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Relativistic perfect fluid spacetimes and Ricci–Yamabe solitons

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Abstract
In this research paper, we determine the Ricci–Yamabe soliton on a perfect fluid spacetime with a torse-forming vector field. Besides this, we evaluate a specific situation when the potential vector field ζ is of the gradient type, we deduce a Poisson and a Liouville equation from the soliton equation. In addition, we explore some harmonic significance of γ-Ricci–Yamabe soliton on perfect fluid spacetime with a harmonic potential function ψ. Finally, we discuss necessary and sufficient conditions for the 1-form γ, which is the g-dual of the vector field ζ on a perfect fluid spacetime to be a solution for the Schrödinger–Ricci equation.

Keywords Ricci–Yamabe soliton · Perfect fluid spacetime · Torse-forming vector field · Schrödinger–Ricci equation

Mathematics Subject Classification 53C44 · 53B30 · 53C50 · 53C80

1 Introduction

The fascinating feature of this universe is symmetry. It is also one of the scientific principles that can describe the laws of nature to other physical phenomena, such as general relativity. Albert Einstein founded in the early nineteenth century, "Theory of General Relativity" (GR). In GR the field of gravity with its source is the spacetime curvature and energy–momentum tensor, respectively. Einstein's field equations

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explain spacetime curvature evolution lead to particles physics especially as nuclear physics, astrophysics and plasma physics. To understand the GR, we discuss the model of relativistic perfect fluids from the view of differential geometry. The GR is based on describing the spacetime as the curved manifold.

The spacetime can be represented as a connected 4-dimensional time oriented Lorentzian manifold, considered as a particular subclass of semi-Riemannian manifold with Lorentzian metric g. The geometry of Lorentzian manifold is established to study the vectors nature on the manifold. As a result, Lorentzian manifold is the best model to investigate GR and cosmology. Alias et al. [1] described the theory of generalized Robertson–Walker spacetime (GRW, in short), which is a generalization of Robertson–Walker (RW) spacetime.

An n-dimensional Lorentzian manifold M is a GRW spacetime with dimension n. According to Sanchez [23], the GRW spacetime is used in inhomogeneous spacetime with an isotropic radiation. In his work, O'Neil [21] stated that RW-spacetime is perfect fluid spacetime (briefly say, $PFST$). For dimension $n = 4$, if it is RW spacetime, GRW spacetime is a $PFST$.

In GR, physical matters symmetry is specially relating to the spacetime geometry. More specifically, the metric of symmetry usually simplifies for the classification of solutions of Einstein's field equations. An important symmetry is soliton that connected to geometrical flow of spacetime geometry. In fact Ricci flow, Yamabe flow and Einstein flow are used to understand the idea of kinematics and thermodynamics in GR. Ricci soliton and Yamabe soliton are focused because curvatures keep the self-similarity.

In [2] authors examined the spacetime in terms of Ricci soliton. Later, Blaga [6] illustrated properties of the $PFST$ with η-Ricci soliton and η-Einstein solitons. Venkatesha and Aruna [28] also discussed Ricci solitons on $PFST$. Several authors (cf. [3,6,10,14,26]) researched spacetime with solitons extensively in different ways.

Consequently, to continue the work as in the mentioned studies, we concentrated here on the geometry of $PFST$ admitting Ricci–Yamabe soliton and γ-Ricci–Yamabe soliton with a torse-forming vector field. With the support of Ricci–Yamabe maps studied in [16], Siddiqi and Akyol have established a novel notion of Ricci–Yamabe soliton and its extension (for more detail see [24]).

2 Emergence of Ricci–Yamabe solitons

In 1988, Hamilton [17] simultaneously presented the theory of Ricci flow and Yamabe flow for the first time. The solution limit of the Ricci flow and Yamabe flow respectively tends to be Ricci soliton and Yamabe soliton. In fact, the Yamabe soliton coincides with the Ricci soliton for dimension $n = 2$, but when $n > 2$, the Yamabe and Ricci solitons are not the same, the Yamabe soliton retains the conformal class.

