Frobenius algebras and root systems: the trigonometric case

Dali Shen[1] ⬤

Abstract

We construct Frobenius structures on the $\mathbb{C}^\times$-bundle of the complement of a toric arrangement associated with a root system, by making use of a one-parameter family of torsion free and flat connections on it. This gives rise to a class of Frobenius manifolds parameterizing Frobenius algebras in terms of root systems in a trigonometric setting. We then determine their potential functions, which amounts to giving the trigonometric solutions of WDVV equations.

Keywords Frobenius manifolds · Root systems · WDVV

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1 Introduction

A remarkable system of third-order nonlinear partial differential equations, now called Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations, emerged from the two-dimensional conformal field theory. It is quite difficult to solve these equations in general, although this overdetermined system does admit exact solutions. The theory of Frobenius manifolds, however, with a more geometric point of view, provides us another route to understand these equations.

There exists a class of rational solutions of WDVV equations, which corresponds to the polynomial Frobenius manifolds. The search of rational solutions dates back to the work of Marshakov, Mironov and Morozov for classical root systems [19,20], the solutions for other root systems were then found by Martini and Gragert [21], and more fully by Veselov in [30] where he introduced the notion of ∨-system generalizing

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✉ Dali Shen
dali.shen@tifr.res.in, dali_shen@hotmail.com

[1] School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400005, India

the configuration of roots to the configuration of vectors. They appear as almost dual potential functions on polynomial Frobenius manifolds introduced by Dubrovin [10].

According to [18], the Frobenius structures on a manifold can also be studied through a family of flat connections. A closely related notion of rational Dunkl system, of this potential relation with Frobenius structure, was introduced and studied by Couwenberg, Heckman and Looijenga in [7], initially presented for some other purpose. Then, later on its relation with the rational solutions of WDVV equations was established by Looijenga in [17]: let V be a complex vector space and denote by $\mathscr{H}$ a finite collection of linear hyperplanes in V, then the potential function on the complement of the hyperplane arrangement is of the form

$$\Phi = \sum_{H \in \mathscr{H}} c_H \alpha_H^2 \log \alpha_H,$$

where $\alpha_H \in V^*$ has zero set H and c_H is a constant depending on the rational Dunkl connection.

Similarly, the author investigated a toric analogue of the rational Dunkl system in [27] where a toric analogue of the rational Dunkl connection was introduced and studied. Therefore, likewise, resorting to a family of torsion free and flat connections of that shape, we can, in this paper, construct a class of Frobenius manifolds which parameterize Frobenius algebras in terms of root systems in a trigonometric setting. Then, looking for the potential functions for these Frobenius structures as well leads us to uncover the trigonometric solutions of WDVV equations.

Let us explain this idea in more detail. Starting from a toric mirror arrangement complement H° defined by a root system R, the author constructed a family of torsion free and flat connections $\tilde{\nabla}^\kappa$ on $H^\circ \times \mathbb{C}^\times$ in [27], depending on a multiplicity parameter

$$\kappa = (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R.$$

Denote by W the Weyl group of R and by $\mathfrak{h}$ the Lie algebra of H. Taking cue from these connections, we define a product structure for each κ on the tangent bundle of $H^\circ \times \mathbb{C}^\times$ as follows:

$$\tilde{X} \cdot_\kappa \tilde{Y} := \frac{1}{2} \sum_{\alpha > 0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} \alpha(X) \alpha(Y) \alpha^\vee + b^\kappa(X, Y)$$

$$+ a^\kappa(X, Y) t \frac{\partial}{\partial t} - \lambda_2 X - \lambda_1 Y - \lambda_1 \lambda_2 t \frac{\partial}{\partial t},$$

where e^α is a character of H, t is the coordinate for $\mathbb{C}^\times$, a^κ is a W-invariant bilinear form on $\mathfrak{h}$, b^κ is a W-equivariant bilinear map on $\mathfrak{h}$, $\tilde{X}$ is a vector field on $H^\circ \times \mathbb{C}^\times$ defined by $\tilde{X} := X + \lambda_1 t \frac{\partial}{\partial t}$, and likewise for $\tilde{Y} := Y + \lambda_2 t \frac{\partial}{\partial t}$ (see Sect. 3 for the precise definitions).

Making use of the so-called structure connection method [18], we can show that the manifold $H^\circ \times \mathbb{C}^\times$ endowed with the structure $(\cdot_\kappa, a^\kappa, -t \frac{\partial}{\partial t})$ is a Frobenius manifold, although we do not discuss the Euler field in our Frobenius structures.

Since we have the Frobenius structures (i.e., the product structure $\cdot_\kappa$ and the bilinear form a^κ) on hand, finding the potential function is reduced to finding the third-order primitive functions of certain functions given by $a^\kappa(-\cdot_\kappa -,-)$. With some effort, we obtain the potential functions as follows:

$$\Phi = \frac{(\log t)^3}{3!} - \frac{\log t}{2} c^\kappa \sum_{\alpha>0} \alpha^2 + \frac{1}{2} \sum_{\alpha>0} k_\alpha \frac{a^\kappa(\alpha^\vee, \alpha^\vee)}{\alpha(\alpha^\vee)} q(\alpha) + d^\kappa \sum_{\alpha>0} \frac{\alpha^2 \alpha'}{2},$$

(1.1)

where $q(\alpha)$ is a trilogarithmic type function

$$q(\alpha) = \frac{1}{6}\alpha^3 - 2Li_3(e^{-\alpha}) = \frac{1}{6}\alpha^3 - 2\sum_{m=1}^{\infty} \frac{e^{-m\alpha}}{m^3}$$

such that

$$q'''(\alpha) = \frac{e^\alpha + 1}{e^\alpha - 1} = \coth\frac{\alpha}{2},$$

and various other notations will be defined and determined accordingly in Sect. 3. Equivalently, these potential functions give the trigonometric solutions of WDVV equations.

Because of the relation of Frobenius structure with quantum cohomology, it is not surprising that the Frobenius structure constructed in this paper has a similar form with the work of Bryan and Gholampour [5] on quantum cohomology of ADE resolutions. The configuration in the total space might be interpreted as an extended $\vee$-system studied by Stedman and Strachan [28]. In fact, searching the trigonomentric solutions of WDVV equations in terms of root systems can be traced back to the work of Hoevenaars and Martini [16,22], and more recently by Feigin [12].

Although the trigonometric solutions corresponding to root systems have already been studied previously for many cases, the general solutions depending on arbitrary W-invariant multiplicities are not obtained until recently, which are presented in this paper (see the formula (3.5)). We note that additional solutions for type A depending on an extra multiplicity parameter, which comes from a W-equivariant bilinear map, are also considered in this paper (a part of Theorem 3.12 and Sect. 4). Moreover, via the route described above, we believe this paper offers a somewhat detour, a geometric way (i.e., the toric Dunkl system), to approach these trigonometric solutions[2].

We illustrate the above geometric way by showing a class of examples, i.e., the so-called toric Lauricella manifolds, to obtain the trigonometric solutions for type A (see the formula (4.1)) in the end. Such Lauricella family of solutions in addition to classical works was also studied recently in the context of WDVV equations, in particular, by Chalykh and Veselov [6], Pavlov [25,26], and Feigin and Veselov [13].

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[2] After the first version of this paper was finished, the author was informed that these solutions have also been obtained by Alkadhem and Feigin [1] recently in a slightly different form, studied from the point of view of the trigonometric $\vee$-system.

We take the occasion to mention that there is also an important class of elliptic solutions of WDVV equation, which starts to attract people's attention, e.g., studied by Bertola [2], Braden et al. [4] and Strachan [29]. An elliptic version of differential equations corresponding to root systems was considered by Etingof et al. [11] based on which an elliptic Dunkl system might be extracted. If these differential equations were understood well, a relation between elliptic solutions of WDVV equations and the elliptic Dunkl systems should be expected to be established. But this is much beyond the scope of this paper.

The paper is organized as follows. We first briefly introduce the definition of Frobenius manifolds in Sect. 2. Then in Sect. 3, we construct Frobenius structures on our $H^\circ \times \mathbb{C}^\times$, and as a result, we determine their potential functions. Finally in Sect. 4, we discuss a class of examples: toric Lauricella manifolds, which descends to our case when all the weights μ_i's are equal.

2 Frobenius manifolds and the structure connection

In this section, we give a quick introduction to the Frobenius structures on a complex manifold. Our main references are [9], [17] and [18].

For us a *Frobenius algebra* is a pair (A, F), where A is a commutative associative $\mathbb{C}$-algebra with unity e, finite dimensional as a $\mathbb{C}$-vector space, and F is a linear function on A, called a *trace map*, if the map

$$a : A \times A \to \mathbb{C}; \ (u, v) \mapsto a(u, v) := F(uv)$$

is a nondegenerate bilinear form. We often write 1 for e.

It is clear that the bilinear form a satisfies the associative law $a(uv, w) = a(u, vw)$. And conversely, any nondegenerate bilinear symmetric map $a : A \times A \to \mathbb{C}$ with the associative law determines a trace map on A. This associativity law of the bilinear form a is also called a *Frobenius condition*.

