-i)^{-1}f$ and we write $\eta \in \mathbb{R}^n$ as the dual variable of y.

Let $\chi \in C_c^\infty((\frac{1}{4},1);[0,1])$ and $\psi \in C_c^\infty(\mathbb{R}^n;[0,1])$ satisfying $\|\chi\|_{L^2(\mathbb{R})} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$ and

$$\chi(t) = 1 \quad \text{on} \quad \frac{1}{2} \leq t \leq 1, \quad \psi(\eta) = 1 \quad \text{on} \quad \frac{1}{2} \leq |\eta| \leq 1.$$

We denote the Fourier transform from the variable η to the variable y by $\mathcal{F}^{-1}_{\eta\to y}$ and set

$$\begin{aligned}&g(t,\eta) := \langle\eta\rangle^{-\frac{n+1}{2}-\varepsilon}\chi\left(\frac{t}{\langle\eta\rangle}\right)e^{ita_1(\eta)} \in L^2(\mathbb{R}^{n+1}), \quad f(t,y) = \mathcal{F}^{-1}_{\eta\to y}g(t,y),\\ &u := (P-i)^{-1}f.\end{aligned}$$

In order to prove Theorem 1.3, it suffices to show

$$\varphi(y)\langle D_y\rangle^{\frac{1}{2}+\varepsilon}u \notin L^2((0,\frac{1}{4})_t \times \mathbb{R}^n_y), \tag{3.3}$$

where $\varphi \in C_c^\infty(\mathbb{R}^n_y)$ is determined as follows: Let $M > 0$ be a constant satisfying $|\partial_\eta a_1(\eta)| \leq M$ for all $\eta \in \mathbb{R}^n$. Take $\varphi, \varphi_1 \in C_c^\infty(\mathbb{R}^n;[0,1])$ such that $\varphi_1(y) = \varphi_1(-y)$ and

$$\varphi(y) = 1 \quad \text{on} \quad |y| \leq M, \quad \varphi_1(y) = 1 \quad \text{on} \quad |y| \leq \frac{M}{8}, \quad \operatorname{supp} \varphi_1 \subset \{|y| \leq \frac{M}{4}\}.$$

As is seen below, $e^{-itA_1}u$ has no oscillation terms and is easy to handle. In order to remove the oscillation factor e^{itA_1}, we will use Egorov's theorem and an elliptic estimate from microlocal analysis. Precisely, we shall prove

$$\varphi_1 \langle D_y \rangle^{\frac{1}{2}+\varepsilon} e^{-itA_1} u \notin L^2((0, \frac{1}{4})_t \times \mathbb{R}^n_y), \tag{3.4}$$

$$\|\varphi_1 \langle D_y \rangle^{\frac{1}{2}+\varepsilon} e^{-itA_1} u\|_{L^2((0,\frac{1}{4})_t \times \mathbb{R}^n_y)} \leq C \|\varphi \langle D_y \rangle^{\frac{1}{2}+\varepsilon} u\|_{L^2((0,\frac{1}{4})_t \times \mathbb{R}^n_y)} + C\|u\|_{L^2(\mathbb{R}^{n+1})} \tag{3.5}$$

which ensure (3.3). Thus, it remains to prove (3.4) and (3.5).

Proof of (3.4)

First, we deal with (3.4). Since $t - s \leq 0$ for $t \leq \frac{1}{4}$ and $s \in \operatorname{supp} \chi(\frac{\cdot}{\langle \eta \rangle})$, it follows from (3.2) that for $t \leq \frac{1}{4}$, we have

$$\begin{aligned}
&\mathcal{F}_{y\to\eta}(\langle D_y \rangle^{\frac{1}{2}+\varepsilon} e^{-itA_1} u)(t, \eta) \\
&\quad = \frac{i}{2} \langle \eta \rangle^{-\frac{n}{2}} a(\eta)^{-1} \int_t^{+\infty} e^{-|t-s|a_2(\eta)} \chi(\frac{s}{\langle \eta \rangle}) ds \\
&\quad = \frac{i}{2} \langle \eta \rangle^{-\frac{n}{2}+1} a(\eta)^{-1} \int_{\langle \eta \rangle^{-1} t}^{+\infty} e^{-|\langle \eta \rangle^{-1} t - s|\langle \eta \rangle a_2(\eta)} \chi(s) ds =: b_t(t, \eta).
\end{aligned}$$