The theory of geometric flows, such as Ricci flow and Yamabe flow, has been the topic of inspiration for many geometers over the last twenty years. In the analysis of singularities of the flows, a certain segment of solutions on which the metric evolves by dilation and diffeomorphisms plays an important role as they appear as feasible singularity models. They are also referred to as soliton solutions.

In 2020, Siddiqi and Akyol [24] have recently addressed the development of Ricci–Yamabe solitons from a geometric flow that is a scalar combination of Ricci and Yamabe flow [16]. This is also referred to as the Ricci–Yamabe flow of the form (α, β). The Ricci–Yamabe flow is an evolution of the semi-Riemannian multiple metrics that are described as

$$\partial_t g(t) = -2\alpha \mathcal{R}ic(t) + \beta \mathcal{R}(t) g(t), \quad g_0 = g(0), \quad t \in (a, b), \tag{2.1}$$

where $\mathcal{R}ic$ and $\mathcal{R}$ denote the Ricci tensor and scalar curvature, respectively. The Ricci–Yamabe flow can also be a Riemannian or semi-Riemannian or singular Riemannian flow due to the indication of involved scalars α and β. In certain geometrical or physical models, such as relativistic theories, this kind of multiple choices can be beneficial. Therefore, Ricci–Yamabe soliton naturally emerges as the limit of the flow of Ricci–Yamabe soliton. This is an important inspiration for learning Ricci–Yamabe solitons. In [9], an interpolated soliton between Ricci and Yamabe soliton is considered named Ricci–Bourguignon soliton corresponding to Ricci–Bourguignon flow.

Ricci–Yamabe solitons are called a soliton in the Ricci–Yamabe flow if it moves only by diffeomorphism and scaling by one parameter group only. A Ricci–Yamabe soliton on Riemannian manifold (M, g) is a data $(g, V, \Omega, \alpha, \beta)$ satisfying

$$\frac{1}{2}\mathcal{L}_V g + \alpha \mathcal{R}ic + \left(\Omega - \beta \frac{\mathcal{R}}{2}\right) g = 0, \tag{2.2}$$

where $\mathcal{L}_V$ indicates the Lie derivative in the direction of vector field V and α and β are scalars. In (M, g), Ricci–Yamabe soliton is called *shrinking*, *expanding* or *steady*, corresponding to $\Omega < 0$, $\Omega > 0$ or $\Omega = 0$, respectively. Therefore, generalization of Ricci soliton and Yamabe soliton is termed as (α, β)-type Ricci–Yamabe soliton in (2.2). Moreover, Ricci–Yamabe soliton is called α-Ricci soliton and β-Yamabe soliton corresponding to $(\alpha, 0)$, and $(0, \beta)$, respectively.

After perturbing the equation (2.2), that specifies a new generalized form of soliton by a multiple of a (0, 2)-tensor field $\gamma \otimes \gamma$, we turn up a new notion, namely, γ-Ricci–Yamabe soliton of type (α, β) defined as [24]:

$$\mathcal{L}_V g + 2\alpha \mathcal{R}ic + (2\Omega - \beta \mathcal{R}) g + 2\mu \gamma \otimes \gamma = 0. \tag{2.3}$$

Again let us note that γ-Ricci–Yamabe soliton of type $(\alpha, 0)$ or, type $(0, \beta)$ are α-γ-Ricci soliton and β-γ-Yamabe soliton, respectively, for more details about these particular cases see ([4,5,11,15,24,25]). In addition, a γ-Ricci–Yamabe soliton is called *shrinking*, *expanding* or *steady*, corresponding to $\Omega < 0$, $\Omega > 0$ or $\Omega = 0$, respectively.

