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Surface defects in gauge theory and KZ equation

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Abstract
We study the regular surface defect in the Ω-deformed four-dimensional supersymmetric gauge theory with gauge group $SU(N)$ with $2N$ hypermultiplets in fundamental representation. We prove its vacuum expectation value obeys the Knizhnik–Zamolodchikov equation for the 4-point conformal block of the $\widehat{\mathfrak{sl}}_N$-current algebra, originally introduced in the context of two-dimensional conformal field theory. The level and the vertex operators are determined by the parameters of the Ω-background and the masses of the hypermultiplets; the cross-ratio of the 4 points is determined by the complexified gauge coupling. We clarify that in a somewhat subtle way the branching rule is parametrized by the Coulomb moduli. This is an example of the BPS/CFT relation.

Keywords KZ equation · Surface defect · qq-character · BPS/CFT correspondence · Supersymmetric gauge theory · Supersymmetry and duality

Mathematics Subject Classification 81T13 · 81T35 · 81T40

1 Introduction

The rich mathematics of quantum field theory has a remarkable feature of admitting, to some extent, an analytic continuation in various parameters, such as momenta, spins etc. This feature is best studied in the examples of two-dimensional conformal field

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theories, where one can observe almost with a naked eye that the building blocks of the correlation functions are analytic in the parameters, such as the central charges, conformal dimensions, weights, spins and so on cf. [53]. Some formulae admit analytic continuation in the level k of the current algebra, cf. [22]. The analytic continuation offers some glimpses of the Langlands duality [12] $(k + h^\vee)(k^\vee + h) = 1$, which suggests an identification of the quantum group parameter q with the modular parameter $\exp(2\pi \mathrm{i}\tau)$ of some elliptic curve [6]. These observations solidified as soon as the connection between the S-duality of four-dimensional supersymmetric theories and the modular invariance of two-dimensional conformal field theories was observed [47]. Localization computations in supersymmetric gauge theories [24, 26, 27, 32–37] showed that the correlation functions of selected observables coincide with conformal blocks of some two-dimensional conformal field theories, or, more generally, are given by the matrix elements of representations of some infinite-dimensional algebras, such as Kac–Moody, Virasoro, or their q-deformations, albeit extended to the complex domain of parameters, typically quantized in the two-dimensional setup. In [26], this phenomenon was attributed to the chiral nature of the tensor field propagating on the worldvolume of the fivebranes. The fivebranes ($M5$ branes in M-theory and $NS5$ branes in the IIA string theory) were used in [21, 49] to engineer, in string theory setup, the supersymmetric systems whose low energy is described by $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. This construction was extended and generalized in [11]. This correspondence, named *the BPS/CFT correspondence* in [29, 30], has been supported by a large class of very detailed examples in [1, 2, 32–37], and more recently in [14, 15, 18, 20].

Finally, in [50, 51], the relation of the quantum group parameter q with the elliptic curves has been brought into the familiar context of the relation of the $\mathcal{N} = 4$ super-Yang–Mills theory to elliptic curves. Hopefully, with this understanding of the analytically continued Chern–Simons theory, the (quasi)-modularity conjectures of [31] could be tested.

In this paper, we shall be studying a particular corner of that theoretical landscape: the $SU(N)$ gauge theory with $2N$ fundamental hypermultiplets. In the BPS/CFT correspondence, it is associated with a zoo of two-dimensional conformal theories living on a 4-punctured sphere, all related to the $\widehat{\mathfrak{sl}}_N$ current algebra, either directly, or through the Drinfeld–Sokolov reduction, producing the W_N-algebra [52], depending on the supersymmetric observables one uses to probe the four-dimensional theory. Two observables are of interest for us. First, the supersymmetric partition function $\mathbf{Z} = \mathbf{Z}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})$ on $\mathbb{R}^4$, which is a function of the vacuum expectation value $\mathbf{a} = \mathrm{diag}\,(a_1, \ldots, a_N)$ of the scalar in the vector multiplet, the masses $\mathbf{m} = \{m_1, m_2, \ldots, m_{2N-1}, m_{2N}\}$, the exponentiated complexified gauge coupling $\mathfrak{q} = \exp(2\pi \mathrm{i}\tau)$,

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi \mathrm{i}}{e^2}$$

and the parameters $\varepsilon = (\varepsilon_1, \varepsilon_2)$ of the Ω-deformation. The latter are the equivariant parameters of the maximal torus $U(1) \times U(1)$ of the Euclidean rotation group $Spin(4)$. In the complex coordinates (z_1, z_2) on $\mathbb{C}^2 \approx \mathbb{R}^4$, the rotational symmetry acts by

$(z_1, z_2) \mapsto (e^{\mathrm{i}t\varepsilon_1} z_1 , e^{\mathrm{i}t\varepsilon_2} z_2)$. Exchanging $\varepsilon_1 \leftrightarrow \varepsilon_2$ is part of the $Spin(4)$ Weyl group; hence, it is a symmetry of $\mathbf{Z}$. The second observable is the partition function $\mathbf{\Psi}$ of the regular surface defect which breaks the gauge group down to its maximal torus $U(1)^{N-1}$ along the surface, which we shall take to be the $z_2 = 0$ plane. This partition function depends on all the parameters $\mathbf{a}$, $\mathbf{m}$, ε, $\mathfrak{q}$ that the bulk partition function $\mathbf{Z}$ depends on and, in addition, it depends on the parameters

$$\mathbf{w} = (w_0 : w_1 : \cdots : w_{N-1}) \in \mathbb{CP}^{N-1}$$

of a two-dimensional theory the defect supports. The physics and mathematics setup of the problem are explained in [36, 37], which the reader may consult for motivations and orientation. However, our exposition is self-contained as a well-posed mathematical problem, which we introduce presently.

Our main result is the proof of a particular case of the BPS/CFT conjecture [29, 30]: the vacuum expectation value $\langle\, \mathcal{S}\, \rangle$ of the surface defect obeys the Knizhnik–Zamolodchikov equation [22], specifically the equation obeyed by the $\left(\widehat{\mathfrak{sl}_N}\right)_k$ current algebra conformal block

$$\Phi = \Big\langle \mathbf{V}_1(0)\mathbf{V}_2(\mathfrak{q})\mathbf{V}_3(1)\mathbf{V}_4(\infty) \Big\rangle^{\mathbf{a}} \tag{1}$$

with the vertex operators corresponding to irreducible infinite-dimensional representations of $\mathfrak{sl}_N$. More specifically, the vertex operators at 0 and ∞ correspond to the generic lowest weight $\mathcal{V}_{\vec{\nu}}$ and highest weight $\tilde{\mathcal{V}}_{\tilde{\vec{\nu}}}$ Verma modules, while the vertex operators at $\mathfrak{q}$ and 1 correspond to the so-called *twisted HW-modules* $\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}, \tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}$. The subscripts $\vec{\nu}, \tilde{\vec{\nu}} \in \mathbb{C}^{N-1}$ and $\mathrm{m}, \tilde{\mathrm{m}} \in \mathbb{C}$ determine the values of the Casimir operators, in correspondence with the $2N$ masses $\mathbf{m}$ and one of the Ω-background parameters ε_1. The superscripts $\vec{\mu}, \tilde{\vec{\mu}} \in \mathbb{C}^{N-1}$ determine the so-called twists of the HW-modules, all defined below, which we express via $\mathbf{m}$, ε_1, and the Coulomb parameters $\mathbf{a}$. In other words, the Coulomb parameters determine the analogue of the "intermediate spin", which we indicate by placing a superscript $\mathbf{a}$ in (1) to label the specific fusion channel. We define these representations and the Knizhnik–Zamolodchikov equation [22].

The appearance of the twisted representations is a curious fact not visible in the rational conformal field theories.

2 Basic setup in four dimensions

First we introduce the setup of the four-dimensional gauge theory calculation.

2.1 Notations

We start by reviewing our notations. The reader is invited to consult [32–37] for the general orientation.

- The parameters of the Ω-deformation: $\varepsilon_1, \varepsilon_2$ – two complex parameters, generating the equivariant cohomology $\mathrm{H}^{\bullet}_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{pt})$. The *twist* part of the Ω-deformation is $\varepsilon = \varepsilon_1 + \varepsilon_2$. The torus $\mathbb{C}^\times \times \mathbb{C}^\times$ is the complexification of the maximal torus of the spin cover of the rotation group $Spin(4)$. We also define

$$\kappa = \frac{\varepsilon_2}{\varepsilon_1} \,. \tag{2}$$

- The Coulomb moduli:

$$\mathbf{a} = (a_b)_{b=1}^N \equiv (a_1, \ldots, a_N) \in \mathbb{C}^N \tag{3}$$

–the equivariant parameters of the *color group*, in other words these are the generators of $\mathrm{H}^{\bullet}_{(\mathbb{C}^\times)^N}(\mathrm{pt})$, on which the symmetric group $S(N)$ acts by permutations.
- The masses:

$$\mathbf{m} = (m_f)_{f=1}^{2N} \equiv (m_1, \ldots, m_{2N}) \in \mathbb{C}^{2N} \tag{4}$$

–the equivariant parameters of the *flavor group*. The symmetric group $S(2N)$ acts on them by permutations. The $S(2N)$-invariants are encoded via the polynomial

$$P(x) = \prod_{f=1}^{2N} (x - m_f) \,. \tag{5}$$

- The splitting of the set of masses into the N "fundamental" and N "anti-fundamental" ones:

$$P(x) = P^+(x) P^-(x) \,, \qquad P^{\pm}(x) = \prod_{f=1}^{N} \left(x - m_f^{\pm} \right) . \tag{6}$$

- *The lattice of equivariant weights* $\Lambda \subset \mathbb{C}$ is defined by:

$$\Lambda := \mathbb{Z} \cdot \varepsilon_1 \oplus \mathbb{Z} \cdot \varepsilon_2 \oplus \bigoplus_{b=1}^{N} \mathbb{Z} \cdot a_b \oplus \bigoplus_{f=1}^{2N} \mathbb{Z} \cdot m_f \,. \tag{7}$$

We assume that all the parameters $\varepsilon_{1,2}$, $\mathbf{a}$, $\mathbf{m}$ are generic, up to the overall translation $a_b \mapsto a_b + s,\ m_f \mapsto m_f + s$, for $s \in \mathbb{C}$. Thus, the rank of Λ is at least $3N + 1$.

Recall that the bulk theory (subject to noncommutative deformation, leading to instanton moduli space being partially compactified to the moduli space $\mathfrak{M}_{k,N}$ of charge k rank N framed torsion-free sheaves on $\mathbb{CP}^2$) is invariant under the nonabelian symmetry group $U(2)$ of rotations, preserving the complex structure of $\mathbb{C}^2 \approx \mathbb{R}^4$. The group $U(N)$ of constant gauge transformations acts on $\mathfrak{M}_{k,N}$ by changing the asymptotics of instantons at infinity. The Coulomb parameters

a represent the maximal torus of $U(N)$; they can be viewed as local coordinates on the Spec $\mathrm{H}^{\bullet}_{U(N)}(\mathrm{pt}) = \mathbb{C}[\mathbf{a}]^{S(N)}$, with $S(N)$ being the Weyl group. Likewise, the parameters $(\varepsilon_1, \varepsilon_2)$ are acted by the Weyl group $\mathbb{Z}_2$ which acts by permuting $\varepsilon_1 \leftrightarrow \varepsilon_2$. The physical theory has a larger rotation symmetry group $Spin(4)$, whose Weyl group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ but we don't see the full symmetry in the **Z**-function. The full symmetry is present once **Z** is divided by the so-called $U(1)$-factor, having to do with decoupling of the $U(1)$-part of gauge group [1].

Finally, the masses represent the equivariant parameters of the flavor group $SU(2N)$ (the physical theory has a larger flavor symmetry group, which we don't see either); hence, the Weyl group $S(2N)$ symmetry making the polynomial $P(x)$ of (5) the good parameter.

The surface defect we are going to study in this paper breaks both the gauge group $U(N)$ to its maximal torus $U(1)^N$ and the flavor group to its maximal torus $U(1)^{2N}$. The group $S(N) \times S(2N)$ acts, therefore, on the space of surface defects. In describing the specific bases in the vector space of surface defects, we keep track of the ordering of the Coulomb and mass parameters.

- The set of vertices of the Young graph $\mathcal{P}$–the set of all Young diagrams (= partitions of nonnegative integers) $\{\lambda\}$. Then

$$\mathcal{P}^N = \left\{ \overline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)}) \,\middle|\, \lambda^{(b)} \in \mathcal{P} \text{ for } 1 \le b \le N \right\} .$$

- For a box $\square = (i, j)$, define its *content* $c(\square)$ by:

$$c(\square) := (i-1)\varepsilon_1 + (j-1)\varepsilon_2 \, . \tag{8}$$

- For $\lambda \in \mathcal{P}$, define:

$$\chi_\lambda := \sum_{\square \in \lambda} e^{c(\square)} \quad \text{and} \quad \chi^*_\lambda := \sum_{\square \in \lambda} e^{-c(\square)} \, . \tag{9}$$

- For $\overline{\lambda} \in \mathcal{P}^N$, define the multiset, i.e., its elements may have multiplicities, *of tangent weights*, $\{\mathrm{w}_t\}_{t \in T_{\overline{\lambda}}} \subset \Lambda$ by the character

$$\sum_{t \in T_{\overline{\lambda}}} e^{\mathrm{w}_t} := \sum_{b,c=1}^{N} e^{a_b - a_c} \left(\chi^*_{\lambda^{(c)}} + e^{\varepsilon} \cdot \chi_{\lambda^{(b)}} - (1-e^{\varepsilon_1})(1-e^{\varepsilon_2}) \cdot \chi_{\lambda^{(b)}} \chi^*_{\lambda^{(c)}} \right) . \tag{10}$$

Remark 2.1 The *duality*: $\{\mathrm{w}_t\}_{t \in T_{\overline{\lambda}}} = \{\varepsilon - \mathrm{w}_t\}_{t \in T_{\overline{\lambda}}}$ is related to the symplectic structure on the instanton moduli space and its completion $\mathfrak{M}_{k,N}$.

- The *pseudo-measure* $\mu = \mu(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) : \mathcal{P}^N \to \mathbb{C}$ on $\mathcal{P}^N$ is defined via:

$$\mu(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})\,|_{\bar{\lambda}} := \frac{1}{\mathbf{Z}^{\text{inst}}} \cdot \left((-1)^N \mathfrak{q}\right)^{|\bar{\lambda}|} \cdot \frac{\prod_{f=1}^{2N} \prod_{b=1}^{N} \prod_{\square \in \lambda^{(b)}} \left(a_b + c(\square) - m_f\right)}{\prod_{t \in T_{\bar{\lambda}}} \mathrm{w}_t}$$
$$= \frac{1}{\mathbf{Z}^{\text{inst}}} \cdot \mathfrak{q}^{|\bar{\lambda}|} \cdot \frac{\prod_{b=1}^{N} \prod_{\square \in \lambda^{(b)}} \left(-P^-(a_b + c(\square))\, P^+(a_b + c(\square))\right)}{\prod_{t \in T_{\bar{\lambda}}} \mathrm{w}_t}, \tag{11}$$

where $|\bar{\lambda}| = \sum_{b=1}^{N} |\lambda^{(b)}|$ with

$$|\lambda^{(b)}| = \sum_i \lambda_i^{(b)}$$

denoting the total number of boxes in $\lambda^{(b)}$, and $\mathbf{Z}^{\text{inst}} = \mathbf{Z}^{\text{inst}}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q})$ is the Taylor series in $\mathfrak{q}$ uniquely determined by the normalization

$$\sum_{\bar{\lambda} \in \mathcal{P}^N} \mu\,|_{\bar{\lambda}} = 1\,.$$

Remark 2.2 The restriction $\deg P(x) = 2N$ comes from the convergence of $\mathbf{Z}^{\text{inst}}$ for generic $\mathbf{a}, \mathbf{m}, \varepsilon_{1,2}$, cf. [9]. When working over the ring $\mathbb{C}[[\mathfrak{q}]]$ of formal power series in $\mathfrak{q}$, the restriction on the degree of $P(x)$, i.e., the number of masses, can be dropped.

- For $\lambda \in \mathcal{P}$, we call $\blacksquare \in \lambda$ a *corner box* if $\lambda \backslash \blacksquare \in \mathcal{P}$ and we call $\square \notin \lambda$ a *growth box* if $\lambda \sqcup \square \in \mathcal{P}$. We denote by $\partial_+\lambda$ the set of all growth boxes of λ, and by $\partial_-\lambda$ the set of all corner boxes of λ. It is easy to check that:

$$\#\partial_+\lambda - \#\partial_-\lambda = 1\,.$$

- For $x \in \mathbb{C}$, we define the function $Y(x)$ on $\mathcal{P}^N$ as follows: its value $Y(x)\,|_{\bar{\lambda}}$ on $\bar{\lambda} \in \mathcal{P}^N$ is equal to

$$Y(x)\,|_{\bar{\lambda}} := \prod_{b=1}^{N} \left((x - a_b) \prod_{\square \in \lambda^{(b)}} \frac{(x - a_b - c(\square) - \varepsilon_1)(x - a_b - c(\square) - \varepsilon_2)}{(x - a_b - c(\square))(x - a_b - c(\square) - \varepsilon)} \right)$$
$$= \prod_{b=1}^{N} \frac{\prod_{\square \in \partial_+\lambda^{(b)}} (x - a_b - c(\square))}{\prod_{\blacksquare \in \partial_-\lambda^{(b)}} (x - a_b - \varepsilon - c(\blacksquare))}, \tag{12}$$

  the second line being obtained from the first one by the simple inspection of the cancelling common factors.
- For $x \in \mathbb{C}$, we define the function $\mathcal{X}(x)$ on $\mathcal{P}^N$, called the *fundamental qq-character*, by specifying its value $\mathcal{X}(x)\,|_{\bar{\lambda}}$ on $\bar{\lambda} \in \mathcal{P}^N$ as follows:

$$\mathcal{X}(x)\,|_{\bar{\lambda}} := Y(x + \varepsilon)\,|_{\bar{\lambda}} + \mathfrak{q}\, \frac{P(x)}{Y(x)\,|_{\bar{\lambda}}}\,. \tag{13}$$

- For a pseudo-measure $\widetilde{\mu}\colon \mathcal{P}^N \to \mathbb{C}$ and a function $g\colon \mathcal{P}^N \to \mathbb{C}(x)$, the *average* $\langle\, g(x)\,\rangle_{\widetilde{\mu}}$ is defined via:

$$\Big\langle g(x) \Big\rangle_{\widetilde{\mu}} := \sum_{\overline{\lambda}\in\mathcal{P}^N} \widetilde{\mu}\,|_{\overline{\lambda}} \cdot g(x)\,|_{\overline{\lambda}}\,. \tag{14}$$

2.2 Dyson–Schwinger equation

The following is the key property of $\mathcal{X}\colon \mathcal{P}^N \to \mathbb{C}(x)$ of (13):

Proposition 2.1 *The average $\langle\, \mathcal{X}(x)\,\rangle_{\mu}$ is a regular function of x.*

This is the simplest case of the general result on the qq-characters as established in [32–37]. For completeness of our exposition, an elementary proof is presented in Appendix A.

2.3 An orbifold version

As explained in [35], there is a very important $\mathbb{Z}_N$-equivariant counterpart of the above story. It is defined in several steps.

First, we change the notations:

$$a_b \mapsto \tilde{a}_b\,, \quad m_f^{\pm} \mapsto \tilde{m}_f^{\pm}\,, \quad (\varepsilon_1,\varepsilon_2)\mapsto(\varepsilon_1,\tilde{\varepsilon}_2)\,, \quad \text{so that} \quad \varepsilon \mapsto \tilde{\varepsilon} := \varepsilon_1 + \tilde{\varepsilon}_2\,.$$

Next, we introduce the $\mathbb{Z}_N$-grading $\lambda \mapsto \mathfrak{S}_\lambda \in \mathbb{Z}_N$ of the lattice Λ via:

$$\begin{gathered}\lambda = k_1\varepsilon_1 + k_2\tilde{\varepsilon}_2 + \sum_b k_b^a\,\tilde{a}_b + \sum_f k_f^{m^+}\,\tilde{m}_f^+ + \sum_f k_f^{m^-}\,\tilde{m}_f^- \;\mapsto \\ \mathfrak{S}_\lambda := k_2 + \sum_{\omega\in\mathbb{Z}_N} \omega \left(\sum_{b\in A_\omega} k_b^a + \sum_{f\in F_\omega^+} k_f^{m^+} + \sum_{f\in F_\omega^-} k_f^{m^-} \right) \bmod N\,,\end{gathered} \tag{15}$$

for some partitions

$$\Big\{1,\ldots,N\Big\} = \bigsqcup_{\omega\in\mathbb{Z}_N} A_\omega = \bigsqcup_{\omega\in\mathbb{Z}_N} F_\omega^{\pm}$$

of the sets of the Coulomb moduli and the fundamental/anti-fundamental masses. Such $\mathbb{Z}_N$-grading is also often called an *N-coloring*. We define:

$$P_\omega^{\pm}(x) = \prod_{f\in F_\omega^{\pm}} (x - \tilde{m}_f^{\pm})\,.$$

The following depends on a choice of a section $\mathbb{Z}_N \to \mathbb{Z}$. We send

$$\mathbb{Z}_N \ni \omega \mapsto 0 \le \omega < N\,, \tag{16}$$

thus, identifying $\mathbb{Z}_N$ with $\{0, \dots, N-1\}$, as a set. An N-coloring is called *regular* iff

$$\#A_\omega = \#F_\omega^+ = \#F_\omega^- = 1 \text{ for all } \omega \in \mathbb{Z}_N \,.$$

For a regular N-coloring, the ω-colored masses are packaged into a degree two polynomial

$$P_\omega(x) = P_\omega^+(x) P_\omega^-(x) =: (x - \varepsilon_1 \mu_\omega - \omega \tilde{\varepsilon}_2)^2 - \varepsilon_1^2 \delta \mu_\omega^2 \,. \tag{17}$$

Also, for a regular N-coloring, assuming (16), we set:

$$\alpha_\omega := -\omega \tilde{\kappa} + \frac{1}{\varepsilon_1} \sum_{b \in A_\omega} \tilde{a}_b \,, \tag{18}$$

where

$$\tilde{\kappa} = \frac{\kappa}{N} \,. \tag{19}$$

We shall also need a few more new notations.