If we are only given a vector space A, a symmetric trilinear map $T : A \times A \times A \to \mathbb{C}$, and an element $e \in A$, such that the bilinear form $(u, v) \in A \times A \mapsto T(u, v, e) \in \mathbb{C}$ is nondegenerate, we must as well require that the following system of so-called *associativity equations* to hold

$$T_{ijp} a^{pq} T_{qkl} = T_{jkp} a^{pq} T_{iql}, \tag{Ass.}$$

in order that the product being associative. So that a Frobenius algebra structure on A with unity e can be recovered by T, where the product uv is defined by the unique element of A with the property that $T(uv, x, e) = T(u, v, x)$ for all $x \in A$. Here, $T_{ijk} := T(u_i, u_j, u_k)$ and (a^{jk}) is the inverse matrix of $(a_{jk} := T_{1jk})$, if $\{u_1, \ldots, u_n\}$ is a basis of A.

Now, let be given a complex manifold M. On its tangent bundle TM we are also given a nondegenerate symmetric bilinear form a and a symmetric trilinear form T, both depending holomorphically on the base point. The product of this bundle can be characterized by the property that $a(XY, Z) = T(X, Y, Z)$, denoted by

$$\cdot : TM \times TM \to TM; \ X \cdot Y \mapsto XY.$$

It is clear that this product is commutative by the symmetry of T.

Let ∇ denote the Levi-Civita connection on TM with respect to a. We can define a one-parameter family of connections $\nabla(\mu)$ on this bundle by

$$\nabla(\mu)_X Y := \nabla_X Y + \mu X \cdot Y, \quad \mu \in \mathbb{C}.$$

The connection $\nabla(\mu)$ is called the *structure connection* of (M, a, T).

By the commutativity of the product, we readily have

$$\nabla(\mu)_X Y - \nabla(\mu)_Y X - [X, Y] = \nabla_X Y - \nabla_Y X - [X, Y] = 0,$$

which shows that $\nabla(\mu)$ is torsion free as well.

If for a local vector field X on M, ι_X denotes the multiplication operator on vector fields:

$$\iota_X(Y) := X \cdot Y,$$

then we can define a new tensor

$$R'(\nabla)(X, Y) := [\nabla_X, \iota_Y] - [\nabla_Y, \iota_X] - \iota_{[X,Y]}$$

which is a holomorphic 2-form taking values in the symmetric endomorphism of TM.

We compute the curvature form of $\nabla(\mu)$ as follows:

$$R(\nabla(\mu))(X, Y) = \nabla(\mu)_X \nabla(\mu)_Y - \nabla(\mu)_Y \nabla(\mu)_X - \nabla(\mu)_{[X,Y]}$$
$$= R(\nabla)(X, Y) + \mu R'(\nabla)(X, Y) + \mu^2(\iota_X \iota_Y - \iota_Y \iota_X).$$

The following proposition is well-known to experts, see, e.g., Theorem 1.5 of [18].

Proposition 2.1 *The following statements are equivalent:*

(i) ∇ *is flat, the product is associative and if* X, Y, Z *are (local) flat vector fields on a domain* $U \subset M$, *then the trilinear form* $T(X, Y, Z)$ *locally is given by* $T(X, Y, Z) = \nabla_X \nabla_Y \nabla_Z \Phi$ *where* $\Phi : U \to \mathbb{C}$ *is a holomorphic function on* U.
(ii) ∇ *is flat, the product is associative and* $R' \equiv 0$.
(iii) The connection $\nabla(\mu)$ *is flat for any* $\mu \in \mathbb{C}$.

Remark 2.2 If we denote the coefficient of μ^2 in $R(\nabla(\mu))$, i.e., the tensor $\iota_X \iota_Y - \iota_Y \iota_X$, by $R''(X, Y)$, then (1) the condition $R' = 0$ is a potential condition, and (2) the condition $R'' = 0$ is an associativity condition.

The function Φ that appears in Statement (i) of Proposition 2.1 is called a (local) *potential function*. It needs not be defined on all of M. The associativity equation (Ass.) now is read as a highly nontrivial system of partial differential equations: if

$(z^1, \ldots, z^n)$ is a system of flat coordinates and $\partial_\gamma := \frac{\partial}{\partial z^\gamma}$, then we require that for all i, j, k, l,

$$(\partial_i \partial_j \partial_p \Phi) a^{pq} (\partial_q \partial_k \partial_l \Phi) = (\partial_j \partial_k \partial_p \Phi) a^{pq} (\partial_i \partial_q \partial_l \Phi). \tag{WDVV}$$

These are known as the *Witten–Dijkgraaf–Verlinde–Verlinde* equations. And the entry in the inverse matrix $(a_{jk} = T_{1jk})$ of (a^{jk}) in the associativity equation (Ass.) is read as $a_{jk} = \partial_e \partial_j \partial_k \Phi$, where ∂_e is the derivative along the direction of the identity field of the corresponding Frobenius algebras.

Then, we are properly prepared to introduce the main notion of this section.

Definition 2.3 A complex manifold M is called a *Frobenius manifold* if its holomorphic tangent bundle is fiberwisely endowed with the structure of a Frobenius algebra $(\cdot, F, e)$ satisfying

(i) the equivalent conditions of Proposition 2.1 are fulfilled for the associated symmetric bilinear and trilinear forms a and T,
(ii) the identity field e on M is flat for the Levi-Civita connection of a.

Remark 2.4 Note that Dubrovin [9] requires an Euler vector field for the definition of a Frobenius manifold. And Manin in his book [18] starts from a $\mathbb{Z}/2\mathbb{Z}$-graded structure sheaf on a manifold (which he called a supermanifold) for his definition of a Frobenius manifold. But in this paper, we do not introduce these notions because we want to focus on the aforementioned more central conditions for our construction of Frobenius algebras associated with root systems and their potentials.

3 Frobenius algebras from toric Dunkl connection

In the previous work of the author [27], starting from a toric mirror arrangement complement, we have constructed affine structures on its $\mathbb{C}^\times$-bundle by showing that there exists a family of torsion free and flat connections on this total space. By regarding them as a structure connection, we can thus define a product structure on the tangent bundle of this manifold. This gives rise to a class of Frobenius manifolds parameterizing Frobenius algebras in terms of root systems in a trigonometric setting.

3.1 Root systems

Let $\mathfrak{a}$ be a real vector space of dimension n, which is further made to be a Euclidean vector space by endowing it with an inner product $(\cdot, \cdot)$. Denote its dual vector space by $\mathfrak{a}^*$. We can identify $\mathfrak{a}$ with $\mathfrak{a}^*$ by the inner product, so that the dual space $\mathfrak{a}^*$ is also endowed with an inner product, denoted by $(\cdot, \cdot)$ as well by abuse of notation.

For a nonzero vector $\alpha \in \mathfrak{a}^*$, there corresponds an orthogonal reflection s_α with the hyperplane perpendicular to α being the mirror. This reflection could be written as

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

for any $\beta \in \mathfrak{a}^*$. We can easily check that

$$s_\alpha(\alpha) = -\alpha \text{ and } s_\alpha(\beta) = \beta \text{ for } (\beta, \alpha) = 0.$$

Then, $s_\alpha^2 = 1$ follows from the above formula directly. We recall the definition of a root system first.

A finite subset R of $\mathfrak{a}^*$ is called a *root system* if it does not contain 0 and spans $\mathfrak{a}^*$ such that any s_α leaves R invariant and $s_\alpha(\beta) \in \beta + \mathbb{Z}\alpha$ for any $\alpha, \beta \in R$. Any vector belonging to R is called a root. The dimension of $\mathfrak{a}^*$ is called the *rank* of the system. The group $W(R)$ generated by the s_α is called the *Weyl group* of R. This root system R is said to be *reduced* if $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for any $\alpha \in R$, and said to be *irreducible* if nonempty R cannot be decomposed as a direct sum of two nonempty root systems.

For each $\alpha \in R$ there exists a coroot $\alpha^\vee \in \mathfrak{a}$ such that $(\alpha, \alpha^\vee) = 2$ and $(\beta, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$, and for any $\alpha \in R$ the reflection $s_\alpha(\gamma) = \gamma - (\gamma, \alpha^\vee)\alpha$ leaves R invariant. The set $R^\vee = \{\alpha^\vee | \alpha \in R\}$ is again a root system in $\mathfrak{a}$, called the coroot system relative to R.

Suppose now we are given a reduced irreducible root system $R \subset \mathfrak{a}^*$. The integral span $Q = \mathbb{Z}R$ of the root system R in $\mathfrak{a}^*$ is called the root lattice, its dual $P^\vee = \text{Hom}(Q, \mathbb{Z})$ in $\mathfrak{a}$ is called the coweight lattice of $R^\vee$. Hence, we have an algebraic torus defined as follows:

$$H = \text{Hom}(Q, \mathbb{C}^\times)$$

with character lattice being Q, sometimes also called an adjoint torus.

We denote by $\mathfrak{h}$ the Lie algebra of H, which is equal to $\mathbb{C} \otimes P^\vee$. First let us consider a W-invariant symmetric bilinear form a and a W-equivariant symmetric bilinear map b on $\mathfrak{h}$ respectively as follows:

$$a : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}, \quad b : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}.$$

We have the following characterization for a and b.

Lemma 3.1 *Let a and b be given as above. If R is irreducible, then*

(1) The W-invariant symmetric bilinear form a is just a multiple of the given inner product.