Let us mention the case of Einstein soliton, which generates self-similar solutions to Einstein flow in such a way that

$$\partial_t g(t) = -2\left(\mathcal{R}ic - \frac{\mathcal{R}}{2} g\right).$$

Therefore, an Einstein soliton occurs as the limit solution of the Einstein flow:

$$\frac{1}{2}\mathcal{L}_V g + \mathcal{R}ic + \left(\Omega - \frac{\mathcal{R}}{2}\right) g = 0. \tag{2.4}$$

Comparing equation (2.2) and (2.4), we find (1, 1) type-Ricci–Yamabe soliton.

3 Preliminaries

Matter tells space how to curve, and curved space tells matter how to move. Distribution of matter is known to be fluid and evolution carrying as a matter of spacetime substance like pressure, volume, thermal amounts, speed, torque, shear and expansion [27]. The matter tensor performs a vital part in standard cosmological models, the material content of the universe is considered to perform like a $PFST$.

In the following, the energy-momentum tensor $\mathcal{T}$ of a $PFST$ is of type ([20], [21])

$$\mathcal{T}(E, F) = \rho g(E, F) + (\sigma + \rho)\gamma(E)\gamma(F) \tag{3.1}$$

where σ, ρ indicates the energy density and isotropic pressure, respectively, of the fluid.

The field equation governing the perfect fluid motion is Einstein's gravitational equation [21]

$$\mathcal{R}ic(E, F) + \left(\lambda - \frac{\mathcal{R}}{2}\right) g(E, F) = \kappa\mathcal{T}(E, F), \tag{3.2}$$

for all $E, F \in \chi(M)$, where λ is the cosmological constant κ is the gravitational constant (which can be taken as $8\pi G$, with the universal gravitational constant of G), $\mathcal{R}ic$ is the Ricci tensor, and $\mathcal{R}$ the scalar curvature.

Now, adopting (3.1) and (3.2) we find the Einstein's equation for a $PFST$:

$$\mathcal{R}ic = \left(-\lambda + \frac{\mathcal{R}}{2} + \kappa\rho\right) g + \kappa(\sigma + \rho)\gamma(E) \otimes \gamma(F). \tag{3.3}$$

4 Perfect fluid spacetimes

In this, section, we discuss the basic ingredients about a $PFST$.

Definition 4.1 [13] An n-dimensional Lorentzian manifold is said to be a $PFST$ if its non-vanishing Ricci tensor $\mathcal{R}ic$ satisfies

$$\mathcal{R}ic = ag + b\gamma \otimes \gamma, \tag{4.1}$$

where a, b are scalar fields (not simultaneously zero), ζ is a vector field metrically equivalent to the 1-form γ, that is $g(E, \zeta) = \gamma(E)$, for all E, and $g(\zeta, \zeta) = -1$. Here the unit timelike vector field ζ is also known as the velocity vector filed of the $PFST$.

Definition 4.2 [30] A vector field ζ on a semi-Riemannian manifold is said to be a torse-forming vector field (TFV) if, for any $E \in \chi(M)$, it satisfies

$$\nabla_E \zeta = \omega E + \gamma(E)\zeta, \tag{4.2}$$

where ω is a scalar function and γ is a non-vanishing 1-form.

In ([11], Theorem 1, Page 3) Chen and Mantica and Molinari ([19], Proposition 3.7, Page 10) have given the following deepest results on GRW spacetime. We mention their results in the form of following theorems:

Theorem 4.3 [11] *A Lorentzian manifold M with $dim(M) \geq 3$ is a GRW spacetime if and only if it admits a timelike concircular vector field.*

Theorem 4.4 [19] *A Lorentzian manifold M with $dim(M) \geq 3$ is a GRW spacetime (GRW) if and only if it admits a unit timelike TFV $\nabla_E \zeta = \omega E + \gamma(E)\zeta$, that is also an eigenvector of the Ricci tensor.*