- For every $\omega \in \mathbb{Z}_N$, define the observable $k_\omega \colon \mathcal{P}^N \to \mathbb{Z}_{\geq 0}$ by:

$$k_\omega |_{\overline{\lambda}} := \sum_{b=1}^{N} \sum_{\Box \in \lambda^{(b)}} \delta^{\omega}_{\mathfrak{S}_{\tilde{a}_b + \tilde{c}(\Box)}} \,, \tag{20}$$

  where $\tilde{c}(i, j) := (i-1)\varepsilon_1 + (j-1)\tilde{\varepsilon}_2$, cf. (8), and $\delta_i^j \equiv \delta_{i,j}$ is the Kronecker delta.
- The *fractional couplings*:

$$\overline{\mathfrak{q}} = (\mathfrak{q}_\omega)_{\omega \in \mathbb{Z}_N} \equiv (\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_{N-1}) \in \mathbb{C}^N \,. \tag{21}$$

- Given $\overline{\mathfrak{q}}$ of (21), define the observable $\underline{\mathfrak{q}} \colon \mathcal{P}^N \to \mathbb{C}$, called the *fractional instanton factor*, as follows:

$$\underline{\mathfrak{q}} |_{\overline{\lambda}} := \prod_{b=1}^{N} \prod_{\Box \in \lambda^{(b)}} \mathfrak{q}_{\mathfrak{S}_{\tilde{a}_b + \tilde{c}(\Box)}} = \prod_{\omega \in \mathbb{Z}_N} \mathfrak{q}_\omega^{k_\omega |_{\overline{\lambda}}} \,. \tag{22}$$

- The pseudo-measure $\mu^{\text{orb}} = \mu^{\text{orb}}(\tilde{\mathbf{a}}, \tilde{\mathbf{m}}, \varepsilon_1, \tilde{\varepsilon}_2; \overline{\mathfrak{q}}) \colon \mathcal{P}^N \to \mathbb{C}$ on $\mathcal{P}^N$ is defined via:

$$\mu^{\text{orb}}(\tilde{\mathbf{a}}, \tilde{\mathbf{m}}, \varepsilon_1, \tilde{\varepsilon}_2; \overline{\mathfrak{q}}) |_{\overline{\lambda}} := \frac{\underline{\mathfrak{q}}|_{\overline{\lambda}}}{\boldsymbol{\Psi}^{\text{inst}}} \cdot \frac{\prod_{f=1}^{N} \prod_{b=1}^{N} \prod_{\Box \in \lambda^{(b)}} \left(\tilde{a}_b + \tilde{c}(\Box) - \tilde{m}_f^+\right)^{\delta^0_{\mathfrak{S}_{\tilde{a}_b + \tilde{c}(\Box) - \tilde{m}_f^+}}} \left(-\tilde{a}_b - \tilde{c}(\Box) + \tilde{m}_f^-\right)^{\delta^0_{\mathfrak{S}_{\tilde{a}_b + \tilde{c}(\Box) - \tilde{m}_f^-}}}}{\prod_{t \in T_{\overline{\lambda}}} \mathrm{w}_t^{\delta^0_{\mathfrak{S}_{\mathrm{w}_t}}}} \,, \tag{23}$$

where the tangent weights $\{\mathrm{w}_t\}_{t\in T_{\overline{\lambda}}}$ are defined via (10) with the substitution $a_b \mapsto \tilde{a}_b$, $\varepsilon_2 \mapsto \tilde{\varepsilon}_2$, and the partition function $\mathbf{\Psi}^{\text{inst}} = \mathbf{\Psi}^{\text{inst}}(\tilde{\mathbf{a}}, \tilde{\mathbf{m}}, \varepsilon_1, \tilde{\varepsilon}_2; \overline{\mathfrak{q}})$ is the formal power series[1] in $(\mathfrak{q}_0, \ldots, \mathfrak{q}_{N-1})$ uniquely determined by the normalization

$$\sum_{\overline{\lambda}\in\mathcal{P}^N} \mu^{\text{orb}}\,|_{\overline{\lambda}} = 1\,. \tag{24}$$

- For every $\omega \in \mathbb{Z}_N$, define the $\mathbb{C}(x)$-valued observable $Y_\omega\colon \mathcal{P}^N \to \mathbb{C}(x)$ via:

$$Y_\omega(x)\,|_{\overline{\lambda}} := \prod_{b=1}^{N} \Bigg((x-\tilde{a}_b)^{\delta^{\omega}_{\mathfrak{S}_{\tilde{a}_b}}} \times \prod_{\square\in\lambda^{(b)}} \left(\frac{x-\tilde{a}_b-\tilde{c}(\square)-\varepsilon_1}{x-\tilde{a}_b-\tilde{c}(\square)}\right)^{\delta^{\omega}_{\mathfrak{S}_{\tilde{a}_b+\tilde{c}(\square)}}} \left(\frac{x-\tilde{a}_b-\tilde{c}(\square)-\tilde{\varepsilon}_2}{x-\tilde{a}_b-\tilde{c}(\square)-\tilde{\varepsilon}}\right)^{\delta^{\omega-1}_{\mathfrak{S}_{\tilde{a}_b+\tilde{c}(\square)}}} \Bigg). \tag{25}$$

- For every $\omega \in \mathbb{Z}_N$, define the $\mathbb{C}(x)$-valued observable $\mathcal{X}_\omega\colon \mathcal{P}^N \to \mathbb{C}(x)$ via:

$$\mathcal{X}_\omega(x)\,|_{\overline{\lambda}} := Y_{\omega+1}(x+\tilde{\varepsilon})\,|_{\overline{\lambda}} + \mathfrak{q}_\omega \frac{P_\omega(x)}{Y_\omega(x)\,|_{\overline{\lambda}}}\,. \tag{26}$$

2.4 Surface defects

Consider a map

$$\pi_N\colon \mathcal{P}^N \longrightarrow \mathcal{P}^N \tag{27}$$

defined via

$$\overline{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(N)}\right) \mapsto \overline{\Lambda} = \left(\Lambda^{(1)}, \ldots, \Lambda^{(N)}\right) \tag{28}$$

with

$$\Lambda^{(b)}_i = \left[\frac{\lambda^{(b)}_i + b - 1}{N}\right], \qquad b = 1, \ldots, N\,. \tag{29}$$

The geometric origin of π_N is explained in [32–37]. Note that $\pi_1 = \mathrm{Id}_{\mathcal{P}}$.

Following [32–37], let us now pass from $\overline{\mathfrak{q}} = (\mathfrak{q}_0, \ldots, \mathfrak{q}_{N-1})$ of (21) to another set of variables, namely $\mathbf{w} = (w_0 : w_1 : \cdots : w_{N-1})$ and $\mathfrak{q}$ via:

$$\mathfrak{q}_0 = w_1/w_0\,,\ \mathfrak{q}_1 = w_2/w_1\,,\ \ldots\,,\ \mathfrak{q}_{N-2} = w_{N-1}/w_{N-2}\,,\ \mathfrak{q}_{N-1} = \mathfrak{q}w_0/w_{N-1}\,, \tag{30}$$

where the *bulk* coupling $\mathfrak{q}$ is recovered by:

$$\mathfrak{q} = \mathfrak{q}_0\mathfrak{q}_1\ldots\mathfrak{q}_{N-1}\,. \tag{31}$$

[1] One can show that this power series converges when all $|\mathfrak{q}_\omega| < 1$, uniformly on compact sets in the complex domain $\tilde{a}_b - \tilde{a}_c + i\varepsilon_1 + j\tilde{\varepsilon}_2 \neq 0$ for all $i, j \geq 1$.

The variables $\mathbf{w}$ are redundant, in the sense that correlation functions are invariant under the simultaneous rescaling of all w's. However, just as the bulk coupling $\mathfrak{q}$ is identified below with the cross-ratio of four points on a sphere, thus revealing a connection to the 4-point function in conformal field theory, the variables w's are identified with the coordinates of N particles, whose dynamics is described by the partition function $\boldsymbol{\Psi}^{\text{inst}}$.

In terms of the $(\mathbf{w}, \mathfrak{q})$-variables, the instanton factor looks as follows (recall that $k_{-1} = k_{N-1}$):

$$\prod_{\omega \in \mathbb{Z}_N} \mathfrak{q}_\omega^{k_\omega} = \mathfrak{q}^{k_{N-1}} \prod_{\omega=0}^{N-1} w_\omega^{k_{\omega-1}-k_\omega} . \tag{32}$$

Evoking (16), we also have an obvious equality

$$\sum_{\omega \in \mathbb{Z}_N} k_\omega = N k_{N-1} + \sum_{i=1}^{N-1} i (k_{i-1} - k_i) . \tag{33}$$

Using the aforementioned map π_N, we define the *Surface defect observable* $\mathcal{S}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q})$ in the statistical model defined by the pseudo-measure μ of (11) via:

$$\mathcal{S}(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathbf{w}, \mathfrak{q}) \,|_{\overline{\Lambda}} := \sum_{\overline{\lambda} \in \pi_N^{-1}(\overline{\Lambda})} \prod_{\omega=0}^{N-1} w_\omega^{(k_{\omega-1}-k_\omega)\,|_{\overline{\lambda}}} \frac{\mu^{\text{orb}}(\tilde{\mathbf{a}}, \tilde{\mathbf{m}}, \varepsilon_1, \tilde{\varepsilon}_2; \overline{\mathfrak{q}}) \,|_{\overline{\lambda}}}{\mu(\mathbf{a}, \mathbf{m}, \varepsilon_1, \varepsilon_2; \mathfrak{q}) \,|_{\overline{\Lambda}}} , \tag{34}$$

where, again with (16) understood,

$$\varepsilon_2 = N \tilde{\varepsilon}_2 , \qquad a_b = \tilde{a}_b - \mathfrak{S}_{\tilde{a}_b} \cdot \tilde{\varepsilon}_2 , \qquad m_f^{\pm} = \tilde{m}_f^{\pm} - \mathfrak{S}_{\tilde{m}_f^{\pm}} \cdot \tilde{\varepsilon}_2 \qquad \text{for } 1 \le b, f \le N . \tag{35}$$

Note that

$$m_f^{\pm} = \varepsilon_1 \left(\mu_{f-1} \pm \delta \mu_{f-1} \right) \tag{36}$$

evoking the notations of (17). The shifts (35) are motivated by the relation between the sheaves on the orbifold $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_N$ and the covering space $\mathbb{C} \times \mathbb{C}$, see [23, 38]. In what follows, we shall not be using the observable (34). Instead, we shall work directly with the pseudo-measure μ^{orb}.

2.5 The key property of $\mathcal{X}_\omega$

The following result [32–37] (whose proof is presented in Appendix A for completeness of our exposition) is a simple consequence of Proposition 2.1:

Proposition 2.2 *The average $\langle \mathcal{X}_\omega(x) \rangle_{\mu^{\text{orb}}}$ is a regular function of x for every $\omega \in \mathbb{Z}_N$.*

For a power series $F(x) = \sum_{\ell=-\infty}^{\infty} F_\ell x^{-\ell}$ and $k \in \mathbb{Z}$, let $\left[x^{-k}\right] F(x)$ denote the coefficient F_k. The regularity property of Proposition 2.2 implies the following result:

$$\left\langle \left[x^{-k}\right] \mathcal{X}_\omega(x) \right\rangle_{\mu^{\mathrm{orb}}} = \left[x^{-k}\right] \left\langle \mathcal{X}_\omega(x) \right\rangle_{\mu^{\mathrm{orb}}} = 0 \qquad \text{for any } k > 0 \text{ and every } \omega \in \mathbb{Z}_N \,. \tag{37}$$

The main point to take home is that the $k = 1$ case of Eq. (37) implies a second-order differential equation on the partition function $\boldsymbol{\Psi}^{\mathrm{inst}}$, viewed as a function of $\mathfrak{q}_0, \ldots, \mathfrak{q}_{N-1}$. This differential equation is the subject of the following subsection.

2.6 The differential operator $\mathcal{D}^{\mathbf{BPS}}$

To apply (37) for $k = 1$, we shall first explicitly compute $[x^{-1}]\mathcal{X}_\omega(x)\,|_{\bar{\lambda}}$. For every $\omega \in \mathbb{Z}_N$, define the observable $c_{\omega,\mathbf{a}} \colon \mathcal{P}^N \to \mathbb{C}$ via:

$$c_{\omega,\mathbf{a}}\,|_{\bar{\lambda}} := \frac{\varepsilon_1}{2} k_\omega|_{\bar{\lambda}} + \sum_{b=1}^{N} \sum_{\square \in \lambda^{(b)}} \delta^{\omega}_{\mathfrak{S}_{\tilde{a}_b + \tilde{c}(\square)}} \cdot (\tilde{a}_b + \tilde{c}(\square)) \,. \tag{38}$$

Recalling (18, 35), so that in particular $a_b = \varepsilon_1 \alpha_{\mathfrak{S}_{\tilde{a}_b}}$ and $\tilde{\kappa} = \tilde{\varepsilon}_2/\varepsilon_1$, we get:

$$\begin{aligned} & Y_\omega(x)\,|_{\bar{\lambda}} \\ & = (x - \varepsilon_1\alpha_\omega - \omega\tilde{\varepsilon}_2) \times \prod_{b=1}^{N} \prod_{\square \in \lambda^{(b)}} \left\{ \left(1 - \frac{\varepsilon_1}{x} - \frac{\varepsilon_1(\tilde{a}_b + \tilde{c}(\square))}{x^2} + O(x^{-3})\right)^{\delta^{\omega}_{\mathfrak{S}_{\tilde{a}_b+\tilde{c}(\square)}}} \right. \\ & \left. \times \left(1 + \frac{\varepsilon_1}{x} + \frac{\varepsilon_1(\tilde{a}_b + \tilde{c}(\square) + \tilde{\varepsilon})}{x^2} + O(x^{-3})\right)^{\delta^{\omega-1}_{\mathfrak{S}_{\tilde{a}_b+\tilde{c}(\square)}}} \right\} , \end{aligned}$$

which implies:

Lemma 2.3 *The large x expansion of the observable $Y_\omega(x)$ has x as a leading term, while the next two coefficients are the observables $\mathcal{P}^N \to \mathbb{C}$ given explicitly by:*

$$\begin{aligned} \varepsilon_1^{-1} \left[x^0\right] Y_\omega(x) &= \mathrm{d}_\omega := k_{\omega-1} - k_\omega - \alpha_\omega - \omega\tilde{\kappa} \,, \\ \varepsilon_1^{-2} \left[x^{-1}\right] Y_\omega(x) &= \frac{\mathrm{d}_\omega^2 - (\alpha_\omega + \omega\tilde{\kappa})^2}{2} + \tilde{\kappa} k_{\omega-1} + \frac{c_{\omega-1,\boldsymbol{a}} - c_{\omega,\boldsymbol{a}}}{\varepsilon_1} \,. \end{aligned}$$

As an immediate corollary, using notations (2, 17, 18), we obtain:

Proposition 2.4 *The observable* $\left[x^{-1}\right]\mathcal{X}_\omega(x) \colon \mathcal{P}^N \to \mathbb{C}$ *is explicitly given by:*

$$\begin{aligned}\varepsilon_1^{-2}\left[x^{-1}\right]\mathcal{X}_\omega(x) = & \frac{\left(c_{\omega,\boldsymbol{a}} - c_{\omega+1,\boldsymbol{a}}\right) - \mathfrak{q}_\omega\left(c_{\omega-1,\boldsymbol{a}} - c_{\omega,\boldsymbol{a}}\right)}{\varepsilon_1} \\ & + \tilde{\kappa}\left(k_\omega - \mathfrak{q}_\omega k_{\omega-1}\right) + \mathfrak{q}_\omega\left((\mathrm{d}_\omega + \mu_\omega + \omega\tilde{\kappa})^2 - \delta\mu_\omega^2 - \mathrm{d}_\omega^2\right) \\ & + \frac{1}{2}\left(\mathrm{d}_{\omega+1}^2 + \mathfrak{q}_\omega \mathrm{d}_\omega^2 + \mathfrak{q}_\omega(\alpha_\omega + \omega\tilde{\kappa})^2 - (\alpha_{\omega+1} + (\omega+1)\tilde{\kappa})^2\right). \end{aligned} \tag{39}$$

To get rid of the observables $c_{\omega,\mathbf{a}}$'s (38) in the right-hand side of (39), we introduce, following [32–37], the functions $\{U_\omega\}_{\omega\in\mathbb{Z}_N}$ via:

$$U_\omega = 1 + \mathfrak{q}_{\omega+1} + \mathfrak{q}_{\omega+1}\mathfrak{q}_{\omega+2} + \cdots + \mathfrak{q}_{\omega+1}\cdots\mathfrak{q}_{\omega-1}\,, \tag{40}$$

with the conventions $U_{\omega+N} = U_\omega$ being used. They provide a (unique up to a common factor) solution of the following linear system:

$$(1+\mathfrak{q}_\omega)\cdot U_\omega - U_{\omega-1} - \mathfrak{q}_{\omega+1}\cdot U_{\omega+1} = 0 \quad \text{for any } \omega\in\mathbb{Z}_N\,. \tag{41}$$

We also note that

$$U_\omega - \mathfrak{q}_{\omega+1}\cdot U_{\omega+1} = 1-\mathfrak{q} \quad \text{for any } \omega\in\mathbb{Z}_N\,.$$

Due to the key property (41) of U_ω's, the coefficient of x^{-1} in the observable $\sum_{\omega\in\mathbb{Z}_N} U_\omega\,\mathcal{X}_\omega(x)$ is a degree-two polynomial in the *instanton charges* $\{k_\omega\}_{\omega\in\mathbb{Z}_N}$. Therefore,

$$\left\langle \left[x^{-1}\right]\left(\sum_{\omega\in\mathbb{Z}_N} U_\omega\,\mathcal{X}_\omega(x)\right)\right\rangle_{\mu^{\mathrm{orb}}} = D^{\mathrm{inst}}\left(\boldsymbol{\Psi}^{\mathrm{inst}}\right)$$

with D^{inst}, a second-order differential operator in $\mathfrak{q}_\omega$'s, naturally arising from the equality

$$\left\langle \prod_{\omega\in\mathbb{Z}_N} k_\omega^{r_\omega}\right\rangle_{\mu^{\mathrm{orb}}} = \frac{\prod_{\omega\in\mathbb{Z}_N}\left(\mathfrak{q}_\omega\frac{\partial}{\partial\mathfrak{q}_\omega}\right)^{r_\omega}\boldsymbol{\Psi}^{\mathrm{inst}}(\tilde{\mathbf{a}},\tilde{\mathbf{m}},\varepsilon_1,\tilde{\varepsilon}_2;\overline{\mathfrak{q}})}{\boldsymbol{\Psi}^{\mathrm{inst}}(\tilde{\mathbf{a}},\tilde{\mathbf{m}},\varepsilon_1,\tilde{\varepsilon}_2;\overline{\mathfrak{q}})}\,, \tag{42}$$

due to (23, 24). We can further express D^{inst} as a differential operator in $\mathfrak{q}$ and w_ω's by using

$$\mathfrak{q}\frac{\partial}{\partial\mathfrak{q}} = \mathfrak{q}_{N-1}\frac{\partial}{\partial\mathfrak{q}_{N-1}}\,, \quad w_\omega\frac{\partial}{\partial w_\omega} = \mathfrak{q}_{\omega-1}\frac{\partial}{\partial\mathfrak{q}_{\omega-1}} - \mathfrak{q}_\omega\frac{\partial}{\partial\mathfrak{q}_\omega} \quad \text{for any } \omega\in\mathbb{Z}_N\,. \tag{43}$$

It is convenient to introduce the normalized partition function $\boldsymbol{\Psi}$ via:

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}^{\mathrm{tree}}\cdot\boldsymbol{\Psi}^{\mathrm{inst}}\,, \tag{44}$$

where

$$\boldsymbol{\Psi}^{\rm tree} \ := \ \mathfrak{q}^{-\frac{1}{2\kappa}\sum_{\omega=0}^{N-1}\alpha_\omega^2} \cdot \prod_{\omega=0}^{N-1} w_\omega^{\mu_\omega-\alpha_\omega} \,. \tag{45}$$

Combining Propositions 2.2, 2.4 with formulae (41) and (42), we get (cf. [32, 37]):

Theorem 2.5 *The normalized partition function* $\boldsymbol{\Psi} = \boldsymbol{\Psi}(\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{m}}, \varepsilon_1, \tilde{\varepsilon}_2; \boldsymbol{w}, \mathfrak{q})$ *of* (44) *satisfies the equation*

$$\mathcal{D}^{\rm BPS}(\boldsymbol{\Psi}) \ = \ 0$$

with the second-order differential operator $\mathcal{D}^{\rm BPS}$ *explicitly given by [cf.* (2)*]*

$$\mathcal{D}^{\rm BPS} \ = \kappa\frac{\partial}{\partial\mathfrak{q}} + \frac{\hat{H}_0}{\mathfrak{q}} + \frac{\hat{H}_1}{\mathfrak{q}-1}\,, \tag{46}$$

where $\hat{H}_0$, $\hat{H}_1$ *are the second-order differential operators in* w_ω*'s, independent of* $\mathfrak{q}$ *and* α_ω*'s:*

$$\begin{aligned} \hat{H}_0 &= \sum_{\omega=0}^{N-1}\left\{\sum_{\omega'=\omega+1}^{N-1}\frac{w_{\omega'}}{w_\omega}\left(D_\omega^2-\delta\mu_\omega^2\right)+\frac{1}{2}\left(D_\omega-\mu_\omega\right)^2\right\}, \\ \hat{H}_1 &= -\sum_{\omega',\omega=0}^{N-1}\frac{w_{\omega'}}{w_\omega}\left(D_\omega^2-\delta\mu_\omega^2\right)\,, \end{aligned} \tag{47}$$

with

$$D_\omega = w_\omega\frac{\partial}{\partial w_\omega}\,. \tag{48}$$

Remark 2.3 Note that $\boldsymbol{\Psi}^{\rm inst}$ is a single-valued homogeneous function of w_ω's. If we wrote the differential equation obeyed by $\boldsymbol{\Psi}^{\rm inst}$ in the original variables $\mathfrak{q}_0, \ldots, \mathfrak{q}_{N-2}, \mathfrak{q}_{N-1}$, it would not contain any ambiguity due to the redundant nature of the variables $w_0, \ldots, w_{N-1}$. However, the equations written in the invariant variables, such as the variables v_i introduced below, look more complicated. Conversely, by introducing more degrees of freedom with additional symmetries, modifying accordingly the prefactor $\boldsymbol{\Psi}^{\rm tree}$, one arrives at a very simple form of the operators $\hat{H}_0$, $\hat{H}_1$, cf. Theorem 3.1. This is known as the *projection method* in the theory of many-body systems [43].

Remark 2.4 The normalized partition function $\boldsymbol{\Psi}$ obeys:

$$\sum_{\omega=0}^{N-1} D_\omega\left(\boldsymbol{\Psi}\right) = \sum_{\omega=0}^{N-1}\left(\mu_\omega-\alpha_\omega\right)\cdot\,\boldsymbol{\Psi}\,. \tag{49}$$

The operators $\hat{H}_0$, $\hat{H}_1$ in (47) are therefore defined up to addition of the second-order differential operators of the form

$$\mathfrak{D}_1\sum_{\omega=0}^{N-1}\left(D_\omega+\alpha_\omega-\mu_\omega\right) \tag{50}$$

with a first-order differential operator $\mathfrak{D}_1$. The choice (47) is uniquely characterized by its α_ω-independence, for any ω.