(2) The W-equivariant symmetric bilinear map b vanishes unless R is of type A_n for $n \geq 2$ in which case there exists a $k' \in \mathbb{C}$ such that

$$b(u, v) = \frac{1}{2}k' \sum_{\alpha > 0} \alpha(u)\alpha(v)\alpha' \text{ for any } u, v \in \mathfrak{h}$$

with $\alpha' = \varepsilon_i + \varepsilon_j - \frac{2}{n+1}\sum_l \varepsilon_l$ if we take the construction of α from Bourbaki [3]: $\alpha = z_i - z_j$ for $1 \leq i < j \leq n+1$, where $\{z_i\}$ is the dual basis of $\{\varepsilon_i\}$ in $(\mathbb{R}^{n+1})^*$.

Proof (1) The given inner product $(\cdot, \cdot)$ on $\mathfrak{a}$ can be extended $\mathbb{C}$-linearly to a nondegenerate symmetric bilinear form on $\mathfrak{h}$, which is invariant under the action of W, still

denoted by $(\cdot, \cdot)$. Since R is irreducible, then the W-invariant symmetric bilinear form a is just a multiple of this given inner product $(\cdot, \cdot)$ by Schur's lemma.

(2) Obviously the W-equivariant symmetric bilinear map b can be identified with an element of $\mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2\mathfrak{a}^*)^W$. By (1), there exists a positive definite generator $g \in (\mathrm{Sym}^2\mathfrak{a}^*)^W$ and we fix it. We choose a line $L \subset \mathfrak{a}$ such that its g-orthogonal complement H, as a hyperplane, is spanned by an irreducible root system $R_H := R \cap H$. Then, we decompose $\mathrm{Sym}^2\mathfrak{a}^* = \mathrm{Sym}^2 H^* \oplus (H^* \otimes L^*) \oplus (L^* \otimes H^*) \oplus (L^*)^{\otimes 2}$. Considering its $W(R_H)$-invariant part, the middle two summands vanishes since $(H^*)^{W(R_H)} = 0$. Then, we have $(\mathrm{Sym}^2\mathfrak{a}^*)^{W(R_H)} = \mathbb{R}g_H \oplus \mathbb{R}g_L = \mathbb{R}g_L \oplus \mathbb{R}g$ where g_H resp. g_L is the restriction of g on H resp. L and so that $g = g_H + g_L$.

Let $f \in \mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2\mathfrak{a}^*)^W$, then we have $f(v) = \mu g_L + \lambda g$ for some $v \in L$ since L lies in the $W(R_H)$-invariant part. If there exists a w such that $w(v) = -v$, then it implies $f = 0$. That is because $f(v) = 0$ by the linearity of f and w preserves both g_L and g so that we must have $\mu = \lambda = 0$. Since the W-orbit of v spans $\mathfrak{a}$, it follows that $f = 0$.

The condition is certainly satisfied if $-1 \in W$. So we only need to consider the remaining cases: E_6, D_{odd} and $A_{n \geq 2}$. For E_6, we take v to be a root, then R_H is of type A_5. For D_{odd}, we take v perpendicular to a subsystem of type D_{n-1}, then there is a w which is a reflection relative to H (in terms of the Bourbaki construction: $v = \varepsilon_1$ and $w = s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_1 + \varepsilon_2}$).

When R is of type $A_{n \geq 2}$, we still use the Bourbaki construction: let $\mathfrak{a}^*$ be the hyperplane in $(\mathbb{R}^{n+1})^*$ defined by $\sum_{i=1}^{n+1} z_i = 0$. Put $\bar{z}_i := z_i | \mathfrak{a}^*$ so that $\sum_i \bar{z}_i = 0$. We take v to be $\bar{\varepsilon}_i$, the orthogonal projection of ε_i to $\mathfrak{a}$ in $\mathbb{R}^{n+1}$. Then, the orthogonal complement of $\bar{\varepsilon}_i$ is spanned by a subsystem of type A_{n-1}, and $f(\bar{\varepsilon}_i) = \mu \bar{z}_i^2 + \lambda g$. We notice that all the $\bar{\varepsilon}_i$'s make up a W-orbit with sum zero. So we sum them up: $0 = \sum_i f(\bar{\varepsilon}_i) = \mu \sum_i \bar{z}_i^2 + (n+1)\lambda g$. Hence, we have $f(\bar{\varepsilon}_i) = \mu(\bar{z}_i^2 - \frac{1}{n+1}\sum_i \bar{z}_i^2)$. This indeed defines an element of $\mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2\mathfrak{a}^*)^W$ and we thus have $\dim(\mathrm{Hom}(\mathfrak{a}, \mathrm{Sym}^2\mathfrak{a}^*)^W) = 1$.

Now let $b_0(u, v) = \sum_{\alpha > 0} \alpha(u)\alpha(v)\alpha'$. Since $w(\alpha') = w(\alpha)'$, we have $wb_0(u, v) = b_0(wu, wv)$ for all $u, v \in \mathfrak{h}$ and $w \in W(A_n) = \mathfrak{S}_{n+1}$. This shows that b_0 is a generator of $\mathrm{Hom}(\mathrm{Sym}^2\mathfrak{h}, \mathfrak{h})^W$. $\square$

Remark 3.2 In fact, for type A_n, besides the construction in the above proof, there is also another way to obtain a generator for b: by taking $v \in \mathfrak{a} \mapsto \partial_v \bar{\sigma}_3$, where $\bar{\sigma}_3 := \sigma_3 | \mathfrak{a}^*$ is an element of $(\mathrm{Sym}^3\mathfrak{a}^*)^W$. This viewpoint will become more clear when we discuss the toric Lauricella case in Sect. 4.

3.2 Frobenius structures

Each root α of R is primitive in Q and determines a character $e^\alpha : H \to \mathbb{C}^\times$. The kernel of the character $H_\alpha = \{z \in H \mid e^\alpha(z) = 1\}$ defines a hypertorus, called the mirror determined by α. Its Lie algebra $\mathfrak{h}_\alpha$ is the zero set of α in $\mathfrak{h}$. The root system is closed under inversion and note that the negative $-\alpha$ determines the same hypertorus as α. The finite collection of these hypertori H_α's is called a *toric mirror arrangement* associated with the root system R, sometimes in this paper also referred as a *toric*

arrangement for short if it leads no confusion. We write H° for the complement of the toric mirror arrangement as follows:

$$H^\circ := H - \cup_{\alpha>0} H_\alpha.$$

For $u \in \mathfrak{h}$ we denote by ∂_u the associated translation invariant vector field on H. Likewise, for $\phi \in \mathfrak{h}^*$, we denote by $d\phi$ the associated translation invariant differential on H. In case $\phi \in Q$, it determines a character of H, $e^\phi : H \to \mathbb{C}^\times$, then we have $d\phi = (e^\phi)^*(\frac{dt}{t})$ with t the coordinate on $\mathbb{C}^\times$. We denote by ∇^0 the flat translation invariant connection on H, so that $\nabla^0_{\partial_v} = \partial_v$. So is $\tilde{\nabla}^0$ on $H \times \mathbb{C}^\times$.

Let κ be a W-invariant function

$$\kappa = (k_\alpha)_{\alpha \in R} \in \mathbb{C}^R,$$

meaning that $k_{w\alpha} = k_\alpha$ for any $w \in W$, called a multiplicity parameter. We write k_i for k_{α_i} if $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of simple roots for R. It is also clear that there are at most two W-orbits if R is reduced and irreducible. So for convenience, we also write k for k_1 and k' for k_n if α_n is not in the W-orbit of α_1. In our situation, the root system R of type A_n is somehow peculiar, it has only one single W-orbit, but we also let k', given in Lemma 3.1, enter into the parameter κ, since for type A_n, there exists a nontrivial W-equivariant symmetric bilinear map. Let

$$a^\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}, \quad b^\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}.$$

be a W-invariant symmetric bilinear form and a W-equivariant symmetric bilinear map on $\mathfrak{h}$, depending on κ, respectively.

Taking cue from the special hypergeometric functions constructed by Heckman and Opdam [14,15,23,24], we consider for $u, v \in \mathfrak{h}$, such a second-order differential operator on $\mathcal{O}_{H^\circ}$ defined by

$$D^\kappa_{u,v} := \partial_u \partial_v + \frac{1}{2} \sum_{\alpha>0} k_\alpha \alpha(u)\alpha(v) X_\alpha + \partial_{b^\kappa(u,v)} + a^\kappa(u,v),$$

where the vector fields X_α's are defined as

$$X_\alpha := \frac{e^\alpha + 1}{e^\alpha - 1} \partial_{\alpha^\vee}.$$

Notice that X_α is invariant under inversion: $X_{-\alpha} = X_\alpha$.

It adds to the main linear second-order term a lower-order perturbation, consisting of a W-equivariant first-order term and a W-invariant constant. Notice that $w D^\kappa_{u,v} w^{-1} = D^\kappa_{wu,wv}$ where $w \in W$.