Thus, we have $\varphi_1(y)(\langle D_y \rangle^{\frac{1}{2}+\varepsilon} e^{-itA_1} u)(t, y) = \varphi_1(y)(\mathcal{F}^{-1}_{\eta\to y} b_t)(y)$. By Lemma 3.1, b_t satisfies

$$|\partial^\alpha_\eta b_t(\eta)| \leq C_\alpha \langle \eta \rangle^{-\frac{n}{2}}, \quad C^{-1} \langle \eta \rangle^{-\frac{n}{2}} \leq |b_t(\eta)| \leq C \langle \eta \rangle^{-\frac{n}{2}} \tag{3.6}$$

uniformly in $t \in (0, \frac{1}{4})$.

Lemma 3.3 *Let $b_t \in C^\infty(\mathbb{R}^n)$ satisfying (3.6) uniformly in t. Then we have $\varphi_1 \mathcal{F}^{-1}_{\eta\to y} b_t \notin L^2(\mathbb{R}^n_y)$ for each t.*

Proof By the second inequality in (3.6), we have $b_t \notin L^2(\mathbb{R}^n_y)$. This implies $\mathcal{F}^{-1}_{\eta\to y} b_t \notin L^2(\mathbb{R}^n_y)$ by the Plancherel theorem. Thus, it suffices to prove $(1 - \varphi_1)\mathcal{F}^{-1}_{\eta\to y} b_t \in L^2(\mathbb{R}^n_y)$. By integrating by parts and by using the first inequality in (3.6), it turns out that $\mathcal{F}^{-1}_{\eta\to y} b_t(y)$ is rapidly decreasing and smooth away from $y = 0$. Since $\varphi_1(y) = 1$ near $y = 0$, then we obtain $(1 - \varphi_1)\mathcal{F}^{-1}_{\eta\to y} b_t \in L^2(\mathbb{R}^n_y)$. □

Now we suppose $\varphi_1(y)\langle D_y \rangle^{\frac{1}{2}+\varepsilon} e^{-itA_1} u \in L^2((0, \frac{1}{4})_t \times \mathbb{R}^n_y)$. Then, we have

$$\varphi_1(y)\langle D_y \rangle^{\frac{1}{2}+\varepsilon} e^{-itA_1} u \in L^2(\mathbb{R}^n_y)$$

for almost all $t \in (0, \frac{1}{4})$. This contradicts the lemma above. We complete the proof of (3.4).

Proof of (3.5)

Next, we prove (3.5). We briefly recall the notion of pseudodifferential operators. For a precise treatment, see [12,§18]. For $k \in \mathbb{R}$ and $a \in C^\infty(\mathbb{R}^{2n})$, we call $a \in S^k$ if

$$|\partial_y^\alpha \partial_\eta^\beta a(y,\eta)| \leq C_{\alpha\beta}\langle\eta\rangle^{k-|\beta|}.$$

We define the Weyl quantization of a function a by

$$\mathrm{Op}(a)u(y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(y-y')\cdot\eta} a(\frac{y+y'}{2}, \xi)u(y')dy'd\eta.$$

We note that if a is real-valued, then $\mathrm{Op}(a)$ is formally self-adjoint and if $a \in S^0$, then $\mathrm{Op}(a)$ is bounded in $L^2(\mathbb{R}^n)$. Moreover, if the function a depends only on the variable $y \in \mathbb{R}^n$, then its quantization $\mathrm{Op}(a)$ is a multiplication operator $a(y)$. We define $\{a, b\} := H_a b := \partial_\eta a \cdot \partial_y b - \partial_y a \cdot \partial_\eta b$. For $a \in S^{k_1}$ and $b \in S^{k_2}$, we have

$$\mathrm{Op}(a)\mathrm{Op}(b) - \mathrm{Op}(ab) \in \mathrm{Op}S^{k_1+k_2-1},\ \ [\mathrm{Op}(a), i\mathrm{Op}(b)] - \mathrm{Op}(\{a,b\}) \in \mathrm{Op}S^{k_1+k_2-2} \tag{3.7}$$

We need the following lemmas.

Lemma 3.4 *[Egorov's theorem] Set* $a_t(y,\eta) := \varphi(y - t\partial_\eta a_1(\eta))$. *Then*

$$e^{-itA_1}\varphi(y)e^{itA_1} + R_t = \mathrm{Op}(a_t),$$

where $R_t \in B(H^{-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ *is an operator locally uniformly bounded in* t.

