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Algebraic classical W-algebras and Frobenius manifolds

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Abstract

We consider Drinfeld–Sokolov bihamiltonian structure associated with a distinguished nilpotent elements of semisimple type and the space of common equilibrium points defined by its leading term. On this space, we construct a local bihamiltonian structure which forms an exact Poisson pencil, defines an algebraic classical W-algebra, admits a dispersionless limit, and its leading term defines an algebraic Frobenius manifold. This leads to a uniform construction of algebraic Frobenius manifolds corresponding to regular cuspidal conjugacy classes in irreducible Weyl groups.

Keywords Classical W-algebra · Frobenius manifolds · Nilpotent orbits in Lie algebras · Common equilibrium points · Exact Poisson pencil · Drinfeld–Sokolov reduction

Mathematics Subject Classification 37K25 · 37k30 · 53D45 · 17B80 · 53D17 · 17B68 · 17B08

Contents

Dedicated to the memory of Boris Dubrovin

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1 Introduction

Frobenius manifold is a marvelous geometric realization introduced by Boris Dubrovin for undetermined partial differential equations known as Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations which describe the module space of two-dimensional topological field theory. Remarkably, Frobenius manifolds are also recognized in many other fields in mathematics like invariant theory, quantum cohomology, integrable systems and singularity theory [24]. Briefly, a Frobenius manifold is a manifold with a smooth structure of Frobenius algebra on the tangent space with certain compatibility conditions. By Frobenius algebra, we mean a commutative associative algebra with unity and an invariant nondegenerate symmetric bilinear form.

Let M be a Frobenius manifold. Then, we require the bilinear form $(.,.)$ to be flat, and the unity vector field e is constant with respect to it. Let $(t^1,\ldots,t^r)$ be flat coordinates for $(.,.)$ where $e=\partial_{t^r}$. Then, the compatibility conditions imply that there exists a function $\mathbb{F}(t^1,\ldots,t^r)$ such that

$$\eta_{ij} := (\partial_{t^i},\partial_{t^j}) = \partial_{t^r}\partial_{t^i}\partial_{t^j}\mathbb{F}(t) \tag{1.1}$$

and the structure constants of the Frobenius algebra are given by

$$C_{ij}^k(t) := \sum_p \eta^{kp}\partial_{t^p}\partial_{t^i}\partial_{t^j}\mathbb{F}(t) \tag{1.2}$$

where the matrix η^{ij} is the inverse of the matrix η_{ij}. Associativity in T_tM implies that $\mathbb{F}(t)$ satisfies WDVV equations [13]:

$$\sum_{k,p}\partial_{t^i}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)\;\eta^{kp}\;\partial_{t^p}\partial_{t^q}\partial_{t^s}\mathbb{F}(t) = \sum_{k,p}\partial_{t^s}\partial_{t^j}\partial_{t^k}\mathbb{F}(t)\;\eta^{kp}\;\partial_{t^p}\partial_{t^q}\partial_{t^i}\mathbb{F}(t), \tag{1.3}$$

for all i, j, q and s. In this article, we consider Frobenius manifolds where the quasi-homogeneity condition for $\mathbb{F}(t)$ can take the form:

$$\sum_{i=1}^r d_i t^i\partial_{t^i}\mathbb{F}(t) = (3-d)\,\mathbb{F}(t);\quad d_r = 1. \tag{1.4}$$

The vector field $E = \sum_{i=1}^{n} d_i t^i \partial_{t_i}$ is known as Euler vector field, and it defines the degrees d_i and the charge d of M. A Frobenius manifold is called algebraic if $\mathbb{F}(t)$ is an algebraic function, and it is called semisimple if $T_t M$ is a semisimple algebra for generic point t.

This work is related to a conjecture due to Dubrovin which states that semisimple irreducible algebraic Frobenius manifolds with positive degrees correspond to primitive (quasi-Coxeter) conjugacy classes of irreducible finite Coxeter groups [28]. A primitive conjugacy class in a Coxeter group is a conjugacy class which has no representative in a proper Coxeter subgroup (see [5] for the classification). Coxeter conjugacy class is an example of a primitive conjugacy class which exists in any Coxeter group. (It is formed by the product of simple reflections in the case of standard reflection representation.) The conjecture arises from studying the algebraic solutions to associated equations of isomonodromic deformation of an algebraic Frobenius manifold [28]. It leads to a primitive conjugacy class in a Coxeter group by considering the classification of finite orbits of the braid group action on tuple of reflections [47]. A stage to verify the conjecture is to show the existence of these algebraic Frobenius manifolds.

Under the conjecture, it is known that polynomial Frobenius manifolds correspond to Coxeter conjugacy classes. Dubrovin constructed these polynomial Frobenius structures on orbit spaces of the standard reflection representations of Coxeter groups [23]. Their isomonodromic deformations lead to Coxeter conjugacy classes [28], and C. Hertling [36] proved (as also conjectured by Dubrovin) that they exhaust the set of all possible polynomial structures up to an equivalence. This classification and other examples reveal a relation between orders and eigenvalues of the conjugacy classes, and charges and degrees of algebraic Frobenius manifolds. More precisely, if the order of a primitive conjugacy class is $\eta_r + 1$ and the eigenvalues are $\exp \frac{2\eta_i \pi \mathbf{i}}{\eta_r+1}$, then the charge of the corresponding Frobenius structure is $\frac{\eta_r-1}{\eta_r+1}$ and the degrees are $\frac{\eta_i+1}{\eta_r+1}$. We depend on this relation in constructing algebraic Frobenius structures.

One of the main methods to obtain examples of Frobenius manifolds exists within the theory of flat pencils of metrics (equivalently, nondegenerate compatible Poisson brackets of hydrodynamics type). Besides, the leading terms of certain type of local compatible Poisson brackets (a local bihamiltonian structure) which admit(s) a dispersionless limit form a flat pencil of metric [25].

One of the main ideas to find algebraic Frobenius structures is to restrict ourselves to irreducible Weyl groups, i.e., crystallographic Coxeter groups, and to consider the associated simple Lie algebras. Then, under the notion of opposite Cartan subalgebra, regular primitive conjugacy classes correspond to certain nilpotent orbits of semisimple type. On the other hand, we can obtain compatible local Poisson brackets for any nilpotent orbit using Drinfeld–Sokolov reduction. These Poisson brackets form an exact Poisson pencil, and one of them is (or satisfies identities leading to) a classical W-algebra. However, they admit a dispersionless limit only when the nilpotent orbit is regular (which corresponds to Coxeter conjugacy class). In this article, we will work with a larger type of conjugacy classes called cuspidal. A cuspidal conjugacy class has no representative in a Coxeter subgroup of smaller rank. Regular cuspidal conjugacy classes correspond to what is called distinguished nilpotent orbits of semisimple type

[12,31]. In other words, we get certain Drinfeld–Sokolov bihamiltonian structures associated with regular cuspidal conjugacy classes in irreducible Weyl groups.

Examples of Frobenius manifolds constructed using Drinfeld–Sokolov bihamiltonian structure can be traced back to the work of I. Krichever [39]. In our terminologies, he treated the case of Coxeter conjugacy classes in Weyl groups of type A_r. (Here, classical W-algebras are known as second Gelfand-Dickey brackets.) In [15], we gave a generalization to all Coxeter conjugacy classes in Weyl groups which, as expected, lead to the polynomial Frobenius manifolds.

For regular primitive non-Coxeter conjugacy classes, we always get algebraic non-polynomial Frobenius structures. Pavlyk obtained the first example which is related to the Weyl group of type D_4 [43]. In [14], we got another example working with Weyl group of type F_4. We added another 3 by giving a uniform construction related to certain conjugacy classes in Weyl groups of type E_r, $r = 6, 7, 8$ [18]. In all these cases, we have to perform Dirac reduction for the Drinfeld–Sokolov bihamiltonian structure to a subspace to get a bihamiltonian structure admitting a dispersionless limit. In this article, we give a slightly better interpretation for this subspace which leads to a uniform construction of algebraic Frobenius structures for all regular cuspidal conjugacy classes. Precisely, we will prove the following theorem.

Theorem 1.1 *Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r. Fix a regular cuspidal conjugacy class $[w]$ in the Weyl group $\mathcal{W}(\mathfrak{g})$ of $\mathfrak{g}$. Assume the order of representatives in $[w]$ is $\eta_r + 1$ and eigenvalues are ϵ^{η_i}, $i = 1, \ldots, r$, where ϵ is a primitive $(\eta_r + 1)$th root of unity. Let $\mathcal{O}_{L_1}$ be the distinguished nilpotent orbit of semisimple type associated with $[w]$ under the notion of opposite Cartan subalgebra. Consider the finite bihamiltonian structure formed by the leading term of Drinfeld-Sokolov bihamiltonian structure associated with a representative L_1 of $\mathcal{O}_{L_1}$. Then, its space of common equilibrium points acquires an algebraic Frobenius manifold structure with charge $\frac{\eta_r-1}{\eta_r+1}$ and degrees $\frac{\eta_i+1}{\eta_r+1}$. This structure depends only on the conjugacy class.*

We explain in some details the major steps to prove Theorem 1.1 which lead us to a construction of algebraic classical W-algebras admitting a dispersionless limit. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r with the Lie bracket $[\cdot, \cdot]$. Define the adjoint representation $\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$ by $\text{ad}_{g_1}(g_2) := [g_1, g_2]$. For $g \in \mathfrak{g}$, let $\mathfrak{g}^g$ denote the centralizer of g in $\mathfrak{g}$, i.e., $\mathfrak{g}^g := \ker \text{ad}_g$. Fix a distinguished nilpotent element L_1 of semisimple type. (More details are given in Sect. 3.) Then, using Jacobson–Morozov theorem, we fix a nilpotent element f and a semisimple element h such that $A := \{L_1, h, f\} \subseteq \mathfrak{g}$ is a sl_2-triple with relations

$$[h, L_1] = L_1, \quad [h, f] = -f, \quad [L_1, f] = 2h. \tag{1.5}$$

We normalize the Killing form on $\mathfrak{g}$ to get an invariant bilinear form $\langle .|. \rangle$ such that $\langle L_1 | f \rangle = 1$.

Let η_r denote the maximal eigenvalue of ad_h acting on $\mathfrak{g}$. By definition, we can (and we will) fix an element K_1 for L_1 such that $\text{ad}_h K_1 = -\eta_r K_1$ and $h' := L_1 + K_1$ is a regular semisimple element. Thus, $\mathfrak{h}' := \ker \text{ad}_{h'}$ is a Cartan subalgebra known as opposite Cartan subalgebra. The adjoint group element $w := \exp \frac{2\pi \mathbf{i}}{\eta_r+1} \text{ad}_h$ acts on $\mathfrak{h}'$

as a representative of regular cuspidal conjugacy class of order $\eta_r + 1$ in the underline Weyl group $\mathcal{W}(\mathfrak{g})$ (see [31], and the appendix of [12]).

Let $\eta_1 \leq \ldots \leq \eta_r$ be natural numbers such that ϵ^{η_i} are eigenvalues of w where ϵ is $(\eta_r + 1)$th root of unity. Let $n = \dim \mathfrak{g}^f$, then using representation theory of sl_2-subalgebras, there exist natural numbers $\eta_{r+1}, \ldots, \eta_n$ such that the eigenvalues of ad_h on $\mathfrak{g}^f$ are $-\eta_i$, $i = 1, \ldots, n$. We list all distinguished nilpotent elements of semisimple type in simple Lie algebras and the numbers η_i in Table 1.