We associate with these data, coming from the above operators, connections $\nabla^\kappa = \nabla^0 + \Omega^\kappa$ on the cotangent bundle of H° with $\Omega^\kappa \in \text{Hom}(\Omega_{H^\circ}, \Omega_{H^\circ} \otimes \Omega_{H^\circ})$ given by

$$\Omega^{\kappa} : \zeta \in \Omega_{H^{\circ}} \mapsto \frac{1}{2} \sum_{\alpha > 0} k_{\alpha} \zeta(X_{\alpha}) d\alpha \otimes d\alpha + (B^{\kappa})^{*}(\zeta), \tag{3.1}$$

where B^{κ} denotes the translation invariant tensor field on H defined by b^{κ}.

Then, following the idea in Proposition 2.2 of [27], we define connections $\tilde{\nabla}^{\kappa} = \tilde{\nabla}^{0} + \tilde{\Omega}^{\kappa}$ on the cotangent bundle of $H^{\circ} \times \mathbb{C}^{\times}$ with $\tilde{\Omega}^{\kappa} \in \mathrm{Hom}(\Omega_{H^{\circ} \times \mathbb{C}^{\times}}, \Omega_{H^{\circ} \times \mathbb{C}^{\times}} \otimes \Omega_{H^{\circ} \times \mathbb{C}^{\times}})$ given by

$$\tilde{\Omega}^{\kappa} : \begin{cases} \zeta \in \Omega_{H^{\circ}} \mapsto \dfrac{1}{2} \sum_{\alpha > 0} k_{\alpha} \zeta(X_{\alpha}) d\alpha \otimes d\alpha + (B^{\kappa})^{*}(\zeta) - \zeta \otimes \dfrac{dt}{t} - \dfrac{dt}{t} \otimes \zeta, \\[2mm] \dfrac{dt}{t} \in \Omega_{\mathbb{C}^{\times}} \mapsto A^{\kappa} - \dfrac{dt}{t} \otimes \dfrac{dt}{t}, \end{cases} \tag{3.2}$$

where t is the coordinate for $\mathbb{C}^{\times}$, and A^{κ} denotes the translation invariant tensor field on H (or $H \times \mathbb{C}^{\times}$) defined by a^{κ}.

According to (3.2), we can write out $\tilde{\Omega}^{\kappa}$ explicitly:

$$\begin{aligned} \tilde{\Omega}^{\kappa} := &\frac{1}{2} \sum_{\alpha > 0} k_{\alpha} d\alpha \otimes d\alpha \otimes X_{\alpha} + (B^{\kappa})^{*} + c^{\kappa} \sum_{\alpha > 0} d\alpha \otimes d\alpha \otimes t \frac{\partial}{\partial t} \\ &- \sum_{\alpha_i \in \mathfrak{B}} d\alpha_i \otimes \frac{dt}{t} \otimes \partial_{p_i} - \frac{dt}{t} \otimes \frac{dt}{t} \otimes t \frac{\partial}{\partial t} - \sum_{\alpha_i \in \mathfrak{B}} \frac{dt}{t} \otimes d\alpha_i \otimes \partial_{p_i}, \end{aligned}$$

where c^{κ} is a constant for each κ such that $A^{\kappa} = c^{\kappa} \sum_{\alpha > 0} d\alpha \otimes d\alpha$, $\mathfrak{B}$ is a fundamental system for R and p_i is the dual basis of $\mathfrak{h}$ to α_i such that $\alpha_i(p_j) = \delta_j^i$ for which δ_j^i is the Kronecker delta.

We determine the constant c^{κ} in the following lemma.

Lemma 3.3 *Assume $a^{\kappa}(u, v) = c^{\kappa} \sum_{\alpha > 0} \alpha(u)\alpha(v) = \delta.(u, v)$ where $(\cdot, \cdot)$ is the inner product used in the Bourbaki construction such that $(\varepsilon_i, \varepsilon_j) = \delta_j^i$. Then, we have*

$$c^{\kappa} = \frac{\delta n}{\sum_{\alpha > 0}(\alpha, \alpha)}.$$

Proof Let $\{e_i\}$ be an orthonormal basis in $\mathfrak{a}^{*}$. Then, we have

$$\begin{aligned} \delta n = \delta \sum_{i=1}^{n} (e_i, e_i) &= \sum_{i=1}^{n} c^{\kappa} \sum_{\alpha > 0} (\alpha, e_i)(\alpha, e_i) \\ &= c^{\kappa} \sum_{\alpha > 0} (\sum_{i=1}^{n} (\alpha, e_i)e_i, \alpha) = c^{\kappa} \sum_{\alpha > 0} (\alpha, \alpha). \end{aligned}$$

$\square$

Since ∇^κ is torsion free: for taking the values in the symmetric tensors, it is clear that $\tilde{\nabla}^\kappa$ is also torsion free. In [27], we prove that

Theorem 3.4 *There exists a bilinear form a^κ for each κ such that $\tilde{\nabla}^\kappa$ is flat.*

Proof See Section 2 of [27], where one can also find an explicit form of a^κ, i.e., the δ, for a given κ as follows:

$$A_n : a^\kappa(u, v) = \frac{(n+1)}{4}(k^2 - k'^2)(u, v);$$

$$B_n : a^\kappa(u, v) = ((n-2)k^2 + kk')(u, v);$$

$$C_n : a^\kappa(u, v) = ((n-2)k^2 + 2kk')(u, v);$$

$$D_n : a^\kappa(u, v) = (n-2)k^2(u, v);$$

$$E_n : a^\kappa(u, v) = ck^2(u, v), \quad c = 6, 12, 30 \text{ for } n = 6, 7, 8;$$

$$F_4 : a^\kappa(u, v) = (k + k')(2k + k')(u, v);$$

$$G_2 : a^\kappa(u, v) = \frac{3}{4}(k + 3k')(k + k')(u, v),$$

for which we use the construction of root systems in Bourbaki and take the inner product $(\cdot, \cdot)$ such that $(\varepsilon_i, \varepsilon_j) = \delta^i_j$. $\qquad\qquad\square$

Remark 3.5 Fixing this a^κ, by [27], we also know that for every sublattice L of the root lattice Q spanned by elements of R, the "linearized connection" on $\mathfrak{h} - \cup_{\alpha \in R \cap L}\mathfrak{h}_\alpha$ defined by the following End($\mathfrak{h}$)-valued differential

$$\Omega_L := \sum_{\alpha \in R \cap L} k_\alpha \frac{d\alpha}{\alpha} \otimes \pi_\alpha$$

is flat, where $\pi_\alpha \in$ End($\mathfrak{h}$) is twice of the orthogonal projection to $\alpha^\vee$ with kernel $\mathfrak{h}_\alpha$. In the meantime, each $u_\alpha := k_\alpha(\alpha^\vee \otimes \alpha)$ is self-adjoint relative to a^κ. Then, (H, R, κ) defines a toric analogue of the Dunkl system in the sense of Couwenberg–Heckman–Looijenga [7]. Hence, the connection $\tilde{\nabla}^\kappa$ defined in (3.2) is usually called a *toric Dunkl connection*.

Fixing the bilinear form a^κ for which $\tilde{\nabla}^\kappa$ is flat, we then consider the dual connections defined on the tangent bundle of $H^\circ \times \mathbb{C}^\times$ instead of the connections $\tilde{\nabla}^\kappa$ defined on the cotangent bundle of $H^\circ \times \mathbb{C}^\times$:

$$(\tilde{\Omega}^\kappa)^* = \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1}d\alpha \otimes \partial_{\alpha^\vee} \otimes d\alpha + ((B^\kappa)^*)' + c^\kappa \sum_{\alpha>0} d\alpha \otimes t\frac{\partial}{\partial t} \otimes d\alpha$$

$$- \sum_{\alpha_i \in \mathfrak{B}} d\alpha_i \otimes \partial_{p_i} \otimes \frac{dt}{t} - \sum_{\alpha_i \in \mathfrak{B}} \frac{dt}{t} \otimes \partial_{p_i} \otimes d\alpha_i - \frac{dt}{t} \otimes t\frac{\partial}{\partial t} \otimes \frac{dt}{t}.$$

That the connection form is $-(\tilde{\Omega}^\kappa)^*$ is because the dual connection is characterized by the property that the pairing between differentials and vector fields is flat. But in what follows, we will still write the dual connection as $\tilde{\nabla}^\kappa$ if no confusion would arise.

For a general manifold M, we note that each summand in a connection form $\Omega :=$ $\sum_{i \in I} \omega_i \otimes \rho_i$ defined on its tangent bundle is an $\mathrm{End}(V)$-valued differential 1-form, if we denote its tangent space at any point by V. Here we write the differential 1-form as ω_i, write the tensor field defined by an endomorphism of V as ρ_i, and I is an index set. So that in general we can naturally define a product structure on the tangent bundle of a manifold M, according to the connection form, by

$$J \cdot K := \sum_{i \in I} \omega_i(J)\rho_i(K),$$

where J and K are a vector field on M, respectively.

We can apply this idea to our situation. Since $T_{(p,t)}(H^\circ \times \mathbb{C}^\times) = T_p H^\circ \oplus T_t \mathbb{C}^\times$, we can write a vector field $\tilde{X}$ on $H^\circ \times \mathbb{C}^\times$ in the following form:

$$\tilde{X} = X(p,t) + \lambda_1(p,t)t\frac{\partial}{\partial t},$$

for which $X(p,t)$ is a vector field on H° and $\lambda_1(p,t)$ is a holomorphic function depending on both p and t. Here, we write $\tilde{X} = X + \lambda_1 t\frac{\partial}{\partial t}$ just for convenience.