Proof Using $(e^{itA_1}\mathrm{Op}(a_t)e^{-itA_1})'|_{t=0} = \varphi(y)$, we have $\mathrm{Op}(a_t) = e^{-itA_1}\varphi(y)e^{itA_1} + R_t$, where

$$\begin{aligned} R_t &= \int_0^t e^{-i(t-s)A_1}(\frac{d}{ds}\mathrm{Op}(a_s) + [A_1, i\mathrm{Op}(a_s)])e^{i(t-s)A_1}ds \\ &= \int_0^t e^{-i(t-s)A_1}L_s e^{i(t-s)A_1}ds \end{aligned}$$

and $L_s = \frac{d}{ds}\mathrm{Op}(a_s) + [A_1, i\mathrm{Op}(a_s)]$. The formula (3.7) implies that $L_s \in \mathrm{Op}S^{-1}$ is locally uniformly bounded in t. Since e^{-itA_1} preserves $H^k(\mathbb{R}^n)$, it follows that $R_t \in B(H^{-1}(\mathbb{R}^n), L^2(\mathbb{R}^n))$ is locally uniformly bounded in t. □

Lemma 3.5 *[Elliptic parametrix] For* $0 \leq t \leq \frac{1}{4}$, *we have*

$$(\mathrm{supp}\,\varphi_1) \times \mathbb{R}^n \subset \{(y,\eta) \in \mathbb{R}^{2n} \mid \varphi(y - t\partial_\eta a_1(\eta)) = 1\}, \tag{3.8}$$

that is, $\varphi(y - t\partial_\eta a_1(\eta))$ is elliptic on $\operatorname{supp}\varphi_1 \times \mathbb{R}^n$ *in the phase space. Moreover, there exist $c_t \in S^0$ and $r_t \in S^{-1}$, which are locally uniformly bounded in t such that*

$$\varphi_1 = \mathrm{Op}(c_t)\mathrm{Op}(a_t) + \mathrm{Op}(r_t) \tag{3.9}$$

where we recall $a_t(y,\eta) := \varphi(y - t\partial_\eta a_1(\eta))$ and the notation φ_1 in the left-hand side of (3.9) is a multiplication operator.

Proof For $0 \le t \le \frac{1}{4}$ and $|y| \le \frac{M}{4}$, we have

$$|y - t\partial_\eta a(\eta)| \le |y| + |t||\partial_\eta a(\eta)| \le \frac{M}{4} + \frac{1}{4}\cdot M < M.$$

which implies (3.8). We note that the smooth function $c_t(y,\eta) = \varphi_1(y)/a_t(y.\eta)$ is well-defined by the support condition (3.8). Since $a_t \in S^0$ is locally uniformly bounded in t, c_t is also locally uniformly bounded in t. The composition formula (3.7) gives (3.9) with a symbol $r_t \in S^{-1}$ locally uniformly bounded in t. □

Now we prove (3.5). Set

$$\begin{aligned} C_1 := \sup_{t\in(0,\frac{1}{4})} &\max(\|\mathrm{Op}(c_t)\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}, \|R_t\|_{H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}, \\ &\|\mathrm{Op}(r_t)\|_{H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}). \end{aligned}$$

By Lemma 3.5 and the unitarity of e^{-itA_1}, we have

$$\begin{aligned} &\|\varphi_1\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} \\ &\le \|\mathrm{Op}(c_t)\mathrm{Op}(a_t)\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} + \|\mathrm{Op}(r_t)\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} \\ &\le C_1\|\mathrm{Op}(a_t)\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} + C_1\|u\|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{3.10}$$

uniformly in $t \in (0,\frac{1}{4})$. Moreover, Lemma 3.5 and the unitarity of e^{-itA_1} imply

$$\begin{aligned} &\|\mathrm{Op}(a_t)\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} \\ &\le \|e^{-itA_1}\varphi e^{itA_1}\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} + \|R_t\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} \\ &\le \|\varphi\langle D_y\rangle^{\frac{1}{2}+\varepsilon}u\|_{L^2(\mathbb{R}^n_y)} + C_1\|u\|_{L^2(\mathbb{R}^n_y)}, \end{aligned} \tag{3.11}$$

where we use the identity $e^{itA_1}\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1} = \langle D_y\rangle^{\frac{1}{2}+\varepsilon}$ at the last line. By inequalities (3.10) and (3.11), we have