2.7 One more coordinate change

For the purpose of the next section, it will be convenient to use the coordinates

$$v_i = \frac{w_{i-1}}{w_0 + w_1 + \cdots + w_{N-1}}\,, \qquad i = 1, \ldots, N-1\,, \tag{51}$$

and the associated quantities

$$u_i = \sum_{j=i+1}^{N} v_j\,, \qquad i = 0, \ldots, N-1\,, \tag{52}$$

with

$$v_N \equiv 1 - \sum_{i=1}^{N-1} v_i \qquad \text{and} \qquad u_N \equiv 0\,. \tag{53}$$

Define the $\mathbb{C}[[v_1^{\pm 1}, v_2^{\pm 1}, \ldots, v_{N-1}^{\pm 1}]]$-valued power series in $\mathfrak{q}$ by:

$$\psi(v_1, v_2, \ldots, v_{N-1}; \mathfrak{q}) = \boldsymbol{\Psi}^{\text{inst}}(v_2/v_1, v_3/v_2, \ldots, v_N/v_{N-1}, \mathfrak{q}v_1/v_N)\,, \tag{54}$$

where we intentionally omit the parameters $\tilde{\mathbf{a}}, \tilde{\mathbf{m}}, \varepsilon_1, \tilde{\varepsilon}_2$ in the right-hand side and note that

$$v_2/v_1 = \mathfrak{q}_0,\ v_3/v_2 = \mathfrak{q}_1,\ \ldots\ ,\ \mathfrak{q}v_1/v_N = \mathfrak{q}_{N-1}\,.$$

The following is a straightforward reformulation of Theorem 2.5 in the present setting:

Theorem 2.6 *The function* $\psi = \psi(v_1, v_2, \ldots, v_{N-1}; \mathfrak{q})$ *satisfies the equation*

$$\nabla^{\text{bps}}(\psi) = 0 \tag{55}$$

with

$$\nabla^{\text{bps}} = \kappa \frac{\partial}{\partial \mathfrak{q}} + \frac{\hat{h}_0^{\text{bps}}}{\mathfrak{q}} + \frac{\hat{h}_1^{\text{bps}}}{\mathfrak{q} - 1} \tag{56}$$

with the residues of the meromorphic connection ∇^{bps} *at* $\mathfrak{q} = 0$ *and* $\mathfrak{q} = 1$ *having the decomposition:*

$$\hat{h}_0^{\text{bps}} = \hat{h}_{0,\text{kin}}^{\text{bps}} + \hat{h}_{0,\text{mag}}^{\text{bps}} + \hat{h}_{0,\text{pot}}^{\text{bps}}\,, \qquad \hat{h}_1^{\text{bps}} = \hat{h}_{1,\text{kin}}^{\text{bps}} + \hat{h}_{1,\text{mag}}^{\text{bps}} + \hat{h}_{1,\text{pot}}^{\text{bps}}\,, \tag{57}$$

with the kinetic, magnetic, and potential terms given by:

$$
\begin{aligned}
&\hat{h}^{\text{bps}}_{0,\text{kin}} = \frac{1}{2}\text{D}^2 + \sum_{i=1}^{N-1} \left(u_i + \frac{v_i}{2}\right)\left(v_i^{-1}\text{D}_i - 2\text{D}\right)\text{D}_i\,, \qquad \hat{h}^{\text{bps}}_{1,\text{kin}} = \text{D}^2 - \sum_{i=1}^{N-1} v_i^{-1}\text{D}_i^2\,,\\
&\hat{h}^{\text{bps}}_{0,\text{mag}} = \left(\alpha_{N-1} + 1 - N + \sum_{i=1}^{N-1}(N - i - \alpha_{N-1})v_i\right)\text{D}\\
&\qquad + 2\sum_{i=1}^{N-1}\left(\mu_{i-1}u_i - \alpha_{i-1}\left(u_i + \frac{v_i}{2}\right)\right)\left(v_i^{-1}\text{D}_i - \text{D}\right),\\
&\hat{h}^{\text{bps}}_{1,\text{mag}} = (N - 1 + 2\mu_{N-1} - 2\alpha_{N-1})\,\text{D} - 2\sum_{i=1}^{N-1}(\mu_{i-1} - \alpha_{i-1})\left(v_i^{-1}\text{D}_i - \text{D}\right),\\
&\hat{h}^{\text{bps}}_{0,\text{pot}} = \sum_{i=1}^{N-1} u_i\,\frac{(\mu_{i-1} - \alpha_{i-1})^2 - \delta\mu_{i-1}^2}{v_i}\,, \qquad \hat{h}^{\text{bps}}_{1,\text{pot}} = -\sum_{a=1}^{N}\frac{(\mu_{a-1} - \alpha_{a-1})^2 - \delta\mu_{a-1}^2}{v_a}\,,
\end{aligned}
\tag{58}
$$

where we defined

$$
\text{D}_\text{i} = v_i\frac{\partial}{\partial v_i}\,, \qquad i = 1, \dots, N-1\,, \tag{59}
$$

and

$$
\text{D} = \sum_{i=1}^{N-1}\text{D}_i\,. \tag{60}
$$

Remark 2.5 The operator ∇^{bps} of (56) depends, explicitly, on $\vec{\mu}$, $\delta\vec{\mu}$, $\vec{\alpha}$. However, Theorem 2.5 shows that the $\vec{\alpha}$ dependence is a pure gauge:

$$
Y^{-1}\,\nabla^{\text{bps}}\,Y \qquad \text{is} \qquad \vec{\alpha}\text{−independent}\,, \tag{61}
$$

where [cf. (45)]

$$
Y = \mathfrak{q}^{\frac{1}{2\kappa}\sum_{\omega=0}^{N-1}\alpha_\omega^2}\cdot\prod_{i=1}^{N} v_i^{\alpha_{i-1}-\mu_{i-1}}\,. \tag{62}
$$

3 The CFT side, or the projection method

The operator $\hat{h}^{\text{bps}}_0/\mathfrak{q} + \hat{h}^{\text{bps}}_1/(\mathfrak{q}-1)$ of (56) can be viewed as a time-dependent Hamiltonian of a quantum mechanical system with $N-1$ degrees of freedom $v_1, \dots, v_{N-1}$. The parameters $\vec{\mu} = (\mu_0, \dots, \mu_{N-1})$, $\delta\vec{\mu} = (\delta\mu_0, \dots, \delta\mu_{N-1})$ play the rôle of the coupling constants, while the parameters $\vec{\alpha} = (\alpha_0, \dots, \alpha_{N-1})$ play the rôle of the spectral parameters, such as the asymptotic momenta of N particles, in the center-of-mass frame, where the interactions between the particles can be neglected.

The BPS/CFT correspondence [29, 30] suggests to look for the representation-theoretic realization of the operators $\hat{h}^{\text{bps}}_0$ and $\hat{h}^{\text{bps}}_1$.

We present such a realization below.

3.1 Flags, co-flags, lines, and co-lines

Let $W \approx \mathbb{C}^N$ be the complex vector space of dimension N, and let W^* denote its dual. Let $F(W)$, $F(W^*)$, $\mathbb{P}(W)$, $\mathbb{P}(W^*)$ denote the space of complete flags in W, the space of complete flags in W^*, the projective space of lines in W, and the projective space of lines in W^*, respectively. The natural action of the general linear group $GL(W)$ on W and W^* gives rise to canonical actions of $GL(W)$ on those four projective varieties. Let J^a_b, $\tilde{J}^a_b$, V^a_b, $\tilde{V}^a_b$, with $a, b = 1, \ldots, N$, denote the vector fields on $F(W)$, $F(W^*)$, $\mathbb{P}(W)$, $\mathbb{P}(W^*)$, respectively, representing those actions. Here, to define those vector fields, we need to choose some basis $\{e_a\}_{a=1}^N$ in W, with the dual basis in W^* denoted by $\{\tilde{e}^b\}_{b=1}^N$, so that the operators

$$T^a_b = e_b \otimes \tilde{e}^a \in \mathrm{End}(W) \tag{63}$$

represent the action of the Lie algebra of $GL(W)$ on W. They obey the $\mathfrak{gl}_N$ commutation relations:

$$\left[T^a_b, T^{a'}_{b'}\right] = \delta^a_{b'} T^{a'}_b - \delta^{a'}_b T^a_{b'} \tag{64}$$

to which we shall refer in what follows.

We define the second-order differential operators $\hat{h}_0$, $\hat{h}_1$ on the product

$$\mathcal{X} = F(W) \times F(W^*) \times \mathbb{P}(W) \times \mathbb{P}(W^*) \tag{65}$$

by

$$\hat{h}_0 = \sum_{a,b=1}^N J^a_b V^b_a , \qquad \hat{h}_1 = \sum_{a,b=1}^N V^a_b \tilde{V}^b_a . \tag{66}$$

These operators are independent of the choice of the basis in W and are globally well defined on $\mathcal{X}$. Furthermore, they commute with the diagonal action of $GL(W)$ on $\mathcal{X}$:

$$\left[J^a_b + \tilde{J}^a_b + V^a_b + \tilde{V}^a_b, \hat{h}_p\right] = 0 , \qquad a, b = 1, \ldots, N , \quad p = 0, 1 . \tag{67}$$

Note that the center of $GL(W)$ acts trivially on $\mathcal{X}$, hence a natural action of $PGL(W)$ on $\mathcal{X}$.

3.2 The *v*-coordinates

Let us now endow W with the volume form $\varpi \in \Lambda^N W^*$. Denote

$$\tilde{\pi}^N = \varpi , \qquad \pi_N = \varpi^{-1} \in \Lambda^N W . \tag{68}$$

Let $H = SL(W, \varpi) \approx SL(N, \mathbb{C})$ denote the group of linear transformations of W preserving ϖ. The center $Z(H) \simeq \mathbb{Z}_N$ of $H \subset GL(W)$ is finite and acts trivially on $\mathcal{X}$. There is an H-invariant open subset $\mathcal{X}^\circ$ (described in (78)) of $\mathcal{X}$, on which the action of $H/Z(H)$ is free. The corresponding quotient $\mathcal{X}^{\circ}/H$ can be coordinatized by the values of $N-1$ functions $v_1, \ldots, v_{N-1}$, defined as follows:

$$v_i\left(w, \tilde{w}, \mathrm{z}, \tilde{\mathrm{z}}\right) = \frac{\left(\tilde{\mathrm{z}} \wedge \tilde{\pi}^{i-1}\right)(\pi_i) \cdot \tilde{\pi}^i\left(\mathrm{z} \wedge \pi_{i-1}\right)}{\tilde{\mathrm{z}}(\mathrm{z}) \cdot \tilde{\pi}^{i-1}(\pi_{i-1}) \cdot \tilde{\pi}^i(\pi_i)}, \qquad i = 1, \ldots, N-1, \tag{69}$$

where

$$\left(w = (W_i)_{i=1}^{N-1}, \; \tilde{w} = \left(\tilde{W}_i\right)_{i=1}^{N-1}, \; \mathrm{z}, \; \tilde{\mathrm{z}}\right) \in \mathcal{X}^{\circ}$$

is the collection consisting of a pair

$$\begin{aligned} w: &\quad 0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_{N-1} \subset W_N \equiv W \in F(W), \\ \tilde{w}: &\quad 0 = \tilde{W}_0 \subset \tilde{W}_1 \subset \tilde{W}_2 \subset \ldots \subset \tilde{W}_{N-1} \subset \tilde{W}_N \equiv W^* \in F(W^*) \end{aligned} \tag{70}$$

of flags in W and W^*, respectively, and another pair

$$\mathbb{C}\mathrm{z} \subset W, \qquad \mathbb{C}\tilde{\mathrm{z}} \subset W^* \tag{71}$$

of lines in W and W^*; and finally,

$$\pi_i = \Lambda^i W_i \subset \Lambda^i W, \qquad \tilde{\pi}^i = \Lambda^i \tilde{W}_i \subset \Lambda^i W^* \tag{72}$$

are the corresponding i-polyvector and the i-form on W, both defined up to a scalar multiplier. Note that these scalar factor ambiguities cancel out in (69).

We can also view v_i's as meromorphic functions on $\mathcal{X}/H$. To this end, we promote $\pi_i, \tilde{\pi}^i, \mathrm{z}, \tilde{\mathrm{z}}$ to global objects, the canonical holomorphic sections of the corresponding vector bundles:

$$\Pi_i \in H^0\left(F(W), \Lambda^i W \otimes \det(W_i)^{-1}\right), \qquad \tilde{\Pi}^i \in H^0\left(F(W^*), \Lambda^i W^* \otimes \det(\tilde{W}_i)\right), \tag{73}$$

and

$$Z \in H^0\left(\mathbb{P}(W), W \otimes \mathcal{O}(1)\right) \approx W \otimes W^*, \qquad \tilde{Z} \in H^0\left(\mathbb{P}(W^*), W^* \otimes \mathcal{O}(1)\right) \approx W^* \otimes W, \tag{74}$$

and define

$$v_i = \frac{\left(\tilde{Z} \wedge \tilde{\Pi}^{i-1}\right)(\Pi_i) \cdot \tilde{\Pi}^i\left(Z \wedge \Pi_{i-1}\right)}{\tilde{Z}(Z) \cdot \tilde{\Pi}^i(\Pi_i) \cdot \tilde{\Pi}^{i-1}(\Pi_{i-1})}, \qquad i = 1, \ldots, N-1. \tag{75}$$

We also note that while (69, 75) can be extended to $i = N$, the corresponding quantity v_N satisfies

$$\sum_{a=1}^{N} v_a = 1, \tag{76}$$

due to the Desnanot–Jacobi–Dodgson–Sylvester theorem, which states that

$$v_{a+1} = u_a - u_{a+1}\,, \qquad u_a = \frac{\left(\tilde{Z} \wedge \tilde{\Pi}^a\right)(Z \wedge \Pi_a)}{\tilde{Z}(Z) \cdot \tilde{\Pi}^a(\Pi_a)}\,, \qquad a = 0, \dots, N-1\,. \tag{77}$$

The open set $\mathfrak{X}^\circ \subset \mathfrak{X}$ has the following description: there exists a basis e_a in W such that

$$\begin{aligned} &W_i = \mathrm{Span}(e_1, \dots, e_i)\,, \qquad \tilde{W}_i = \mathrm{Span}(\tilde{e}^1, \dots, \tilde{e}^i)\,, \\ &Z = \sum_{a=1}^{N} \xi_a e_a\,, \qquad \tilde{Z} = \sum_{a=1}^{N} \xi_a \tilde{e}^a\,, \qquad \xi_a \neq 0. \end{aligned} \tag{78}$$

We note that the aforementioned equality (76) is obvious in this basis, since

$$v_a = \frac{\xi_a^2}{\xi_1^2 + \dots + \xi_N^2}\,, \qquad a = 1, \dots, N. \tag{79}$$

Remark 3.1 The flag varieties $F(W)$ and $F(W^*)$ are isomorphic. For example, the assignment $W_i = \tilde{W}_{N-i}^{\perp}$ gives rise to an isomorphism $F(W^*) \xrightarrow{\sim} F(W)$. Alternatively, fixing the volume form $\varpi \in \Lambda^N W^*$, we have an $SL(W)$-equivariant isomorphism $F(W) \xrightarrow{\sim} F(W^*)$ given by:

$$\tilde{\pi}^i = \varpi(\pi_{N-i})\,, \qquad i = 1, \dots, N-1\,. \tag{80}$$

Remark 3.2 In the $N = 2$ case, we have $F(W) \simeq F(W^*) \simeq \mathbb{P}(W) \simeq \mathbb{P}(W^*)$, and the only nontrivial coordinate v_1 of (69) is determined by the usual cross-ratio of four points on $\mathbb{CP}^1$. More precisely, if $z_1, z_2, z_3, z_4 \in W$ are defined (each up to a scalar multiplier) by:

$$z_1 = \pi_1\,,\ \varpi(z_2, \cdot) = \tilde{z}\,,\ \varpi(z_3, \cdot) = \tilde{\pi}^1\,,\ z_4 = z\,, \tag{81}$$

then

$$v_1 = \frac{\varpi(z_2, z_1)\varpi(z_3, z_4)}{\varpi(z_3, z_1)\varpi(z_2, z_4)} \tag{82}$$

depends only on the four points $\mathbb{C}z_i \in \mathbb{P}(W)$.

3.3 The $\mathfrak{L}$-twist

Let $L_1, \dots, L_{N-1}$ denote the tautological line bundles over $F(W)$, the fiber of L_i over the point $0 = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_{N-1} \subset W_N \equiv W$ being

$$L_i = W_i / W_{i-1}\,, \qquad i = 1, \dots, N-1\,. \tag{83}$$

Similarly, let $\tilde{L}^1, \dots, \tilde{L}^{N-1}$ denote the tautological line bundles over $F(W^*)$, and

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(W)}(-1)\,, \qquad \tilde{\mathcal{L}} = \mathcal{O}_{\mathbb{P}(W^*)}(-1) \tag{84}$$

be the tautological line bundles over $\mathbb{P}(W)$, $\mathbb{P}(W^*)$, respectively. We note that

$$\det(W_a) = \Lambda^a W_a \simeq \bigotimes_{i=1}^{a} L_i\,, \qquad \det(\tilde{W}_a) = \Lambda^a \tilde{W}_a \simeq \bigotimes_{i=1}^{a} \tilde{L}^i\,, \qquad a = 1, \dots, N-1\,.$$

All these line bundles are $GL(W)$-equivariant. By abuse of notation, we shall use the same notations for the pull-backs of the aforementioned line bundles to $\mathcal{X}$ of (65) under the natural projections. The line bundles $\tilde{\mathcal{L}}^{-1} \otimes \Lambda^{a-1}\tilde{W}_{a-1} \otimes (\Lambda^a W_a)^{-1}$, $\Lambda^a \tilde{W}_a \otimes \mathcal{L}^{-1} \otimes (\Lambda^{a-1} W_{a-1})^{-1}$, $\Lambda^i \tilde{W}_i \otimes (\Lambda^i W_i)^{-1}$, and $\tilde{\mathcal{L}}^{-1} \otimes \mathcal{L}^{-1}$ on $\mathcal{X}$ are H-invariant (and those with $a < N$ are actually $GL(W)$-invariant). Furthermore, each factor in formula (75) can be viewed as a holomorphic section of one of those line bundles. For example,

$$\tilde{\Pi}^a (Z \wedge \Pi_{a-1}) \tag{85}$$

is a holomorphic section of $\det(\tilde{W}_a) \otimes \mathcal{L}^{-1} \otimes \det(W_{a-1})^{-1}$. Its zeroes determine the locus in $\mathcal{X}$ where the plane W_{a-1}, the line $\mathbb{C}z$, and the plane $\tilde{W}_a^{\perp} \subset W$ are not in general position, i.e., their linear span does not coincide with the entire W. Let $\Sigma \subset \mathcal{X}^{\circ}$ denote the union of vanishing loci of $\tilde{\Pi}^a(Z \wedge \Pi_{a-1})$, $(\tilde{Z} \wedge \tilde{\Pi}^{a-1})(\Pi_a)$, $\tilde{\Pi}^i(\Pi_i)$ for $a = 1, \dots, N$ and $i = 1, \dots, N-1$.

For $\vec{n}, \vec{\tilde{n}} \in \mathbb{C}^N$, $\vec{\gamma} \in \mathbb{C}^{N-1}$, consider the tensor product of "complex powers of line bundles"

$$\begin{aligned}
\mathfrak{L} &= \bigotimes_{i=1}^{N-1} \Big(\det(W_i)\Big)^{-\nu_i} \otimes \bigotimes_{i=1}^{N-1} \Big(\det(\tilde{W}_i)\Big)^{\tilde{\nu}_i} \otimes \mathcal{L}^{-\mathrm{m}} \otimes \tilde{\mathcal{L}}^{-\tilde{\mathrm{m}}} \\
&= \bigotimes_{a=1}^{N} \Big(\tilde{\mathcal{L}}^{-1} \otimes \det(\tilde{W}_{a-1}) \otimes \det(W_a)^{-1}\Big)^{\tilde{n}_a} \otimes \Big(\det(\tilde{W}_a) \otimes \mathcal{L}^{-1} \otimes \det(W_{a-1})^{-1}\Big)^{n_a} \\
&\quad \otimes \bigotimes_{i=1}^{N-1} \Big(\det(\tilde{W}_i) \otimes \det(W_i)^{-1}\Big)^{\gamma_i - n_i - \tilde{n}_i}
\end{aligned} \tag{86}$$

defined on any simply-connected open domain $\mathcal{U} \subset (\mathcal{X}^{\circ} \backslash \Sigma)\,/H$. Here, the complex numbers $\mathrm{m}, \tilde{\mathrm{m}} \in \mathbb{C}$ and the vectors $\vec{\nu}, \vec{\tilde{\nu}} \in \mathbb{C}^{N-1}$ are defined via:

$$\begin{aligned}
&\mathrm{m} = \sum_{a=1}^{N} n_a\,, \quad \tilde{\mathrm{m}} = \sum_{a=1}^{N} \tilde{n}_a\,, \\
&\nu_i = n_{i+1} - n_i + \gamma_i\,, \qquad \tilde{\nu}_i = \tilde{n}_{i+1} - \tilde{n}_i + \gamma_i\,, \qquad i = 1, \dots, N-1\,.
\end{aligned} \tag{87}$$

Our main result is:

Theorem 3.1 *The operators* $\hat{h}_0^{\mathrm{bps}}, \hat{h}_1^{\mathrm{bps}}$ *of* (56) *coincide with the operators* $\hat{h}_0^{\mathrm{cft}}, \hat{h}_1^{\mathrm{cft}}$*, which are* $\hat{h}_0, \hat{h}_1$ *of* (66)*, viewed now as the differential operators on* $\mathfrak{X}^{\circ}/H$*, twisted by the "line bundle"* $\mathfrak{L}$*:*

$$\hat{h}_p^{\mathrm{cft}} = \Upsilon^{-1}\, \hat{h}_p\, \Upsilon\,, \qquad p = 0, 1\,, \tag{88}$$

where

$$\Upsilon = \prod_{a=1}^{N} \left(\frac{\left(\tilde{Z} \wedge \tilde{\Pi}^{a-1}\right)(\Pi_a)}{\tilde{\Pi}^a(\Pi_a)} \right)^{\tilde{n}_a} \cdot \left(\frac{\tilde{\Pi}^a\,(Z \wedge \Pi_{a-1})}{\tilde{\Pi}^a(\Pi_a)} \right)^{n_a} \cdot \prod_{i=1}^{N-1} \left(\tilde{\Pi}^i\,(\Pi_i)\right)^{\gamma_i} \tag{89}$$

is the holomorphic section of $\mathfrak{L}$ *on* $\mathfrak{U}$*. The parameters* $\vec{n}, \vec{\tilde{n}}, \vec{\gamma}$ *are related to the parameters* $\vec{\mu}, \delta\vec{\mu}$ *and* $\vec{\alpha}$ *(which encode the mass parameters* $\boldsymbol{m}$ *and the Coulomb parameters* $\boldsymbol{a}$ *via* (17, 36) *and* (18)*, respectively) as follows:*

$$\begin{aligned} n_b &= \mu_{b-1} + \delta\mu_{b-1} - \alpha_{b-1}\,, \\ \tilde{n}_b &= \mu_{b-1} - \delta\mu_{b-1} - \alpha_{b-1}\,, \\ \gamma_i &= -1 - \alpha_{i-1} + \alpha_i\,, \end{aligned} \tag{90}$$

for $b = 1, \ldots, N$ *and* $i = 1, \ldots, N-1$.

For future use, let us record the relation between the parameters of the gauge theory and the parameters $\vec{\nu}, \vec{\tilde{\nu}}, \mathrm{m}, \tilde{\mathrm{m}}$ of (87):

$$\begin{aligned} \varepsilon_1 \nu_i &= m_{i+1}^{+} - m_i^{+} - \varepsilon_1\,, & \varepsilon_1 \tilde{\nu}_i &= m_{i+1}^{-} - m_i^{-} - \varepsilon_1\,, \\ \varepsilon_1 \mathrm{m} &= \sum_{f=1}^{N} m_f^{+} - \sum_{b=1}^{N} a_b\,, & \varepsilon_1 \tilde{\mathrm{m}} &= \sum_{f=1}^{N} m_f^{-} - \sum_{b=1}^{N} a_b\,, \end{aligned} \tag{91}$$

where we used (18, 36) and the second formula of (35).