We fix Slodowy slice $Q := L_1 + \mathfrak{g}^f$ as a transverse subspace to the orbit space of L_1 at L_1. Let $\mathfrak{L}(\mathfrak{g}^f)$ denote the space of smooth functions from the circle to $\mathfrak{g}^f$. The affine loop space $\mathcal{Q} := L_1 + \mathfrak{L}(\mathfrak{g}^f)$ carries compatible local Poisson structures (Drinfeld–Sokolov bihamiltonian structure formed by) $\mathbb{B}_2^{\mathcal{Q}}$ and $\mathbb{B}_1^{\mathcal{Q}}$, where $\mathbb{B}_2^{\mathcal{Q}}$ is a classical W-algebra [33,34] and $\mathbb{B}_1^{\mathcal{Q}}$ is related to a 2-cocycle on $\mathfrak{g}$ provided by K_1. They depend only on the adjoint orbit of L_1, and they can be obtained equivalently by using Drinfeld–Sokolov reduction, bihamiltonian reduction and Dirac reduction [19]. Note that performing any of these reductions, we need to fix a transverse subspace. However, taking a different subspace than Q will lead to isomorphic bihamiltonian structures. As it is already known by experts, we will prove in Proposition 5.4 that $\mathbb{B}_2^{\mathcal{Q}}$ and $\mathbb{B}_1^{\mathcal{Q}}$ form an exact Poisson pencil.

We identify Slodowy slice Q with the subspace of constant loops of $\mathcal{Q}$. We can (and will) fix coordinates $(z^1, \ldots, z^n)$ for Q such that

$$Q = L_1 + \sum z^i \gamma_i, \;\; \gamma_i \in \mathfrak{g}^f, \mathrm{ad}_h \gamma_i = -\eta_i \gamma_i, \;\; i = 1, \ldots, n \tag{1.6}$$

where $\gamma_1 = f$ and for $q \in Q$, $z^1 = \langle L_1 | q \rangle$. Then, the leading terms of $\mathbb{B}_m^{\mathcal{Q}}$, $m = 1, 2$, can be written as follows:

$$\begin{aligned} \{z^i(x), z^j(y)\}_m^{[-1]} &= F_m^{ij}(z(x))\delta(x-y), \\ \{z^i(x), z^j(y)\}_m^{[0]} &= \Omega_m^{ij}(z(x))\delta'(x-y) + \sum_k \Gamma_{k,m}^{ij}(z(x)) z_x^k \delta(x-y). \end{aligned} \tag{1.7}$$

Such a local Poisson bracket admits a dispersionless limit iff $F_m^{ij} = 0$. In general, $F_2^{ij}(z)$ and $F_1^{ij}(z)$ define compatible Poisson structures B_2^Q and B_1^Q, respectively, on Q. Moreover, B_2^Q can be identified with the transverse Poisson structure of Lie-Poisson structure on $\mathfrak{g}$ [19]. We assign $\deg z^i = \eta_i + 1$. Then, after certain normalization, we will prove the following theorem

Theorem 1.2 *There exists a quasihomogeneous change of coordinates on Q in the form:*

$$t^i = \begin{cases} z^1, & i=1, \\ z^i + \text{non linear terms}, & i=2, \ldots, r, \\ z^i, & i=r+1, \ldots, n. \end{cases} \tag{1.8}$$

such that

1. $\deg t^i = \deg z^i = \eta_i + 1$
2. $t^1, \ldots, t^r$ *form a complete set of Casimirs of* B_1^Q *and they are in involution with respect to* B_2^Q.

We will keep the notations $(t^1, \ldots, t^n)$ for the coordinates obtained in the last theorem. (Except in Sect. 6, we can and will assume they are flat coordinates of the resulted Frobenius structure.)

We are interested in the space of common equilibrium points N of the bihamiltonian structure formed by B_2^Q and B_1^Q. Combining results from [3] and [20], we explain in theorem 4.10 that the argument shift method leads to a completely integrable system for B_2^Q and

$$N = \{q \in Q : \ker B_1^Q(q) = \ker B_2^Q(q)\}. \tag{1.9}$$

Using Chevalley's theorem, we fix homogeneous set of generators $P_1, \ldots, P_r$ of the ring of invariant polynomials of $\mathfrak{g}$ under the adjoint group action. Let $\overline{P}_i^0$ denote the restriction of P_i to Q. We can choose $P_1, \ldots, P_r$ such that the following theorem is valid. Here, we assume L_1 is of type $Z_r(a_s)$ where Z_r is the type of $\mathfrak{g}$.

Theorem 1.3 *The space of common equilibrium points N is given by*

$$N = \{t : F_2^{i\beta}(t) = 0;\ i = 1, \ldots, r,\ \beta = r+1, \ldots, n\}, \tag{1.10}$$
$$= \{t : \partial_{t^\beta} \overline{P}_j^0(t) = 0;\ j = r-s+1, \ldots, r,\ \ \beta = r+1, \ldots, n\}. \tag{1.11}$$

Moreover, $(t^1, \ldots, t^r)$ provide local coordinates around generic points of N. In addition, Dirac reduction of the Poisson pencil $B_\lambda^Q := B_2^Q + \lambda B_1^Q$ to N is well defined and leads to the trivial Poisson bracket.

Then, we construct compatible local Poisson brackets on the loop space $\mathcal{N} = \mathfrak{L}(N)$.

Theorem 1.4 *The Dirac reduction of the Poisson pencil $\mathbb{B}_\lambda^Q := \mathbb{B}_2^Q + \lambda \mathbb{B}_1^Q$ to $\mathcal{N}$ is well defined and leads to compatible local Poisson brackets $\{.,.\}_\alpha^{\mathcal{N}}$, $\alpha = 1, 2$ which admit a dispersionless limit and form an exact Poisson pencil. Moreover, $\{.,.\}_2^{\mathcal{N}}$ is an algebraic classical W-algebra.*

Let us emphasize that Theorem 1.4 implies that the leading terms of the local Poisson brackets on $\mathcal{N}$ are Poisson brackets of hydrodynamic types, i.e.,

$$\{t^u(x), t^v(y)\}_\alpha^{[0]} = \Omega_\alpha^{uv}(t(x))\delta'(x-y) + \Gamma_{\alpha k}^{uv}(t(x))t_x^k\delta(x-y),\ u, v = 1, \ldots r,\ \alpha = 1, 2. \tag{1.12}$$

where $t^k, k > r$ are solutions of the polynomial equations (1.10) defining N.

One of the important steps on the construction is to prove that the matrices $\Omega_\alpha^{uv}(t)$ are nondegenerate. We will show that this condition follows from the fact that the restriction of the Killing form on $\mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}'$ is nondegenerate (see Proposition 5.6).

Then, we will prove the following.

Theorem 1.5 *The two metrics* Ω_1^{uv} *and* Ω_2^{uv} *form a flat pencil of metrics on* N *which is regular quasihomogeneous of degree* $d = \frac{\eta_r - 1}{\eta_r + 1}$.

In the end, using Theorem 2.8 due to Dubrovin, we get the proof of Theorem 1.1.

We organize the article as follows. In Sect. 2, we fix notations and terminologies within the theory of local Poisson brackets, flat pencils of metrics and Frobenius manifolds. We review the classification of distinguished nilpotent orbits of semisimple type in simple Lie algebras in Sect. 3, and we will drive some algebraic properties associated with them. In Sect. 4, we will study the space N of common equilibrium points and prove Theorems 1.2 and 1.3. We review the Drinfeld–Sokolov reduction in Sect. 5 and prove Theorem 1.4. In Sect. 6, we will prove Theorem 1.1 and we give examples. The notations given in the introduction are in agreement with the flow of the article.

2 Preliminaries

In this section, we recall relations between local bihamiltonian structures, flat pencils of metrics and Frobenius manifolds. We also review the notion of Dirac reduction for local Poisson brackets.

2.1 Contravariant metrics and local Poisson brackets

Let M be a smooth manifold of dimension n and fix local coordinates $(u^1, \dots, u^n)$ on M. Here, and in what follows, summation with respect to repeated upper and lower indices is assumed, i.e., we will adopt Einstein summation convention.

Definition 2.1 A symmetric bilinear form $(.,.)$ on T^*M is called a contravariant metric if it is invertible on an open dense subset $M_0 \subseteq M$. We define the contravariant Levi–Civita connection or Christoffel symbols Γ_k^{ij} for a contravariant metric $(.,.)$ by

$$\Gamma_k^{ij} := -g^{is}\Gamma_{sk}^j \tag{2.1}$$

where Γ_{sk}^j are the Christoffel symbols of the metric $< .,. >$ defined on TM_0 by the inverse of the matrix $\Omega^{ij}(u) = (du^i, du^j)$. We say the metric $(.,.)$ is flat if $< .,. >$ is flat.

Let $(.,.)$ be a contravariant metric on M and set $\Omega^{ij}(u) = (du^i, du^j)$. Then, we will use Ω^{ij} to refer to both the metric and the entries defined by the metric. In particular, Lie derivative of $(.,.)$ along a vector field X will be written $\mathfrak{L}_X \Omega^{ij}$, while $X\Omega^{ij}$ means the vector field X acting on the entry Ω^{ij}.

The loop space $\mathfrak{L}(M)$ of M is the space of smooth functions from the circle to M. A local Poisson bracket $\{.,.\}$ is a certain bracket on the space of local functional on $\mathfrak{L}(M)$ [29]. We can write $\{.,.\}$ as a finite summation of the form:

$$\{u^i(x), u^j(y)\} = \sum_{k=-1}^{\infty} \{u^i(x), u^j(y)\}^{[k]}$$
$$\{u^i(x), u^j(y)\}^{[k]} = \sum_{l=0}^{k+1} A_{k,l}^{i,j}(u(x))\delta^{(k-l+1)}(x-y), \tag{2.2}$$

where $A_{k,l}^{i,j}(u(x))$ are quasihomogeneous polynomials in $\partial_x^m u^i(x)$ of degree l when we assign degree $\partial_x^m u^i(x)$ equals m, and $\delta(x-y)$ is the Dirac delta function defined by

$$\int_{S^1} f(y)\delta(x-y)dy = f(x). \tag{2.3}$$

Definition 2.2 [34] A local Poisson bracket $\{.,.\}$ in the form (2.2) is called a classical W-algebra if there exist local coordinates $(z^1, \ldots, z^n)$ such that

$$\{z^1(x), z^1(y)\} = c\delta'''(x-y) + 2z^1(x)\delta'(x-y) + z_x^1\delta(x-y),$$
$$\{z^1(x), z^i(y)\} = (\eta_i + 1)z^i(x)\delta'(x-y) + \eta_i z_x^i\delta(x-y), \tag{2.4}$$

for nonzero constant c.

Let us fix a local Poisson bracket $\{.,.\}$ on $\mathfrak{L}(M)$. The first terms can be written as follows:

$$\{u^i(x), u^j(y)\}^{[-1]} = F^{ij}(u(x))\delta(x-y),$$
$$\{u^i(x), u^j(y)\}^{[0]} = \Omega^{ij}(u(x))\delta'(x-y) + \Gamma_k^{ij}(u(x))u_x^k\delta(x-y),$$
$$\{u^i(x), u^j(x)\}^{[k]} = S_k^{ij}(u(x))\delta^{k+1}(x-y) + \ldots, \quad k > 0. \tag{2.5}$$

Note that M can be defined as the subspace of constant loops of $\mathfrak{L}(M)$. Then, $\Omega^{ij}(u)$, $F^{ij}(u)$, $S_k^{ij}(u)$ and $\Gamma_k^{ij}(u)$ are smooth functions on M. Moreover, the matrix $F^{ij}(u)$ represents a finite-dimensional Poisson structure on M. This gives a bridge between finite-dimensional and local Poisson structures.