It is observed that a unit timelike TFV unit is F on an n-dimensional Lorentzian manifold M satisfies [30]:

$$\nabla_E F = \omega[E + \gamma(E)F], \tag{4.3}$$

where γ is a 1-form such that $g(E, F) = \gamma(E)$, for all E. For more details (see [11], [19]). Now, following by the above equations, the Lie differentiation of g for $PFST$ along with a unit timelike TFV ζ, is

$$(\mathcal{L}_\zeta g)(E, F) = 2\omega[g(E, F) + \gamma(E)\gamma(F)]. \tag{4.4}$$

Let (M^4, g) be a $PFST$ satisfying (3.3). Contracting (3.3) and considering that $g(\zeta, \zeta) = -1$, we obtain

$$\mathcal{R}ic(E, F) = \left(\lambda - \frac{\kappa}{2}\{\rho - \sigma\}\right) g(E, F) + \kappa(\sigma + \rho)\gamma(E)\gamma(F), \tag{4.5}$$

$$QE = aE + b\gamma(E)\zeta, \tag{4.6}$$

$$\mathcal{R} = \kappa(\sigma - 3\rho) + 4\lambda, \tag{4.7}$$

where $a = \left(\lambda - \frac{\kappa}{2}(\rho - \sigma)\right)$ and $b = \kappa(\sigma + \rho)$. Also

$$\mathcal{R}ic(\zeta, \zeta) = \frac{\kappa}{2}(3p + \sigma) - \lambda. \tag{4.8}$$

5 Perfect fluid spacetimes admitting Ricci–Yamabe soliton

In this section, we study Ricci–Yamabe soliton of (α, β)-type in a $PFST$ with a timelike TFV ζ.

Now, adopting $V = \zeta$, equation (2.2) becomes

$$(\mathcal{L}_{\zeta} g)(E, F) + 2\alpha \mathcal{R}ic(E, F) + (2\Omega - \beta\mathcal{R})g(E, F) = 0. \tag{5.1}$$

Using (4.4), we turn up

$$\mathcal{R}ic(E, F) = -\left(\frac{\Omega}{\alpha} - \frac{\beta\mathcal{R}}{2\alpha} + \omega\right) g(E, F) - \omega\gamma(E)\gamma(F). \tag{5.2}$$

Thus, we can conclude the following results.

Theorem 5.1 *A Lorentzian manifold M $(dim(M) \geq 4)$ admitting a Ricci–Yamabe soliton of type (α, β), whose soliton field is a unit timelike TFV, is a $PFST$, provided $\alpha \neq 0$ and $\beta \neq 0$.*

Now, in the light of the Theorems (4.4) and (5.1) we turn up the following corollary.

Corollary 5.2 *A GRW spacetime admitting a Ricci–Yamabe soliton of type (α, β) is a $PFST$, provided $\alpha \neq 0$ and $\beta \neq 0$.*

Now, putting $E = F = \zeta$ in (5.2) and using (4.8), we get

$$\Omega = \frac{\beta\mathcal{R}}{2} + \alpha\left(\frac{\kappa}{2}(\sigma + 3\rho) - \lambda\right). \tag{5.3}$$

Thus, we have the following results.

Theorem 5.3 *If a $PFST$ with a unit timelike TFV ζ admits a Ricci–Yamabe soliton (g, ζ, Ω) of type (α, β), then the Ricci–Yamabe soliton is expanding.*

Corollary 5.4 *If a $PFST$ with a unit timelike TFV ζ admits a Ricci–Yamabe soliton (g, ζ, Ω) of type $(0, \beta)$, then the β-Yamabe soliton is expanding.*

Corollary 5.5 *If a $PFST$ with a unit timelike TFV ζ admits a Ricci–Yamabe solitons (g, ζ, Ω) of type $(\alpha, 0)$, then the Ricci–Yamabe soliton is expanding, steady and shrinking according as*

(1) $\lambda < \frac{\kappa}{2}(\sigma + 3\rho)$,
(2) $\lambda = \frac{\kappa}{2}(\sigma + 3\rho)$ *and*
(3) $\lambda > \frac{\kappa}{2}(\sigma + 3\rho)$, *respectively.*

Remark 5.6 According to the above Corollaries 5.4 and 5.5, we can easily obtain similar results for GRW-spacetime with Ricci–Yamabe solitons.