Likewise, write $\tilde{Y} = Y + \lambda_2 t\frac{\partial}{\partial t}$. Inspired by the idea above, according to the flat connection $\tilde{\nabla}^\kappa$, we define a product for each κ on the tangent bundle of $H^\circ \times \mathbb{C}^\times$ by

$$
\begin{aligned}
\tilde{X} \cdot_\kappa \tilde{Y} :=& \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)\alpha^\vee + b^\kappa(X,Y) + c^\kappa \sum_{\alpha>0}\alpha(X)\alpha(Y)t\frac{\partial}{\partial t} \\
& - \sum_{\alpha_i \in \mathfrak{B}} \alpha_i(X)\lambda_2 p_i - \sum_{\alpha_i \in \mathfrak{B}} \lambda_1 p_i \alpha_i(Y) - \lambda_1 \lambda_2 t\frac{\partial}{\partial t} \\
=& \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)\alpha^\vee + b^\kappa(X,Y) + a^\kappa(X,Y)t\frac{\partial}{\partial t} \\
& - \lambda_2 X - \lambda_1 Y - \lambda_1 \lambda_2 t\frac{\partial}{\partial t}.
\end{aligned}
\tag{3.3}
$$

We already know that a^κ is a symmetric bilinear form on $\mathfrak{h}$:

$$a^\kappa : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}.$$

We can extend a^κ to be a symmetric bilinear form on $\mathfrak{h} \oplus \mathbb{C}$, the tangent space of $H^\circ \times \mathbb{C}^\times$ at (p,t), by defining

$$
\begin{cases}
a^\kappa(X, t\dfrac{\partial}{\partial t}) = 0 \\
a^\kappa(t\dfrac{\partial}{\partial t}, t\dfrac{\partial}{\partial t}) = -1.
\end{cases}
$$

Now is $a^\kappa(\alpha^\vee, \cdot)$ a linear form whose zero set is the hyperplane which is perpendicular to α and therefore it is proportional to α. By evaluating both sides on $\alpha^\vee$, we

see that

$$a^\kappa(\alpha^\vee, \cdot) = \frac{a^\kappa(\alpha^\vee, \alpha^\vee)}{\alpha(\alpha^\vee)}\alpha.$$

Remark 3.6 We also notice that

$$\left(-t\frac{\partial}{\partial t}\right) \cdot_\kappa \tilde{Y} = Y + \lambda_2 t\frac{\partial}{\partial t} = \tilde{Y},$$

from which we can see that $-t\frac{\partial}{\partial t}$ plays a role of identity in this algebra.

Theorem 3.7 *The product structure $\cdot_\kappa$ defined on $T(H^\circ \times \mathbb{C}^\times)$ by (3.3) endows each fiber of $T(H^\circ \times \mathbb{C}^\times)$ with a Frobenius algebra structure.*

Proof In order to see this product structure indeed defines a Frobenius algebra on each fiber of the tangent bundle of $H^\circ \times \mathbb{C}^\times$, we need to verify three properties:

1. the product is commutative,
2. the product satisfies the associativity law with respect to the symmetric bilinear form a^κ, by which property the trace map can also be determined,
3. the product is associative.

1. Commutativity of the product.
  This is quite obvious since the expression for $\tilde{X} \cdot_\kappa \tilde{Y}$ is symmetric in $\{\tilde{X}, \tilde{Y}\}$.
2. Frobenius condition.
  Write $\tilde{Z} = Z + \lambda_3 t\frac{\partial}{\partial t}$, then we have

$$a^\kappa(\tilde{X} \cdot_\kappa \tilde{Y}, \tilde{Z})$$
$$= \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1}\alpha(X)\alpha(Y)a^\kappa(\alpha^\vee, \tilde{Z}) + a^\kappa(b^\kappa(X,Y), Z) + a^\kappa(X,Y)a^\kappa(t\frac{\partial}{\partial t}, \tilde{Z})$$
$$\quad - \lambda_2 a^\kappa(X, \tilde{Z}) - \lambda_1 a^\kappa(Y, \tilde{Z}) - \lambda_1\lambda_2 a^\kappa(t\frac{\partial}{\partial t}, \tilde{Z})$$
$$= \frac{1}{2}\sum_{\alpha>0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} \cdot \frac{a^\kappa(\alpha^\vee, \alpha^\vee)}{\alpha(\alpha^\vee)}\alpha(X)\alpha(Y)\alpha(Z) + a^\kappa(b^\kappa(X,Y), Z)$$
$$\quad - \lambda_3 a^\kappa(X,Y) - \lambda_2 a^\kappa(X,Z) - \lambda_1 a^\kappa(Y,Z) + \lambda_1\lambda_2\lambda_3.$$

From this, we can see that

$$a^\kappa(\tilde{X} \cdot_\kappa \tilde{Y}, \tilde{Z}) = a^\kappa(\tilde{X}, \tilde{Y} \cdot_\kappa \tilde{Z}),$$

when $b^\kappa = 0$ since the above expression is fully symmetric in $\{\tilde{X}, \tilde{Y}, \tilde{Z}\}$ in this case. For type A_n where b^κ can be nonzero, the symmetry of $a^\kappa(b^\kappa(X,Y), Z)$ can be seen from Remark 4.2.
3. Associativity of the product.

Let us look at the connection $\tilde{\nabla}^\kappa(\mu)$ defined by

$$\tilde{\nabla}^\kappa(\mu)_{\tilde{X}}\tilde{Y} := \tilde{\nabla}^0_{\tilde{X}}\tilde{Y} + \mu\tilde{X}\cdot_\kappa\tilde{Y}.$$

Written out,

$$\tilde{\nabla}^\kappa(\mu)_{\tilde{X}}\tilde{Y} = \tilde{\nabla}^0_{\tilde{X}}\tilde{Y} + \frac{1}{2}\mu\sum_{\alpha>0}k_\alpha\frac{e^\alpha+1}{e^\alpha-1}\alpha(X)\alpha(Y)\alpha^\vee + \frac{1}{2}\mu k'\sum_{\alpha>0}\alpha(X)\alpha(Y)\alpha'$$

$$+\mu c^\kappa\sum_{\alpha>0}\alpha(X)\alpha(Y)t\frac{\partial}{\partial t} - \mu\lambda_2 X - \mu\lambda_1 Y - \mu\lambda_1\lambda_2 t\frac{\partial}{\partial t}.$$

Note that the term $\frac{1}{2}\mu k'\sum_{\alpha>0}\alpha(X)\alpha(Y)\alpha'$ only exists for A_n case.

The connection form of $\tilde{\nabla}^\kappa(\mu)$ is a holomorphic differential 1-form on $H^\circ\times\mathbb{C}^\times$ taking values in $\mathrm{End}(\mathfrak{h}\oplus\mathbb{C})$. Upon replacing these endomorphisms, denoted by ρ_α or ρ_t, by their μ multiplication $\mu\rho_\alpha$ or $\mu\rho_t$, we see that it suffices to prove the flatness of $\tilde{\nabla}^\kappa(1)$. But $\tilde{\nabla}^\kappa(1)$ is just $\tilde{\nabla}^\kappa$, and we already know that $\tilde{\nabla}^\kappa$ is flat by Theorem 3.4, so we can see that $\tilde{\nabla}^\kappa(\mu)$ is also flat for all $\mu\in\mathbb{C}$. Therefore, the associativity of the product follows by Proposition 2.1.   □

Remark 3.8 In fact, our Frobenius algebra given above includes the Frobenius algebra constructed by Bryan and Gholampour in [5] as a special case, which requires $k'=0$ for type A_n and $k=k'$ for type $BCFG$. They provided a proof for the associativity of the product from a point of view of Gromov–Witten theory.

Corollary 3.9 *The Weyl group acts on the tangent bundle by automorphisms. Namely, if we define*

$$w(e^\alpha) = e^{w(\alpha)}$$

for $w\in W$, then for $\tilde{X},\tilde{Y}\in\Gamma(T(H^\circ\times\mathbb{C}^\times))$, we have

$$w(\tilde{X}\cdot_\kappa\tilde{Y}) = w(\tilde{X})\cdot_\kappa w(\tilde{Y}).$$

Proof Let s_β be the reflection about the hyperplane orthogonal to β. By [3], s_β permutes the positive roots other than β. And since the terms

$$\frac{e^\alpha+1}{e^\alpha-1}\partial_{\alpha^\vee} \quad\text{and}\quad \alpha(X)\alpha(Y)$$

remain unchanged under $\alpha \to -\alpha$, the effect of s_β to the formula for $\tilde{X} \cdot_\kappa \tilde{Y}$ is to permute the order of the sum:

$$
\begin{aligned}
&s_\beta(\tilde{X} \cdot_\kappa \tilde{Y}) \\
&= \frac{1}{2} \sum_{\alpha>0} k_{s_\beta(\alpha)} \frac{e^{s_\beta(\alpha)}+1}{e^{s_\beta(\alpha)}-1} s_\beta(\alpha)(s_\beta X) s_\beta(\alpha)(s_\beta Y) \partial_{s_\beta(\alpha^\vee)} + b^\kappa(s_\beta X, s_\beta Y) \\
&\quad + a^\kappa(s_\beta X, s_\beta Y) s_\beta(t \frac{\partial}{\partial t}) - s_\beta(\lambda_2 X) - s_\beta(\lambda_1 Y) - s_\beta(\lambda_1 \lambda_2 t \frac{\partial}{\partial t}) \\
&= \frac{1}{2} \sum_{\alpha>0} k_\alpha \frac{e^\alpha+1}{e^\alpha-1} \alpha(s_\beta X) \alpha(s_\beta Y) \partial_{\alpha^\vee} + b^\kappa(s_\beta X, s_\beta Y) \\
&\quad + a^\kappa(s_\beta X, s_\beta Y) t \frac{\partial}{\partial t} - \lambda_2 s_\beta X - \lambda_1 s_\beta Y - \lambda_1 \lambda_2 t \frac{\partial}{\partial t} \\
&= (s_\beta X + \lambda_1 t \frac{\partial}{\partial t}) \cdot_\kappa (s_\beta Y + \lambda_2 t \frac{\partial}{\partial t}) \\
&= (s_\beta X + s_\beta(\lambda_1 t \frac{\partial}{\partial t})) \cdot_\kappa (s_\beta Y + s_\beta(\lambda_2 t \frac{\partial}{\partial t})) \\
&= s_\beta(\tilde{X}) \cdot_\kappa s_\beta(\tilde{Y})
\end{aligned}
$$

since $s_\beta(\lambda_i t \frac{\partial}{\partial t}) = \lambda_i t \frac{\partial}{\partial t}$. Then, the corollary follows. $\square$

We thus construct a W-invariant fiberwise Frobenius algebra on $H^\circ \times \mathbb{C}^\times$. We then have the following theorem.

Theorem 3.10 *The manifold $H^\circ \times \mathbb{C}^\times$ endowed with the structure $(\cdot_\kappa, a^\kappa, -t\frac{\partial}{\partial t})$ is a Frobenius manifold.*

Proof By Remark 3.6, the vector field $-t\frac{\partial}{\partial t}$ is the identity of this algebra. Then, we know that $(\cdot_\kappa, a^\kappa, -t\frac{\partial}{\partial t})$ endows with a Frobenius algebra on $H^\circ \times \mathbb{C}^\times$ fiberwisely, since the trace map F can be determined by the bilinear form a^κ.

We then check the conditions of Definition 2.3. Condition (1) is satisfied since it is already proved that $\tilde{\nabla}^\kappa(\mu)$ is flat for any $\mu \in \mathbb{C}$. Condition (2) is also clear because the vector field $-t\frac{\partial}{\partial t}$ is flat with respect to the Levi-Civita connection $\tilde{\nabla}^0$ of a^κ. Therefore, $(H^\circ \times \mathbb{C}^\times, \cdot_\kappa, a^\kappa, -t\frac{\partial}{\partial t})$ is a Frobenius manifold. $\square$

Although we do not discuss the Euler field for our Frobenius struture, we still have the dilatation field on $H^\circ \times \mathbb{C}^\times$ as follows. Here, a holomorphic vector field D is called a *dilatation filed* on a complex manifold M with factor $\nu \in \mathbb{C}$, assuming an affine structure on M is given by the torsion free and flat connection ∇, if for every local field J on M, we have $\nabla_J(D) = \nu J$.

Corollary 3.11 *Suppose an affine structure on $H^\circ \times \mathbb{C}^\times$ is given by the torsion free flat connection $\tilde{\nabla}^\kappa$ defined by (3.2), then the vector field $t\frac{\partial}{\partial t}$ is in fact a dilatation field on $H^\circ \times \mathbb{C}^\times$ with factor $\nu = 1$.*

Proof It is a straightforward computation. Suppose a local vector field $\tilde{X}$ on $H^\circ \times \mathbb{C}^\times$ is of the form $\tilde{X} := X + \lambda t \frac{\partial}{\partial t}$, where X is a vector field on H°, we have

$$
\tilde{\nabla}^\kappa_{\tilde{X}}(t\frac{\partial}{\partial t}) = \tilde{\nabla}^0_{X+\lambda t \frac{\partial}{\partial t}}(t\frac{\partial}{\partial t}) - \tilde{\Omega}^{\kappa,*}_{X+\lambda t \frac{\partial}{\partial t}}(t\frac{\partial}{\partial t})
$$

$$
= 0 + \sum_{\alpha_i \in \mathfrak{B}} \alpha_i(X)\partial_{p_i} + \lambda t \frac{\partial}{\partial t}
$$

$$
= \tilde{X}
$$

since $t\frac{\partial}{\partial t}$ is flat with respect to $\tilde{\nabla}^0$. $\qquad\square$

Now let us try to find the (local) potential function for this Frobenius structure. In order to find this potential function Φ, we require that for $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ being flat vector fields on $H^\circ \times \mathbb{C}^\times$, we should have

$$
\tilde{\nabla}^\kappa_{\tilde{X}} \tilde{\nabla}^\kappa_{\tilde{Y}} \tilde{\nabla}^\kappa_{\tilde{Z}} \Phi = a^\kappa(\tilde{X} \cdot_\kappa \tilde{Y}, \tilde{Z})
$$

$$
= \frac{1}{2} \sum_{\alpha>0} k_\alpha \frac{e^\alpha + 1}{e^\alpha - 1} \cdot \frac{a^\kappa(\alpha^\vee, \alpha^\vee)}{\alpha(\alpha^\vee)} \alpha(X)\alpha(Y)\alpha(Z) + a^\kappa(b^\kappa(X,Y), Z)
$$

$$
- \lambda_1 a^\kappa(Y, Z) - \lambda_2 a^\kappa(X, Z) - \lambda_3 a^\kappa(X, Y) + \lambda_1 \lambda_2 \lambda_3.
$$

So let us analyze these terms one by one. For terms $\lambda_1 \lambda_2 \lambda_3$ and $-\lambda_1 a^\kappa(Y, Z) - \lambda_2 a^\kappa(X, Z) - \lambda_3 a^\kappa(X, Y)$, we can easily find their potential functions as follows:

$$
\frac{(\log t)^3}{3!} \quad \text{and} \quad -\frac{\log t}{2} c^\kappa \sum_{\alpha>0} \alpha^2,
$$

i.e.,

$$
\tilde{\nabla}^\kappa_{\tilde{X}} \tilde{\nabla}^\kappa_{\tilde{Y}} \tilde{\nabla}^\kappa_{\tilde{Z}}(\frac{(\log t)^3}{3!}) = \lambda_1 \lambda_2 \lambda_3
$$

$$
\tilde{\nabla}^\kappa_{\tilde{X}} \tilde{\nabla}^\kappa_{\tilde{Y}} \tilde{\nabla}^\kappa_{\tilde{Z}}(-\frac{\log t}{2} c^\kappa \sum_{\alpha>0} \alpha^2) = -\lambda_1 a^\kappa(Y, Z) - \lambda_2 a^\kappa(X, Z) - \lambda_3 a^\kappa(X, Y).
$$

By the discussion in Lemma 3.1, we can write the term $a^\kappa(b^\kappa(X, Y), Z) = d^\kappa \sum_{\alpha>0} \alpha(X)\alpha(Y)\alpha'(Z)$ (here we regard α' in $\mathfrak{a}^*$, i.e., $\alpha'(Z) := (\alpha', Z)$) where d^κ is a constant when κ is given. So for term $a^\kappa(b^\kappa(X, Y), Z)$, we have its potential function:

$$
d^\kappa \sum_{\alpha>0} \frac{\alpha^2 \alpha'}{2},
$$

i.e.,

$$\tilde{\nabla}^{\kappa}_{\tilde{X}} \tilde{\nabla}^{\kappa}_{\tilde{Y}} \tilde{\nabla}^{\kappa}_{\tilde{Z}} \left(d^{\kappa} \sum_{\alpha>0} \frac{\alpha^2 \alpha'}{2} \right) = a^{\kappa}(b^{\kappa}(X, Y), Z).$$

We determine d^{κ} for type A_n:

$$a^{\kappa}(b^{\kappa}(u, v), w) = a^{\kappa}(\frac{1}{2}k' \sum_{\alpha>0} \alpha(u)\alpha(v)\alpha', w)$$

$$= \frac{1}{2}k' \sum_{\alpha>0} \alpha(u)\alpha(v)a^{\kappa}(\alpha', w) = \frac{1}{2}k'\delta \sum_{\alpha>0} \alpha(u)\alpha(v)\alpha'(w),$$

so $d^{\kappa} = \frac{1}{2}k'\delta$.