3.4 Proof of Theorem 3.1

The vector fields $V_b^a, \tilde{V}_b^a$ can be explicitly written in the homogeneous coordinates $(\mathrm{z}^1 : \mathrm{z}^2 : \cdots : \mathrm{z}^N)$ on $\mathbb{P}(W)$ and $(\tilde{\mathrm{z}}_1 : \tilde{\mathrm{z}}_2 : \cdots : \tilde{\mathrm{z}}_N)$ on $\mathbb{P}(W^*)$:

$$V_a^b = -\mathrm{z}^b \frac{\partial}{\partial \mathrm{z}^a}\,, \qquad \tilde{V}_a^b = \tilde{\mathrm{z}}_a \frac{\partial}{\partial \tilde{\mathrm{z}}_b}\,, \tag{92}$$

so that $\hat{h}_1$ of (66) is explicitly given by:

$$\hat{h}_1 = -\tilde{\mathrm{z}}(\mathrm{z}) \cdot \sum_{a=1}^{N} \frac{\partial^2}{\partial \mathrm{z}^a \partial \tilde{\mathrm{z}}_a}\,, \tag{93}$$

where

$$\tilde{z}(z) = \sum_{a=1}^{N} \tilde{z}_a z^a \,. \tag{94}$$

The minus sign in (92) in the formula for V_a^b does match the commutation relations (64). This minus sign is due to the fact that the vector space of polynomials in z^a's is the symmetric algebra built on W^*, while that of polynomials in $\tilde{z}_a$'s is built on W. Thus, (92) is the infinitesimal version of the group action, where $h \in GL(W)$ acts on $f = f(z)$, $\tilde{f} = \tilde{f}(\tilde{z})$ via $f \mapsto f^h$, $\tilde{f} \mapsto \tilde{f}^h$:

$$f^h(z) = f(h^{-1} \cdot z) \,, \qquad \tilde{f}^h(\tilde{z}) = \tilde{f}(\tilde{z} \cdot h) \,. \tag{95}$$

As for J_b^a, $\tilde{J}_b^a$, let us first recall the quiver description of the flag varieties $F(W)$, $F(W^*)$. Let $F_1, \tilde{F}_1, \ldots, F_{N-1}, \tilde{F}_{N-1}$ be the sequence of complex vector spaces with $\dim F_i = \dim \tilde{F}_i = i$. Consider the vector spaces of linear maps:

$$\mathcal{A} = \bigoplus_{i=1}^{N-1} \mathrm{Hom}(F_i, F_{i+1}) \,, \tag{96}$$

$$\tilde{\mathcal{A}} = \bigoplus_{i=1}^{N-1} \mathrm{Hom}(\tilde{F}_{i+1}, \tilde{F}_i) \,, \tag{97}$$

where we set $F_N = W$ and $\tilde{F}_N = W$. Consider the groups

$$\mathcal{G} = \prod_{i=1}^{N-1} GL(F_i) \,, \qquad \tilde{\mathcal{G}} = \prod_{i=1}^{N-1} GL(\tilde{F}_i) \tag{98}$$

of linear transformations of the respective vector spaces. The groups $\mathcal{G}$, $\tilde{\mathcal{G}}$ act on $\mathcal{A}$, $\tilde{\mathcal{A}}$, respectively, in the natural way:

$$\begin{aligned} (g_i)_{i=1}^{N-1} &: (U_i)_{i=1}^{N-1} \in \mathcal{A} \mapsto \left(g_{i+1} U_i g_i^{-1} \right)_{i=1}^{N-1} \in \mathcal{A} \,, \\ (\tilde{g}_i)_{i=1}^{N-1} &: \left(\tilde{U}_i \right)_{i=1}^{N-1} \in \tilde{\mathcal{A}} \mapsto \left(\tilde{g}_i \tilde{U}_i \tilde{g}_{i+1}^{-1} \right)_{i=1}^{N-1} \in \tilde{\mathcal{A}} \,, \end{aligned} \tag{99}$$

where $g_i \in GL(F_i)$, $U_i : F_i \to F_{i+1}$, $\tilde{g}_i \in GL(\tilde{F}_i)$, $\tilde{U}_i : \tilde{F}_{i+1} \to \tilde{F}_i$, and $g_N, \tilde{g}_N$ are vacuous. Then, the flag variety $F(W)$ is the quotient of the open subvariety $\mathcal{A}^s$ of $\mathcal{A}$, consisting of the collections $(U_i)_{i=1}^{N-1}$ for which the composition $U_{N-1} U_{N-2} \cdots U_i : F_i \to W$ has no kernel for any $i = 1, \ldots, N-1$, by the free action of $\mathcal{G}$:

$$F(W) = \mathcal{A}^s / \mathcal{G} \,. \tag{100}$$

We can represent the π_i's of (72), in coordinates, as:

$$\pi_i \;=\; \sum_{1\le a_1<a_2<\ldots<a_i\le N} \mathrm{Det}\Big\| \Big[U_{N-1}U_{N-2}\cdots U_i\Big]_{\ell}^{a_k} \Big\|_{k,\ell=1}^{i} \, e_{a_1}\wedge\cdots\wedge e_{a_i}\,. \tag{101}$$

Here, $\Big[U_{N-1}U_{N-2}\cdots U_i\Big]_{\ell}^{a_k}$ denote the matrix coefficients of the corresponding linear operator with respect to some bases $\{\varepsilon_\ell^{(i)}\}_{\ell=1}^{i}$ in F_i and the chosen basis $\{e_a\}_{a=1}^{N}$ in W. Note that the group $\mathcal{G}$ acts on $\mathcal{A}^s$ by the changes of bases $\{\varepsilon_\ell^{(i)}\}_{\ell=1}^{i}$ in each F_i: $\varepsilon_\ell^{(i)} \mapsto \sum_{m=1}^{i} g_{i|\ell}^{m}\varepsilon_m^{(i)}$. This results in $U_{N-1}U_{N-2}\cdots U_i$ being multiplied on the right by g_i^{-1}; hence, according to (101), the π_i's are transformed via:

$$\pi_i \mapsto \pi_i \cdot \det(g_i)^{-1}\,, \tag{102}$$

thus justifying the $\det(W_i)^{-1}$ factor in (73). The group $GL(W)$ acts on $\mathcal{A}$ via:

$$h\cdot(U_{N-1},U_{N-2},\ldots,U_1) \;=\; (h\,U_{N-1},U_{N-2},\ldots,U_1)\,. \tag{103}$$

This $GL(W)$-action preserves $\mathcal{A}^s \subset \mathcal{A}$ and also commutes with the $\mathcal{G}$-action. The resulting action of $GL(W)$ on $\mathcal{A}^s/\mathcal{G}$ clearly coincides with the natural action of $GL(W)$ on $F(W)=\mathcal{A}^s/\mathcal{G}$. Accordingly, the $GL(W)$-action on functions on $F(W)$ is given by:

$$h\colon f\mapsto f^h\,, \qquad f^h\big[U_{N-1},U_{N-2},\ldots,U_1\big]=f\big[h^{-1}U_{N-1},U_{N-2},\ldots,U_1\big]. \tag{104}$$

This means that the vector field $J_a^b \in Vect(F(W))$ representing the action of the element $T_a^b = e_a\otimes\tilde{e}^b \in \mathfrak{gl}(W)$ on functions on $F(W)$ is given by (cf. the first formula of (92)):

$$J_a^b \;=-\sum_{m=1}^{N-1} U_{N-1|m}^{b}\frac{\partial}{\partial U_{N-1|m}^{a}}\,, \tag{105}$$

where $U_{N-1|m}^{a}$ are the matrix coefficients of $U_{N-1}\colon F_{N-1}\to W$ defined via:

$$U_{N-1}\varepsilon_m^{(N-1)} \;=\; \sum_{a=1}^{N} U_{N-1|m}^{a}\, e_a\,. \tag{106}$$

Up to a compensating infinitesimal g_i-transformation, the vector field J_a^b acts on π_i (more precisely, on functions of π_i viewed as functions on $F(W)$) by:

$$J_a^b\pi_i \;=-e_a\wedge\tilde{e}^b\,\pi_i\,. \tag{107}$$

To clarify, the right-hand side of (105) should be viewed as a descent of the $\mathcal{G}$-equivariant vector field on $\mathcal{A}^s$, given by the same formula, to the quotient space

$\mathcal{A}^s/\mathcal{G} = F(W)$. The attentive reader will be content to see that the minus sign in (105) is needed to match the commutation relations (64).

Likewise, the flag variety $F(W^*)$ admits the quotient realization:

$$F(W^*) = \tilde{\mathcal{A}}^s/\tilde{\mathcal{G}}\,, \tag{108}$$

where the open subvariety $\tilde{\mathcal{A}}^s$ of $\tilde{\mathcal{A}}$ consists of the collections $\left(\tilde{U}_i\right)_{i=1}^{N-1}$ for which the composition $\tilde{U}_i\tilde{U}_{i+1}\cdots\tilde{U}_{N-1}\colon W \to \tilde{F}_i$ has no cokernel (i.e., has the maximal rank) for any $i = 1,\ldots,N-1$, and the action of $\tilde{\mathcal{G}}$ on $\tilde{\mathcal{A}}^s$ is free. We can represent the $\tilde{\pi}^i$'s of (72), in coordinates, as:

$$\tilde{\pi}^i = \sum_{1\le a_1<a_2<\ldots<a_i\le N} \mathrm{Det}\left\| \left[\tilde{U}_i\tilde{U}_{i+1}\cdots\tilde{U}_{N-2}\tilde{U}_{N-1}\right]^{\ell}_{a_k} \right\|^{i}_{k,\ell=1} \tilde{e}^{a_1}\wedge\cdots\wedge\tilde{e}^{a_i}\,. \tag{109}$$

Here, $\left[\tilde{U}_i\tilde{U}_{i+1}\cdots\tilde{U}_{N-2}\tilde{U}_{N-1}\right]^{\ell}_{a_k}$ denote the matrix coefficients of the corresponding linear operator with respect to some bases $\{\tilde{\varepsilon}^{(i)}_{\ell}\}^{i}_{\ell=1}$ in $\tilde{F}_i$ and the bases $\{e_a\}_{a=1}^N$ in W which is dual to the chosen basis $\{\tilde{e}^a\}_{a=1}^N$ in W^*. Note that the group $\tilde{\mathcal{G}}$ acts on $\tilde{\mathcal{A}}^s$ by the changes of bases $\{\tilde{\varepsilon}^{(i)}_{\ell}\}^{i}_{\ell=1}$ in each $\tilde{F}_i$: $\tilde{\varepsilon}^{(i)}_{\ell} \mapsto \sum_{m=1}^{i} \tilde{g}^{m}_{i|\ell}\tilde{\varepsilon}^{(i)}_{m}$. This results in $\tilde{U}_i\tilde{U}_{i+1}\cdots\tilde{U}_{N-2}\tilde{U}_{N-1}$ being multiplied on the left by $\tilde{g}_i$; hence, according to (109), the $\tilde{\pi}^i$'s are transformed via:

$$\tilde{\pi}^i \mapsto \tilde{\pi}^i\cdot\det(\tilde{g}_i)\,, \tag{110}$$

thus justifying the $\det(\tilde{W}_i)$ factor in (73). The group $GL(W)$ acts on $\tilde{\mathcal{A}}$ via:

$$h\cdot\left(\tilde{U}_{N-1},\tilde{U}_{N-2},\ldots,\tilde{U}_1\right) = \left(\tilde{U}_{N-1}h^{-1},\tilde{U}_{N-2},\ldots,\tilde{U}_1\right)\,. \tag{111}$$

This action preserves $\tilde{\mathcal{A}}^s \subset \tilde{\mathcal{A}}$ and also commutes with the $\tilde{\mathcal{G}}$-action. The resulting action of $GL(W)$ on $\tilde{\mathcal{A}}^s/\tilde{\mathcal{G}}$ clearly coincides with the natural action of $GL(W)$ on $F(W^*) = \tilde{\mathcal{A}}^s/\tilde{\mathcal{G}}$, see (108). Therefore, the vector field $\tilde{J}^b_a \in Vect(F(W^*))$ representing the action of the element $T^b_a = e_a\otimes\tilde{e}^b \in \mathfrak{gl}(W)$ on $F(W^*)$ is given by (cf. the second formula of (92)):

$$\tilde{J}^b_a = \sum_{m=1}^{N-1}\tilde{U}^m_{N-1|a}\frac{\partial}{\partial\tilde{U}^m_{N-1|b}}\,, \tag{112}$$

where $\tilde{U}^m_{N-1|a}$ are the matrix coefficients of $\tilde{U}_{N-1}\colon W\to\tilde{F}_{N-1}$ defined via:

$$\tilde{U}_{N-1}e_a = \sum_{m=1}^{N-1}\tilde{U}^m_{N-1|a}\,\varepsilon^{(N-1)}_m\,. \tag{113}$$

To clarify, the right-hand side of (112) should be viewed as a descent of the $\tilde{\mathcal{G}}$-equivariant vector field on $\tilde{\mathcal{A}}^s$, given by the same formula, to the quotient space $\tilde{\mathcal{A}}^s/\tilde{\mathcal{G}} = F(W^*)$. The attentive reader will be content to see that the commutation relations (64) are obeyed by $\tilde{J}_a^b$ of (112).

3.5 End of proof of Theorem 3.1

It remains to compute the action of the operators $\Upsilon^{-1}\hat{h}_p\Upsilon$ in the coordinates v_i, and then to compare formulas (264, 265) in Appendix B to formulas (57, 58). We leave this straightforward computation to the interested reader.

4 Representation theory

Let us now explain the representation-theoretic meaning of the main Theorem 3.1. Namely, we identify the function Φ, given by

$$\Phi = \Upsilon\left(U, \tilde{U}, \mathrm{z}, \tilde{\mathrm{z}}\right) \cdot \psi(v_1, \ldots, v_{N-1}; \mathfrak{q}), \tag{114}$$

for any $\mathfrak{q}$, with the $\mathfrak{sl}_N$-invariant in the completed tensor product

$$\Phi \in \left(V_1 \hat{\otimes} V_2 \hat{\otimes} V_3 \hat{\otimes} V_4\right)^{\mathfrak{sl}_N} \tag{115}$$

of four irreducible infinite-dimensional representations $\{V_i\}_{i=1}^4$ of the Lie algebra $\mathfrak{sl}_N$.

We shall actually define V_i's as representations of $\mathfrak{gl}_N$. Let us denote the generators of $\mathfrak{gl}_N$ by $\mathbf{J}_a^b$, with $a, b = 1, \ldots, N$. These obey the commutation relations (64):

$$\left[\mathbf{J}_b^a, \mathbf{J}_{b'}^{a'}\right] = \delta_{b'}^a \mathbf{J}_b^{a'} - \delta_b^{a'} \mathbf{J}_{b'}^a . \tag{116}$$

Notation 4.1 For a Lie algebra $\mathfrak{g}$, its element $\xi \in \mathfrak{g}$, and a representation R of $\mathfrak{g}$, we denote by $T_R(\xi) \in \mathrm{End}(R)$ the linear operator in R, corresponding to ξ.

It is well known that (116) implies that the *Casimir operators*

$$\mathcal{C}_k = \sum_{a_1, a_2, \ldots, a_k = 1}^{N} \mathbf{J}_{a_1}^{a_2} \mathbf{J}_{a_2}^{a_3} \ldots \mathbf{J}_{a_k}^{a_1} \in U(\mathfrak{gl}_N) \tag{117}$$

commute with all generators $\mathbf{J}_b^a$, so that in every irreducible $\mathfrak{gl}_N$-representation R the operator $\mathcal{C}_k$ acts via a multiplication by a scalar $c_k(R)$, also commonly known as the *k-th Casimir of R*:

$$\sum_{a_1, a_2, \ldots, a_k = 1}^{N} T_R\left(\mathbf{J}_{a_1}^{a_2}\right) T_R\left(\mathbf{J}_{a_2}^{a_3}\right) \ldots T_R\left(\mathbf{J}_{a_k}^{a_1}\right) = c_k(R) \cdot \mathbf{1}_R . \tag{118}$$

Notation 4.2 The Lie algebra $\mathfrak{sl}_N$ is a subalgebra of $\mathfrak{gl}_N$ with a basis consisting of $\mathbf{J}_a^b$, with $a \neq b$, and

$$\mathfrak{h}_i = \mathbf{J}_i^i - \mathbf{J}_{i+1}^{i+1}, \qquad i = 1, \ldots, N-1. \tag{119}$$

Notation 4.3 The Chevalley generators of $\mathfrak{sl}_N$ are formed by $\mathfrak{h}_i$'s, and

$$\mathfrak{f}_i = \mathbf{J}_{i+1}^i, \qquad \mathfrak{e}_i = \mathbf{J}_i^{i+1}, \tag{120}$$

also for $i = 1, \ldots, N-1$.

The elements $\mathfrak{e}_i$ generate, via commutators, the Lie subalgebra $\mathfrak{n}_+$ of $\mathfrak{sl}_N$. As a vector space, $\mathfrak{n}_+$ has a basis consisting of $\mathbf{J}_a^b$ with $b > a$. Likewise, the elements $\mathfrak{f}_i$ generate the Lie subalgebra $\mathfrak{n}_-$ which, as a vector space, has a basis consisting of $\mathbf{J}_a^b$ with $b < a$.

Remark 4.1 With a slight abuse of notation, when this does not lead to a confusion, below we shall also denote by $\mathfrak{h}_i, \mathfrak{f}_i, \mathfrak{e}_i$ the corresponding operators

$$T_R\left(\mathbf{J}_i^i\right) - T_R\left(\mathbf{J}_{i+1}^{i+1}\right), \; T_R\left(\mathbf{J}_{i+1}^i\right), \; T_R\left(\mathbf{J}_i^{i+1}\right) \tag{121}$$

in a $\mathfrak{gl}_N$-module R.

4.1 Verma modules

4.1.1 Lowest weight module

For a generic $\vec{\nu} \in \mathbb{C}^{N-1}$, the lowest weight Verma $\mathfrak{sl}_N$-module $\mathcal{V}_{\vec{\nu}}$ is defined, algebraically, as follows. There is a vector $\Omega_{\vec{\nu}} \in \mathcal{V}_{\vec{\nu}}$, which obeys:

$$\mathbf{J}_b^a \Omega_{\vec{\nu}} = 0, \qquad a < b, \tag{122}$$

and:

$$\mathfrak{h}_i \, \Omega_{\vec{\nu}} \; = \; -\nu_i \, \Omega_{\vec{\nu}}, \qquad i = 1, \ldots, N-1, \tag{123}$$

and which generates $\mathcal{V}_{\vec{\nu}}$, i.e., $\mathcal{V}_{\vec{\nu}}$ is spanned by polynomials in $\mathbf{J}_b^a$, with $a > b$, acting on $\Omega_{\vec{\nu}}$. Geometrically, $\mathcal{V}_{\vec{\nu}}$ can be realized as the space of analytic functions Ψ of $(U_i)_{i=1}^{N-1}$, obeying:

$$\Psi\left[g_{i+1} U_i g_i^{-1}\right]_{i=1}^{N-1} \prod_{i=1}^{N-1} \det(g_i)^{\nu_i} = \Psi\left[U_i\right]_{i=1}^{N-1}, \qquad (g_i)_{i=1}^{N-1} \in \mathcal{G}^{\text{formal}}, \tag{124}$$

where g_N is vacuous and $\mathcal{G}^{\text{formal}}$ denotes the group of formal exponents $g_i = \exp h\xi_i$ with $\xi_i \in \text{End}(F_i)$ and h being a nilpotent parameter.

Remark 4.2 For $\vec{\nu} \in \mathbb{Z}^{N-1}$, the equation (124) makes sense for $(g_i)_{i=1}^{N-1} \in \mathcal{G}$. For $\vec{\nu} \in \mathbb{Z}_{\geq 0}^{N-1}$, the polynomial solutions to the equation (124) are in one-to-one correspondence with the holomorphic sections of the following line bundle on the complete flag variety $F(W)$:

$$\mathbb{L}_{W,\vec{\nu}} = \bigotimes_{i=1}^{N-1} \det(W_i)^{-\nu_i} \,. \tag{125}$$

For our chosen basis $\{e_a\}_{a=1}^{N}$ of W, consider the i-form $\tilde{\pi}_0^i$ defined via:

$$\tilde{\pi}_0^i = \tilde{e}^1 \wedge \tilde{e}^2 \wedge \cdots \wedge \tilde{e}^i \,. \tag{126}$$

Then,

$$\Omega_{\vec{\nu}} := \prod_{i=1}^{N-1} \left(\tilde{\pi}_0^i(\pi_i)\right)^{\nu_i} = \prod_{i=1}^{N-1} \left(\mathrm{Det} \left\| \left[U_{N-1} U_{N-2} \cdots U_i \right]_a^b \right\|_{a,b=1}^{i} \right)^{\nu_i} \tag{127}$$

(here, the index b runs through the labels of the first i basis vectors e_b in W, while the index a runs through the labels of a basis $\varepsilon_a^{(i)}$ in F_i) clearly satisfies (124). Furthermore, using $\tilde{\pi}_0^i\left(e_a \wedge \tilde{e}^b \pi_i\right) = 0$ unless $i \geq a$ and $b > i$ for $a \neq b$, we get (122) and (123), due to (107).

The Lie algebra $\mathfrak{gl}_N$ acts on the space of analytic functions $\Psi = \Psi[U_i]$ by vector fields, viewed as the first-order differential operators, via (105):

$$T_{\mathcal{V}_{\vec{\nu}}}\left(\mathbf{J}_a^b\right) \Psi = \mathrm{Lie}_{J_a^b}(\Psi) \,. \tag{128}$$

We can easily compute the first two Casimirs of $\mathcal{V}_{\vec{\nu}}$:

$$\begin{aligned} c_1(\mathcal{V}_{\vec{\nu}}) &= -\sum_{i=1}^{N-1} i\nu_i \,, \\ c_2(\mathcal{V}_{\vec{\nu}}) &= \sum_{i=1}^{N-1} i\nu_i \left(N - i + \nu_i + 2 \sum_{j=i+1}^{N-1} \nu_j \right) . \end{aligned} \tag{129}$$

Now, obviously $\Omega_{\vec{\nu}}$ is not well-defined for arbitrary U_i's. We need first to impose:

$$\tilde{\pi}_0^i(\pi_i) \neq 0 \,, \qquad i = 1, \ldots, N-1 \,. \tag{130}$$

On the open set of U_i's obeying (130) $\Omega_{\vec{\nu}}$ is not single-valued. We can, however, view it as an analytic function in the neighborhood $F(W)^\circ$ of the point where, in some $\mathcal{G}$-gauge, $\pi_i = \pi_i^0$ with the i-polyvector π_i^0 defined via:

$$\pi_i^0 = e_1 \wedge \cdots \wedge e_i \,. \tag{131}$$

To parametrize $F(W)^{\circ}$, we use:

$$u_k^{(i)} = \frac{\tilde{\pi}_0^i\left(e_k \wedge \tilde{e}^{i+1}\pi_i\right)}{\tilde{\pi}_0^i(\pi_i)} = \frac{\mathrm{Det}\|\ (U_{N-1}\cdots U_i)_\ell^{a_m}\ \|_{m,\ell=1}^i}{\mathrm{Det}\|\ (U_{N-1}\cdots U_i)_\ell^{m}\ \|_{m,\ell=1}^i}\,, \qquad 1 \le k \le i \le N-1\,, \tag{132}$$

where $a_m = m$ for $m \neq k$ while $a_k = i+1$, so that the vectors

$$e_\ell^{(i)}\,, \qquad 1 \le \ell \le i\,, \tag{133}$$

form the unique basis in $W_i = \mathrm{Im}\,(U_{N-1}U_{N-2}\cdots U_i)$, $i = 1, \ldots, N-1$, obeying:

$$\begin{aligned} &\pi_i = e_1^{(i)} \wedge e_2^{(i)} \wedge \cdots \wedge e_i^{(i)}\,, \\ &e_\ell^{(i)} = e_\ell^{(i+1)} + u_\ell^{(i)} e_{i+1}^{(i+1)}\,, \qquad 1 \le \ell \le i \le N-1\,, \end{aligned} \tag{134}$$

with $e_a^{(N)} := e_a$. Therefore, we have:

$$\begin{aligned} &e_\ell^{(i)} = e_\ell + \sum_{j=1}^{N-i} \mathbf{U}_\ell^{i|j} e_{i+j}\,, \\ &\mathbf{U}_\ell^{i|j} = u_\ell^{(i)}\delta_j^1 + \mathbf{U}_\ell^{i+1|j-1} + u_\ell^{(i)}\mathbf{U}_{i+1}^{i+1|j-1}\,, \end{aligned} \tag{135}$$

with $\mathbf{U}_\ell^{i|j}$ polynomial in $u_k^{(m)}$, $m \ge i$, nonzero only for $1 \le j \le N-i$, $1 \le \ell \le i$. Explicitly,

$$\begin{aligned} &\mathbf{U}_\ell^{i|1} = u_\ell^{(i)}\,, \qquad \mathbf{U}_\ell^{i|2} = u_\ell^{(i+1)} + u_\ell^{(i)}u_{i+1}^{(i+1)}\,, \\ &\mathbf{U}_\ell^{i|3} = u_\ell^{(i+2)} + u_\ell^{(i+1)}u_{i+2}^{(i+2)} + u_\ell^{(i)}\left(u_{i+1}^{(i+2)} + u_{i+1}^{(i+1)}u_{i+2}^{(i+2)}\right)\,, \quad \ldots \end{aligned} \tag{136}$$