Definition 2.3 We say a local Poisson bracket $\{.,.\}$ in the form (2.5) admits a dispersionless limit if $F^{ij}(u) = 0$ and $\{.,.\}^{[0]} \neq 0$. In this case $\{.,.\}^{[0]}$ defines a local Poisson bracket on $\mathfrak{L}(M)$ known as Poisson bracket of hydrodynamic type. We call it nondegenerate if $\det \Omega^{ij} \neq 0$ on an open dense subset of M.

The following theorem, due to Dubrovin and Novikov, relates contravariant metrics on a manifold M to theory of local Poisson brackets on $\mathfrak{L}(M)$.

Theorem 2.4 [27] *In the notations of formulas (2.5), if* $\{.,.\}^{[0]}$ *is a nondegenerate Poisson brackets of hydrodynamic type, then the matrix* $\Omega^{ij}(u)$ *defines a contravariant flat metric on* M *and* $\Gamma^{ij}_k(u)$ *are its contravariant Christoffel symbols.*

We recall the notion of Dirac reduction of a local Poisson bracket to loop spaces of certain sub-manifolds. Let us fix a submanifold $M' \subset M$ of dimension r. We assume M' is defined by the equations $u^\alpha = 0$ for $\alpha = r+1, \dots, n$. We introduce three types of indices: capital letters $I, J, K, \dots = 1, \dots, n$, small letters $i, j, k, \dots = 1, \dots, r$ which parameterize the submanifold M' and Greek letters $\alpha, \beta, \gamma, \delta, \dots = r+1, \dots, n$.

Proposition 2.5 [19] *In the notations of equations (2.5), assume the minor matrix* $F^{\alpha\beta}$ *is nondegenerate. Then, Dirac reduction is well defined on* $\mathfrak{L}(M')$*, and it gives a local Poisson bracket. If we write the leading terms of the reduced Poisson bracket in the form:*

$$\{u^i(x), u^j(y)\}^{[-1]}_{M'} = \widetilde{F}^{ij}(u)\delta(x-y), \tag{2.6}$$

$$\{u^i(x), u^j(y)\}^{[0]}_{M'} = \widetilde{\Omega}^{ij}(u)\delta'(x-y) + \widetilde{\Gamma}^{ij}_k u^k_x \delta(x-y),$$

$$\{u^i(x), u^j(x)\}^{[k]}_{M'} = \widetilde{S}^{ij}_k(u)\delta^{k+1}(x-y) + \dots, \ k > 0. \tag{2.7}$$

Then,

$$\begin{aligned}
\widetilde{F}^{ij} &= F^{ij} - F^{i\beta}F_{\beta\alpha}F^{\alpha j},\\
\widetilde{\Omega}^{ij} &= \Omega^{ij} - \Omega^{i\beta}F_{\beta\alpha}F^{\alpha j} + F^{i\beta}F_{\beta\alpha}\Omega^{\alpha\varphi}F_{\varphi\gamma}F^{\gamma j} - F^{i\beta}F_{\beta\alpha}\Omega^{\alpha j},\\
\widetilde{\Gamma}^{ij}_k u^k_x &= \left(\Gamma^{ij}_k - \Gamma^{i\beta}_k F_{\beta\alpha}F^{\alpha j} + F^{i\lambda}F_{\lambda\alpha}\Gamma^{\alpha\beta}_k F_{\beta\varphi}F^{\varphi j} - F^{i\beta}F_{\beta\alpha}\Gamma^{\alpha j}_k\right)u^k_x\\
&\quad -\left(\Omega^{i\beta} - F^{i\lambda}F_{\lambda\alpha}\Omega^{\alpha\beta}\right)\partial_x(F_{\beta\varphi}F^{\varphi j}),
\end{aligned} \tag{2.8}$$

while other higher terms could be found by solving certain recursive equations.

Corollary 2.6 $\widetilde{F}^{ij}$ *is the Dirac reduction of the finite-dimensional Poisson structure* F^{IJ} *on* M *to* M'*. If the entries* $F^{i\alpha} = 0$ *on* M'*, then the reduced Poisson bracket on* $\mathfrak{L}(M')$ *has the same leading terms, i.e.,*

$$\widetilde{F}^{ij} = F^{ij}, \ \ \widetilde{\Omega}^{ij} = \Omega^{ij}, \ \ \widetilde{\Gamma}^{ij}_k = \Gamma^{ij}_k, \ \text{ and } \ \widetilde{S}^{ij}_k = S^{ij}_k. \tag{2.9}$$

2.2 From bihamiltonian structures to Frobenius manifolds

We use the notations given in Sect. 2.1 to bring a relations between local bihamiltonian structures and Frobenius manifolds.

Definition 2.7 [25] Let Ω_1^{ij} and Ω_2^{ij} be two flat contravariant metrics on M with Christoffel symbols Γ^{ij}_{2k} and Γ^{ij}_{1k}, respectively. Then, they form a flat pencil of metrics if $\Omega^{ij}_\lambda := \Omega^{ij}_2 + \lambda\Omega^{ij}_1$ defines a flat metric on T^*M for generic λ and the Christoffel symbols of Ω^{ij}_λ satisfy $\Gamma^{ij}_{\lambda k} = \Gamma^{ij}_{2k} + \lambda\Gamma^{ij}_{1k}$. Such flat pencil of metrics is called quasihomogeneous of degree d if there exists a function τ on M such that the vector fields

$$E := \nabla_2 \tau, \quad E^i = \Omega_2^{ij} \partial_{u^j} \tau$$
$$e := \nabla_1 \tau, \quad e^i = \Omega_1^{ij} \partial_{u^j} \tau \tag{2.10}$$

satisfy the following properties

$$[e, E] = e, \quad \mathfrak{L}_E \Omega_2^{ij} = (d-1)\Omega_2^{ij}, \quad \mathfrak{L}_e \Omega_2^{ij} = \Omega_1^{ij} \text{ and } \mathfrak{L}_e \Omega_1^{ij} = 0. \tag{2.11}$$

In addition, the quasihomogeneous flat pencil of metrics is called **regular** if the (1,1)-tensor

$$R_i^j = \frac{d-1}{2}\delta_i^j + \nabla_{1i} E^j \tag{2.12}$$

is nondegenerate on M.

The connection between the theory of Frobenius manifolds and flat pencil of metrics is encoded in the following theorem due to Dubrovin.

Theorem 2.8 [25] *A contravariant quasihomogeneous regular flat pencil of metrics of degree d on a manifold M defines a Frobenius structure on M of charge d.*

It is well known that from a Frobenius manifold we always have a flat pencil of metrics but it does not necessarily satisfy the regularity condition (2.12) [25]. Locally, in the coordinates defining equations (1.3) and (1.4), the flat pencil of metrics is found by setting

$$\Omega_1^{ij} = \eta^{ij},$$
$$\Omega_2^{ij} = (d-1+d_i+d_j)\eta^{i\alpha}\eta^{j\beta}\partial_{t^\alpha}\partial_{t^\beta}\mathbb{F}. \tag{2.13}$$

This flat pencil of metric is quasihomogeneous of degree d with $\tau = t^1$. Furthermore, we have

$$E = \sum_i d_i t^i \partial_{t^i}; \quad e = \partial_{t^r}. \tag{2.14}$$

There is a source of flat pencil of metric within the theory of local bihamiltonian structures.

Definition 2.9 Two local Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$ on $\mathfrak{L}(M)$ form a bihamiltonian structure or they are compatible if the Poisson pencil $\{.,.\}_\lambda := \{.,.\}_2 + \lambda\{.,.\}_1$ is a Poisson bracket for generic constant λ. Compatible Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$ form an exact Poisson pencil if there exists a vector field X such that

$$\{.,.\}_1 = \mathfrak{L}_X\{.,.\}_2; \quad \mathfrak{L}_X\{.,.\}_1 = 0. \tag{2.15}$$

In this case, we call X Liouville vector field.

For recent developments about the theory of exact Poisson pencil see [32] and [41].

Let us fix compatible local Poisson brackets $\{.,.\}_2$ and $\{.,.\}_1$ on $\mathfrak{L}(M)$ and write their leading terms in the form:

$$\{u^i(x), u^j(y)\}_\alpha^{[-1]} = F_\alpha^{ij}(u(x))\delta(x-y), \alpha = 1, 2$$
$$\{u^i(x), u^j(y)\}_\alpha^{[0]} = \Omega_\alpha^{ij}(u(x))\delta'(x-y) + \Gamma_{\alpha k}^{ij}(u(x))u_x^k\delta(x-y). \quad (2.16)$$

Suppose that $\{.,.\}_1$ and $\{.,.\}_2$ admit a dispersionless limit. (We also say the bihamiltonian structure admits a dispersionless limit.) In addition, assume the corresponding Poisson brackets of hydrodynamics type are nondegenerate as well as the dispersionless limit of $\{.,.\}_\lambda$ for generic λ. Then, using Theorem 2.4, the matrices Ω_1^{ij} and Ω_2^{ij} define a flat pencil of metrics on M.

3 Nilpotent elements of semisimple type

In this section, we collect properties of the so-called distinguished nilpotent elements of semisimple type in simple Lie algebras. Then, we derive important identities needed to prove our main results.

3.1 Background

We fix a complex simple Lie algebra $\mathfrak{g}$ of rank r. We refer to the type of $\mathfrak{g}$ by Z_r. For $g \in \mathfrak{g}$, let $\mathcal{O}_g$ denotes the orbit of g under the adjoint group action. The element g is called nilpotent if ad_g is nilpotent in $\text{End}(\mathfrak{g})$ and it is called regular if $\dim \mathfrak{g}^g = r$. Any simple Lie algebra contains regular nilpotent elements.

We fix a nilpotent element L_1 in $\mathfrak{g}$. (Later, we will assume it is distinguished.) Let $A := \{L_1, h, f\} \subseteq \mathfrak{g}$ be an associated sl_2-triple satisfying the relations (1.5). It follows from representation theory of sl_2-algebra that the eigenvalues of ad_h are integers and half integers. Consider Dynkin grading associated with L_1

$$\mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i; \quad \mathfrak{g}_i := \{g \in \mathfrak{g} : \text{ad}_h g = ig\}. \quad (3.1)$$

We retrieve from [6] the following definitions concerning nilpotent orbits and their classification. If L_1 is regular, then $\mathcal{O}_{L_1}$ is called regular nilpotent orbit, and it is equal to the set of all regular nilpotent elements in $\mathfrak{g}$. The nilpotent orbit $\mathcal{O}_{L_1}$ is called distinguished, and hence also L_1, if $\mathcal{O}_{L_1}$ has no representative in a proper Levi subalgebra of $\mathfrak{g}$. It turns out that L_1 is distinguished iff $\dim \mathfrak{g}_0 = \dim \mathfrak{g}_1$. Moreover, if L_1 is distinguished, then the eigenvalues of ad_h are all integers. The regular nilpotent orbit in $\mathfrak{g}$ is distinguished.

Distinguished nilpotent orbits, along with other nilpotent orbits, are classified by using weighted Dynkin diagrams. In the case $\mathfrak{g}$ is an exceptional Lie algebra, distinguished nilpotent orbits are listed in the form $Z_r(a_i)$ where i is the number of vertices of weight 0 in the corresponding weighted Dynkin diagram. If there is another orbit of the same number i of 0's, then the notation $Z_r(b_i)$ is used. For all simple Lie algebras, the type of the regular nilpotent orbit is $Z_r(a_0)$.