Let us assume that the spacetime without cosmological constant, i.e., $\lambda = 0$. Then from equation (4.8), we turn up $\mathcal{R}ic(\zeta, \zeta) = \frac{\kappa(3\rho+\sigma)}{2}$. In case if the spacetime satisfies the *timelike convergence condition* (TCC), i.e., $\mathcal{R}ic(\zeta, \zeta) > 0$, then $(3\rho + \sigma > 0$, the spacetime satisfies cosmic strong energy condition (briefly say, SEC) [12]. Therefore, in light of the above fact and by means of equation (5.3), we conclude the following:

Theorem 5.7 *If a $PFST$ admits a Ricci–Yamabe soliton (g, ζ, Ω) of type (α, β) with a unit timelike TFV ζ and obeying TCC, then the Ricci–Yamabe soliton is expanding.*

Corollary 5.8 *If a $PFST$ admits a Yamabe soliton (g, ζ, Ω) of type $(0, \beta)$ with a unit timelike TFV ζ and obeying TCC, then the Yamabe soliton is expanding.*

6 γ-Ricci–Yamabe soliton on *PFST*

Assume the expression

$$(\mathcal{L}_\zeta g)(E, F) + 2\alpha \mathcal{R}ic(E, F) + (2\Omega - \beta\mathcal{R})g(E, F) + 2\mu\gamma(E)\gamma(F) = 0, \tag{6.1}$$

where g is a Lorentzian metric, $\mathcal{R}ic$ is the Ricci curvature, ζ is a timelike vector field, γ is a 1-form and Ω and μ are real constant.

Using the definition of Lie derivative $\mathcal{L}_\zeta g$ and (6.1), we turn up

$$\begin{aligned}\alpha\mathcal{R}ic(E, F) = & -\left(\Omega - \frac{\beta\mathcal{R}}{2}\right) g(E, F) - \mu\gamma(E)\gamma(F)\\ & -\frac{1}{2}[g(\nabla_E \zeta, F) + g(E, \nabla_F \zeta)],\end{aligned} \tag{6.2}$$

for any $E, F \in \chi(M)$.

Now, using contraction in (6.2) we obtain

$$-(\alpha - \beta)\mathcal{R} = -4\Omega + \mu - div(\zeta). \tag{6.3}$$

Let the data (g, ζ, Ω, μ) be an γ-Ricci–Yamabe soliton on a $PFST$ (M^4, g). From (4.5) and (6.2) we find

$$\begin{aligned}&\alpha\left[\lambda - \frac{\kappa(\rho - \sigma)}{2} + \left(\Omega - \frac{\beta\mathcal{R}}{2}\right)\right] g(E, F) + [\alpha\kappa(\sigma + \rho) + \mu]\gamma(E)\gamma(F)\\ &+\frac{1}{2}[g\nabla_E \zeta, F) + g(E, \nabla_F \zeta] = 0,\end{aligned} \tag{6.4}$$

for any $E, F \in \chi(M)$. Let $\{\mathcal{I}_i\}_{1 \le i \le 4}$ be an orthonormal frame and $\zeta = \sum_{i=1}^{4} \zeta^i \mathcal{I}_i$. We have $\sum_{i=1}^{4} \varepsilon_{ii}(\zeta^i)^2 = -1$ and $\eta(\mathcal{I}_i) = \varepsilon_{ii}\zeta^i$, where $\varepsilon_{ii} = 1, i \in \{2, 3, 4\}$, $\varepsilon_{ij} = 0, i, j \in \{1, 2, 3, 4\}$.