Invoking (134) and the first equality of (135), we obtain the following analogue of (132):

$$\mathbf{U}_b^{i|a-i} = \frac{\tilde{\pi}_0^i\,(e_b \wedge \tilde{e}^a\pi_i)}{\tilde{\pi}_0^i(\pi_i)}\,, \qquad 1 \le b \le i < a \le N\,. \tag{137}$$

Since the local coordinates $u_k^{(i)}$ are $\mathcal{G}$-invariant, the general solution to (124) can be written as:

$$\Psi\,[U_i] = \psi\left[u_k^{(i)}\right]\cdot\,\Omega_{\vec{\nu}} \tag{138}$$

with some analytic functions ψ. We amend the definition of $\mathcal{V}_{\vec{\nu}}$ given prior to Remark 4.2 by rather defining $\mathcal{V}_{\vec{\nu}}$ as the space of analytic functions Ψ, obeying (124), such that the corresponding functions ψ (138) are polynomials in $u_k^{(i)}$'s. Using the equality (based on (137))

$$\mathbf{J}_b^a \Omega_{\vec{\nu}} = -\left(\delta_b^a \sum_{i \geq a} \nu_i + \sum_{i=b}^{a-1} \nu_i \mathbf{U}_b^{i|a-i} \right) \cdot \Omega_{\vec{\nu}} \,, \tag{139}$$

the generators $\mathbf{J}_b^a$ can be expressed as the first-order differential operators in $u_k^{(i)}$:

$$\mathbf{J}_b^a = - \sum_{1 \leq k \leq i \leq N-1} \left(\delta_k^a + \mathbf{U}_k^{i|a-i} \right) \left(\delta_b^{i+1} - u_b^{(i)} \right) \frac{\partial}{\partial u_k^{(i)}} - \delta_b^a \sum_{i \geq a} \nu_i - \sum_{i=b}^{a-1} \nu_i \mathbf{U}_b^{i|a-i} \,, \tag{140}$$

with polynomial in $u_k^{(i)}$'s coefficients. In particular, the Cartan generators of $\mathfrak{gl}_N$ act by:

$$\mathbf{J}_a^a = - \sum_{k<a} \left(u_k^{(a-1)} \frac{\partial}{\partial u_k^{(a-1)}} \right) + \sum_{k \geq a} \left(u_a^{(k)} \frac{\partial}{\partial u_a^{(k)}} - \nu_k \right), \tag{141}$$

hence, the Cartan generators of $\mathfrak{sl}_N$ act by:

$$\begin{aligned} \mathfrak{h}_i &= -\nu_i + 2u_i^{(i)} \frac{\partial}{\partial u_i^{(i)}} - \sum_{k<i} \left(u_k^{(i-1)} \frac{\partial}{\partial u_k^{(i-1)}} - u_k^{(i)} \frac{\partial}{\partial u_k^{(i)}} \right) \\ &\quad + \sum_{k>i} \left(u_i^{(k)} \frac{\partial}{\partial u_i^{(k)}} - u_{i+1}^{(k)} \frac{\partial}{\partial u_{i+1}^{(k)}} \right) \\ &= -\nu_i - \deg_{u_*^{(i-1)}} + \deg_{u_*^{(i)}} + \deg_{u_i^{(*)}} - \deg_{u_{i+1}^{(*)}} \,. \end{aligned} \tag{142}$$

With the natural definition of the order on the weights, it is not difficult to show that the positive degree polynomials in $u_k^{(i)}$'s have higher weights than the vacuum, the state $\psi = 1$. According to (140), the generators $\mathfrak{f}_i = \mathbf{J}_{i+1}^i$ act by:

$$\mathfrak{f}_i = -\frac{\partial}{\partial u_i^{(i)}} + \sum_{k>i} u_{i+1}^{(k)} \frac{\partial}{\partial u_i^{(k)}} \,, \tag{143}$$

thus annihilating the vacuum, the state $\psi = 1$, as they should. Likewise, according to (140), the generators $\mathfrak{e}_i = \mathbf{J}_i^{i+1}$ act by:

$$\mathfrak{e}_i = - \sum_{k<i} u_k^{(i)} \frac{\partial}{\partial u_k^{(i-1)}} + \sum_{k>i} u_i^{(k)} \frac{\partial}{\partial u_{i+1}^{(k)}} - u_i^{(i)} \left(\sum_{k<i} u_k^{(i-1)} \frac{\partial}{\partial u_k^{(i-1)}} - \sum_{k \leq i} u_k^{(i)} \frac{\partial}{\partial u_k^{(i)}} + \nu_i \right), \tag{144}$$

which generate the whole module, as we can see using $[\mathfrak{e}_i, \mathfrak{e}_{i+1}] = \mathbf{J}_i^{i+2}$, etc.

4.1.2 Highest weight module

For a generic $\vec{\tilde{\nu}} \in \mathbb{C}^{N-1}$, the highest weight Verma $\mathfrak{sl}_N$-module $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ is defined similarly, so we'd be brief. Algebraically, $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ is generated by a vector $\tilde{\Omega}_{\vec{\tilde{\nu}}}$, obeying:

$$\mathbf{J}^a_b \tilde{\Omega}_{\vec{\tilde{\nu}}} = 0 \,, \qquad a > b \,, \tag{145}$$

and:

$$\mathfrak{h}_i \, \tilde{\Omega}_{\vec{\tilde{\nu}}} \; = \; \tilde{\nu}_i \, \tilde{\Omega}_{\vec{\tilde{\nu}}} \,, \qquad i = 1, \ldots, N-1 \,. \tag{146}$$

Geometrically, $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ can be realized in the space of analytic functions $\tilde{\Psi}$ of $\left(\tilde{U}_i\right)_{i=1}^{N-1}$, obeying:

$$\tilde{\Psi}\left[\tilde{g}_i \tilde{U}_i \tilde{g}_{i+1}^{-1}\right]_{i=1}^{N-1} \prod_{i=1}^{N-1} \det(\tilde{g}_i)^{-\tilde{\nu}_i} = \tilde{\Psi}\left[\tilde{U}_i\right]_{i=1}^{N-1} \,, \qquad (\tilde{g}_i)_{i=1}^{N-1} \in \tilde{\mathcal{G}}^{\text{formal}} \,, \tag{147}$$

where $\tilde{g}_N$ is vacuous and $\tilde{\mathcal{G}}^{\text{formal}}$ denotes the group of formal exponents $\tilde{g}_i = \exp h\tilde{\xi}_i$ with $\tilde{\xi}_i \in \text{End}(\tilde{F}_i)$ and h being a nilpotent parameter. Again, we take:

$$\tilde{\Omega}_{\vec{\tilde{\nu}}} := \prod_{i=1}^{N-1} \left(\tilde{\pi}^i(\pi_i^0)\right)^{\tilde{\nu}_i} \,, \tag{148}$$

which clearly satisfies (145, 146). Then, $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ is realized in the space of analytic functions $\tilde{\Psi}$, obeying (147), of the form $\tilde{\Psi}[\tilde{U}_i] = \tilde{\psi}[\tilde{u}^k_{(i)}] \cdot \tilde{\Omega}_{\vec{\tilde{\nu}}}$ with $\tilde{\psi}$ polynomial in the $\tilde{\mathcal{G}}$-invariant coordinates

$$\tilde{u}^k_{(i)} = \frac{\tilde{e}^k \wedge \iota_{e_{i+1}} \tilde{\pi}^i \left(\pi_i^0\right)}{\tilde{\pi}^i(\pi_i^0)} \,, \qquad 1 \le k \le i \le N-1 \,, \tag{149}$$

on the open domain $F(W^*)^\circ$, where $\tilde{\pi}^i(\pi_i^0) \neq 0$ for $i = 1, \ldots, N-1$.

Remark 4.3 The identification of the vector space of representation $\mathcal{V}_{\vec{\nu}}$ with the space of polynomials in $u_k^{(i)}$'s, and similarly for $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$, is known mathematically under the name of the Poincare–Birkhoff–Witt theorem [3, 45, 48] (apparently proven in the case of our interest by A. Capelli).

Remark 4.4 The genericity assumption on $\vec{\nu} \in \mathbb{C}^{N-1}$ (resp. $\vec{\tilde{\nu}} \in \mathbb{C}^{N-1}$) guarantees that the Verma $\mathfrak{sl}_N$-module $\mathcal{V}_{\vec{\nu}}$ (resp. $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$) is irreducible, and thus is the unique lowest (resp. highest) weight module of the given lowest (resp. highest) weight, up to an isomorphism.

4.2 Twisted HW-modules

For generic $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{C}^N$ and $\tilde{\mathbf{n}} = (\tilde{n}_1, \ldots, \tilde{n}_N) \in \mathbb{C}^N$, let us define the *HW-modules* $H_{\mathbf{n}}$ and $\tilde{H}_{\tilde{\mathbf{n}}}$ of $\mathfrak{gl}_N$ (for W. Heisenberg and H. Weyl) by making $\mathbf{J}^b_a$ act via the first-order differential operators in N complex variables. In other words, the generators of $GL(N)$ in its defining N-dimensional representation W or its dual W^* act on the space of appropriately twisted functions on $\mathrm{Hom}(\mathtt{F}, W)$ or $\mathrm{Hom}(W, \tilde{\mathtt{F}})$, where $\mathtt{F} \approx \mathbb{C}$, $\tilde{\mathtt{F}} \approx \mathbb{C}$ denote complex lines.

Explicitly, let $(\mathrm{z}^a)_{a=1}^N$ and $(\tilde{\mathrm{z}}_a)_{a=1}^N$ denote the coordinates on $\mathrm{Hom}(\mathtt{F}, W)$ and $\mathrm{Hom}(W, \tilde{\mathtt{F}})$, respectively, in the dual bases $(e_a)_{a=1}^N$, $(\tilde{e}^a)_{a=1}^N$ of W, W^* we used in the previous section and in the dual bases $\mathrm{e} \in \mathtt{F}$, $\tilde{\mathrm{e}} \in \mathtt{F}^*$. Then, the underlying vector spaces $H_{\mathbf{n}}$, $\tilde{H}_{\tilde{\mathbf{n}}}$ of the HW-modules are the spaces of homogeneous (i.e., degree zero) Laurent polynomials in $\{\mathrm{z}^a\}$, $\{\tilde{\mathrm{z}}_a\}$, respectively:

$$H_{\mathbf{n}} = \mathbb{C}[\mathrm{z}^a, (\mathrm{z}^a)^{-1}]^{\mathbb{C}^\times}, \qquad \tilde{H}_{\tilde{\mathbf{n}}} = \mathbb{C}[\tilde{\mathrm{z}}_a, \tilde{\mathrm{z}}_a^{-1}]^{\mathbb{C}^\times}, \tag{150}$$

while the generators of $\mathfrak{gl}_N$ are represented by the following differential operators:

$$T_{H_{\mathbf{n}}}\left(\mathbf{J}_b^a\right) = -\omega_{\mathbf{n}}^{-1}\left(\mathrm{z}^a \partial_{\mathrm{z}^b}\right)\omega_{\mathbf{n}} \tag{151}$$

and

$$T_{\tilde{H}_{\tilde{\mathbf{n}}}}\left(\mathbf{J}_b^a\right) = \tilde{\omega}_{\tilde{\mathbf{n}}}^{-1}\left(\tilde{\mathrm{z}}_b \partial_{\tilde{\mathrm{z}}_a}\right)\tilde{\omega}_{\tilde{\mathbf{n}}} \tag{152}$$

with

$$\omega_{\mathbf{n}} = \prod_{a=1}^N \left(\mathrm{z}^a\right)^{n_a}, \qquad \tilde{\omega}_{\tilde{\mathbf{n}}} = \prod_{a=1}^N \tilde{\mathrm{z}}_a^{\tilde{n}_a}. \tag{153}$$

Remark 4.5 For $\tilde{\mathbf{n}} = (s, \ldots, s)$, the module $\tilde{H}_{\tilde{\mathbf{n}}}$ coincides with V_s of [7, Sect. 1], as $\mathfrak{sl}_N$-modules.

In general, $\tilde{H}_{\tilde{\mathbf{n}}}$ is a *twisted* version of $V_{(\tilde{n}_1+\ldots+\tilde{n}_N)/N}$, with underlying vector spaces being isomorphic. We thus shall use the following notation:

Notation 4.4 For $\mathrm{m} \in \mathbb{C}$ and $\vec{\mu} \in \mathbb{C}^{N-1}$, define:

$$\mathcal{H}_{\mathrm{m}}^{\vec{\mu}} := \omega_{\mathbf{n}} \cdot H_{\mathbf{n}} \tag{154}$$

with

$$\mathrm{m} = \sum_{a=1}^N n_a, \qquad \mu_i = n_i - n_{i+1}, \qquad i = 1, \ldots, N-1. \tag{155}$$

The action of $\mathfrak{gl}_N$ on $\mathcal{H}_{\mathrm{m}}^{\vec{\mu}}$ is represented by the ordinary vector fields:

$$T_{\mathcal{H}_{\mathrm{m}}^{\vec{\mu}}}\left(\mathbf{J}_a^b\right) = -\mathrm{z}^b \frac{\partial}{\partial \mathrm{z}^a}. \tag{156}$$

Notation 4.5 For $\tilde{\mathrm{m}} \in \mathbb{C}$ and $\vec{\tilde{\mu}} \in \mathbb{C}^{N-1}$, define:

$$\tilde{\mathcal{H}}_{\tilde{\mathrm{m}}}^{\vec{\tilde{\mu}}} := \tilde{\omega}_{\tilde{\mathbf{n}}} \cdot \tilde{H}_{\tilde{\mathbf{n}}} \tag{157}$$

with

$$\tilde{\mathrm{m}} = \sum_{a=1}^{N} \tilde{n}_a \,, \qquad \tilde{\mu}_i = \tilde{n}_i - \tilde{n}_{i+1} \,, \qquad i = 1, \ldots, N-1 \,. \tag{158}$$

The action of $\mathfrak{gl}_N$ on $\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}$ is represented by the ordinary vector fields:

$$T_{\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}}\left(\mathbf{J}^b_a\right) = \tilde{z}_a \frac{\partial}{\partial \tilde{z}_b} \,. \tag{159}$$

Remark 4.6 (a) It is clear that the Casimirs $c_k\left(\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}\right)$ and $c_k\left(\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}\right)$, defined by (118), depend only on m and $\tilde{\mathrm{m}}$, respectively.

(b) The $\mathfrak{gl}_N$-weight subspaces, i.e., the joint eigenspaces of a commuting family $\{\mathbf{J}^a_a\}_{a=1}^N$, of $\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}$ and $\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}$ are all one-dimensional, the corresponding sets of weights being $-\mathbf{n} + \Lambda_0 \subset \mathbb{C}^N$ and $\tilde{\mathbf{n}} + \Lambda_0 \subset \mathbb{C}^N$, respectively, where Λ_0 denotes the lattice $\Lambda_0 = \left\{(r_1, \ldots, r_N) \in \mathbb{Z}^N \,\middle|\, \sum_{i=1}^N r_i = 0\right\}$.

(c) The vectors $\Omega_{\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}} := \omega_{\mathbf{n}} \in \mathcal{H}^{\vec{\mu}}_{\mathrm{m}}$, $\tilde{\Omega}_{\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}} := \tilde{\omega}_{\tilde{\mathbf{n}}} \in \tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}$ have the following $\mathfrak{sl}_N$-weights:

$$\mathfrak{h}_i \cdot \Omega_{\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}} = -\mu_i \cdot \Omega_{\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}} \,, \qquad \mathfrak{h}_i \cdot \tilde{\Omega}_{\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}} = \tilde{\mu}_i \cdot \tilde{\Omega}_{\tilde{\mathcal{H}}^{\tilde{\vec{\mu}}}_{\tilde{\mathrm{m}}}} \,, \qquad i = 1, \ldots, N-1 \,. \tag{160}$$

4.3 Vermas and HW-modules in the $N = 2$ case

The generators $\mathfrak{e} \equiv \mathfrak{e}_1, \mathfrak{f} \equiv \mathfrak{f}_1, \mathfrak{h} \equiv \mathfrak{h}_1$ of $\mathfrak{sl}_2$, see (119, 120), obey the standard relations:

$$[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h} \,, \quad [\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e} \,, \quad [\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f} \,. \tag{161}$$

For $a, s \in \mathbb{C}$ and $i \in \{-1, 0, 1\}$, consider the differential operators:

$$L_i = -z^{i+1}\partial_z + (a + (i+1)s)z^i \,, \tag{162}$$

obeying the commutation relations:

$$[L_i, L_j] = (i - j)L_{i+j} \,. \tag{163}$$

The assignments

$$\mathfrak{e} \mapsto -L_{-1} \,, \quad \mathfrak{f} \mapsto L_1 \,, \quad \mathfrak{h} \mapsto 2L_0 \,, \tag{164}$$

or

$$\mathfrak{e} \mapsto -L_1 \,, \quad \mathfrak{f} \mapsto L_{-1} \,, \quad \mathfrak{h} \mapsto -2L_0 \,, \tag{165}$$

represent $\mathfrak{sl}_2$ by the first-order differential operators on a line.

The modules we defined in the general N case can be described quite explicitly. Specifically, the highest/lowest weight Verma and the twisted HW $\mathfrak{sl}_2$-modules are all realized in the spaces of the twisted tensors:

$$f(z)z^{-a}dz^{-s}\,, \tag{166}$$

with $f(z)$ being a single-valued function of $z \in \mathbb{C}^\times$, so that the operators (162) are the infinitesimal fractional linear transformations:

$$z \mapsto \frac{Az+B}{Cz+D}\,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{C})\,. \tag{167}$$

To make this relation precise, let us start with the geometric descriptions of the Verma modules.

In the geometric realization of the lowest weight Verma modules, we have a two-component vector

$$U_1 = \left(U^1_{1|1}, U^2_{1|1}\right) =: \left(u^1, u^2\right), \tag{168}$$

which is acted upon by the gauge $\mathbb{C}^\times$-symmetry via $(u^1,u^2) \mapsto (t^{-1}u^1, t^{-1}u^2)$. We look at the space of the locally defined functions $\Psi = \Psi(u^1,u^2)$ which transform with weight $-\nu$ under the Lie algebra of the gauge $\mathbb{C}^\times$-symmetry. More precisely, following (138) and the succeeding discussion, we look at Ψ of the form:

$$\Psi(u^1,u^2) = \psi(z)\cdot\left(u^1\right)^\nu, \tag{169}$$

where ψ is a polynomial and $z = u^2/u^1$ is the only coordinate $u^{(1)}_1$ (132) in the present setting. One can perceive the right-hand side of (169) as the local section of a complex power of a line bundle $\mathcal{O}(1)$ over a neighborhood of $z=0$ in $\mathbb{CP}^1$, defined near the slice $u^1 = 1$. The generators of $\mathfrak{sl}_2$ act via:

$$\begin{aligned} \mathfrak{e} &= -u^2\frac{\partial}{\partial u^1} = z^2\partial_z - \nu z\,, \\ \mathfrak{f} &= -u^1\frac{\partial}{\partial u^2} = -\partial_z\,, \\ \mathfrak{h} &= u^2\frac{\partial}{\partial u^2} - u^1\frac{\partial}{\partial u^1} = 2z\partial_z - \nu\,, \end{aligned} \tag{170}$$

where the differential operators in the middle act on Ψ while the rightmost ones act on $\psi = \psi(z)$. The vacuum is:

$$\Omega_\nu = (u^1)^\nu, \tag{171}$$

corresponding to $\psi = 1$, and the lowest weight Verma module is:

$$\mathcal{V}_\nu = \mathbb{C}[\mathfrak{e}]\Omega_\nu\,. \tag{172}$$

The weight (eigenvalue of $\mathfrak{h}$) of the state z^n is $2n-\nu$. Note that the fractional linear transformation (167) transforms $(u^1,u^2) \mapsto (Cu^2+Du^1, Au^2+Bu^1)$, hence it maps the vacuum to (again, we are working infinitesimally):

$$(Cu^2 + Du^1)^\nu = (Cz + D)^\nu \Omega_\nu \,. \tag{173}$$

The formula (173) allows us to match:

$$\Omega_\nu \sim dz^{-\frac{\nu}{2}} \,. \tag{174}$$

Thus, the lowest weight Verma module $\mathcal{V}_\nu$ corresponds to the realization (165, 166) with:

$$a = 0, \qquad s = \frac{\nu}{2} \,, \tag{175}$$

and with polynomial f in (166).

In the geometric realization of the highest weight Verma modules, we have a two-component covector

$$\tilde{U}_1 = \left(\tilde{U}^1_{1|1}, \tilde{U}^1_{1|2}\right) =: (v_1, v_2) \,, \tag{176}$$

which is acted upon by the gauge $\mathbb{C}^\times$-symmetry via $(v_1, v_2) \mapsto (tv_1, tv_2)$. We are looking at the space of locally defined functions $\tilde{\Psi} = \tilde{\Psi}(v_1, v_2)$, which transform with weight $\tilde{\nu}$ under the Lie algebra of the gauge $\mathbb{C}^\times$-symmetry. More precisely, following (148, 149), we look at $\tilde{\Psi}$ of the form:

$$\tilde{\Psi}(v_1, v_2) = \tilde{\psi}(\tilde{z}) \cdot (v_1)^{\tilde{\nu}} \,, \tag{177}$$

where $\tilde{\psi}$ is a polynomial and $\tilde{z} = v_2/v_1$ is the only coordinate $\tilde{u}^1_{(1)}$ (149) in the present setting. The generators of $\mathfrak{sl}_2$ act via:

$$\begin{aligned} \mathfrak{e} &= v_1 \frac{\partial}{\partial v_2} = \partial_{\tilde{z}} \,, \\ \mathfrak{f} &= v_2 \frac{\partial}{\partial v_1} = -\tilde{z}^2 \partial_{\tilde{z}} + \tilde{\nu} \tilde{z} \,, \\ \mathfrak{h} &= v_1 \frac{\partial}{\partial v_1} - v_2 \frac{\partial}{\partial v_2} = -2\tilde{z}\partial_{\tilde{z}} + \tilde{\nu} \,, \end{aligned} \tag{178}$$

where the differential operators in the middle act on $\tilde{\Psi}$, while the rightmost ones act on $\tilde{\psi} = \tilde{\psi}(\tilde{z})$. The vacuum is:

$$\tilde{\Omega}_{\tilde{\nu}} = (v_1)^{\tilde{\nu}} \,, \tag{179}$$

corresponding to $\tilde{\psi} = 1$, and the highest weight Verma module is:

$$\tilde{\mathcal{V}}_{\tilde{\nu}} = \mathbb{C}[\mathfrak{f}]\tilde{\Omega}_{\tilde{\nu}} \,. \tag{180}$$

The weight of the state $\tilde{z}^n$ is $-2n + \tilde{\nu}$. Note that under the $SL(2, \mathbb{C})$ fractional linear transformation (167) the covector (v_1, v_2) transforms via $(v_1, v_2) \mapsto (-Bv_2 +$

$Av_1, Dv_2 - Cv_1)$ with $AD - BC = 1$, so that the pairing $\tilde{U}_1 \cdot U_1 = v \cdot u \equiv u^1 v_1 + u^2 v_2$ is invariant, leading to:

$$\tilde{z} \mapsto \frac{D\tilde{z} - C}{-B\tilde{z} + A} . \tag{181}$$

Thus, the vacuum $\tilde{\Omega}_{\tilde{\nu}}$ is transformed via:

$$\tilde{\Omega}_{\tilde{\nu}} \mapsto (Av_1 - Bv_2)^{\tilde{\nu}} = (A - B\tilde{z})^{\tilde{\nu}}\ \tilde{\Omega}_{\tilde{\nu}} , \tag{182}$$

which allows us to match:

$$\tilde{\Omega}_{\tilde{\nu}} \sim d\tilde{z}^{-\frac{\tilde{\nu}}{2}} . \tag{183}$$

Hence, the highest weight Verma module $\tilde{\mathcal{V}}_{\tilde{\nu}}$ corresponds to the realization (164, 166) with:

$$a = 0, \qquad s = \frac{\tilde{\nu}}{2} , \tag{184}$$

and with polynomial f in (166).