In case $\mathfrak{g}$ is a classical Lie algebra, nilpotent orbits are also classified through partitions of the dimension of the fundamental representation of $\mathfrak{g}$. In this article, by $B_{2m}(a_m)$, we refer to the distinguished nilpotent orbit corresponding to the partition $[2m+1, 2m-1, 1]$ when the Lie algebra $\mathfrak{g}$ is so_{4m+1} (type B_{2m}). While, as usual in the literature, $D_{2m}(a_{m-1})$ denotes the distinguished nilpotent orbit corresponding to the partition $[2m+1, 2m-1]$ when $\mathfrak{g}$ is so_{4m} (type D_{2m}).

From [31], we recall the following definition and properties. The nilpotent element L_1 is of semisimple type, and so its orbit, if there exists an element g of the minimal eigenvalue of ad_h such that $L_1 + g$ is semisimple. In this case $L_1 + g$ is called a cyclic element. If L_1 is also distinguished then $L_1 + g$ will be regular. The list of distinguished nilpotent elements of semisimple types is (*idid*, Lemma 5.1 and see the appendix of [12]):

1. All regular nilpotent orbits in simple Lie algebras (those of type $Z_r(a_0)$)
2. Subregular nilpotent orbits $F_4(a_1)$, $E_6(a_1)$, $E_7(a_1)$ and $E_8(a_1)$.
3. Nilpotent orbits of type $B_{2m}(a_m)$ and $D_{2m}(a_{m-1})$.
4. Nilpotent orbits of type $F_4(a_2)$, $F_4(a_3)$, $E_6(a_3)$, $E_7(a_5)$, $E_8(a_2)$,$E_8(a_4)$, $E_8(a_6)$ and $E_8(a_7)$.

From now on, we assume that L_1 is a distinguished nilpotent element of semisimple type and we refer to its type by $Z_r(a_s)$. Let η_r denote the maximal eigenvalue of ad_h. Thus, we can (and will) fix an element $K_1 \in \mathfrak{g}_{-\eta_r}$ such that the cyclic element $\Lambda_1 := L_1 + K_1$ is regular semisimple.

In what follows, we give a general setup associated with the cyclic element Λ_1. It was initiated by Kostant for the case of regular nilpotent elements [37] and obtained for distinguished nilpotent elements of semisimple type in [12]. Let $\mathfrak{h}' := \mathfrak{g}^{\Lambda_1}$ be the Cartan subalgebra containing Λ_1 which is known as the opposite Cartan subalgebra. Then, the adjoint group element w defined by

$$w := \exp \frac{2\pi \mathbf{i}}{\eta_r + 1} \mathrm{ad}_h \tag{3.2}$$

acts on $\mathfrak{h}'$ as a representative of a regular cuspidal conjugacy class $[w]$ in the Weyl group $\mathcal{W}(\mathfrak{g})$ of $\mathfrak{g}$ of order $\eta_r + 1$. We recall that a conjugacy class $[w'] \subset \mathcal{W}(\mathfrak{g})$ is called cuspidal [35] (resp. primitive [5]) if $\det(w' - I) \neq 0$ (resp. $\det(w' - I) = \det \mathbb{K}$, $\mathbb{K}$ is the Cartan matrix of $\mathcal{W}(\mathfrak{g})$). Also, $[w']$ is called regular if w' has an eigenvector not fixed by any non-identity element in $\mathcal{W}(g)$ (see [46] for the classification of regular conjugacy classes). We emphasize that the results in this article depend on the nilpotent orbit $\mathcal{O}_{L_1}$ and not on the particular representative L_1 of $\mathcal{O}_{L_1}$.

3.2 Normalization and identities

The element Λ_1 is an eigenvector of w of eigenvalue $\epsilon = \exp \frac{2\pi \mathbf{i}}{\eta_r + 1}$. We define the multiset $\mathrm{E}(L_1)$ which consists of natural numbers η_i, $i = 1, \ldots, r$ such that ϵ^{η_i}'s is an eigenvalue of the action of w on $\mathfrak{h}'$. We call $\mathrm{E}(L_1)$ the exponents of the nilpotent element L_1. When L_1 is a regular nilpotent element, $\mathrm{E}(L_1)$ equals the exponents $\mathrm{E}(\mathfrak{g})$ of the Lie algebra [37]. In Table 1, we list elements of $\mathrm{E}(L_1)$ in the second column. We calculated them by combining the results of [12,31] and [46]. Note that $\mathrm{E}(\mathfrak{g})$ is listed in Table 1 as $\mathrm{E}(L_1)$ when L_1 is of type $Z_r(a_0)$. We denote throughout this article, the elements of $\mathrm{E}(L_1)$ by η_i and elements of $\mathrm{E}(\mathfrak{g})$ by ν_i and we assume they are given in a non-decreasing order, i.e.,

$$\eta_1 \leq \eta_2 \leq \ldots \leq \eta_r \text{ and } \nu_1 \leq \nu_2 \leq \ldots \leq \nu_r. \tag{3.3}$$

The following lemma summarizes an important relation between $\mathrm{E}(\mathfrak{g})$ and $\mathrm{E}(L_1)$.

Lemma 3.1 *For* $i = 1, \ldots, r$,

$$\eta_i + \eta_{r-i+1} = \eta_r + 1. \tag{3.4}$$

Moreover, there exists a unique non-negative integer μ_i *such that* $\nu_i - \mu_i(\eta_r + 1)$ *belongs to* $\mathrm{E}(L_1)$. *Furthermore, the multiset formed by the numbers* $\nu_i - \mu_i(\eta_r + 1)$ *equals the multiset* $\mathrm{E}(L_1)$. *In addition, the number of* μ_i'*s which are zero equals* $r - s$.

Proof The proof is obtained by examining the multisets $\mathrm{E}(L_1)$ and $\mathrm{E}(\mathfrak{g})$ for each nilpotent orbit listed in Table 1. □

Example 3.2 In case L_1 is of type $E_7(a_5)$. Then, $\eta_7 = 5$ and the values of u_i'*s* are given in the following table The last row is just the elements of $\mathrm{E}(L_1)$ (not in order).

i	1	2	3	4	5	6	7
ν_i	1	5	7	9	11	13	17
μ_i	0	0	1	1	1	2	2
$\nu_i - \mu_i(\eta_7 + 1)$	1	5	1	3	5	1	5

We keep the notations $\mu_i, i = 1, \ldots, r$ for the non-negative numbers introduced in the last lemma. Many formulas below depend on these numbers. We list them in the fourth column of Table 1 using conventional notation for repetitions. For example, we write $[0^2, 1^3, 2^2]$ instead of [0, 0, 1, 1, 1, 2, 2].

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_r$ be a basis of $\mathfrak{h}'$ of eigenvectors of w such that $w(\Lambda_i) = \epsilon^{\eta_i}\Lambda_i$. Then, Λ_i has the form:

$$\Lambda_i = L_i + K_i; \quad L_i \in \mathfrak{g}_{\eta_i}, \ K_i \in \mathfrak{g}_{\eta_i-(\eta_r+1)}, \quad L_i \neq 0 \neq K_i, \quad i = 1, \ldots, r. \tag{3.5}$$

We normalized the invariant nondegenerate bilinear form $\langle . | . \rangle$ on $\mathfrak{g}$ such that $\langle L_1 | f \rangle = 1$. Then, the following lemma is valid.

Table 1 Exponents and weights of distinguished nilpotent elements of semisimple type

	W(L_1)		
$Z_r(a_s)$	$\eta_1 \leq \ldots \leq \eta_r$	$\eta_{r+1} \leq \ldots \leq \eta_n$	$[\mu_1, \ldots, \mu_r]$
$A_r(a_0)$	$1, 2, \ldots, r$	–	$[0^r]$
$B_r(a_0)$	$1, 3, \ldots, 2r-1$	–	$[0^r]$
$B_{2m}(a_m)$	$1, 1, 3, 3, \ldots, 2m-1, 2m-1$	$1, 2, \cdots, m-1; m-1,$ $m; m, m+1, \ldots, 2m-2$	$[0^{m+1}, 1^{m-1}]$
$C_r(a_0)$	$1, 3, \ldots, 2r-1$	–	$[0^r]$
$D_r(a_0)$	$1, 3, \ldots, r-1; r-1, r-3, \ldots, 2r-3$	–	$[0^r]$
$D_{2m}(a_{m-1})$	$1, 1, 3, 3, \ldots, 2m-1, 2m-1$	$1, 2, \cdots, 2m-2$	$[0^{m+1}, 1^{m-1}]$
$E_6(a_0)$	$1, 4, 5, 7, 8, 11$	–	$[0^n]$
$E_6(a_1)$	$1, 2, 4, 5, 7, 8$	$3, 5$	$[0^{n-1}, 1]$
$E_6(a_3)$	$1, 1, 2, 4, 5, 5$	$1, 2, 2, 3, 3, 4$	$[0^3, 1^3]$
$E_7(a_0)$	$1, 5, 7, 9, 11, 13, 17$	–	$[0^7]$
$E_7(a_1)$	$1, 3, 5, 7, 9, 11, 13$	$5, 8$	$[0^6, 1]$
$E_7(a_5)$	$1, 1, 1, 3, 5, 5, 5$	$1, 1, 1, 2, 2, 2, 2,$ $3, 3, 3, 3, 4, 4, 4$	$[0^2, 1^3, 2^2]$
$E_8(a_0)$	$1, 7, 11, 13, 17, 19, 23, 29$	–	$[0^8]$
$E_8(a_1)$	$1, 5, 7, 11, 13, 17, 19, 23$	$9, 14$	$[0^7, 1]$
$E_8(a_2)$	$1, 3, 7, 9, 11, 13, 17, 19$	$5, 8, 11, 14$	$[0^6, 1^2]$
$E_8(a_4)$	$1, 2, 4, 7, 8, 11, 13, 14$	$3, 5, 5, 7, 7, 9, 9, 11$	$[0^5, 1^3]$
$E_8(a_5)$	$1, 1, 5, 5, 7, 7, 11, 11$	$1, 2, 3, 4, 5, 5, 6, 6, 7, 8, 9, 10$	$[0^3, 1^4, 2]$
$E_8(a_6)$	$1, 1, 3, 3, 7, 7, 9, 9$	$1, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 8$	$[0^2, 1^4, 2^2]$

Table 1 continued

	W(L_1)		
$E_8(a_7)$	1, 1, 1, 1, 5, 5, 5, 5	1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4	$[0, 1^2, 2^2, 3^2, 4]$
$F_4(a_0)$	1,5,7,11	–	$[0^4]$
$F_4(a_1)$	1,3,5,7	2,5	$[0^3, 1]$
$F_4(a_2)$	1,1,5,5	1,2,3,4	$[0^2, 1^2]$
$F_4(a_3)$	1,1,3,3	1,1,1,1,2,2,2,2	$[0, 1^2, 2]$
$G_2(a_0)$	1, 5	–	$[0^2]$
	E(L_1)	$\overline{\text{E}}(L_1)$	

Lemma 3.3 *The matrix* $T_{ij} := \langle \Lambda_i | \Lambda_j \rangle$ *is nondegenerate and antidiagonal with respect to* $\mathrm{E}(L_1)$*, i.e.,*

$$T_{ij} = 0, \text{ if } \eta_i + \eta_j \neq \eta_r + 1.$$

Moreover, the elements $\Lambda_i,\ i > 1$ *can be normalized such that*

$$\langle \Lambda_i | \Lambda_j \rangle = (\eta_r + 1)\delta_{i+j,r+1}. \tag{3.6}$$

Proof The first part follows from the fact that the restriction of $\langle .|. \rangle$ to a Cartan subalgebra is nondegenerate. Therefore, for any element Λ_i there exists an element Λ_j such that $\langle \Lambda_i | \Lambda_j \rangle \neq 0$. But for the Weyl group element w defined in (3.2), we have the equality

$$\langle \Lambda_i | \Lambda_j \rangle = \langle w\Lambda_i | w\Lambda_j \rangle = \exp \frac{2(\eta_i + \eta_j)\pi \mathbf{i}}{\eta_r + 1} \langle \Lambda_i | \Lambda_j \rangle$$

which forces $\eta_i + \eta_j = \eta_r + 1$ in case $\langle \Lambda_i | \Lambda_j \rangle \neq 0$. For the second part of the lemma, recursively, we can define a change of basis with linear combination upon the elements Λ_i which have the same eigenvalue such that the matrix T_{ij} transform to the anti-diagonal form: $(\eta_r + 1)\delta_{i+j,r+1}$.