Multiplying (6.4) by ε_{ii} and using summation over i, for $E = F = \mathcal{I}_i$, we get

$$4\alpha\Omega - \mu = -(\alpha + 2\alpha\beta)4\lambda + \kappa[(\alpha + 6\alpha\beta)\rho - (3\alpha + 2\alpha\beta)\sigma] - div(\zeta). \tag{6.5}$$

Writing (6.4) for $E = F = \zeta$, we infer

$$\alpha\Omega - \mu = -(\alpha + 2\alpha + \beta)\lambda - \frac{\kappa}{2}[(\alpha + 3\alpha\beta)\rho + (\alpha + \alpha\beta)\sigma]. \tag{6.6}$$

Therefore,

$$\begin{cases}\Omega = -(1+2\beta)\lambda + \frac{\kappa}{\alpha}(c_1\rho - c_2\sigma) - \frac{div(\zeta)}{3\alpha} \\ \mu = \frac{\kappa}{3}(3\alpha\rho + c_3\sigma) - \frac{div(\zeta)}{3},\end{cases} \tag{6.7}$$

where $c_1 = (\frac{\alpha}{2} + \frac{3\alpha\beta}{2})$, $c_2 = (7\alpha - \alpha\beta)$ and $c_3 = (5\alpha + 4\alpha\beta)$. Using (6.7) we can state the following results.

Theorem 6.1 *Let M be a relativistic $PFST$ and let γ be the g-dual 1-form of the gradient vector field $\zeta = grad(\psi)$. If (6.1) defines a γ-Ricci–Yamabe soliton with nonzero α and β in M, then the Poisson equation satisfied by ψ becomes*

$$\Delta(\psi) = -3[\mu - \frac{\kappa}{3}(3\alpha\rho + c_3\sigma)]. \tag{6.8}$$

Corollary 6.2 *Let M be a relativistic $PFST$ and let γ be the g-dual 1-form of the gradient vector field $\zeta = grad(\psi)$. If (6.1) defines a γ-Yamabe soliton with $\alpha = 0$ in M, then the Poisson equation satisfied by ψ becomes*

$$\Delta(\psi) = -3\mu. \tag{6.9}$$

Example 6.3 A γ-Ricci–Yamabe soliton $(g, \zeta, \Omega, \mu, \alpha, \beta)$ in a $PFST$ is given by

$$\begin{cases}\Omega = -(1+2\beta)\lambda + \frac{\kappa}{\alpha}(c_1\rho - c_2\sigma) - \frac{div(\zeta)}{3\alpha} \\ \mu = \frac{\kappa}{3}(3\alpha\rho + c_3\sigma) - \frac{div(\zeta)}{3}.\end{cases}$$

If $\mu = 0$, then we turn up the Ricci–Yamabe soliton of (α, β)-type with $\Omega = -(1 + 2\beta)\lambda + \frac{\kappa}{\alpha}(c_1\rho - c_2\sigma)$ which is steady if $\rho = \frac{(\alpha+2\alpha\beta)}{c_1}\left(\frac{\lambda}{\kappa}\right) + \frac{c_2}{c_1}\sigma$, expanding if $\rho > \frac{(\alpha+2\alpha\beta)}{c_1}\left(\frac{\lambda}{\kappa}\right) + \frac{c_2}{c_1}\sigma$, and shrinking if $\rho < \frac{(\alpha+2\alpha\beta)}{c_1}\left(\frac{\lambda}{\kappa}\right) + \frac{c_2}{c_1}\sigma$. Thus, we have the following result.

Theorem 6.4 *Let $(g, \zeta, \Omega, \mu, \alpha, \beta)$ be a γ-Ricci–Yamabe soliton of (α, β)-type in a relativistic $PFST$. Then the soliton is expanding, steady and shrinking according as*

(1) $\rho > \frac{(\alpha+2\alpha\beta)}{c_1}\left(\frac{\lambda}{\kappa}\right) + \frac{c_2}{c_1}\sigma$,
(2) $\rho = \frac{(\alpha+2\alpha\beta)}{c_1}\left(\frac{\lambda}{\kappa}\right) + \frac{c_2}{c_1}\sigma$ *and*
(3) $\rho < \frac{(\alpha+2\alpha\beta)}{c_1}\left(\frac{\lambda}{\kappa}\right) + \frac{c_2}{c_1}\sigma$, *respectively.*