Now we have only one term left: $\frac{1}{2} \sum_{\alpha>0} k_{\alpha} \frac{e^{\alpha}+1}{e^{\alpha}-1} \cdot \frac{a^{\kappa}(\alpha^{\vee},\alpha^{\vee})}{\alpha(\alpha^{\vee})} \alpha(X)\alpha(Y)\alpha(Z)$. For this, we have the trilogarithmic type function $q(\alpha)$ given as follows:

$$q(\alpha) = \frac{1}{6}\alpha^3 - 2Li_3(e^{-\alpha}) = \frac{1}{6}\alpha^3 - 2 \sum_{m=1}^{\infty} \frac{e^{-m\alpha}}{m^3} \tag{3.4}$$

such that

$$q'''(\alpha) = \frac{e^{\alpha}+1}{e^{\alpha}-1} = \coth \frac{\alpha}{2}.$$

Therefore, we have

$$\tilde{\nabla}^{\kappa}_{\tilde{X}} \tilde{\nabla}^{\kappa}_{\tilde{Y}} \tilde{\nabla}^{\kappa}_{\tilde{Z}}(\frac{1}{2} \sum_{\alpha>0} k_{\alpha} \frac{a^{\kappa}(\alpha^{\vee}, \alpha^{\vee})}{\alpha(\alpha^{\vee})} q(\alpha))$$

$$= \frac{1}{2} \sum_{\alpha>0} k_{\alpha} \frac{e^{\alpha}+1}{e^{\alpha}-1} \cdot \frac{a^{\kappa}(\alpha^{\vee}, \alpha^{\vee})}{\alpha(\alpha^{\vee})} \alpha(X)\alpha(Y)\alpha(Z).$$

Finally, adding them up, we have the potential function for this Frobenius structure as follows.

Theorem 3.12 *For the Frobenius structure given as above, the potential function is given by*

$$\Phi = \frac{(\log t)^3}{3!} - \frac{\log t}{2} c^{\kappa} \sum_{\alpha>0} \alpha^2 + \frac{1}{2} \sum_{\alpha>0} k_{\alpha} \frac{a^{\kappa}(\alpha^{\vee}, \alpha^{\vee})}{\alpha(\alpha^{\vee})} q(\alpha) + d^{\kappa} \sum_{\alpha>0} \frac{\alpha^2 \alpha'}{2}, \tag{3.5}$$

where $q(\alpha)$ is given by (3.4), and a^{κ}, c^{κ} and d^{κ} are summarized in Table 1.

Table 1 a^κ, c^κ and d^κ

	δ (i.e., a^κ)	c^κ	d^κ
A_n	$\frac{n+1}{4}(k^2 - k'^2)$	$\frac{1}{4}(k^2 - k'^2)$	$\frac{n+1}{8}k'(k^2 - k'^2)$
B_n	$(n-2)k^2 + kk'$	$\frac{1}{2n-1}((n-2)k^2 + kk')$	–
C_n	$(n-2)k^2 + 2kk'$	$\frac{1}{2(n+1)}((n-2)k^2 + 2kk')$	–
D_n	$(n-2)k^2$	$\frac{(n-2)}{2(n-1)}k^2$	–
E_6	$6k^2$	$\frac{1}{2}k^2$	–
E_7	$12k^2$	$\frac{2}{3}k^2$	–
E_8	$30k^2$	k^2	–
F_4	$(k+k')(2k+k')$	$\frac{1}{9}(k+k')(2k+k')$	–
G_2	$\frac{3}{4}(k+3k')(k+k')$	$\frac{1}{16}(k+3k')(k+k')$	–

This means, we have

$$\tilde{\nabla}^\kappa_{\tilde{X}} \tilde{\nabla}^\kappa_{\tilde{Y}} \tilde{\nabla}^\kappa_{\tilde{Z}} \Phi = a^\kappa(\tilde{X} \cdot_\kappa \tilde{Y}, \tilde{Z}).$$

Equivalently, for each κ, this potential function gives a trigonometric solution of WDVV equations.

Remark 3.13 These solutions have also appeared recently in [1] in a slightly different form, where arbitrary W-invariant multiplicities are treated as well, see Formula (2.1) therein for the precise form. They also dealt with the nonreduced type BC_n which depends on three parameters. The corresponding identity field ∂_y therein for which the coordinate y could be thought of as the "linearized" coordinate of our coordinate t, comparing to our identity field $-t\frac{\partial}{\partial t}$. However, for type A_n, we have an extra term $d^\kappa \sum_{\alpha>0} \frac{\alpha^2 \alpha'}{2}$, which comes from a W-equivariant bilinear map on $\mathfrak{h}$.

For type A_n, such solutions were also already found in [16,22, both Theorem 2.1] for constant multiplicity functions (but without considering the extra multiplicity parameter).

4 An example: toric Lauricella manifolds

In this section, we give an explicit class of Frobenius manifolds, falling into the discussion of the preceding section. We refer this class of examples as *toric Lauricella manifolds*. They are called by this name because their relation to the Lauricella hypergeometric functions [8].

Let N be an index set $\{1, 2, \ldots, n+1\}$ and associate with each $i \in N$ a real number $\mu_i \in (0, +\infty)$. Denote the standard basis of $\mathbb{C}^{n+1}$ by $\varepsilon_1, \ldots, \varepsilon_{n+1}$. We endow $\mathbb{C}^{n+1}$ with a symmetric bilinear form as $a(z, w) := \sum_{i=1}^{n+1} \mu_i z^i w^i$ for which z is defined by $z := \sum z^i \varepsilon_i$. Let $\mathfrak{h}$ be the quotient of $\mathbb{C}^{n+1}$ by its main diagonal $\Delta_N := \mathbb{C} \sum \varepsilon_i$. Since the generator $\varepsilon_N = \sum \varepsilon_i$ of the main diagonal has a self-product $a(\varepsilon_N, \varepsilon_N) = \sum \mu_i \neq$

0, its orthogonal complement is nondegenerate. Thus, we can often identify $\mathfrak{h}$ with this orthogonal complement, that is, the hyperplane defined by $\sum \mu_i z^i = 0$. We take our α's to be the collection $\alpha_{i,j} := (z_i - z_j)_{i \neq j}$ where z_i is the dual basis of ε_i in $(\mathbb{C}^{n+1})^*$. We associate with each $\alpha_{i,j}$ a hyperplane $H_{i,j}$ in $\mathbb{C}^{n+1}$ defined by $\{z^i - z^j = 0\}$, and its orthogonal complement is spanned by the vector $v_{i,j} := v_{z_i - z_j} := \mu_j \varepsilon_i - \mu_i \varepsilon_j$. It is clear that $v_{i,j} \in \mathfrak{h}$. We denote by $\mathfrak{h}_{i,j}$ the intersection of $H_{i,j}$ with $\mathfrak{h}$.

We immediately notice that the set $R := \{\alpha_{i,j}\}$ generates a discrete subgroup of $\mathfrak{h}^*$ whose $\mathbb{R}$-linear span defines a real form $\mathfrak{h}(\mathbb{R})$ of $\mathfrak{h}$. It is easy to show that $a(v_{i,j}, \beta) = 0$ for any $\beta \in \ker(\alpha_{i,j})$. According to [7], if $u_{\alpha_{i,j}}$ is the self-adjoint map of $\mathfrak{h}$ defined by $u_{\alpha_{i,j}}(z) = \alpha_{i,j}(z) v_{i,j}$ (with trace $\mu_i + \mu_j$), the system $(\mathfrak{h}, \{\mathfrak{h}_{i,j}\}, \{\mu_i + \mu_j\})$ defines a Dunkl system.

As already mentioned in Remark 3.2, there actually exists a nonzero cubic form in this case. Let $\tilde{f} : \mathbb{C}^{n+1} \to \mathbb{C}$ be defined by $\tilde{f}(z) := \sum \mu_i (z^i)^3$, and denote by $f : \mathfrak{h} \to \mathbb{C}$ its restriction to $\mathfrak{h}$. The partial derivative of $\tilde{f}$ with respect to $v_{i,j}$ is $3\mu_j \mu_i (z_i^2 - z_j^2)$, which is divisible by $\alpha_{i,j}$.

The symmetric bilinear map $\tilde{b} : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is defined by $\tilde{b}(\varepsilon_i, \varepsilon_j) := \delta_j^i \varepsilon_i$. Then, the map $b : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ is given as the restriction of $\tilde{b}$ to $\mathfrak{h} \times \mathfrak{h}$ followed by $\pi : \mathbb{C}^{n+1} \to \mathfrak{h}$ the orthogonal projection from $\mathbb{C}^{n+1}$ to $\mathfrak{h}$, namely $b := \pi \circ \tilde{b}|_{\mathfrak{h} \times \mathfrak{h}}$. We then have that $a(\tilde{b}(z, z), z) = \tilde{f}(z)$ and $a(b(z, z), z) = f(z)$. So if we write $\tilde{b}_i(z)$ for $\tilde{b}(\varepsilon_i, z)$, we can write $\tilde{b}_i$ as $\tilde{b}_i = \varepsilon_i \otimes z_i$. If we write $b_{i,j}(z)$ for $b(v_{i,j}, z)$, then

$$b_{i,j} = \mu_j \varepsilon_i \otimes z_i - \mu_i \varepsilon_j \otimes z_j - \frac{\mu_i \mu_j}{\mu_N} \varepsilon_N \otimes (z_i - z_j),$$

where $\mu_N := \sum_i \mu_i$. If we write $a_{i,j}(z)$ for $a(v_{i,j}, z)$, then $a_{i,j} = \mu_i \mu_j (z_i - z_j)$, we can verify that $[b_{i,j}, b_{k,l}] = -\mu_N^{-1}(v_{i,j} \otimes a_{k,l} - v_{k,l} \otimes a_{i,j})$. Hence, we have $[b_z, b_w] = -\mu_N^{-1}(z \otimes a_w - w \otimes a_z)$.