□

We assume from now on that the basis Λ_i of $\mathfrak{h}'$ is normalized and satisfies the hypothesis of the previous lemma. Then, we get the following identities.

Corollary 3.4

$$\langle L_i | K_j \rangle = \eta_j \delta_{i+j,r+1}, \quad i, j = 1, \ldots, r. \tag{3.7}$$

Proof Recall that

$$\Lambda_i = L_i + K_i; \quad L_i \in \mathfrak{g}_{\eta_i}, \ K_i \in \mathfrak{g}_{\eta_i - (\eta_r + 1)}. \tag{3.8}$$

Using the relation $0 = [\Lambda_i, \Lambda_j] = [L_i, K_j] + [K_i, L_j]$ with the invariant bilinear form yields

$$0 = \langle h | [L_i, K_j] + [K_i, L_j] \rangle = (\eta_i)\langle L_i | K_j \rangle + (\eta_i - (\eta_r + 1))\langle K_i | L_j \rangle. \tag{3.9}$$

This equation with the normalization $\langle \Lambda_i | \Lambda_j \rangle = \langle L_i | K_j \rangle + \langle K_i | L_j \rangle = (\eta_r + 1)\delta_{i+j,r+1}$ leads to the required identity. □

Corollary 3.5

$$\langle [K_1, L_j] | \mathrm{ad}_f L_i \rangle = \eta_i \eta_j \delta_{i+j,r+1}, \quad i, j = 1, \ldots, r.$$

Proof The identity $[\Lambda_1, \Lambda_j] = 0$ leads to $[L_1, K_j] = -[K_1, L_j]$. Then,

$$\begin{aligned}\langle [K_1, L_j] | \mathrm{ad}_f L_i \rangle &= -\langle [L_1, K_j] | [f, L_i] \rangle = \langle K_j | [L_1, [f, L_i]] \rangle = \langle K_j | [L_i, [f, L_1]] \rangle \\ &= -\langle K_j | [L_i, h] \rangle = \eta_i \langle K_j | L_i \rangle = \eta_i \eta_j \delta_{i+j,r+1}. \end{aligned} \tag{3.10}$$

□

The commutators $[\Lambda_i, \Lambda_j] = 0$ imply that the set $\{L_1, \ldots, L_r\}$ generates a commutative subalgebra of $\mathfrak{g}^{L_1}$. We consider the restriction of the adjoint representation to the sl_2-subalgebra $\mathcal{A}$ generated by $\{L_1, h, f\}$. Then, the vectors L_i are maximal weight vectors of irreducible $\mathcal{A}$-submodules $\mathcal{V}_i$ of dimension $2\eta_i + 1$. We set $n = \dim \mathfrak{g}^{L_1}$, and we fix the following decomposition of $\mathfrak{g}$ into irreducible $\mathcal{A}$-submodules

$$\mathfrak{g} = \bigoplus_{j=1}^{n} \mathcal{V}_j, \quad \dim \mathcal{V}_j = 2\eta_j + 1, \ L_j \in \mathcal{V}_j, \mathrm{ad}_{L_1} L_j = 0, \ \mathrm{ad}_h L_j = \eta_j L_j. \tag{3.11}$$

Note that, for convenience, we extend the notation L_j to cover all maximal eigenvectors, i.e., L_j's form a basis for $\mathfrak{g}^{L_1}$. The numbers $\eta_1, \ldots, \eta_n$ are given in Table 1 as the collection of the numbers in the second and fourth columns. We refer to them as the weights of the nilpotent element L_1. We could not find them in the literature, and we had to calculate them explicitly. See [21] for a procedure to find the weights of a distinguished nilpotent element and the calculation for the nilpotent element of type $D_{2m}(a_{m-1})$. After calculating the weights, we observe the following:

Corollary 3.6 $n = r + 2\sum \mu_i$.

Let $\overline{\mathrm{E}}(L_1)$ denotes the multiset consisting of the numbers η_i, $i = r+1, \ldots, n$ and assume they are given in non-decreasing order, i.e.,

$$\eta_{r+1} \leq \eta_{r+1} \leq \ldots \leq \eta_n.$$

Then, from Table 1, we get

Corollary 3.7 $\eta_{r+i} + \eta_{n-i+1} = \eta_r$ *for* $i = 1, \ldots, n-r$.

We use the fact that $\mathfrak{g}^f$ is the dual of $\mathfrak{g}^{L_1}$ under $\langle . | . \rangle$ [48] to fix a basis γ_i of $\mathfrak{g}^f$ such that

$$\langle \gamma_i | L_j \rangle = \delta_{ij}, \ i = 1, \ldots, n. \tag{3.12}$$

Then, $\mathrm{ad}_h \gamma_i = -\eta_i \gamma_i$. Let us introduce the following basis for $\bigoplus_{i\leq 0} \mathfrak{g}_i$

$$\gamma_i, \mathrm{ad}_{L_1} \gamma_i, \ldots, \frac{1}{\eta_i!} \mathrm{ad}_{L_1}^{\eta_i} \gamma_i, \quad i := 1, \ldots, n, \tag{3.13}$$

and similarly a basis for $\bigoplus_{i\geq 0}\mathfrak{g}_i$

$$L_i, \mathrm{ad}_f L_i, \ldots, \mathrm{ad}_f^{\eta_i} L_i, \quad i := 1, \ldots, n. \tag{3.14}$$

Lemma 3.8

$$\langle \frac{1}{I!}\mathrm{ad}_{L_1}^I \gamma_i | ad_f^J L_j\rangle = (-1)^I \binom{\eta_i}{I}\delta_{ij}\delta^{IJ}; \ I = 0, 1, \ldots, \eta_i; \ J = 0, 1, \ldots, \eta_j. \tag{3.15}$$

Proof For $I = J = 1$, we get

$$\langle \mathrm{ad}_{L_1}\gamma_i | ad_f L_j\rangle = -\langle \gamma_i | ad_{L_1}\mathrm{ad}_f L_j\rangle = \langle \gamma_i | [L_j, h]\rangle = -\eta_i\delta_{ij}. \tag{3.16}$$

Hence, by induction for $I > 1$,

$$\begin{aligned}\langle \frac{1}{I!}\mathrm{ad}_{L_1}^I \gamma_i | ad_f^I L_j\rangle &= \langle \frac{1}{I!}\mathrm{ad}_{L_1}^{I-1}\gamma_i | [ad_f^{I-1}L_j, h]\rangle \\ &= -\frac{\eta_j - I + 1}{I}\langle \frac{1}{(I-1)!}\mathrm{ad}_{L_1}^{I-1}\gamma_i | ad_f^{I-1}L_j\rangle = (-1)^I\binom{\eta_i}{I}\delta_{ij}.\end{aligned} \tag{3.17}$$

Suppose $I > J$. Then, we can recursively equate the value $\langle \mathrm{ad}_{L_1}^I\gamma_i | ad_f^J L_j\rangle$ to constant multiplication of the zero valued $\langle \mathrm{ad}_{L_1}^{I-J-1}\gamma_i | ad_f L_j\rangle$. □

Corollary 3.9 $\gamma_r = K_1$.

Proof Recall that $K_1 \in \mathfrak{g}_{-\eta_r}$. It follows from the Dynkin grading that $K_1 \in \mathfrak{g}^f$. Then, for $j \leq r$, it follows from Corollary 3.4 that $\langle K_1 | L_j\rangle = \delta_{jr}$. While for $j > r$, we get from Dynkin grading and the fact that $\eta_j < \eta_r$ that $\langle K_1 | L_j\rangle = 0$. Thus, by construction $\gamma_r = K_1$. □

4 The space of common equilibrium points

In this section, we fix Slodowy slice Q as a transverse subspace to the orbit space of L_1. We discuss the integrability of the transverse Poisson structure at L_1 of Lie-Poisson structure on $\mathfrak{g}$ which leads to the definition of the space of common equilibrium points N. Then, we will introduce special coordinates on Q and give alternative definitions for N.

4.1 Background

Let us define the gradient $\nabla H : \mathfrak{g} \to \mathfrak{g}$ for a function H on $\mathfrak{g}$ by

$$\frac{d}{dt}_{|t=0} H(g + tv) = \langle \nabla H(g)|v\rangle, \ \forall g, v \in \mathfrak{g}. \tag{4.1}$$

We fix the following standard compatible Poisson structures on $\mathfrak{g}$ which consists of the frozen Lie-Poisson structure $B_1^{\mathfrak{g}}$ and the standard Lie-Poisson structure $B_2^{\mathfrak{g}}$. We denote their Poisson brackets by $\{.,.\}_1^{\mathfrak{g}}$ and $\{.,.\}_2^{\mathfrak{g}}$, respectively. For any two functions H and G on $\mathfrak{g}$, and $v \in T_g^*\mathfrak{g} \cong \mathfrak{g}$, we set

$$\begin{aligned} \{H, G\}_1^{\mathfrak{g}}(g) &= \langle [\nabla G(g), \nabla H(g)]|K_1\rangle; \quad B_1^{\mathfrak{g}}(v) = [K_1, v], \\ \{H, G\}_2^{\mathfrak{g}}(g) &= \langle [\nabla G(g), \nabla H(g)]|g\rangle; \quad B_2^{\mathfrak{g}}(v) = [g, v]. \end{aligned} \tag{4.2}$$

We use $B_i^{\mathfrak{g}}$, $i = 1, 2$ to refer to both the Poisson structures (tensors) and the corresponding Poisson brackets. Then, the Hamiltonian vector field χ_H of a function H under $B_2^{\mathfrak{g}}$ at a point $g \in \mathfrak{g}$ is defined by

$$\chi_H(g) := -\mathrm{ad}_{\nabla H(g)} g = [g, \nabla H(g)]. \tag{4.3}$$

It is known that [1] the symplectic leaf through $g \in \mathfrak{g}$ coincides with the adjoint orbit $\mathcal{O}_g$ and invariant polynomials under the adjoint group action are global Casimirs of $B_2^{\mathfrak{g}}$.