Remark 6.5 Also, for $\Psi \in C^\infty(M)$ and a vector field ζ, a straight forward calculation gives

$$\text{div}(\Psi\zeta) = \zeta(d\Psi) + \Psi\text{div}\zeta. \tag{6.10}$$

The function $\Psi \in C^\infty(M)$ is said to be a last multiplier of vector field ζ with respect to g if $\text{div}(\Psi\zeta) = 0$. The corresponding equation

$$\zeta(d\ln\Psi) = -\text{div}(\zeta) \tag{6.11}$$

is called the **Liouville equation** of the vector field ζ with respect to g (for more details see [18,22]).

Now, infer the above remark and equation (6.7), we obtain the following result.

Theorem 6.6 *Let M be a relativistic $PFST$ and γ be the g-dual 1-form of the gradient vector field $\zeta = grad(\psi)$, $\Psi \in C^{\infty}(M)$. If (6.1) defines a γ-Ricci–Yamabe soliton with non-vanishing α and β, then the Liouville equation satisfied by Ψ and ζ becomes*

$$\zeta(d\ln\Psi) = -3\left(\mu - \frac{\kappa}{3}(3\alpha\rho + c_3\sigma)\right). \tag{6.12}$$

7 Harmonic significance of γ-Ricci–Yamabe soliton on a *PFST*

Let η be the g-dual 1-form of the given vector field ζ, with $g(E,\zeta) = \gamma(E)$ and $g(\zeta,\zeta) = -1$. Then ζ is called the solution of the *Schrödinger–Ricci* equation if it satisfies

$$\mathrm{div}(\mathcal{L}_\zeta g) = 0, \tag{7.1}$$

where $\mathcal{L}_F g$ is the Lie derivative in the direction of vector field ζ. Chow et al. [8] studied the divergence of the Lie derivative such that

$$\mathrm{div}(\mathcal{L}_\zeta g) = (\Delta + \mathcal{R}ic)(\zeta) + d(\mathrm{div}(\zeta)), \tag{7.2}$$

where Δ represents the Laplace–Hodge operator with respect to the metric g and $\mathcal{R}ic$ is the Ricci curvature tensor field. Now, consider the equation

$$\mathcal{L}_\zeta g + 2\alpha\mathcal{R}ic + (2\Omega - \beta\mathcal{R})g + 2\mu\gamma\otimes\gamma = 0. \tag{7.3}$$

Taking the trace of the equation (7.3) we find

$$\mathrm{div}(\zeta) + (\alpha - 4\beta)\mathcal{R} + 4\Omega + \mu\,|\zeta|^2 = 0, \tag{7.4}$$

where $\mathcal{R}$ is the scalar curvature. By the direct computation we turn up

$$\mathrm{div}(\gamma\otimes\gamma) = \mathrm{div}(\zeta)\gamma + \nabla_\zeta\gamma. \tag{7.5}$$

By computing the divergence of (7.3) and using (7.2), we turn up

$$\mathrm{div}(\mathcal{L}_\xi g) + (\alpha - 4\beta)d(\mathcal{R}) + 2\mu[\mathrm{div}(\zeta)\gamma + \nabla_\zeta\gamma] = 0. \tag{7.6}$$

Schrödinger–Ricci solution. We state that a 1-form π is a solution of the *Schrödinger–Ricci* equation, if

$$(\Delta + \mathcal{R}ic)(\pi) + d(\mathrm{div}(\pi)) = 0. \tag{7.7}$$

Thus, we have the following.