We note that the transformations (167) and (181) are related via $\tilde{z}z = -1$, so that we get an equivalent representation (165, 166) with:

$$a = \tilde{\nu}, \qquad s = \frac{\tilde{\nu}}{2} . \tag{185}$$

Finally, to describe the twisted HW-modules $H_{\mathbf{n}}$, $\tilde{H}_{\tilde{\mathbf{n}}}$ with $\mathbf{n} = (n_1, n_2)$, $\tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2)$, we recall the notation (153):

$$\omega_{\mathbf{n}} = \left(\mathrm{z}^1\right)^{n_1} \left(\mathrm{z}^2\right)^{n_2} , \qquad \tilde{\omega}_{\tilde{\mathbf{n}}} = \tilde{\mathrm{z}}_1^{\tilde{n}_1} \tilde{\mathrm{z}}_2^{\tilde{n}_2} . \tag{186}$$

The vector space underlying $H_{\mathbf{n}}$ is the space of Laurent polynomials ψ in $z = \mathrm{z}^2/\mathrm{z}^1$. Analogously, the vector space underlying $\tilde{H}_{\tilde{\mathbf{n}}}$ is the space of Laurent polynomials $\tilde{\psi}$ in $\tilde{z} = \tilde{\mathrm{z}}_2/\tilde{\mathrm{z}}_1$.

In the first case, the generators of $\mathfrak{sl}_2$ act via:

$$\begin{aligned} \mathfrak{e} &= -\omega_{\mathbf{n}}^{-1} \left(\mathrm{z}^2 \frac{\partial}{\partial \mathrm{z}^1} \right) \omega_{\mathbf{n}} = z^2 \partial_z - n_1 z , \\ \mathfrak{f} &= -\omega_{\mathbf{n}}^{-1} \left(\mathrm{z}^1 \frac{\partial}{\partial \mathrm{z}^2} \right) \omega_{\mathbf{n}} = -\partial_z - n_2 z^{-1} , \\ \mathfrak{h} &= \omega_{\mathbf{n}}^{-1} \left(\mathrm{z}^2 \frac{\partial}{\partial \mathrm{z}^2} - \mathrm{z}^1 \frac{\partial}{\partial \mathrm{z}^1} \right) \omega_{\mathbf{n}} = 2z\partial_z + n_2 - n_1 . \end{aligned} \tag{187}$$

Thus, the twisted HW-module $H_{\mathbf{n}} \sim \mathcal{H}_{2s}^{2(s+a)}$ corresponds to the realization (165, 166) with:

$$a = -n_2, \qquad s = \frac{n_1 + n_2}{2} . \tag{188}$$

In the second case, analogously, the generators of $\mathfrak{sl}_2$ act via:

$$\begin{aligned}\mathfrak{e} &= \tilde{\omega}_{\tilde{\mathbf{n}}}^{-1}\left(\tilde{z}_1\frac{\partial}{\partial\tilde{z}_2}\right)\tilde{\omega}_{\tilde{\mathbf{n}}} = \partial_{\tilde{z}} + \tilde{n}_2\tilde{z}^{-1}\,,\\ \mathfrak{f} &= \tilde{\omega}_{\tilde{\mathbf{n}}}^{-1}\left(\tilde{z}_2\frac{\partial}{\partial\tilde{z}_1}\right)\tilde{\omega}_{\tilde{\mathbf{n}}} = -\tilde{z}^2\partial_{\tilde{z}} + \tilde{n}_1\tilde{z}\,,\\ \mathfrak{h} &= \tilde{\omega}_{\tilde{\mathbf{n}}}^{-1}\left(\tilde{z}_1\frac{\partial}{\partial\tilde{z}_1} - \tilde{z}_2\frac{\partial}{\partial\tilde{z}_2}\right)\tilde{\omega}_{\tilde{\mathbf{n}}} = -2\tilde{z}\partial_{\tilde{z}} + \tilde{n}_1 - \tilde{n}_2\,.\end{aligned} \tag{189}$$

Thus, the twisted HW-module $\tilde{H}_{\tilde{\mathbf{n}}} \sim \tilde{\mathcal{H}}_{2s}^{2(s+a)}$ corresponds to the realization (164, 166) with:

$$a = -\tilde{n}_2, \qquad s = \frac{\tilde{n}_1 + \tilde{n}_2}{2}\,. \tag{190}$$

4.4 Tensor products and invariants

Let us recall the following $SL(2,\mathbb{C})$-invariants (under the fractional linear action) on the configurations of 2, 3, and 4 points on $\mathbb{CP}^1$:

$$\upsilon(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \tag{191}$$

is an invariant $(1,0)\otimes(1,0)$–form on $\mathbb{CP}^1\times\mathbb{CP}^1$,

$$\frac{z_2 - z_1}{(z_3 - z_1)(z_3 - z_2)}dz_3 = \left(\frac{\upsilon(z_1,z_3)\otimes\upsilon(z_2,z_3)}{\upsilon(z_1,z_2)}\right)^{\frac{1}{2}} \tag{192}$$

is an invariant $0\otimes 0\otimes(1,0)$–form on $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$, and finally, the cross-ratio

$$[z_1, z_2; z_3, z_4] := \frac{z_2 - z_1}{z_3 - z_1}\cdot\frac{z_4 - z_3}{z_4 - z_2} = \left(\frac{\upsilon(z_1,z_3)\otimes\upsilon(z_2,z_4)}{\upsilon(z_1,z_2)\otimes\upsilon(z_3,z_4)}\right)^{\frac{1}{2}} \tag{193}$$

is an invariant meromorphic function on $\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$.

Thus,

$$I_\nu^{(2)} = \upsilon(z_1,z_2)^{-\frac{\nu}{2}} = (1 + z_1\tilde{z}_2)^\nu\,(dz_1)^{-\frac{\nu}{2}}\otimes(d\tilde{z}_2)^{-\frac{\nu}{2}} \tag{194}$$

is an $\mathfrak{sl}_2$-invariant element in the completed tensor product $\mathcal{V}_\nu\hat{\otimes}\tilde{\mathcal{V}}_\nu$. More precisely, we need to view (194) as a power series in $z_1, \tilde{z}_2 = -z_2^{-1}$ in the domain $z_1 \to 0$, $z_2 \to \infty$:

$$I_\nu^{(2)}|_{|z_1|\ll|z_2|} \in \left(\mathcal{V}_\nu\hat{\otimes}\tilde{\mathcal{V}}_\nu\right)^{\mathfrak{sl}_2}. \tag{195}$$

For another domain of convergence, e.g., $z_1 \to \infty$, $z_2 \to 0$, the expression (194) would define an invariant in the completed tensor product $\tilde{\mathcal{V}}_\nu\hat{\otimes}\mathcal{V}_\nu$ instead:

$$I_\nu^{(2)}|_{|z_2|\ll|z_1|} \in \left(\tilde{\mathcal{V}}_\nu\hat{\otimes}\mathcal{V}_\nu\right)^{\mathfrak{sl}_2}. \tag{196}$$

Finally, invoking (171, 174, 179, 183), we can express $I_\nu^{(2)}$ (194) in terms of $U_1, \tilde{U}_1$ (168, 176):

$$I_\nu^{(2)} = \left(\tilde{U}_1 \cdot U_1 \equiv U^1_{1|1}\tilde{U}^1_{1|1} + U^2_{1|1}\tilde{U}^1_{1|2} \equiv u^1 v_1 + u^2 v_2\right)^\nu$$
$$= \Omega_\nu \tilde{\Omega}_\nu \times \left(\text{power series in } z = u^2/u^1, \tilde{z} = v_2/v_1\right). \tag{197}$$

The benefit of formula (197) is that it admits a natural generalization to the general N:

$$I_{\vec{\nu}}^{(2)} = \prod_{i=1}^{N-1} \tilde{\pi}^i(\pi_i)^{\nu_i} = \Omega_{\vec{\nu}}\tilde{\Omega}_{\vec{\nu}} \times \left(\text{power series in } u_k^{(i)}, \tilde{u}^k_{(i)}\right) \in \left(\mathcal{V}_{\vec{\nu}}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}}\right)^{\mathfrak{gl}_N}. \tag{198}$$

Remark 4.7 In coordinates, we have:

$$\tilde{\pi}^i(\pi_i) = \mathrm{Det}\left(\tilde{U}_i\tilde{U}_{i+1}\ldots\tilde{U}_{N-1}U_{N-1}\ldots U_{i+1}U_i\right). \tag{199}$$

Remark 4.8 The formula (198) determines the unique $\mathfrak{gl}_N$-invariant bilinear pairing:

$$(\cdot,\cdot)_{\vec{\nu}} \colon \mathcal{V}_{\vec{\nu}} \times \tilde{\mathcal{V}}_{\vec{\nu}} \longrightarrow \mathbb{C} \tag{200}$$

such that

$$\left(\Omega_{\vec{\nu}}, \tilde{\Omega}_{\vec{\nu}}\right)_{\vec{\nu}} = 1. \tag{201}$$

One can present $(\cdot,\cdot)_{\vec{\nu}}$ as an integral over $F(W)$, but the quicker way is the following: the matrix $G_{\vec{n},\vec{\tilde{n}}}$ inverse to

$$\left(\prod_{k\le i}\left(u_k^{(i)}\right)^{n_k^{(i)}}\Omega_{\vec{\nu}}\,,\ \prod_{k\le i}\left(\tilde{u}^k_{(i)}\right)^{\tilde{n}^k_{(i)}}\tilde{\Omega}_{\vec{\nu}}\right)_{\vec{\nu}} \tag{202}$$

is given by the coefficients of the expansion

$$\mathcal{I}_{\vec{\nu}} = \prod_{i=1}^{N-1}\left(\frac{\tilde{\pi}^i(\pi_i)}{\tilde{\pi}^i(\pi_i^0)\cdot\tilde{\pi}^i_0(\pi_i)}\right)^{\nu_i} = \sum_{\vec{n},\vec{\tilde{n}}} G_{\vec{n},\vec{\tilde{n}}} \prod_{1\le k\le i\le N-1}\left(u_k^{(i)}\right)^{n_k^{(i)}}\left(\tilde{u}^k_{(i)}\right)^{\tilde{n}^k_{(i)}} = 1+\cdots \tag{203}$$

Let us now similarly produce an $\mathfrak{sl}_2$-invariant in the completed tensor product of three $\mathfrak{sl}_2$-representations: the lowest weight and the highest weight Vermas, as well as the twisted HW-module. To this end, we consider:

$$I^{(3)}_{\nu_1,\nu_2,\nu_3} = \upsilon(z_1,z_2)^{-\frac{\nu_1+\nu_2-\nu_3}{4}}\upsilon(z_1,z_3)^{-\frac{\nu_1+\nu_3-\nu_2}{4}}\upsilon(z_2,z_3)^{-\frac{\nu_2+\nu_3-\nu_1}{4}}. \tag{204}$$

By invoking (175, 185, 188) and expanding (204) in the region $|z_1| \ll |z_2| \ll |z_3|$, we arrive at the following interpretation:

$$I^{(3)}_{\nu_1,\nu_2,\nu_3}\big|_{|z_1|\ll|z_2|\ll|z_3|} \in \left(\mathcal{V}_{\nu_1}\hat{\otimes}\mathcal{H}^{\nu_3-\nu_1}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\nu_3}\right)^{\mathfrak{sl}_2}. \tag{205}$$

Finally, in the $\left(u^1, u^2\right)$, $\left(\mathrm{z}^1, \mathrm{z}^2\right)$, (v_1, v_2)-realizations, this invariant takes the following form:

$$\begin{aligned} I^{(3)}_{\nu_1,\nu_2,\nu_3} &= \left(u^1\mathrm{z}^2 - u^2\mathrm{z}^1\right)^{\frac{\nu_1+\nu_2-\nu_3}{2}} \left(v_1\mathrm{z}^1 + v_2\mathrm{z}^2\right)^{\frac{\nu_2+\nu_3-\nu_1}{2}} \left(u^1 v_1 + u^2 v_2\right)^{\frac{\nu_1+\nu_3-\nu_2}{2}} \\ &= \Omega_{\nu_1}\tilde{\Omega}_{\nu_3}(\mathrm{z}^1)^{n_1}(\mathrm{z}^2)^{n_2} \times \left(\text{power series in } z = u^2/u^1,\, \tilde{z} = v_2/v_1,\, (\mathrm{z}^2/\mathrm{z}^1)^{\pm 1}\right) \end{aligned} \tag{206}$$

with

$$n_1 = \frac{\nu_2+\nu_3-\nu_1}{2}, \qquad n_2 = \frac{\nu_1+\nu_2-\nu_3}{2}, \tag{207}$$

where we matched $z_1 \sim z,\ z_2 \sim \mathrm{z}^2/\mathrm{z}^1,\ z_3 \sim -1/\tilde{z}$. We note that the last two factors in (206) are $\mathfrak{gl}_2$-invariant, while the first one is only $\mathfrak{sl}_2$-invariant.

The formula (206) admits a natural generalization to the general N, with the triple ν_1, ν_2, ν_3 being replaced with $\vec{\nu}_1, \vec{\nu}_3 \in \mathbb{C}^{N-1},\ \nu_2 \in \mathbb{C}$. In this case, we have a unique invariant (cf. (68)):

$$\begin{aligned} I^{(3)}_{\vec{\nu}_1,\nu_2,\vec{\nu}_3} &= \prod_{a=1}^{N} \tilde{\pi}^a\,(\pi_{a-1}\wedge \mathbf{z})^{n_a} \cdot \prod_{i=1}^{N-1} \tilde{\pi}^i\,(\pi_i)^{\nu_{3,i}-n_i} \\ &= \Omega_{\vec{\nu}_1}\left(\prod_{a=1}^{N}(\mathrm{z}^a)^{n_a}\right)\tilde{\Omega}_{\vec{\nu}_3} \times \left(\text{power series in } u^{(i)}_k,\ \tilde{u}^k_{(i)},\ \mathrm{z}^a/\mathrm{z}^b\right) \\ &\in \left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\mathcal{H}^{\vec{\nu}_3-\vec{\nu}_1}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_3}\right)^{\mathfrak{sl}_N}, \end{aligned} \tag{208}$$

where the vector $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{C}^N$ is determined from

$$\sum_{a=1}^{N} n_a = \nu_2 \tag{209}$$

and

$$n_{i+1} - n_i = \nu_{1,i} - \nu_{3,i}\,, \qquad i = 1, \ldots, N-1\,. \tag{210}$$

Similarly to the $N = 2$ case, the factor $\tilde{\pi}^N\,(\pi_{N-1}\wedge \mathbf{z})^{n_N}$ is only $\mathfrak{sl}_N$-invariant, while all other factors in (208) are naturally $\mathfrak{gl}_N$-invariant.

Another generalization of (206) is the invariant

$$\begin{aligned}\tilde{I}^{(3)}_{\vec{\nu}_1,\nu_2,\vec{\nu}_3} &= \prod_{a=1}^{N} \left(\tilde{\pi}^{a-1} \wedge \tilde{\mathbf{z}}\,(\pi_a)\right)^{\tilde{n}_a} \cdot \prod_{i=1}^{N-1} \tilde{\pi}^{i}\,(\pi_i)^{\nu_{1,i}-\tilde{n}_i} \\ &= \Omega_{\vec{\nu}_1}\left(\prod_{a=1}^{N} \tilde{\mathrm{z}}_a^{\tilde{n}_a}\right)\tilde{\Omega}_{\vec{\nu}_3} \times \left(\text{power series in } u_k^{(i)},\ \tilde{u}_{(i)}^{k},\ \tilde{\mathrm{z}}_b/\tilde{\mathrm{z}}_a\right) \\ &\in \left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\tilde{\mathcal{H}}^{\vec{\nu}_1-\vec{\nu}_3}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_3}\right)^{\mathfrak{sl}_N},\end{aligned} \tag{211}$$

where the vector $\tilde{\mathbf{n}} = (\tilde{n}_1, \dots, \tilde{n}_N) \in \mathbb{C}^N$ is determined from

$$\sum_{a=1}^{N} \tilde{n}_a = \nu_2 \tag{212}$$

and

$$\tilde{n}_{i+1} - \tilde{n}_i = \nu_{3,i} - \nu_{1,i}\,, \qquad i = 1, \dots, N-1\,. \tag{213}$$

Remark 4.9 The examples (208, 211) demonstrate the need for twists in the definition of the HW-modules in Sect. 4.2.

To prove that $I^{(2)}$ of (198), $I^{(3)}$ of (208), and $\tilde{I}^{(3)}$ of (211) are the only invariants in the corresponding (completed) tensor products of 2 and 3 modules of $\mathfrak{sl}_N$, see Corollary 4.9, let us recall the realization of the corresponding spaces of invariants as the weight subspaces.

Notation 4.6 For an $\mathfrak{sl}_N$-module W and $\vec{\lambda} \in \mathbb{C}^{N-1}$, we denote by $W[\vec{\lambda}]$ the weight $\vec{\lambda}$ subspace:

$$w \in W[\vec{\lambda}] \Leftrightarrow \mathfrak{h}_i(w) = \lambda_i \cdot w\,, \qquad i = 1, \dots, N-1\,. \tag{214}$$

Remark 4.10 We have (cf. Remark 4.6):

$$\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}[-\vec{\mu}] = \mathbb{C}\cdot\omega_{\mathbf{n}}\,, \qquad \tilde{\mathcal{H}}^{\vec{\tilde{\mu}}}_{\tilde{\mathrm{m}}}[\vec{\tilde{\mu}}] = \mathbb{C}\cdot\tilde{\omega}_{\tilde{\mathbf{n}}}\,. \tag{215}$$

To Verma modules $\mathcal{V}_{\vec{\nu}}, \tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ defined in Sects. 4.1.1 and 4.1.2, we associate the *restricted dual* modules $\mathcal{V}^*_{\vec{\nu}}, \tilde{\mathcal{V}}^*_{\vec{\tilde{\nu}}}$. These are defined as the submodules of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_{\vec{\nu}}, \mathbb{C})$, $\mathrm{Hom}_{\mathbb{C}}(\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}, \mathbb{C})$, respectively, whose underlying vector spaces are direct sums of the spaces, dual to the $\mathfrak{sl}_N$-weight subspaces of $\mathcal{V}_{\vec{\nu}}, \tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$. The following is well known:

Lemma 4.7 *If $\mathcal{V}_{\vec{\nu}}$ (resp. $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$) is an irreducible $\mathfrak{sl}_N$-module, then $\mathcal{V}^*_{\vec{\nu}} \simeq \tilde{\mathcal{V}}_{\vec{\nu}}$ (resp. $\tilde{\mathcal{V}}^*_{\vec{\tilde{\nu}}} \simeq \mathcal{V}_{\vec{\tilde{\nu}}}$).*

For any $\mathfrak{sl}_N$-module W, we define the completed tensor products $\mathcal{V}_{\vec{\nu}}\hat{\otimes}W$ and $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\hat{\otimes}W$ via:

$$\mathcal{V}_{\vec{\nu}}\hat{\otimes}W := \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{V}^*_{\vec{\nu}}\,,\; W\right)\,, \qquad \tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\hat{\otimes}W := \mathrm{Hom}_{\mathbb{C}}\left(\tilde{\mathcal{V}}^*_{\vec{\tilde{\nu}}}\,,\; W\right), \tag{216}$$

both of which have natural structure of $\mathfrak{sl}_N$-modules.

Now we are ready to invoke the standard interpretation of the space of $\mathfrak{sl}_N$-invariants in the tensor product, completed in the sense of (216), of $\mathfrak{sl}_N$-modules involving both the highest weight and the lowest weight Verma modules (cf. the proof of [7, Proposition 1.1]):

Lemma 4.8 *If the lowest weight Verma $\mathcal{V}_{\vec{\nu}}$ and the highest weight Verma $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ modules of $\mathfrak{sl}_N$ are irreducible, then the space of $\mathfrak{sl}_N$-invariants in $\mathcal{V}_{\vec{\nu}}\hat{\otimes}W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ can be described as follows:*

$$\left(\mathcal{V}_{\vec{\nu}}\hat{\otimes}W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)^{\mathfrak{sl}_N} \simeq W\left[\vec{\nu}-\vec{\tilde{\nu}}\right]. \tag{217}$$

Proof This follows from the following sequence of canonical identifications:

$$\begin{aligned}&\left(\mathcal{V}_{\vec{\nu}}\hat{\otimes}W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)^{\mathfrak{sl}_N} \simeq \mathrm{Hom}_{\mathfrak{sl}_N}\left(\mathcal{V}^*_{\vec{\nu}},\, W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)\\ &\simeq \mathrm{Hom}_{\mathfrak{sl}_N}\left(\tilde{\mathcal{V}}_{\vec{\nu}},\, W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right) \simeq \left(W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)^{\mathfrak{n}_+}[\vec{\nu}]\\ &\simeq \mathrm{Hom}_{\mathfrak{n}_+}\left(\tilde{\mathcal{V}}^*_{\vec{\tilde{\nu}}},\, W\right)[\vec{\nu}] \simeq \mathrm{Hom}_{\mathfrak{n}_+}\left(\mathcal{V}_{\vec{\tilde{\nu}}},\, W\right)[\vec{\nu}] \simeq W\left[\vec{\nu}-\vec{\tilde{\nu}}\right]\end{aligned} \tag{218}$$

by using the conventions (216), Lemma 4.7, and Frobenius reciprocity. ◻

Remark 4.11 Putting together the identifications (218), we see that the resulting vector space isomorphism

$$\Xi\colon \left(\mathcal{V}_{\vec{\nu}}\hat{\otimes}W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)^{\mathfrak{sl}_N} \xrightarrow{\sim} W\left[\vec{\nu}-\vec{\tilde{\nu}}\right] \tag{219}$$

is obtained by pairing an element of $\left(\mathcal{V}_{\vec{\nu}}\hat{\otimes}W\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)^{\mathfrak{gl}_N}$ with $\tilde{\Omega}_{\vec{\nu}}\otimes\Omega_{\vec{\tilde{\nu}}}\in\tilde{\mathcal{V}}_{\vec{\nu}}\otimes\mathcal{V}_{\vec{\tilde{\nu}}}$ with respect to $(\cdot,\cdot)_{\vec{\nu}}$ and $(\cdot,\cdot)_{\vec{\tilde{\nu}}}$ in the first and third tensor factors, cf. Remark 4.8 and Lemma 4.7.

Applying Lemma 4.8 to the trivial and the twisted HW-modules of $\mathfrak{sl}_N$, we obtain:

Corollary 4.9 *(a) For the trivial $\mathfrak{sl}_N$-module $W=\mathbb{C}$, the space of invariants $\left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_2}\right)^{\mathfrak{sl}_N}$ vanishes if $\vec{\nu}_1\neq\vec{\nu}_2$, and is one-dimensional (hence, is spanned by $I^{(2)}_{\vec{\nu}_1}$ of (198)) if $\vec{\nu}_1=\vec{\nu}_2$.*

(b) For the twisted HW-modules $W=\mathcal{H}^{\vec{\mu}}_{\nu_2},\tilde{\mathcal{H}}^{\vec{\tilde{\mu}}}_{\nu_2}$, the spaces of invariants $\left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\mathcal{H}^{\vec{\mu}}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_3}\right)^{\mathfrak{sl}_N}$ and $\left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\tilde{\mathcal{H}}^{\vec{\tilde{\mu}}}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_3}\right)^{\mathfrak{sl}_N}$ are at most one-dimensional, and they vanish if $\vec{\mu}+\vec{\nu}_1-\vec{\nu}_3\notin\mathbb{Z}^{N-1}$, $\vec{\tilde{\mu}}+\vec{\nu}_3-\vec{\nu}_1\notin\mathbb{Z}^{N-1}$, respectively.