Using Chevalley's theorem, we fix a complete system of homogeneous generators $P_1, \ldots, P_r$ of the ring of invariant polynomials under the adjoint group action. We assume that degree P_i equals $\nu_i + 1$. These generators give a complete set of global Casimir functions of $B_2^{\mathfrak{g}}$. In particular,

$$\nabla P_i(g) \in \mathfrak{g}^g, \ \forall g \in \mathfrak{g}, \ i = 1, \ldots, r. \tag{4.4}$$

Moreover, the functions $P_i(g + \lambda K_1)$ form a complete set of independent global Casimirs of the Poisson pencil $B_\lambda^{\mathfrak{g}} := B_2^{\mathfrak{g}} + \lambda B_1^{\mathfrak{g}}$ for any $\lambda \in \mathbb{C}$ [3].

Define Slodowy slice Q to be the affine space

$$Q := L_1 + \mathfrak{g}^f. \tag{4.5}$$

Then, Q is a transverse subspace to the symplectic leaf $\mathcal{O}_{L_1}$ of $B_2^{\mathfrak{g}}$ through L_1. The following proposition is a special version of Theorem 5.1 stated below.

Proposition 4.1 *[19] The space Q inherits compatible Poisson structures B_1^Q, B_2^Q from $B_1^{\mathfrak{g}}$, $B_2^{\mathfrak{g}}$, respectively. Moreover, B_2^Q is the transverse Poisson structure at L_1 of Lie-Poisson structure $B_2^{\mathfrak{g}}$. Furthermore, for any $\lambda \in \mathbb{C}$, $B_\lambda^Q := B_2^Q + \lambda B_1^Q$ can be obtained from $B_\lambda^{\mathfrak{g}}$ using Dirac reduction.*

Let $\overline{P}_i^0$ denotes the restriction of the invariant polynomial P_i to Q. Since B_λ^Q can be obtained by Dirac reduction, we have the following standard consequence.

Proposition 4.2 *For $\lambda \in \mathbb{C}$, $\overline{P}_1^0(q+\lambda K_1), \ldots, \overline{P}_r^0(q+\lambda K_1)$ form a complete set of independent Casimirs of the Poisson pencil B_λ^Q.*

Following the argument shift method ([2,40]), we consider the family of functions

$$\mathbf{F} := \cup_{\lambda \in \overline{\mathbb{C}}} \{P'_\lambda : P'_\lambda \text{ is a Casimir of } B_\lambda^Q\}. \tag{4.6}$$

This family commutes pairwise with respect to both Poisson brackets ([2], section 1.3). Let us consider the coefficient $\overline{P}_i^j$ of Taylor expansions

$$\overline{P}_i^0(q+\lambda K_1) = \sum_{j \geq 0} \lambda^j \overline{P}_i^j(q),\ q \in Q. \tag{4.7}$$

Then, the functions $\overline{P}_i^j$ functionally generate $\mathbf{F}$. Moreover, $\overline{P}_i^0$ are Casimirs of B_2^Q, the highest non-constant term $\overline{P}_i^{\varrho_i}$ are Casimirs of B_1^Q, and all functions $\overline{P}_i^j$ are in involution with respect to both Poisson structures. In Proposition 4.8, we will show that $\varrho_i = \mu_i$.

The main propose for applying argument shift method is to show that $\mathbf{F}$ contains enough number of functionally independent functions in order to get a completely integrable system for B_2^Q. We explored this problem in [20] for arbitrary nilpotent elements in $\mathfrak{g}$, and we proved the following theorem

Theorem 4.3 [20] *Suppose L_1 belongs to one of the following distinguished nilpotent orbits of semisimple type: $D_{2m}(a_{m-1})$, $B_{2m}(a_m)$, $F_4(a_2)$, $E_6(a_3)$, $E_8(a_2)$ and $E_8(a_4)$. Then, the set of all functions $\overline{P}_i^j$ result from the expansion (4.7) are functionally independent and form a polynomial completely integrable system under B_2^Q.*

In what follows, a point $q \in Q$ is generic if rank $B_2^Q(q) = n - r$. From [3] we get the following theorem

Theorem 4.4 [3] *The family $\mathbf{F}$ is complete (contains a completely integrable system) if and only if, at a generic point $q \in Q$,* rank $B_\lambda^Q(q) =$ rank $B_\zeta^Q(q)$ *for all $\lambda, \zeta \in \overline{\mathbb{C}}$.*

We are concerned about the space of common equilibrium points N of the family $\mathbf{F}$ which is defined by

$$N := \{q \in Q : B_\lambda^Q(dP')(q) = 0,\ \forall P' \in \mathbf{F}, \lambda \in \overline{\mathbb{C}}\}. \tag{4.8}$$

The following theorem gives an equivalent definition.

Theorem 4.5 [3] *A point $q \in Q$ is a common equilibrium point if and only if* ker $B_\lambda^Q(q) =$ ker $B_\zeta^Q(q)$ *for all generic $\lambda, \zeta \in \overline{\mathbb{C}}$.*

Equivalently, for q to be in N, it is sufficient to require that the kernel of just two generic brackets at q coincides, i.e., ker $B_\lambda^Q(q) =$ ker $B_\zeta^Q(q)$ with $\lambda \neq \zeta$ [3].

4.2 Special coordinates

Let us consider the adjoint quotient map

$$\Psi : \mathfrak{g} \to \mathbb{C}^r, \quad \Psi(g) = (P_1(g), \ldots, P_r(g)). \tag{4.9}$$

Kostant proved in [38] that the rank of Ψ at g equals r if and only if g is a regular element in $\mathfrak{g}$ and it is known that the set of regular element is open and dense in $\mathfrak{g}$. Later, Slodowy proved that the rank of Ψ is $r-1$ at subregular nilpotent elements [45]. Finally, Richarson [42] obtained the ranks of Ψ at distinguished nilpotent elements except for the nilpotent elements of type $E_8(a_2)$. Results in this section are built on and inspired by the articles mentioned in this paragraph.

We fix a basis $e_0, e_1, e_2, \ldots$ for $\mathfrak{g}$ such that

1. The elements $e_0, e_1, \ldots, e_{n+r}$ are $K_r, L_1, L_2, \ldots, L_n, K_1, K_2, \ldots, K_{r-1}$, respectively. Recall that $\Lambda_i = L_i + K_i$ are normalized according to lemma 3.3.
2. $\langle e_i | \Lambda_1 \rangle \neq 0$ if and only if $i = 0$ or $i = r$.

It is not hard to show that such a basis exists. Let us define on $\mathfrak{g}$ the linear coordinates

$$z^i(g) = \langle e_i | g \rangle, \quad i = 0, 1, 2, \ldots. \tag{4.10}$$

Then, by definition, $\nabla H = \sum \frac{\partial H}{\partial z^i} e_i$ for any function H on $\mathfrak{g}$. Note that the rank of Ψ at g equals the dimension of the vector space generated by $\nabla P_i(g)$. In particular, since Λ_1 is regular, the gradients $\nabla P_i(\Lambda_1)$ are linearly independent and form a basis for the opposite Cartan subalgebra $\mathfrak{h}'$. We use these remarks in the following lemma.

Lemma 4.6 *The matrix with entries $\frac{\partial P_i}{\partial z^j}(\Lambda_1)$, $i, j = 1, \ldots, r$, is non-degenerate. Moreover, P_i have the following form:*

$$P_i = R_i^1 + R_i^2 + R_i^3 \tag{4.11}$$

where

$$R_i^1 = \sum_{a(\eta_r+1)=\nu_i-\eta_r} \theta_{i,a} (z^r)^{a+1} (z^0)^{\nu_i - a},$$
$$R_i^2 = \sum_{a=0}^{\nu_i-1} \sum_{j=1}^{r-1} c_{i,j,a} (z^r)^a (z^0)^{\nu_i - a} (z^j + z^{j+n}), \quad \frac{\partial R_i^3}{\partial z^k}(\Lambda_1) = 0, \forall k. \tag{4.12}$$

Here, $c_{i,j,a}$ and $\theta_{i,a}$ are complex numbers.

Proof Since $\nabla P_i(\Lambda_1) \in \mathfrak{g}^{\Lambda_1} = \mathfrak{h}'$ and $\mathfrak{h}'$ has basis $\Lambda_i = L_i + K_i$, we get

$$\nabla P_i(\Lambda_1) = \sum_{j=1}^{r} C_{i,j}\Lambda_j$$
$$= \sum_{j=1}^{r} C_{i,j}(L_j + K_j) = C_{i,r}(e_0 + e_r) + \sum_{j=1}^{r-1} C_{i,j}(e_j + e_{n+j}). \quad (4.13)$$

Hence,

$$C_{i,j} = \begin{cases} \frac{\partial P_i}{\partial z^j}(\Lambda_1) = \frac{\partial P_i}{\partial z^{j+n}}(\Lambda_1),\ 0 < j < r; \\ \frac{\partial P_i}{\partial z^r}(\Lambda_1) = \frac{\partial P_i}{\partial z^0}(\Lambda_1), \quad j = r; \end{cases} \quad (4.14)$$

and $\frac{\partial P_i}{\partial z^j}(\Lambda_1) = 0$ for other values of j. By definition of the coordinates and Corollary 3.4, $z^j(\Lambda_1)$ are all zero except $z^r(\Lambda_1) = 1$ and $z^0(\Lambda_1) = \eta_r$. For $0 < j < r$, imposing the condition $\frac{\partial P_i}{\partial z^j}(\Lambda_1) \neq 0$ and using the homogeneity of P_i, we find that $\frac{\partial P_i}{\partial z^j}$ must contain the polynomial

$$\sum_{a=0}^{\nu_i - 1} c_{i,j,a}(z^r)^a(z^0)^{\nu_i - a - 1}, \quad c_{i,j,a} \in \mathbb{C}. \quad (4.15)$$

This gives the formula for R_i^2. Note that $\frac{\partial R_i^2}{\partial z^r}(\Lambda_1) = 0$ since $z^j(\Lambda_1) = 0$ for $j \neq 0$ and $j \neq r$. Thus, for $\frac{\partial P_i}{\partial z^r}(\Lambda_1)$ to be nonzero, P_i must contain terms of the form $\Xi_{i,a} = (z^r)^{a+1}(z^0)^{\nu_i - a}$. But then a is constrained by the identity

$$\frac{\partial \Xi_{i,a}}{\partial z^r}(\Lambda_1) = (a+1)(\eta_r)^{\nu_i - a} = \frac{\partial \Xi_{i,a}}{\partial z^0}(\Lambda_1) = (\nu_i - a)(\eta_r)^{\nu_i - a - 1}. \quad (4.16)$$

This leads to the formula for R_i^1. The condition on R_i^3 is a direct consequence from our analysis. Finally, the non-degeneracy condition follows from the fact that the vectors $\nabla P_i(\Lambda_1)$ are a basis for $\mathfrak{h}'$. □

For Slodowy slice Q, we observe that $z^0(q) = \langle K_r | L_1 \rangle = \eta_r \neq 0$ for every $q \in Q$ and $(z^1, \ldots, z^n)$ define global coordinates on Q. The values of these coordinates at $\Lambda_1 \in Q$ are $z^i = \delta^{ir}$. We set degree z^i equals $\eta_i + 1$ and recall the following quasihomogeneity theorem due to Slodowy.

Theorem 4.7 ([45], section 2.5) *The restriction $\overline{P}_i^0$ of P_i to Q is quasi-homogeneous polynomial of degree $\nu_i + 1$.*

This theorem leads to the following refinement of the last lemma.