Theorem 7.1 *Let* $(g, \zeta, \Omega, \mu, \alpha, \beta)$ *be a* γ*-Ricci–Yamabe soliton in a relativistic* $PFST$ *with* γ *the* g*-dual of the vector field* ζ*. Then* γ *is a solution of the Schrödinger–Ricci equation if and only if*

$$\kappa d(\sigma - 3\rho) = \frac{2\mu}{(\alpha - 4\beta)} \left\{[4(\Omega + \lambda) - \mu - 3k\rho]\gamma - \nabla_{\zeta}\gamma\right\}. \tag{7.8}$$

Proof Applying, (7.3), (7.4), (7.5), (4.7) and infer the formula

$$2\mathrm{div}(\mathcal{R}ic) = (\alpha - 4\beta)d(\mathcal{R})$$

it proceeds that γ is a solution of the Schrödinger–Ricci equation if and only if (7.6) holds. □

Now, from Theorem (7.1) we can state the following corollaries.

Corollary 7.2 *Let* $(g, \xi, \Omega, \mu, \alpha, 0)$ *be a* γ*-Ricci soliton in a relativistic* $PFST$ *with* γ *the* g*-dual of the vector field* ζ*. Then* γ *is a solution of the Schrödinger–Ricci equation if and only if*

$$\kappa d(\sigma - 3\rho) = \frac{2\mu}{\alpha} \left\{[4(\Omega + \lambda) - \mu - 3k\rho]\gamma - \nabla_{\zeta}\gamma\right\}. \tag{7.9}$$

Corollary 7.3 *Let* $(g, \xi, \Omega, \mu, 0, \beta)$ *be a* γ*-Yamabe soliton in a relativistic* $PFST$ *with* γ*-the* g *dual of the vector field* ζ*. Then* γ *is a solution of the Schrödinger–Ricci equation if and only if*

$$\kappa d(\sigma - 3\rho) = -\frac{\mu}{2\beta} \left\{[4(\Omega + \lambda) - \mu - 3k\rho]\gamma - \nabla_{\zeta}\gamma\right\}. \tag{7.10}$$

Schrödinger–Ricci harmonic forms. We state that a 1-form π is a *Schrödinger–Ricci harmonic form* if [7]

$$(\Delta + \mathcal{R}ic)(\pi) = 0. \tag{7.11}$$

In addition, if $\sigma = 3\rho$, then the fluid is a radiation fluid if and only if $\mu = 0$, which yields the Ricci–Yamabe soliton or

$$\nabla_{\zeta}\gamma = \frac{1}{(\alpha - 4\beta)}[4(\Omega + \lambda) - \mu - 3k\rho]\gamma \tag{7.12}$$

which implies that $\mu = 4(\Omega + \lambda) - 3k\rho$. Thus, we have the following results.

Theorem 7.4 *Let* $(g, \zeta, \Omega, \mu, \alpha, \beta)$ *be a* γ*-Ricci–Yamabe soliton in a relativistic* $PFST$ *with* γ *the* g*-dual of the vector field* ζ*. Then* γ *is a the Schrödinger–Ricci harmonic form if and only if* $\mu = 0$*, which yields Ricci–Yamabe soliton or*

$$\nabla_\zeta \gamma = \frac{1}{(\alpha - 4\beta)}[4(\Omega + \lambda) - \mu - 3k\rho]\gamma \tag{7.13}$$

which implies that $\mu = 4(\Omega + \lambda) - 3k\rho$.

Corollary 7.5 *Let* $(g, \zeta, \Omega, \mu, 0, \beta)$ *be a* γ*-Yamabe soliton in a relativistic* $PFST$ *with* γ *the* g*-dual of the vector field* ζ*. Then* γ *is a Schrödinger–Ricci harmonic form if and only if* $\mu = 0$*, which yields Yamabe soliton or*

$$\nabla_\zeta \gamma = \frac{1}{-4\beta}[4(\Omega + \lambda) - \mu - 3k\rho]\gamma \tag{7.14}$$

which implies that $\mu = 4(\Omega + \lambda) - 3k\rho$.

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Declarations

Conflict of interest: The authors that they have no competing interest.

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