$$\|\varphi_1\langle D_y\rangle^{\frac{1}{2}+\varepsilon}e^{-itA_1}u\|_{L^2(\mathbb{R}^n_y)} \le C_1\|\varphi\langle D_y\rangle^{\frac{1}{2}+\varepsilon}u\|_{L^2(\mathbb{R}^n_y)} + (C_1 + C_1^2)\|u\|_{L^2(\mathbb{R}^n_y)}$$

uniformly in $(0,\frac{1}{4})$. Integrating this inequality in $t \in (0,\frac{1}{4})$, we obtain (3.5).

A short proof of smoothing effects on the Minkowski spacetimes

In this appendix, we give a short proof of (1.1) and (1.2) using the explicit formula (3.2).

Lemma A.1 *Set*

$$I(t,y)=\int_{\mathbb{R}} e^{-i|t-s|A} f(s,y)ds.$$

Then, for $\varepsilon>0$ *we have*

$$\|\langle t\rangle^{-\frac{1}{2}-\varepsilon} I\|_{L^2(\mathbb{R}^{n+1})}\le C\|\langle D_y\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}^{n+1})},\quad \|\langle t\rangle^{-\frac{1}{2}-\varepsilon} I\|_{L^2(\mathbb{R}^{n+1})}$$
$$\le C\|\langle t\rangle^{\frac{1}{2}+\varepsilon} f\|_{L^2(\mathbb{R}^{n+1})}$$

Proof The second inequality immediately follows from Hölder's inequality. Thus, we shall show the first inequality. We write $\hat{f}(t,\eta)=\mathcal{F}_{y\to\eta} f(t,\eta)$ and $a_2:=-\mathrm{Im}\, a=-\mathrm{Im}\sqrt{|\eta|^2-i}$, where $\mathcal{F}_{y\to\eta}$ denotes the Fourier transform from the variable y to the variable η. Fourier transforming in $y\to\eta$ and using Young's inequality, we have

$$|\hat{I}(t,\eta)|\le\left|\int_{\mathbb{R}} e^{-(t-s)a_2}|\hat{f}(s,\eta)|ds\right|\le\|e^{-sa_2}\|_{L^2(\mathbb{R}_s)}\|\hat{f}(s,\eta)\|_{L^2(\mathbb{R}_s)}$$
$$\le C\|a_2^{-\frac{1}{2}}\hat{f}(s,\eta)\|_{L^2(\mathbb{R}_s)}.$$

This calculation gives $\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\hat{I}(t,\eta)\|_{L^2_t}\le\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\|_{L^2_t}\|\hat{I}(t,\eta)\|_{L^\infty_t}\le C\|a_2^{-\frac{1}{2}}\hat{f}(t,\eta)\|_{L^2_t}$. Plancherel's theorem and (3.1) imply

$$\|\langle t\rangle^{-\frac{1}{2}-\varepsilon} I\|_{L^2(\mathbb{R}^{n+1}_{t,y})}\le C\|a_2^{-\frac{1}{2}}\hat{f}\|_{L^2(\mathbb{R}^{n+1}_{t,\eta})}\le C\|\langle D_y\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}^{n+1}_{t,y})}.$$

□

Now we shall prove (1.1) or a stronger bound: $\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\langle D_x\rangle^{\frac{1}{2}}(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})}\le C\|f\|_{L^2(\mathbb{R}^{n+1})}$. To do this, it suffices to prove

$$\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\langle D_y\rangle^{\frac{1}{2}}(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})}\le C\|f\|_{L^2(\mathbb{R}^{n+1})}. \tag{4.1}$$

where we recall $x=(t,y)\in\mathbb{R}\times\mathbb{R}^n$. In fact, we take $\varphi(D_x)=\psi(P/(-\Delta_x+1))$, where $\psi\in C_c^\infty(\mathbb{R};[0,1])$ satisfies $\psi(s)=1$ on $|s|\le\frac{1}{4}$ and $\mathrm{supp}\,\psi\subset\{|s|\le\frac{1}{2}\}$. Moreover, we set $\chi(D_x)=\varphi(D_x)\langle D_x\rangle^{\frac{1}{2}}\langle D_y\rangle^{-\frac{1}{2}}$. Since P is elliptic on the essential support of $1-\varphi(D_x)$, we have $\|(1-\varphi(D_x))\langle D_x\rangle^{\frac{1}{2}}(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})}\le C\|f\|_{L^2(\mathbb{R}^{n+1})}$. Moreover, since $c^{-1}|\xi|\le|\eta|\le c|\xi|$ on the essential support of $\varphi(D_x)$ with a constant $c\ge1$, it turns out that $\chi(D_x)$ is bounded in $L^2(\mathbb{R}^{n+1})$. Hence,