Proposition 4.8 *The restrictions* $\overline{P}_i^0$ *of the invariant polynomials* P_i *to* Q *in the coordinates* $(z^1,\dots,z^n)$ *take the form:*

$$\overline{P}_i^0(z^1,\dots,z^n)=\sum_{\nu_i-\eta_j=\mu_i(\eta_r+1)}\widetilde{c}_{i,j}(z^r)^{\mu_i}z^j+\overline{R}_i^3(z),\ \ \widetilde{c}_{i,j}\in\mathbb{C},\tag{4.17}$$

where $\frac{\overline{R}_i^3}{\partial z^k}(\Lambda_1)=0$ *for* $k=1,\dots,n$. *Moreover, the square matrix* $\frac{\partial\overline{P}_i^0}{\partial z^j}(\Lambda_1)$, $i,j=1,\dots,r$ *is nondegenerate.*

Proof The restriction $\overline{P}_i^0$ of P_i to Q is obtained by setting $z^0=\eta_r$ and $z^k=0$ for $k>n$ in the form (4.11). From the quasihomogeneity of $\overline{P}_i^0$ and lemma 4.6

$$\overline{P}_i^0(z^1,\dots,z^n)=\sum_{a=0}^{\nu_i-1}\sum_{\deg P_i-\deg z^j=a(\eta_r+1)}\widetilde{c}_{i,j,a}(z^r)^a z^j+\overline{R}_i^3(z),\ \ \widetilde{c}_{i,j,a}\in\mathbb{C}\tag{4.18}$$

where $\overline{R}_i^3$ is the restriction of R_i^3 to Q. The expressions given in (4.12) imply that $\frac{\partial\overline{R}_i^3}{\partial z^k}(\Lambda_1)=0$, $k=1,\dots,n$. Note that $\deg P_i-\deg z^j=\nu_i-\eta_j=a(\eta_r+1)$. Using the relation between the multisets $\mathrm{E}(\mathfrak{g})$ and $\mathrm{E}(L_1)$ observed in lemma 3.1, a can only equal μ_i and the values of η_j are uniquely determined and depends on i. On other words the constants $c_{i,j,a}$ in (4.11) are nonzero only if $a=\mu_i$. This gives the form (4.17). For the nondegeneracy condition, note that the only possible value for the index a in (4.11) is $a=\mu_i$ and so z^0 appear only with the power $\nu_i-\mu_i$. This implies that $\frac{\partial P_i}{\partial z^j}(\Lambda_1)=\frac{\partial\overline{P}_i^0}{\partial z^j}(\Lambda_1)$. Thus, the required matrix is nondegenerate. □

Now we give a proof for Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2 Writing $\overline{P}_i^0$ in the form (4.7) and using the last proposition, we get $\overline{P}_i^0(q+\lambda K_1)=\overline{P}_i^0(z^1+\lambda\delta_{1r},\dots,z^n+\lambda\delta_{nr})$ and $\varrho_i=\mu_i$. We observe that each $\partial_{z^r}^{\mu_i}\overline{P}_i^0$ is a constant multiple of $\overline{P}_i^{\mu_i}$. Hence, the functions $\partial_{z^r}^{\mu_i}\overline{P}_i^0$ are Casimirs of B_1^Q and are in involution with respect to B_2^Q . Furthermore, $\partial_{z^r}^{\mu_i}\overline{P}_i^0$ has the form

$$\partial_{z^r}^{\mu_i}\overline{P}_i^0=\sum_{\eta_i-\eta_j=\mu_i(\eta_r+1)}\overline{c}_{i,j}z^j+\partial_{z^r}^{\mu_i}\overline{R}_i^3(z),\ \overline{c}_{i,j}\in\mathbb{C},\tag{4.19}$$

where $\partial_{z_j}\partial_{z^r}^{\mu_i}\overline{R}_i^3$ equals 0 at the origin ($z^k=0,\forall k$). Thus,

$$\partial_{z^j}\partial_{z^r}^{\mu_i}\overline{P}_i^0(0)=\frac{1}{\mu_i!}\frac{\partial\overline{P}_i^0}{\partial z^j}(\Lambda_1),\ \ i,j=1,\dots,r.\tag{4.20}$$

We conclude, using proposition 4.8, that the matrix $\partial_{z^j}\partial_{z^r}^{\mu_i}\overline{P}_i^0$ is nondegenerate. Hence, $\partial_{z^r}^{\mu_i}\overline{P}_i^0$ can replace the coordinates z^i on Q for $i=1,\dots,r$ up to some permutation related to the repetition on $\mathrm{E}(L_1)$. Moreover, using simple linear elimination, we can get the required normalization $t^j = z^j +$ (non linear terms) where t^j is a Casimir of B_1^Q. From Theorem 4.10, it follows that $t^1, \dots, t^r$ form a complete set of Casimirs for B_1^Q. The fact that $t^1 = z^1$ follows from identifying t^1 with the Casimir function $\langle Q|Q\rangle$ and using $\langle \gamma_1|L_1\rangle = 1$. □

We fix the notations $(t^1, \dots, t^n)$ for the coordinates obtained in Theorem 1.2. Recall that $Z_r(a_s)$ denotes the type of L_1.

Corollary 4.9 *The functions $\overline{P}_1^0, \dots, \overline{P}_{r-s}^0$ are quasihomogeneous polynomials on $t^1, \dots, t^r$ only.*

Proof This follows from the fact that $\mu_i = 0$ for $i = 1, \dots, r-s$ and the construction of the coordinates $(t^1, \dots, t^r)$. □

4.3 Integrability and alternative definitions

We combine the theorems stated in Sect. 4.1 to get the following useful result.

Theorem 4.10 *The family* $\mathbf{F}$ *is complete for every distinguished nilpotent element of semisimple type. In particular,* rank $B_1^Q = n - r$ *and*

$$N = \{q \in Q : \ker B_1^Q(q) = \ker B_2^Q(q)\}. \tag{4.21}$$

Proof For regular, subregular and nilpotent elements stated in Theorem 4.3, the family $\mathbf{F}$ is complete [20]. Suppose L_1 belongs to the nilpotent orbit $E_7(a_5)$, $E_8(a_5)$, $E_8(a_6)$, $E_8(a_7)$ or $F_4(a_3)$. We will check that rank $B_\lambda^Q = n - r$ for every $\lambda \in \overline{\mathbb{C}}$ and use Theorem 4.4. It is not hard to show that rank $B_\lambda^Q = n - r$ for $\lambda \in \mathbb{C}$ [20]. We need to show that rank $B_1^Q = n - r$. We verify the equality by direct computations using proposition 5.8 given below. More precisely, we fixed arbitrary basis L_i for $\mathfrak{g}^f$ and K_1 such that $L_1 + K_1$ is regular semisimple. Then, we found that the rank of the matrix $\langle L_i|[K_1, L_j]\rangle$ equals $n - r$. The last statement follows from Theorem 4.5. □

Let us use the special coordinates on Q and denote the entries of the matrix of the reduced Poisson structures by

$$F_\alpha^{ij}(t) := \{t^i, t^j\}_\alpha^Q, \quad \alpha = 1, 2. \tag{4.22}$$

Then, we prove Theorem 1.3 stated in the introduction.

Proof of Theorem 1.3 The first definition (1.10) of N follows directly from the structure of the matrices of the Poisson brackets under the coordinates $(t^1, \dots, t^n)$. For the second definition (1.11), we observe that $dt^1, \dots, dt^r$ are a basis of $\ker B_1^Q$, while $d\overline{P}_1^0, \dots, d\overline{P}_r^0$ are basis for $\ker B_2^Q$. However, by construction $\overline{P}_1^0, \dots, \overline{P}_{r-s}^0$ are polynomials in $t^1, \dots, t^r$ only. Hence, the two kernels coincide exactly on the defined set.

Now we consider the restriction of the adjoint quotient map

$$\Psi^Q(t^1, t^2, \ldots, t^n) = (\overline{P}_1^0, \ldots, \overline{P}_r^0). \tag{4.23}$$

and let $\mathbf{J}\Psi^Q := \frac{\partial \overline{P}_i^0}{\partial t^j}$ denotes its Jacobian matrix. Then, N is defined by the set of points t where the lower-right $s \times (n-r)$ minor of $\mathbf{J}\Psi^Q$ is identically 0. From Corollary 4.9, the upper-right $(r-s) \times (n-r)$ minor also vanishes by Corollary 4.9. Since regular points of Ψ^Q are Zariski dense in Q, there exists open dense set $N_0 \subseteq N$ such that the left $r \times r$ minor of $\mathbf{J}\Psi^Q$ is nondegenerate. In particular, $\overline{P}_1^0, \ldots, \overline{P}_r^0$ are independent functions on N_0. Hence, $\overline{P}_1^0, \ldots, \overline{P}_r^0$ are a part of local coordinates and $\dim N_0 \geq r$. However, the second definition (1.11) of N with Corollary 4.9 implies that $\dim N \leq r$. Thus, $\dim N_0 = r$ and $(t^1, \ldots, t^r)$ acts as local coordinates around each point of N_0.

Recall that B_λ^Q, $\lambda \in \mathbb{C}$, is of rank $n-r$. Since N_0 consists of regular points, the lower-right $(n-r)\times(n-r)$ minor $F_\lambda^{\alpha\beta}$ of F_λ^{ij} is nondegenerate. Thus, Dirac reduction is well defined on N_0. However, applying Corollary 2.6, the reduced Poisson structure is zero as $t^1, \ldots, t^r$ are in involution with respect to the pencil B_λ^Q. □

5 Algebraic classical *W*-algebra

In this section, we summarize the construction of Drinfeld–Sokolov bihamiltonian structure associated with the nilpotent element L_1 and K_1. Then, we will apply Dirac reduction to get a local bihamiltonian structure admitting a dispersionless limit on the loop space $\mathcal{N} := \mathfrak{L}(N)$. This leads to an algebraic classical W-algebra on $\mathcal{N}$.

5.1 Drinfeld–Sokolov reduction

We consider the loop algebra $\mathfrak{L}(\mathfrak{g})$ and we extend the bilinear form $\langle .|.\rangle$ on $\mathfrak{g}$ to $\mathfrak{L}(\mathfrak{g})$ by setting

$$(g_1|g_2) = \int_{S^1} \langle g_1(x)|g_2(x)\rangle dx; \quad g_1, g_2 \in \mathfrak{L}(\mathfrak{g}). \tag{5.1}$$

We use $(.|.)$ to identify $\mathfrak{L}(\mathfrak{g})$ with $\mathfrak{L}(\mathfrak{g})^*$. We define the gradient $\delta\mathcal{F}(g)$ for a functional $\mathcal{F}$ on $\mathfrak{L}(\mathfrak{g})$ to be the unique element in $\mathfrak{L}(\mathfrak{g})$ satisfying

$$\frac{d}{d\theta}\mathcal{F}(g+\theta \mathrm{w}) \mid_{\theta=0} = \big(\delta\mathcal{F}(g)|\mathrm{w}\big) \quad \text{for all } \mathrm{w} \in \mathfrak{L}(\mathfrak{g}). \tag{5.2}$$

Then, we introduce standard compatible local Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$ on $\mathfrak{L}(\mathfrak{g})$ defined for any functionals $\mathcal{I}$ and $\mathcal{F}$ on $\mathfrak{L}(\mathfrak{g})$ by

$$\{\mathcal{F}, \mathcal{I}\}_1(g(x)) := \int_{S_1} \langle [\delta\mathcal{I}(g(x)), K_1] | \delta\mathcal{F}(g(x))\rangle dx,$$

$$\{\mathcal{F}, \mathcal{I}\}_2(g(x)) := \int_{S^1} \langle \partial_x \delta\mathcal{I}(g(x)) + [\delta\mathcal{I}(g(x)), g(x)] | \delta\mathcal{F}(g(x))\rangle dx. \quad (5.3)$$

We denote their Poisson structures by $\mathbb{B}_1$ and $\mathbb{B}_2$, respectively. We mention that $\mathbb{B}_2$ can be interpreted as the restriction to $\mathfrak{L}(\mathfrak{g})$ of Lie-Poisson structure on the untwisted affine Kac–Moody algebra associated with $\mathfrak{g}$. In particular, if we expand these Poisson brackets as in (2.16), the leading term $\{.,.\}_1^{[-1]}$ is the frozen Lie-Poisson structure $B_1^{\mathfrak{g}}$ and $\{.,.\}_2^{[-1]}$ defines the Lie-Poisson structure $B_2^{\mathfrak{g}}$ on $\mathfrak{g}$. Moreover, it is easy to show that these Poisson structures form an exact Poisson pencil with Liouville vector field ∂_{z^r} in the coordinates defined by (4.10), i.e.,

$$\{.,.\}_1 = \mathfrak{L}_{\partial_{z^r}}\{.,.\}_2, \quad \mathfrak{L}_{\partial_{z^r}}\{.,.\}_1 = 0. \quad (5.4)$$

Let us define the affine loop space

$$\mathcal{Q} := L_1 + \mathfrak{L}(\mathfrak{g}^f). \quad (5.5)$$

Then, Slodowy slice Q is identified with the subspace of constant loops of $\mathcal{Q}$.