Lemma 4.1 *The expression $a(b(z_1, z_2), z_3)$ is symmetric in its arguments if all μ_i's are equal.*

Proof The lemma is equivalent to saying that for every $z \in \mathfrak{h}$, b_z is self-adjoint relative to a.

Now we let all μ_i be equal to 1 in the above example, then the above example becomes the case of a root system of type A_n. Since we already know that the dimension of $\text{Hom}(\text{Sym}^2 \mathfrak{h}, \mathfrak{h})^W$ is just 1, then the $b_0 = \sum_{\alpha > 0} \alpha \otimes \alpha \otimes \alpha'$ given in Lemma 3.1 differs the bilinear map b in the above example just by a scalar. We thus have $b_{i,j}(z) = z^i \varepsilon_i - z^j \varepsilon_j - \frac{1}{n+1}(z^i - z^j)\varepsilon_N$. If $i < j < k$, then

$$a(b_{i,j}(z), \varepsilon_j - \varepsilon_k) = -z^j = a(b_{j,k}(z), \varepsilon_i - \varepsilon_j);$$

if i, j, k, l are pairwise distinct, then

$$a(b_{i,j}(z), \varepsilon_k - \varepsilon_l) = 0.$$

Since $\{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, n\}$ is a basis of $\mathfrak{h}$, the lemma follows. $\square$

Remark 4.2 Since a^κ and b^κ must be a multiple of a and b respectively, the expression $a^\kappa(b^\kappa(X, Y), Z)$ is also fully symmetric in its arguments.

Each $\alpha_{i,j}$ now determines a character $e^{\alpha_{i,j}}$ associated to the exponential map

$$\exp : \mathfrak{h} \to H = \mathfrak{h}/2\pi\sqrt{-1}P^\vee$$

to our torus H for which $P^\vee$ is the cocharacter lattice relative to the character lattice spanned by $\{\alpha_{i,j}\}$. Suppose all μ_i being equal now, then the above example becomes our toric case associated to a root system of type A_n: $k = \mu_i$. Once the symmetric W-equivariant bilinear map b^κ is chosen, or equivalently, the parameter k' is given. Then, we can define a connection $\tilde{\nabla}^\kappa$ on the (co)tangent bundle of $H^\circ \times \mathbb{C}^\times$ as in (3.2), and by Theorem 3.4 there exists a corresponding bilinear form a^κ, a multiple of a, such that the connection $\tilde{\nabla}^\kappa$ is torsion free and flat. Thus by Theorem 3.10, we have a class of Frobenius manifolds: toric Lauricella manifolds. Their fiberwise Frobenius algebra and potential functions are given as in (3.3) and (3.5), respectively.

We write out the potential functions for our toric Lauricella manifolds, according to (3.5), as follows:

$$\Phi = \frac{(\log t)^3}{3!} - \frac{(k^2 - k'^2)}{8}(\log t)\sum_{\alpha>0}\alpha^2 + \frac{n+1}{8}k(k^2-k'^2)\sum_{\alpha>0}q(\alpha)$$
$$+ \frac{n+1}{8}k'(k^2-k'^2)\sum_{\alpha>0}\frac{\alpha^2\alpha'}{2}, \tag{4.1}$$

where $q(\alpha)$ is such that $q'''(\alpha) = \coth\frac{\alpha}{2}$. Similar solutions were also found by Alkadhem and Feigin from the trigonometric $\vee$-system [1] and by Pavlov from reductions of Egorov hydrodynamic chains [25]. But as already pointed out in Remark 3.13, there appears an extra term $d^\kappa\sum_{\alpha>0}\frac{\alpha^2\alpha'}{2}$ in our solutions, due to the existence of a W-equivariant bilinear map on $\mathfrak{h}$.

References

1. Alkadhem, M., Feigin, M.: Trigonometric $\vee$-systems and solutions of WDVV equations. J. Phys. A **54**(2), 024002, 34 (2021)
2. Bertola, M.: Frobenius manifold structure on orbit space of Jacobi groups. I. Differ. Geom. Appl. **13**(1), 19–41 (2000)
3. Bourbaki, N.: Groupes et Algèbres de Lie, Chapitres 4,5 et 6. Number 1337 in Actualités Scientifiques et Industrielles. Hermann, Paris (1968)
4. Braden, H.W., Marshakov, A., Mironov, A., Morozov, A.: WDVV equations for 6d Seiberg–Witten theory and bi-elliptic curves. Acta Appl. Math. **99**(3), 223–244 (2007)
5. Bryan, J., Gholampour, A.: Root systems and the quantum cohomology of ADE resolutions. Algebra Number Theory **2**(4), 369–390 (2008)

6. Chalykh, O.A., Veselov, A.P.: Locus configurations and ∨-systems. Phys. Lett. A **285**(5–6), 339–349 (2001)
7. Couwenberg, W., Heckman, G., Looijenga, E.: Geometric structures on the complement of a projective arrangement. Publ. Math. Inst. Hautes Études Sci. **101**, 69–161 (2005)
8. Deligne, P., Mostow, G.D.: Monodromy of hypergeometric functions and non-lattice integral monodromy. Publ. Math. Inst. Hautes Études Sci. **63**, 5–89 (1986)
9. Dubrovin, B.: Geometry of 2D topological field theories. In: Integrable Systems and Quantum Groups. Lectures Given at the 1st Session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 14-22, 1993, pp. 120–348. Springer, Berlin (1996)
10. Dubrovin, B.: On almost duality for Frobenius manifolds. In: Geometry, Topology, and Mathematical Physics. Selected papers from S. P. Novikov's seminar held in Moscow, Russia, 2002–2003, pp. 75–132. Providence, RI: American Mathematical Society (AMS) (2004)
11. Etingof, P., Felder, G., Ma, X., Veselov, A.: On elliptic Calogero–Moser systems for complex crystallographic reflection groups. J. Algebra **329**(1), 107–129 (2011)
12. Feigin, M.V.: Trigonometric solutions of WDVV equations and generalized Calogero–Moser-Sutherland systems. SIGMA, Symmetry Integrability Geom. Methods Appl. **5**, paper088, 10 (2009)
13. Feigin, M.V., Veselov, A.P.: ∨-systems, holonomy Lie algebras, and logarithmic vector fields. Int. Math. Res. Not. **2018**(7), 2070–2098 (2018)
14. Heckman, G.J.: Root systems and hypergeometric functions. II. Compos. Math. **64**, 353–373 (1987)
15. Heckman, G.J., Opdam, E.M.: Root systems and hypergeometric functions. I. Compos. Math. **64**, 329–352 (1987)
16. Hoevenaars, L.K., Martini, R.: On the WDVV equations in five-dimensional gauge theories. Phys. Lett. B **557**(1–2), 94–104 (2003)
17. Looijenga, E.: Introduction to Frobenius manifolds. http://www.staff.science.uu.nl/~looij101/frobenius.pdf
18. Manin, Y.I.: Frobenius manifolds, quantum cohomology, and moduli spaces., vol. 47. Providence, RI: American Mathematical Society (1999)
19. Marshakov, A., Mironov, A., Morozov, A.: WDVV-like equations in $\mathcal{N} = 2$ SUSY Yang-Mills theory. Phys. Lett. B **389**(1), 43–52 (1996)
20. Marshakov, A., Mironov, A., Morozov, A.: More evidence for the WDVV equations in $\mathcal{N}$=2 SUSY Yang–Mills theories. Int. J. Mod. Phys. A **15**(8), 1157–1206 (2000)
21. Martini, R., Gragert, P.K.H.: Solutions of WDVV equations in Seiberg–Witten theory from root systems. J. Nonlinear Math. Phys. **6**(1), 1–4 (1999)
22. Martini, R., Hoevenaars, L.K.: Trigonometric solutions of the WDVV equations from root systems. Lett. Math. Phys. **65**(1), 15–18 (2003)
23. Opdam, E.M.: Root systems and hypergeometric functions. III. Compos. Math. **67**(1), 21–49 (1988)
24. Opdam, E.M.: Root systems and hypergeometric functions. IV. Compos. Math. **67**(2), 191–209 (1988)
25. Pavlov, M.V.: Explicit solutions of the WDVV equation determined by the "flat" hydrodynamic reductions of the Egorov hydrodynamic chains. https://arxiv.org/abs/nlin/0606008
26. Pavlov, M.V.: Integrable hydrodynamic chains. In: Nonlinear Physics: Theory and Experiment II. Proceedings of the Workshop, Gallipoli, Italy, June 27–July 6, 2002, pp. 101–107. River Edge, NJ: World Scientific (2003)
27. Shen, D.: Geometric structures on the complement of a toric mirror arrangement. Math. Z. (2021). https://doi.org/10.1007/s00209-021-02771-1
28. Stedman, R., Strachan, I.A.B.: Extended ⋁-systems and trigonometric solutions to the WDVV equations. J. Math. Phys. **62**(2), 022301, 21 (2021)
29. Strachan, I.A.B.: Weyl groups and elliptic solutions of the WDVV equations. Adv. Math. **224**(5), 1801–1838 (2010)
30. Veselov, A.P.: Deformations of the root systems and new solutions to generalised WDVV equations. Phys. Lett. A **261**(5–6), 297–302 (1999)