$$\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\varphi(D_x)\langle D_x\rangle^{\frac{1}{2}}(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})} = \|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\chi(D_x)\langle D_y\rangle^{\frac{1}{2}}(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})} \le C\|\chi(D_x)f\|_{L^2(\mathbb{R}^{n+1})} \le C\|f\|_{L^2(\mathbb{R}^{n+1})},$$

due to (4.1). Combining these inequalities with (4.2), we obtain (1.1). Now we turn to the proof of (4.1). Lemma A.1 with the formula (3.2) immediately implies

$$\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})} \le C\|A^{-1}\langle D_y\rangle^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^{n+1})} \le C\|\langle D_y\rangle^{-\frac{1}{2}}f\|_{L^2(\mathbb{R}^{n+1})}, \tag{4.2}$$

which shows (4.1).

Finally, we prove (1.2) or a stronger bound: $\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\langle D_x\rangle(P-i)^{-1}\langle t\rangle^{-\frac{1}{2}-\varepsilon} f\|_{L^2(\mathbb{R}^{n+1})} \le C\|f\|_{L^2(\mathbb{R}^{n+1})}$. Lemma A.1 with the formula (3.2) implies

$$\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\langle D_y\rangle(P-i)^{-1}\langle t\rangle^{-\frac{1}{2}-\varepsilon}f\|_{L^2(\mathbb{R}^{n+1})} \le C\|f\|_{L^2(\mathbb{R}^{n+1})},$$

where we recall $x=(t,y)\in\mathbb{R}\times\mathbb{R}^n$. Let φ be as in the proof of (4.1) and set $\chi_1(D_x)=\varphi(D_x)\langle D_x\rangle\langle D_y\rangle^{-1}$. We repeat an argument similar to the proof of (4.1): Since P is elliptic on the essential support of $1-\varphi(D_x)$, we have $\|(1-\varphi(D_x))\langle D_x\rangle(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})} \le C\|f\|_{L^2(\mathbb{R}^{n+1})}$. Moreover, since $c^{-1}|\xi| \le |\eta| \le c|\xi|$ on the essential support of $\varphi(D_x)$ with a constant $c\ge 1$, it turns out that $\chi_1(D_x)$ is bounded in $L^2(\mathbb{R}^{n+1})$. Since $\chi_1(D_x)$ is a pseudodifferential operator of order 0, then $[\langle t\rangle^{-\frac{1}{2}-\varepsilon},\chi_1(D_x)]$ is a pseudodifferential operator of order -1. Hence, $[\langle t\rangle^{-\frac{1}{2}-\varepsilon},\chi_1(D_x)]\langle D_y\rangle$ is bounded on $L^2(\mathbb{R}^{n+1})$. Thus, we have

$$\begin{aligned}
&\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\varphi(D_x)\langle D_x\rangle(P-i)^{-1}\langle t\rangle^{-\frac{1}{2}-\varepsilon}f\|_{L^2(\mathbb{R}^{n+1})}\\
&= \|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\chi_1(D_x)\langle D_y\rangle(P-i)^{-1}\langle t\rangle^{-\frac{1}{2}-\varepsilon}f\|_{L^2(\mathbb{R}^{n+1})}\\
&\le \|[\langle t\rangle^{-\frac{1}{2}-\varepsilon},\chi_1(D_x)]\langle D_y\rangle\|_{L^2\to L^2}\|(P-i)^{-1}f\|_{L^2(\mathbb{R}^{n+1})}\\
&+ C\|\langle t\rangle^{-\frac{1}{2}-\varepsilon}\langle D_y\rangle(P-i)^{-1}\langle t\rangle^{-\frac{1}{2}-\varepsilon}f\|_{L^2(\mathbb{R}^{n+1})}\\
&\le C\|f\|_{L^2(\mathbb{R}^{n+1})},
\end{aligned}$$

This completes the proof of (1.2). See also the proof of [4,Theorem C.1].

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