In particular, the invariants $I^{(3)}_{\vec{\nu}_1,\nu_2,\vec{\nu}_3} \in \left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\mathcal{H}^{\vec{\nu}_3-\vec{\nu}_1}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_3}\right)^{\mathfrak{sl}_N}$ *and* $\tilde{I}^{(3)}_{\vec{\nu}_1,\nu_2,\vec{\nu}_3} \in \left(\mathcal{V}_{\vec{\nu}_1}\hat{\otimes}\tilde{\mathcal{H}}^{\vec{\nu}_1-\vec{\nu}_3}_{\nu_2}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\nu}_3}\right)^{\mathfrak{sl}_N}$ *of* (208) *and* (211) *are unique, up to scalar multipliers.*

4.5 Our quartet

We are now finally ready to relate (89, 114) to the invariants in the completed tensor products of four $\mathfrak{sl}_N$-modules: the two Vermas and the two twisted HW-modules.

Let us fix $\vec{\nu}, \vec{\tilde{\nu}}, \vec{\gamma} \in \mathbb{C}^{N-1}$, and $\mathrm{m}, \tilde{\mathrm{m}} \in \mathbb{C}$. Let us specify four $\mathfrak{sl}_N$-representations as follows:

$$V_1 = \mathcal{V}_{\vec{\nu}}\,,\ V_2 = \mathcal{H}^{\vec{\gamma}-\vec{\nu}}_{\mathrm{m}}\,,\ V_3 = \tilde{\mathcal{H}}^{\vec{\gamma}-\vec{\tilde{\nu}}}_{\tilde{\mathrm{m}}}\,,\ V_4 = \tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\,. \tag{220}$$

We shall work with the completion

$$V_1\hat{\otimes}V_2\hat{\otimes}V_3\hat{\otimes}V_4\,,$$

so defined (cf. (216)) that it contains the power series expansion in $u^{(i)}_k$, $\tilde{u}^k_{(i)}$, z^a, $\tilde{\mathrm{z}}_a$ of Υ given by (89).

Let us now apply Lemma 4.8 to the case $W = V_2 \otimes V_3$. Noticing that

$$W \simeq \left\{ f \,\middle|\, f \in \mathbb{C}\left[(\mathrm{z}^1)^{\pm 1}, \ldots, (\mathrm{z}^N)^{\pm 1}, \tilde{\mathrm{z}}_1^{\pm 1}, \ldots, \tilde{\mathrm{z}}_N^{\pm 1} \right],\ \deg_{\mathrm{z}}(f) = \deg_{\tilde{\mathrm{z}}}(f) = 0 \right\}, \tag{221}$$

with the $\mathfrak{sl}_N$-action (151, 152) twisted by the factors (153), we get the following identification:

$$\left(V_1\hat{\otimes}V_2\hat{\otimes}V_3\hat{\otimes}V_4\right)^{\mathfrak{sl}_N} \simeq W\left[\vec{\nu} - \vec{\tilde{\nu}}\right] \simeq \mathbb{C}\left[\eta_1^{\pm 1}, \ldots, \eta_{N-1}^{\pm 1}\right], \tag{222}$$

where the variables η_i's are defined via:

$$\eta_i := \frac{\mathrm{z}^{i+1}\tilde{\mathrm{z}}_{i+1}}{\mathrm{z}^i\tilde{\mathrm{z}}_i}\ , \qquad 1 \le i \le N-1\,. \tag{223}$$

The above vector space isomorphism $\mathbb{C}\left[\eta_1^{\pm 1}, \ldots, \eta_{N-1}^{\pm 1}\right] \xrightarrow{\sim} \left(V_1\hat{\otimes}V_2\hat{\otimes}V_3\hat{\otimes}V_4\right)^{\mathfrak{sl}_N}$ is constructive. Explicitly, given $\vec{r} = (r_1, \ldots, r_{N-1}) \in \mathbb{Z}^{N-1}$, define the $\mathfrak{sl}_N$-weight $\vec{\delta} = (\delta_1, \ldots, \delta_{N-1}) \in \mathbb{Z}^{N-1}$ via $\delta_i = r_{i-1} - 2r_i + r_{i+1}$ with $r_0 = r_N = 0$. According to Lemma 4.8, the spaces of invariants $\left(\mathcal{V}_{\vec{\nu}}\hat{\otimes}\mathcal{H}^{\vec{\gamma}-\vec{\nu}}_{\mathrm{m}}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\gamma}+\vec{\delta}}\right)^{\mathfrak{sl}_N}$ and $\left(\mathcal{V}_{\vec{\gamma}+\vec{\delta}}\hat{\otimes}\tilde{\mathcal{H}}^{\vec{\gamma}-\vec{\tilde{\nu}}}_{\tilde{\mathrm{m}}}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\right)^{\mathfrak{sl}_N}$ are one-dimensional (for $\vec{r} = \vec{0}$, they are spanned by $I^{(3)}_{\vec{\nu},\mathrm{m},\vec{\gamma}}$ and $\tilde{I}^{(3)}_{\vec{\gamma},\tilde{\mathrm{m}},\vec{\tilde{\nu}}}$). Equivalently, there are unique $\mathfrak{sl}_N$-module homomorphisms:

$$\begin{aligned} \varphi_1 &: \mathcal{V}_{\vec{\gamma}+\vec{\delta}} \longrightarrow \mathcal{V}_{\vec{\nu}}\hat{\otimes}\mathcal{H}^{\vec{\gamma}-\vec{\nu}}_{\mathrm{m}}\,, \\ \varphi_2 &: \tilde{\mathcal{V}}_{\vec{\gamma}+\vec{\delta}} \longrightarrow \tilde{\mathcal{H}}^{\vec{\gamma}-\vec{\tilde{\nu}}}_{\tilde{\mathrm{m}}}\hat{\otimes}\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}\,, \end{aligned} \tag{224}$$

such that

$$\left(\varphi_1(\Omega_{\vec{\gamma}+\vec{\delta}}), \tilde{\Omega}_{\vec{\nu}}\right)_{\vec{\nu}} = \prod_{a=1}^{N} \left(\mathrm{z}^a\right)^{r_{a-1}-r_a} \cdot \omega_{\mathbf{n}}\,, \qquad \left(\Omega_{\tilde{\vec{\nu}}}, \varphi_2(\tilde{\Omega}_{\vec{\gamma}+\vec{\delta}})\right)_{\tilde{\vec{\nu}}} = \prod_{a=1}^{N} \tilde{\mathrm{z}}_a^{r_{a-1}-r_a} \cdot \tilde{\omega}_{\tilde{\mathbf{n}}}\,, \tag{225}$$

cf. (153–155, 157, 158), where we used Lemma 4.7 and the pairing $(\cdot,\cdot)_{\vec{\nu}}, (\cdot,\cdot)_{\tilde{\vec{\nu}}}$ of Remark 4.8 on the first and second components, respectively. Hence, we get an $\mathfrak{sl}_N$-module homomorphism:

$$\varphi := \varphi_1 \otimes \varphi_2 \colon \mathcal{V}_{\vec{\gamma}+\vec{\delta}} \hat{\otimes} \tilde{\mathcal{V}}_{\vec{\gamma}+\vec{\delta}} \longrightarrow V_1 \hat{\otimes} V_2 \hat{\otimes} V_3 \hat{\otimes} V_4\,. \tag{226}$$

Invoking the $\mathfrak{sl}_N$-invariant $I^{(2)}_{\vec{\gamma}+\vec{\delta}} \in \left(\mathcal{V}_{\vec{\gamma}+\vec{\delta}} \hat{\otimes} \tilde{\mathcal{V}}_{\vec{\gamma}+\vec{\delta}}\right)^{\mathfrak{sl}_N}$, we obtain the sought-after $\mathfrak{sl}_N$-invariant

$$\varphi\left(I^{(2)}_{\vec{\gamma}+\vec{\delta}}\right) \in \left(V_1 \hat{\otimes} V_2 \hat{\otimes} V_3 \hat{\otimes} V_4\right)^{\mathfrak{sl}_N}\,, \tag{227}$$

which exactly corresponds to $\eta_1^{r_1}\eta_2^{r_2}\cdots\eta_{N-1}^{r_{N-1}}$ under the identification (222).

Remark 4.12 The realization (222) corresponds to the family (over $\mathfrak{q}$) of maps

$$\begin{aligned} \Phi = \Upsilon(U,\tilde{U},\mathrm{z},\tilde{\mathrm{z}}) \cdot \psi\left(v_1(U,\tilde{U},\mathrm{z},\tilde{\mathrm{z}}), \ldots, v_{N-1}(U,\tilde{U},\mathrm{z},\tilde{\mathrm{z}}); \mathfrak{q}\right) \mapsto \\ \boldsymbol{\Psi}^{\mathrm{inst}}\left(\eta_1,\eta_2,\ldots,\eta_{N-1}, \frac{\mathfrak{q}}{\eta_1\eta_2\ldots\eta_{N-1}}\right) \end{aligned} \tag{228}$$

which consists, in detail, of restricting to $\pi_i \to \pi_i^0$ (131), $\tilde{\pi}^i \to \tilde{\pi}_0^i$ (126), and dropping the factor

$$\Upsilon(U_0,\tilde{U}_0,\mathrm{z},\tilde{\mathrm{z}}) = \prod_{a=1}^{N} \tilde{\mathrm{z}}_a^{\tilde{n}_a}(\mathrm{z}^a)^{n_a} \sim \boldsymbol{\Psi}^{\mathrm{tree}} \cdot \prod_{a=1}^{N}\left(\frac{\mathrm{z}^a}{\tilde{\mathrm{z}}_a}\right)^{\delta\mu_{a-1}}\,. \tag{229}$$

5 Knizhnik–Zamolodchikov equations

5.1 KZ equations

Let us recall the notion of Knizhnik–Zamolodchikov (KZ) equations [22] associated with the following data:

(a) $\mathfrak{g}$ –a semisimple Lie algebra,
(b) t –a non-degenerate ad-invariant bilinear form on $\mathfrak{g}$, that is:

$$t([a,b],c) = t(a,[b,c]) \quad \text{for any } a,b,c \in \mathfrak{g}\,,$$

(c) $V_1,\ldots,V_n$ –representations of $\mathfrak{g}$,

(d) $\kappa \in \mathbb{C}^\times$–a nonzero constant.

Define the Casimir tensor $\hat{C} \in \mathfrak{g} \otimes \mathfrak{g}$ and the Casimir element Cas $\in U(\mathfrak{g})$ via:

$$\hat{C} := \sum_{A,B \in I} t^{AB} X_A \otimes X_B \tag{230}$$

and

$$\mathrm{Cas} := \sum_{A,B \in I} t^{AB} X_A X_B\,, \tag{231}$$

where $\{X_A\}_{A\in I}$ is a basis of $\mathfrak{g}$, $\|t^{AB}\|$ is the matrix inverse to $\|t(X_A, X_B)\|$.

Define the configuration space $\Sigma_n \subset \mathbb{C}^n$ via:

$$\Sigma_n := \left\{ (p_1, \ldots, p_n) \in \mathbb{C}^n \,\middle|\, p_i \neq p_j \text{ for } i \neq j \right\}. \tag{232}$$

A function $F\colon \Sigma_n \to V_1 \otimes \cdots \otimes V_n$ is said to satisfy the *KZ equations* [22] if:

$$\kappa \frac{dF}{dp_i} + \sum_{j \neq i} \frac{\hat{C}_{ij} \cdot F}{p_i - p_j} = 0\,, \qquad i = 1, \ldots, n\,, \tag{233}$$

where $\hat{C}_{ij}$ denotes[2] the action of $\hat{C}$ (230) on the i-th and j-th factors of $V_1 \otimes \cdots \otimes V_n$.

Remark 5.1 Note that the KZ equations essentially depend only on the ad-invariant form $\frac{t}{\kappa}$.

5.2 $\mathfrak{g}$-invariance and $n = 4$ case

A function $F\colon \Sigma_n \to V_1 \otimes \cdots \otimes V_n$ is called $\mathfrak{g}$-*invariant* if:

$$F(\mathrm{p}) \in (V_1 \otimes \cdots \otimes V_n)^{\mathfrak{g}}\,, \qquad \forall\, \mathrm{p} = (p_1, \ldots, p_n) \in \Sigma_n\,. \tag{234}$$

Let $n = 4$. Recall the cross-ratio (193) of 4 points, which can be thought of as a map:

$$\pi\colon \Sigma_4 \longrightarrow \mathbb{C}^\times\,, \qquad \mathrm{p} = (p_1, p_2, p_3, p_4) \mapsto [p_1, p_2; p_3, p_4] := \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}\,.$$

This map can be naturally extended to a map $\bar{\pi}\colon \bar{\Sigma}_4 \to \mathbb{CP}^1$, where $\bar{\Sigma}_4 \subset (\mathbb{CP}^1)^4$ is the locus of points with pairwise distinct coordinates. The map $\bar{\pi}$ is the quotient map for the natural free action of $H = SL(2, \mathbb{C})$ on $\bar{\Sigma}_4$ (the diagonal action by the fractional linear transformations). In particular, for any $\mathrm{p} \in \Sigma_4$ the points $\mathrm{p} = (p_1, p_2, p_3, p_4)$

[2] A more pedantic notation would be:

$$\hat{C}_{ij} = \sum_{A,B \in I} t^{AB}\, \mathbf{1}_{V_1} \otimes \cdots \otimes T_{V_i}(X_A) \otimes \cdots \otimes T_{V_j}(X_B) \otimes \cdots \otimes \mathbf{1}_{V_n}\,.$$

and $(0, \mathfrak{q} = [p_1, p_2; p_3, p_4], 1, \infty)$ of $\bar{\Sigma}_4$ lie in the same H-orbit. Naturally the four KZ equations (233) on a $\mathfrak{g}$-invariant function F reduce to a single equation on a $(V_1 \otimes V_2 \otimes V_3 \otimes V_4)^{\mathfrak{g}}$-valued function of $\mathfrak{q}$:

Proposition 5.1 *Assume that the Casimir element Cas (231) acts on V_i as a multiplication by $\Delta_i \in \mathbb{C}$ for any $1 \leq i \leq 4$. Choose constants $\{\mathrm{d}_{ij} \mid 1 \leq i \neq j \leq 4\}$ so that $\mathrm{d}_{ij} = \mathrm{d}_{ij}$ and $\sum_{j \neq i} \mathrm{d}_{ij} = \Delta_i$ for any $1 \leq i \leq 4$.*[3] *Then, $F \colon \Sigma_4 \to (V_1 \otimes V_2 \otimes V_3 \otimes V_4)^{\mathfrak{g}}$ satisfies all four KZ equations (233) if and only if*[4]

$$F(p_1, p_2, p_3, p_4) = \prod_{i<j} (p_i - p_j)^{\frac{\mathrm{d}_{ij}}{\kappa}} \cdot \Phi\Big([p_1, p_2; p_3, p_4]\Big) \tag{235}$$

with $\Phi \colon \mathbb{C}^{\times} \backslash \{1\} \to (V_1 \otimes V_2 \otimes V_3 \otimes V_4)^{\mathfrak{g}}$ satisfying the following equation:

$$\kappa \frac{d\Phi}{d\mathfrak{q}} + \left(\frac{\mathrm{d}_{23}}{\mathfrak{q} - 1} + \frac{\mathrm{d}_{12}}{\mathfrak{q}} \right) \Phi + \left(\frac{\hat{C}_{23}}{\mathfrak{q} - 1} + \frac{\hat{C}_{12}}{\mathfrak{q}} \right) \Phi = 0 \, . \tag{236}$$

The proof of this result is elementary.

5.3 Our KZ setup

Let us now apply the above discussion to $\mathfrak{g} = \mathfrak{sl}_N$ endowed with an ad-invariant bilinear form $t(a, b) = \mathrm{tr}_{\mathbb{C}^N}(ab)$, and the $n = 4$ modules V_i $(1 \leq i \leq 4)$ as in (220):

$$V_1 = \mathcal{V}_{\vec{\nu}} \, , \;\; V_2 = \mathcal{H}_{\mathrm{m}}^{\vec{\gamma} - \vec{\nu}} \, , \;\; V_3 = \tilde{\mathcal{H}}_{\tilde{\mathrm{m}}}^{\vec{\gamma} - \vec{\tilde{\nu}}} \, , \;\; V_4 = \tilde{\mathcal{V}}_{\vec{\tilde{\nu}}} \, .$$

According to Lemma 4.8 and the identification (222), we have:

$$\left(V_1 \hat{\otimes} V_2 \hat{\otimes} V_3 \hat{\otimes} V_4 \right)^{\mathfrak{sl}_N} \simeq \mathbb{C}\left[\eta_1^{\pm 1}, \ldots, \eta_{N-1}^{\pm 1} \right] ,$$

with η_i's defined in (223). Hence, functions F and Φ of Proposition 5.1 can be thought of as:

$$F \colon \Sigma_4 \longrightarrow \mathbb{C}\left[\eta_1^{\pm 1}, \ldots, \eta_{N-1}^{\pm 1} \right] \quad \text{and} \quad \Phi \colon \mathbb{C}^{\times} \backslash \{1\} \longrightarrow \mathbb{C}\left[\eta_1^{\pm 1}, \ldots, \eta_{N-1}^{\pm 1} \right] . \tag{237}$$

Our next goal is to rewrite Eq. (236) on Φ as a differential equation in $\mathfrak{q}, \eta_1, \ldots, \eta_{N-1}$.

5.4 The differential operator $\widehat{H}^{\mathrm{KZ}}$

Choose the basis $\{X_A\}$ of $\mathfrak{g} = \mathfrak{sl}_N$ as follows:

$$\{X_A\} = \{\mathbf{J}_a^b \mid 1 \leq a \neq b \leq N\} \sqcup \{\mathfrak{h}_i \mid i = 1, \ldots, N-1\} \, .$$

[3] Such $\{\mathrm{d}_{ij}\}$ exist and are unique for an arbitrary choice of d_{12} and d_{13}.

[4] On any simply connected region in $\left(\mathbb{CP}^1\right)^4 \backslash \{\text{diagonals}\}$.

Then, the Casimir tensor (230) has the following form:

$$\hat{C} = \sum_{a\neq b} \mathbf{J}_a^b \otimes \mathbf{J}_b^a + \sum_{i,j=1}^{N-1} C^{ij}\mathfrak{h}_i \otimes \mathfrak{h}_j \in \mathfrak{sl}_N \otimes \mathfrak{sl}_N\,, \tag{238}$$

where $\|C^{ij}\|$ is the matrix inverse to the Cartan matrix $\|(2\delta_i^j - \delta_i^{j+1} - \delta_i^{j-1})\|$ of $\mathfrak{sl}_N$. To simplify the calculations, it is convenient to consider a natural embedding $\iota\colon \mathfrak{sl}_N \hookrightarrow \mathfrak{gl}_N$, so that:

$$(\iota\otimes\iota)\left(\sum_{i,j=1}^{N-1} C^{ij}\mathfrak{h}_i \otimes \mathfrak{h}_j\right) = \sum_{a=1}^{N} \mathbf{J}_a^a \otimes \mathbf{J}_a^a - \frac{1}{N}\mathcal{C}_1 \otimes \mathcal{C}_1\,, \tag{239}$$

where $\mathcal{C}_1 = \sum_{a=1}^N \mathbf{J}_a^a \in \mathfrak{gl}_N$ is the first Casimir operator (117). Similarly, the image of the Casimir element Cas (231) under the induced embedding $\iota\colon U(\mathfrak{sl}_N) \hookrightarrow U(\mathfrak{gl}_N)$ is given by:

$$(\iota\otimes\iota)(\mathrm{Cas}) = \mathcal{C}_2 - \frac{\mathcal{C}_1^2}{N}\,. \tag{240}$$

Define

$$\widehat{H}^{\mathrm{KZ}} = \frac{\hat{C}_{12}}{\mathfrak{q}} + \frac{\hat{C}_{23}}{\mathfrak{q}-1}\,. \tag{241}$$

The operators

$$\begin{aligned} \hat{C}_{12} &= \sum_{a,b=1}^{N} T_{\mathcal{V}_{\vec{\nu}}}(\mathbf{J}_a^b) \otimes T_{\mathcal{H}_{\mathrm{m}}^{\vec{\gamma}-\vec{\nu}}}(\mathbf{J}_b^a) + \frac{\mathrm{m}c_1(\mathcal{V}_{\vec{\nu}})}{N}\,,\\ \hat{C}_{23} &= \sum_{a,b=1}^{N} T_{\mathcal{H}_{\mathrm{m}}^{\vec{\gamma}-\vec{\nu}}}(\mathbf{J}_a^b) \otimes T_{\tilde{\mathcal{H}}_{\tilde{\mathrm{m}}}^{\vec{\gamma}-\vec{\tilde{\nu}}}}(\mathbf{J}_b^a) + \frac{\mathrm{m}\tilde{\mathrm{m}}}{N} \end{aligned} \tag{242}$$

coincide with $\hat{h}_0^{\mathrm{cft}}, \hat{h}_1^{\mathrm{cft}}$ of (88), respectively, which in turn coincide with $\hat{h}_0^{\mathrm{bps}}, \hat{h}_1^{\mathrm{bps}}$ of (56), according to Theorem 3.1. This concludes the proof of our main result: the vacuum expectation value $\langle\, \mathcal{S}\,\rangle$ of the surface defect obeys the Knizhnik–Zamolodchikov equation [22], specifically the equation obeyed by the $\left(\widehat{\mathfrak{sl}}_N\right)_k$ current algebra conformal block

$$\Phi = \Big\langle \mathbf{V}_1(0)\mathbf{V}_2(\mathfrak{q})\mathbf{V}_3(1)\mathbf{V}_4(\infty)\Big\rangle^{\mathbf{a}} \tag{243}$$

with the vertex operators at 0 and ∞ corresponding to the generic lowest weight $\mathcal{V}_{\vec{\nu}}$ and highest weight $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$ Verma modules, while the vertex operators at $\mathfrak{q}$ and 1 correspond to the *twisted* HW-modules $\mathcal{H}_{\mathrm{m}}^{\vec{\mu}}$ and $\tilde{\mathcal{H}}_{\tilde{\mathrm{m}}}^{\vec{\tilde{\mu}}}$.

6 Conclusions and further directions

In this paper, we established that the vacuum expectation value of the regular surface defect in $SU(N)$ gauge theory in four dimensions with $\mathcal{N}=2$ supersymmetry, with $2N$ fundamental hypermultiplets, obeys the analytical continuation of Knizhnik–Zamolodchikov equation for the four-point conformal block $\langle V_1 V_2 V_3 V_4 \rangle$ of the two-dimensional $\mathfrak{sl}_N$ current algebra at the level

$$k = \frac{\varepsilon_2}{\varepsilon_1} - N\,. \tag{244}$$

The surprising feature we discovered is the need to *twist* the irreducible representations corresponding to the *middle* vertex operators V_2 and V_3.

Our result has been anticipated for many years, see [29, 30]. In particular, in the specific limit $m_i \to \infty$, $\mathfrak{q} \to 0$, with

$$\Lambda^{2N} = \mathfrak{q} \prod_{f=1}^{2N} m_f \tag{245}$$

the equation (56) becomes the non-stationary version of the periodic Toda equation:

$$\kappa \Lambda \frac{\partial}{\partial \Lambda} \Psi = \left(\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \Lambda^2 \sum_{i=1}^{N} e^{x_i - x_{i+1}} \right) \Psi\,, \qquad x_{N+1} = x_1\,, \tag{246}$$

where

$$\mathfrak{q}_\omega m_\omega^+ m_\omega^- = \Lambda^2 e^{x_{\omega+1} - x_{\omega+2}}\,. \tag{247}$$

It was shown in [9] that the equation (246) is obeyed by the J-function of the affine flag variety, which in [29, 30] was interpreted as the vev of the surface defect in the pure $\mathcal{N} = 2$ super-Yang–Mills theory with $SU(N)$ gauge group. However, the method of [9] does not generalize to the theories with matter. In [32–37] the equations, obeyed by the surface defects of certain quiver gauge theories, were derived.

In the limit $\varepsilon_1 \to 0$ and/or $\varepsilon_2 \to 0$, the differential operator (56) becomes the equation describing certain Lagrangian submanifolds in the complex symplectic manifolds, which are related to the moduli spaces [46] of vacua of the four-dimensional gauge theory we started with, compactified on a circle. These moduli spaces can be also identified with the moduli space of solutions of some partial differential equations, describing monopoles and instantons in some auxiliary gauge theory [11, 44].

In this paper, we studied the simplest case of the asymptotically conformal $\mathcal{N} = 2$ gauge theory, corresponding to the A_1-type quiver. There exist various quiver generalizations, whose Seiberg–Witten geometry can be exactly computed [39]. The orbifold surface defects of the A_r-generalizations conjecturally obey the KZ equations corresponding to the $r + 3$-point conformal blocks of the $\widehat{\mathfrak{su}(N)}_k$ current algebra, with two Verma modules and $r + 1$ twisted HW-modules. One can also study the intersecting surface defects. For example, in the companion paper [17] a 5-point conformal block corresponding to the infinite-dimensional modules $\mathcal{V}_{\vec{\nu}}$, $\mathcal{H}^{\vec{\mu}}_{\mathrm{m}}$, $\tilde{\mathcal{H}}^{\vec{\tilde{\mu}}}_{\tilde{\mathrm{m}}}$, $\tilde{\mathcal{V}}_{\vec{\tilde{\nu}}}$, and the N-dimensional standard representation is associated with the intersecting surface defect of the orbifold type studied in this paper, and the orthogonal surface defect corresponding to the Q-observable of gauge theory [32–37, 40]

Perhaps the most interesting continuation of our work would be a translation of the connection between the conformal blocks of two-dimensional current algebra $\left(\widehat{\mathfrak{sl}_N}\right)_k$ to the surface defect partition function of four-dimensional gauge theory that we firmly established, to the A_{N-1} (0, 2)-theory in six dimensions.

For integral level k and the weights $\vec{\nu}$, $\vec{\tilde{\nu}}$, m, $\tilde{\text{m}}$ the current algebra conformal blocks have a familiar Chern–Simons interpretation. It can be represented as the path integral in the $SU(N)$ gauge theory on a three-ball B^3 with the action

$$\frac{k}{4\pi}\int_{B^3}\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right) \tag{248}$$

with the gauge fields having a curvature singularity along an embedded graph Γ, as in Fig. 1. The edges of the graph are labelled by the conjugacy classes of the monodromy of connection around the small loop linking the edge. We need an extension, or an analytic continuation, to the case of complex levels and weights. The paper [50] offers such a continuation for the Chern–Simons level. The analytic continuation of Chern–Simons theory in the representation parameters of Wilson and 't Hooft lines is not yet available, but our results strongly suggest it should be possible. We are familiar with the Wilson line operators $W_R(C)$, associated with the representation of the gauge group G and its representation R,

$$W_R(C)=\mathrm{Tr}_R\,T_R\left(P\exp\oint_C A\right). \tag{249}$$

More generally, a tri-valent orientation graph Γ, with oriented edges e labelled by representations R_e, with the understanding that the change of the orientation flips the representation $R_{\bar{e}}=R_e^*$, and vertices labelled by the invariants

$$I_v\in\left(R_{e_1}\otimes R_{e_2}\otimes R_{e_3}\right)^G \tag{250}$$

with the edges e_1, e_2, e_3 coming out of the vertex v, corresponds to the Wilson graph observable

$$W_{R_e,I_v}(\Gamma)=\prod_l \mathrm{Tr}_{R_l}\prod_v I_v\left(\bigotimes_e T_{R_e}\left(P\exp\int_e A\right)\right) \tag{251}$$

where l labels the loops, i.e., the edges with coinciding ends.

In the case the graph has tails, i.e., 1-valent vertices, which are placed at the boundary ∂B, the path integral takes values in the Hilbert space obtained by quantizing the moduli space of flat G-connections on $\Sigma^2=\partial B^3$ with singularities at the end-points, with fixed conjugacy classes of monodromies around those. In the case of B^3, $\Sigma^2\approx S^2$ this Hilbert space is isomorphic to the space of invariants in the tensor product of representations attached to the edges ending at the tails. For the graph Γ in Fig. 1, this would be

$$\left(R_1\otimes R_2\otimes R_3\otimes R_4\right)^G\ . \tag{252}$$

Having the invariants $I_1\in\left(R_1\otimes R_2\otimes R\right)^G$, $I_2\in\left(R^*\otimes R_3\otimes R_4\right)^G$ at the two internal vertices of Γ identifies the conformal block with the channel of the tensor

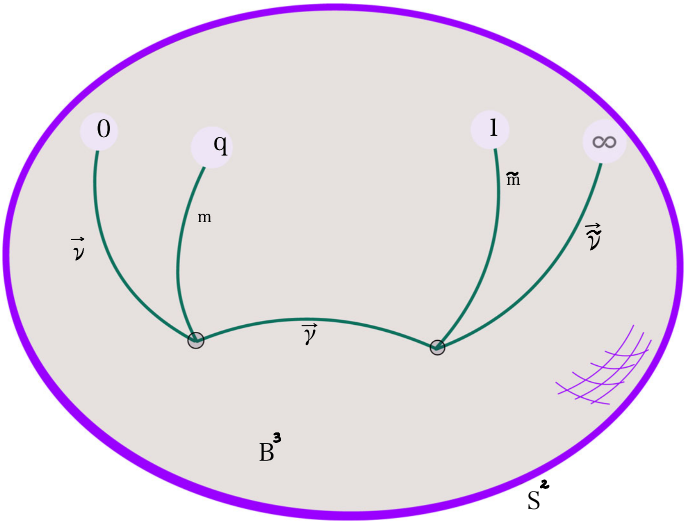

Fig. 1 Wilson graph corresponding to the 4-point conformal block

product decomposition (252) corresponding to the intermediate representation $R \in R_3 \otimes R_4$, $R^* \in R_1 \otimes R_2$.

All this, to a limited extent, generalizes to the infinite-dimensional $\mathfrak{g}$-representations, although the expression (251) does not literally make sense. Nevertheless, the form

$$\Upsilon = I^{(3)}_{\vec{\nu},\mathrm{m},\vec{\gamma}}(\tilde{U}', \mathrm{z}, U) \cdot I^{(2)}_{-\vec{\gamma}}(U'', \tilde{U}'') \cdot \tilde{I}^{(3)}_{\vec{\gamma},\tilde{\mathrm{m}},\vec{\tilde{\nu}}}(\tilde{U}, \tilde{\mathrm{z}}, U')\Big|_{\mathrm{diag}}$$

$$\equiv \prod_{a=1}^{N} \left(\frac{\left(\tilde{Z} \wedge \tilde{\Pi}^{a-1}\right)(\Pi'_a)}{\tilde{\Pi}^a(\Pi'_a)} \right)^{\tilde{n}_a} \left(\frac{\tilde{\Pi}'^a\,(Z \wedge \Pi_{a-1})}{\tilde{\Pi}'^a(\Pi_a)} \right)^{n_a}$$

$$\times \prod_{i=1}^{N-1} \left(\frac{\tilde{\Pi}^i\,(\Pi'_i) \cdot \tilde{\Pi}'^i\,(\Pi_i)}{\tilde{\Pi}''^i\,(\Pi''_i)} \right)^{\gamma_i} \Bigg|_{U=U'=U'',\, \tilde{U}=\tilde{U}'=\tilde{U}''} \qquad (253)$$

of our basic invariant Υ (89), and moreover, the $\mathfrak{q} \to 0$ asymptotics of the surface defect partition function (44), which can be analyzed [23] rather explicitly, are suggestive of some sort of three-dimensional interpretation with the graph Γ, with some intermediate $\mathfrak{sl}_N$-module with the highest/lowest/middle weight $\vec{\gamma}$.

It does not seem to be possible to analytically continue (251) as a line operator in the analytically continued Chern–Simons theory, as in [50]. However, it might be possible to analytically continue the S-dual 't Hooft operator, as a surface defect in the topologically twisted $\mathcal{N} = 4$ theory on a four-dimensional manifold with corners, which locally looks like $B^3 \times I$.

On the other hand, the surface defect in four dimensions can be related [42] to boundary conditions in the two-dimensional sigma model valued in the moduli space of vacua of the theory, compactified on a circle, which in the present case is believed

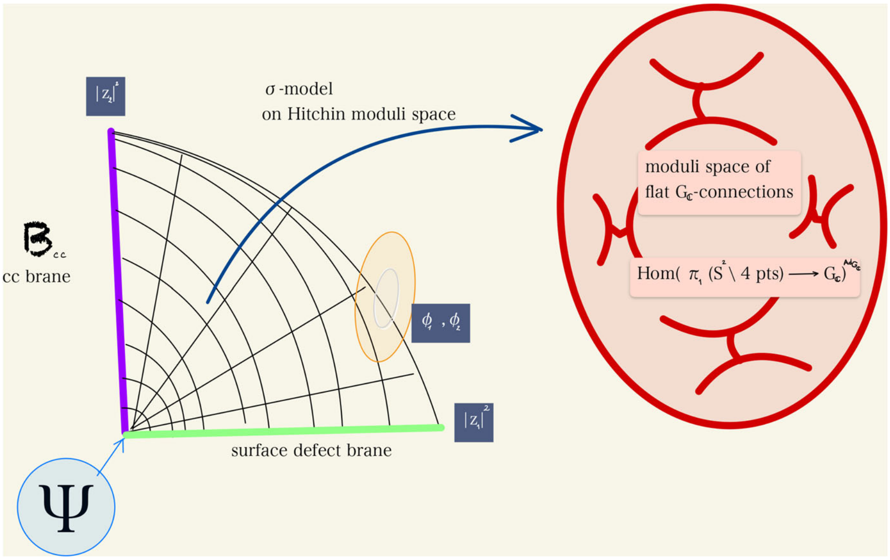

Fig. 2 Four-dimensional gauge theory in two-dimensional presentation

to be the moduli space $\mathcal{M}_N\left(S^2\backslash 4\,\text{pts}\,;\,\vec{\nu},\,\mathrm{m},\,\tilde{\mathrm{m}},\,\vec{\tilde{\nu}}\right)$ of $SU(N)$ Higgs pairs on a 4-punctured sphere with the regular punctures at 0 and ∞, and the minimal punctures at $\mathfrak{q}$ and 1, see Fig. 2. The homotopy between these two representatives of a cohomology class of an intrinsic operator in the six-dimensional theory proceeds by viewing the two-dimensional sigma model, with the worldsheet C as a long distance limit of the four-dimensional $\mathcal{N}=2$ Ω-deformed theory compactified on a two-torus T^2 as in [42], which, in turn, is a limit of the A_{N-1} (0, 2)-theory compactified on $\left(S^2\backslash 4\,\text{pts}\right)\times T^2$, which, finally, can be reinterpreted, as the $\mathcal{N}=4$ theory on $C\times\left(S^2\backslash 4\,\text{pts}\right)$. As in [42], the canonical parameter [19] Ψ (not to be confused with the vev of our surface defect) is identified with the ratio κ of the Ω-deformation parameters. With C having the topology of the corner $\mathbb{R}^2_+$, as in Fig. 2, the $\mathcal{N}=4$ theory on $C\times\left(S^2\backslash 4\,\text{pts}\right)$ looks very much like a gradient flow theory of the analytically continued Chern–Simons theory on $\mathbb{R}_+\times\left(S^2\backslash 4\,\text{pts}\right)$, with certain boundary conditions. We plan to discuss this duality in detail elsewhere.

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Appendix A: Analyticity properties

In this Appendix, we provide proofs of the regularity properties from Sect. 2.

Proof of Proposition 2.1 By inspecting the right-hand side of (12), we see that, for generic $\mathbf{a}, \varepsilon_1, \varepsilon_2$, and for any $\overline{\lambda} \in \mathcal{P}^N$, the rational functions $\left(Y(x)\,|_{\overline{\lambda}}\right)^{\pm 1}$ have only simple poles in x. Moreover, all the poles of $Y(x+\varepsilon)\,|_{\overline{\lambda}}$ and $\left(Y(x)\,|_{\overline{\lambda}}\right)^{-1}$ belong to the set

$$\bigsqcup_{1\le b\le N} \left\{ a_b + i\cdot\varepsilon_1 + j\cdot\varepsilon_2 \,\middle|\, i, j \in \mathbb{Z}_{\ge 0} \right\}. \tag{254}$$

Hence, to prove the regularity of $\langle\, \mathcal{X}(x)\,\rangle_\mu$, it suffices to verify that it has no poles at the above locus (254). Fix $1 \le b \le N,\ i \ge 0,\ j \ge 0$, and set

$$x_0 := a_b + i\cdot\varepsilon_1 + j\cdot\varepsilon_2. \tag{255}$$

The function $Y(x+\varepsilon)\,|_{\overline{\lambda}}$ has a pole at $x = x_0$ iff $\square = (i+1, j+1) \in \partial_-\lambda^{(b)}$, while the function $\left(Y(x)\,|_{\overline{\lambda}}\right)^{-1}$ has a pole at $x = x_0$ iff $\square = (i+1, j+1) \in \partial_+\lambda^{(b)}$. Note that

$$\overline{\lambda} \mapsto \overline{\lambda}' := \overline{\lambda}\backslash\square^b_{(i+1,j+1)} \tag{256}$$

(where $\square^b_{(i+1,j+1)}$ denotes the (i,j)-th box in the b-th Young diagram) establishes a bijection between the loci of $\overline{\lambda}$ satisfying the first condition and the loci of $\overline{\lambda}'$ satisfying the second condition. Finally, for any $\overline{\lambda}$ from the first locus, a straightforward computation shows that:

$$\mu\,|_{\overline{\lambda}} \cdot \mathrm{Res}_{x=x_0}\, Y(x+\varepsilon)\,|_{\overline{\lambda}} \;=\; -\,\mathfrak{q}\,\cdot \mu\,|_{\overline{\lambda}'} \cdot \mathrm{Res}_{x=x_0} \left(\frac{P(x)}{Y(x)\,|_{\overline{\lambda}'}} \right). \tag{257}$$

This completes our proof of the proposition. □

This result admits the following multi-parameter generalization [32–37]:

Proposition A.1 *For arbitrary parameters $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m) \in \mathbb{C}^m$, define the $\mathbb{C}(x)$-valued observable $\mathcal{X}(x;\boldsymbol{\nu})\colon \mathcal{P}^N \to \mathbb{C}(x)$ via:*

$$\mathcal{X}(x;\boldsymbol{\nu})\,|_{\overline{\lambda}} := \sum_{I\sqcup J=\{1,\ldots,m\}} \mathfrak{q}^{|J|}\cdot \prod_{i\in I}^{j\in J} R(\nu_i-\nu_j)\cdot \prod_{i\in I} Y(x-\nu_i+\varepsilon)\,|_{\overline{\lambda}}\cdot \prod_{j\in J} \frac{P(x-\nu_j)}{Y(x-\nu_j)\,|_{\overline{\lambda}}}, \tag{258}$$

where $R(z) = \frac{(z-\varepsilon_1)(z-\varepsilon_2)}{z(z-\varepsilon_1-\varepsilon_2)}$. Then, the average $\langle\, \mathcal{X}(x;\boldsymbol{\nu})\,\rangle_\mu$ is a regular function of x.

As for $m = 1$ and $\nu_1 = 0$, we have $\mathcal{X}(x;0) = \mathcal{X}(x)$, this result generalizes Proposition 2.1.

Proof of Proposition A.1 The proof is similar to the previous one. For generic $(\boldsymbol{\nu}, \mathbf{a}, \varepsilon_1, \varepsilon_2)$, each summand of (258) is a rational function in x with simple poles, all belonging to the set

$$\bigsqcup_{1\le b\le N} \left\{ a_b + \nu_r + i\cdot\varepsilon_1 + j\cdot\varepsilon_2 \,\middle|\, 1 \le r \le m\,,\ i, j \in \mathbb{Z}_{\ge 0} \right\}. \tag{259}$$

Moreover, for a fixed quadruple $(b, r, i, j) \in \{1, \ldots, N\} \times \{1, \ldots, m\} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ as in (259), the (I, J)-th summand of $\mathcal{X}(x; \boldsymbol{\nu}) |_{\overline{\lambda}}$ (258) has a pole at

$$x_0 := a_b + \nu_r + i \cdot \varepsilon_1 + j \cdot \varepsilon_2 \tag{260}$$

iff either of the following two conditions hold:

(I) $r \in I$ and $\square = (i+1, j+1) \in \partial_{-}\lambda^{(b)}$,
(II) $r \in J$ and $\square = (i+1, j+1) \in \partial_{+}\lambda^{(b)}$.

Clearly, the map

$$\left\{(I, J), \overline{\lambda}\right\} \mapsto \left\{\Big(I' := I \backslash \{r\}, J' := J \sqcup \{r\}\Big), \overline{\lambda}' := \overline{\lambda} \backslash \square^b_{(i+1, j+1)}\right\} \tag{261}$$

establishes a bijection between the loci of $\overline{\lambda}$ satisfying the first condition (I) and the loci of those satisfying the second condition (II), while a straightforward computation shows that:

$$\mu\, |_{\overline{\lambda}} \cdot \mathrm{Res}_{x=x_0}\, \mathcal{X}(x; \boldsymbol{\nu})\, |_{\overline{\lambda}} \; = \; -\, \mu\, |_{\overline{\lambda}'} \cdot \mathrm{Res}_{x=x_0}\, \mathcal{X}(x; \boldsymbol{\nu})\, |_{\overline{\lambda}'} \,. \tag{262}$$

The regularity of $\langle\, \mathcal{X}(x; \boldsymbol{\nu})\, \rangle_{\mu}$ follows. □

Finally, let us prove the analyticity in the orbifold/colored setup.

Proof of Proposition 2.2 It follows immediately from the proof of Proposition 2.1 presented above. The key observation is that, while each *non-colored* residue of $Y(x+\varepsilon)\, |_{\overline{\lambda}}$ and $\frac{P(x)}{Y(x)\, |_{\overline{\lambda}'}}$ at $x = x_0$ (255) is a product of elements from the lattice Λ (7) and their inverses, the corresponding *colored* residues of $Y_{\omega+1}(x+\varepsilon)\, |_{\overline{\lambda}}$ and $\frac{P_{\omega}(x)}{Y_{\omega}(x)\, |_{\overline{\lambda}'}}$ at $x = x_0$ are zero unless $\mathfrak{S}_{x_0} = \omega$, while in the latter case they are obtained from their *non-colored* counterparts by disregarding all factors from Λ with a nonzero $\mathbb{Z}_N$-grading. Likewise, all elements of the lattice Λ that appear in $\mu^{\mathrm{orb}}\, |_{\overline{\lambda}}$ (23) are obtained from those that appear in $\mu\, |_{\overline{\lambda}}$ (11) by disregarding all factors from Λ with a nonzero $\mathbb{Z}_N$-grading.

Therefore, for each pair $(\overline{\lambda}, \overline{\lambda}')$ from the proof of Proposition 2.1, see (256), we get (cf. (257)):

$$\mu^{\mathrm{orb}}\, |_{\overline{\lambda}} \cdot \mathrm{Res}_{x=x_0}\, \mathcal{X}_{\omega}(x)\, |_{\overline{\lambda}} \; = \; -\, \mu^{\mathrm{orb}}\, |_{\overline{\lambda}'} \cdot \mathrm{Res}_{x=x_0}\, \mathcal{X}_{\omega}(x)\, |_{\overline{\lambda}'} \,. \tag{263}$$

The regularity of $\langle\, \mathcal{X}_{\omega}(x)\, \rangle_{\mu^{\mathrm{orb}}}$ follows. □

Appendix B: Some technical computations

The following equations are used in the proof of Theorem 3.1:

$$\begin{aligned}
&\tilde{\mathrm{z}}(\mathrm{z}) \sum_{a=1}^{N} \left(\frac{\partial^2 v_i}{\partial \mathrm{z}^a \partial \tilde{\mathrm{z}}_a} \,,\, \frac{\partial v_i}{\partial \mathrm{z}^a} \frac{\partial v_j}{\partial \tilde{\mathrm{z}}_a} \,,\, \frac{\partial^2 \log \Upsilon}{\partial \mathrm{z}^a \partial \tilde{\mathrm{z}}_a} \right) = \left(1 - N v_i \,,\, v_i \delta_i^j - v_i v_j \,,\, 0 \right) \\
&\tilde{\mathrm{z}}(\mathrm{z}) \sum_{a=1}^{N} \left(\frac{\partial \log \Upsilon}{\partial \mathrm{z}^a} \frac{\partial v_i}{\partial \tilde{\mathrm{z}}_a} \,,\, \frac{\partial \log \Upsilon}{\partial \tilde{\mathrm{z}}_a} \frac{\partial v_i}{\partial \mathrm{z}^a} \right) = \left(n_i - \left(\sum_{a=1}^{N} n_a \right) v_i \,,\, \tilde{n}_i - \left(\sum_{a=1}^{N} \tilde{n}_a \right) v_i \right) \\
&\tilde{\mathrm{z}}(\mathrm{z}) \sum_{a=1}^{N} \frac{\partial \log \Upsilon}{\partial \mathrm{z}^a} \frac{\partial \log \Upsilon}{\partial \tilde{\mathrm{z}}_a} = \sum_{a=1}^{N} \frac{n_a \tilde{n}_a}{v_a}
\end{aligned} \tag{264}$$

and

$$\begin{aligned}
&\sum_{a,b=1}^{N} \mathrm{z}^b J_b^a \left(\frac{\partial v_i}{\partial \mathrm{z}^a} \right) = v_i (v_i + i - 2) + u_i (2 v_i - 1) \\
&\sum_{a,b=1}^{N} \mathrm{z}^b J_b^a \left(\frac{\partial \log \Upsilon}{\partial \mathrm{z}^a} \right) = \sum_{a=1}^{N} (a-1) n_a \\
&\sum_{a,b=1}^{N} \mathrm{z}^b J_b^a (\log \Upsilon) \frac{\partial v_i}{\partial \mathrm{z}^a} = v_i \left(\sum_{j=1}^{i-1} (\gamma_j - n_j) + \sum_{j=1}^{N-1} (n_j + \tilde{n}_j - \gamma_j) u_j \right) - \tilde{n}_i u_i \\
&\sum_{a,b=1}^{N} \mathrm{z}^b J_b^a (v_i) \frac{\partial (\log \Upsilon)}{\partial \mathrm{z}^a} = \left(\sum_{a=1}^{i-1} n_a \right) v_i - n_i u_i \\
&\sum_{a,b=1}^{N} \mathrm{z}^b J_b^a (v_i) \frac{\partial v_j}{\partial \mathrm{z}^a} = v_i v_j \left(\delta_{j<i} + 2 u_i - 1 + v_i \right) - u_i v_i \delta_i^j \\
&\sum_{a,b=1}^{N} \mathrm{z}^b J_b^a (\log \Upsilon) \frac{\partial (\log \Upsilon)}{\partial \mathrm{z}^a} = \sum_{1 \le a \le b \le N} n_a n_b - \sum_{1 \le a \le b \le N-1} n_a \gamma_b - \sum_{a=1}^{N} n_a \tilde{n}_a \frac{u_a}{v_a}
\end{aligned} \tag{265}$$

with u_i's defined in (77) and satisfying the equality $v_{i+1} = u_i - u_{i+1}$ of *loc.cit*.

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