Theorem 5.1 [19] *The space $\mathcal{Q}$ inherits compatible local Poisson structures $\mathbb{B}_2^{\mathcal{Q}}$ and $\mathbb{B}_1^{\mathcal{Q}}$ from $\mathbb{B}_2$ and $\mathbb{B}_1$, respectively. They can be obtained equivalently by using the bihamiltonian reduction with Poisson tensor procedure, Dirac reduction and the generalized Drinfeld–Sokolov reduction. Moreover, the leading terms of the bihamiltonian structure on $\mathcal{Q}$ can be identified with the bihamiltonian structure B_2^Q and B_1^Q on Q.*

Details on bihamiltonian reduction can be found in [7]. Drinfeld–Sokolov reduction is initiated and applied for regular nilpotent elements in [22]. Generalizations to other nilpotent elements is obtained in [4,33] (see also [14]). The relation between Drinfeld–Sokolov reduction and bihamiltonian reduction in the case of regular nilpotent elements is treated in [8] and [44]. In [8], the Poisson tensor procedure is also initiated (also called the method of transverse subspace in [41]). The relation between Drinfeld–Sokolov reduction and Dirac reduction is also proved in [33]. See [11] and references therein, for more recent development and tools used to study Drinfeld–Sokolov reduction.

We let $\{.,.\}_1^{\mathcal{Q}}$ and $\{.,.\}_2^{\mathcal{Q}}$ denote the Poisson brackets defined by $\mathbb{B}_1^{\mathcal{Q}}$ and $\mathbb{B}_2^{\mathcal{Q}}$, respectively.

In what follows, we review Drinfeld–Sokolov reduction. We identify $\mathfrak{L}(\mathfrak{g})$ with the space of operators of the form $\partial_x + g$, $g \in \mathfrak{L}(\mathfrak{g})$, and $\mathcal{Q}$ with the subspace of operators of the form $\partial_x + q + L_1$, $q \in \mathfrak{L}(\mathfrak{g}^f)$. Let $\mathcal{B}$ denote the subspace of operators of the form:

$$\mathcal{L} = \partial_x + b + L_1 \qquad \text{where } b \in \mathfrak{L}(\mathfrak{b}), \quad \mathfrak{b} := \bigoplus_{i \leq 0} \mathfrak{g}_i. \quad (5.6)$$

There is a natural action of the adjoint group of $\mathfrak{L}(\mathfrak{n})$, $\mathfrak{n} := \bigoplus_{i<0} \mathfrak{g}_i$, on $\mathcal{B}$ defined by

$$(\mathrm{w}, \mathcal{L}) \to (\exp \operatorname{ad} \mathrm{w})\, \mathcal{L} \text{ for all } \mathrm{w} \in \mathfrak{L}(\mathfrak{n}) \text{ and } \mathcal{L} \in \mathcal{B}. \quad (5.7)$$

Moreover, for any operator $\mathcal{L} \in \mathcal{B}$ there is a unique element $\mathrm{w} \in \mathfrak{L}(\mathfrak{n})$ such that

$$\mathcal{L}^c := \partial_x + q + L_1 = (\exp \mathrm{ad}\, \mathrm{w})\mathcal{L} \tag{5.8}$$

where $q \in \mathfrak{L}(\mathfrak{g}^f)$. Hence, q and w are differential polynomials in the coordinates of b. The entries of q give a set of generators of the ring R of differential polynomials invariant under the action (5.7) . More precisely, if we write

$$b = \sum_{i=1}^{n}\sum_{I=0}^{\eta_i} b_I^i(x)\frac{1}{I!}\mathrm{ad}_{L_1}^I\gamma_i, \quad q = \sum_{i=1}^{n} z^i(x)\gamma_i \ \text{ and } \mathrm{w} = \sum_{i=1}^{n}\sum_{I=1}^{\eta_i} \mathrm{w}_I^i(x)\frac{1}{I!}\mathrm{ad}_{L_1}^I\gamma_i, \tag{5.9}$$

then equation (5.8) reads

$$q - [\mathrm{w}, L_1] = b - \mathrm{w}_x + [\mathrm{w}, b] + \sum_{i>0}\frac{1}{i+1!}\mathrm{ad}_{\mathrm{w}}^i(-\mathrm{w}_x + [\mathrm{w}, b] + [\mathrm{w}, L_1]). \tag{5.10}$$

Using Dynkin grading and the fact that $\mathfrak{g}^f \oplus [\mathfrak{n}, L_1] = \mathfrak{b}$, we get recursive equations defining the coordinates of q as differential polynomials on the coordinates of b. Moreover, if we assign degree $\partial_x^k b_J^i$ equals $k+\eta_i-J+1$, then $z^i(x)$ is a quasihomogeneous polynomial of degree η_i+1. The set of functionals $\mathcal{R}$ on $\mathcal{Q}$ are the functionals on $\mathcal{B}$ with densities belonging to the ring R. It follows that $\mathcal{R}$ is closed Poisson subalgebra with respect to the Poisson brackets $\{.,.\}_2$ and $\{.,.\}_1$. Thus, the reduced Poisson pencil $\{.,.\}_\lambda^{\mathcal{Q}} := \{.,.\}_2^{\mathcal{Q}} + \lambda\{.,.\}_1^{\mathcal{Q}}$ can be obtained by apply the Leibniz rule

$$\{z^u(x), z^v(y)\}_\lambda^{\mathcal{Q}} := \frac{\partial z^u(x)}{\partial (b_I^i)^{(k)}}\partial_x^k\Big(\frac{\partial z^v(y)}{\partial (b_J^j)^{(l)}}\partial_y^n\big(\{b_I^i(x), b_J^j(y)\}_\lambda\big)\Big) \tag{5.11}$$

where

$$\begin{aligned}\{b_I^i(x), b_J^j(y)\}_\lambda &= \frac{1}{\Theta_I^i}\frac{1}{\Theta_J^j}\Big(\langle \mathrm{ad}_f^J L_j|\mathrm{ad}_f^I L_i\rangle\partial_x + \langle b(x)+\lambda K_1|[\mathrm{ad}_f^J L_j, \mathrm{ad}_f^I L_i]\rangle\Big)\delta(x-y)\\ &= \frac{1}{\Theta_I^i}\frac{1}{\Theta_J^j}\Big(\langle \mathrm{ad}_f^J L_j|\mathrm{ad}_f^I L_i\rangle\partial_x + \langle \mathrm{ad}_f^I L_i|[b+\lambda K_1, \mathrm{ad}_f^J L_j]\rangle\Big)\delta(x-y)\end{aligned} \tag{5.12}$$

and $\Theta_J^j := (-1)^J\binom{\eta_j}{J}$. We will use these formulas in the next sections to analyze the leading terms of $\mathbb{B}_2^{\mathcal{Q}}$ and $\mathbb{B}_1^{\mathcal{Q}}$.

We end this section by finding the linear terms of the generators of the invariant ring R.

Proposition 5.2 *The linear terms of each $z^i(x)$ equal*

$$\sum_{I=0}^{\eta_i} \frac{(-1)^I}{I!} \partial_x^I b_I^i. \tag{5.13}$$

In particular, $z^r(x)$ is the only generator of R depends on $b_0^r(x)$ and this dependence is linear. Moreover, all $z^i(x)$ do not depend on derivatives of $b_0^r(x)$.

Proof The second part of the statement follows from the quasihomogeneity of the generators $z^i(x)$ of R. To find linear terms of each z^i, we introduce spectral parameter ϵ and set $\mathcal{L}(\epsilon) = \partial_x + \epsilon b + L_1$. Let $\mathrm{w}(\epsilon)$ and $\mathcal{L}^c(\epsilon)$ be the corresponding operators. Then, $\mathcal{L}(0) = \partial_x + L_1$, $\mathrm{w}(0) = 0$ and $\mathcal{L}^c(0) = \mathcal{L}(0)$. Therefore, differentiating the relation

$$\mathcal{L}^c(\epsilon) = \mathcal{L}(\epsilon) + [\mathfrak{n}(\epsilon), \mathcal{L}(\epsilon)] + \frac{1}{2}[\mathfrak{n}(\epsilon), [\mathfrak{n}(\epsilon), \mathcal{L}(\epsilon)]] + \dots \tag{5.14}$$

with respect to ϵ and evaluating at $\epsilon = 0$ we get

$$\begin{aligned} q'(0) &= b + [\mathrm{w}(0), \mathcal{L}'(0)] + [\mathrm{w}'(0), \partial_x + L_1] \\ &= b + [\mathrm{w}'(0), \partial_x + L_1] \\ &= b - \mathrm{w}'_x(0) + [\mathrm{w}'(0), L_1]. \end{aligned} \tag{5.15}$$

Note that $[\mathrm{w}'(0), L_1]$ does not contribute to $q'(0)$. Then, the coordinate of γ_i gives

$$(z^i)'(0) = b_0^i - (\mathrm{w}'_x(0))_0^i \tag{5.16}$$

where we write $\mathrm{w}'(0) = \sum_{i=1}^n \sum_{I>0} (\mathrm{w}'(0))_I^i \frac{1}{I!} \mathrm{ad}_{L_1}^I \gamma_i$. Then, the coefficients of $\frac{1}{I!} ad_{L_1}^I \gamma_i$ for $I > 0$ give the recursive relations

$$[(\mathrm{w}'(0))_{I-1}^i, L_1] - (\mathrm{w}'_x(0))_I^i + b_I^i = 0 \tag{5.17}$$

which leads to

$$(\mathrm{w}'(0))_{I-1}^i = \frac{1}{I+1}(-(\mathrm{w}'_x(0))_I^i + b_I^i). \tag{5.18}$$

For example,

$$\begin{aligned} (\mathrm{w}'(0))_{\eta_i-1}^i &= \frac{1}{\eta_i}(b_{\eta_i}^i), \\ (\mathrm{w}'(0))_{\eta_i-2}^i &= \frac{1}{\eta_i - 1}(-\frac{1}{\eta_i}(\partial_x b_{\eta_i}^i) + b_{\eta_i-1}^i). \end{aligned} \tag{5.19}$$

These recursive relations lead to

$$(z^i)'(0) = \sum_{I=0}^{\eta_i} \frac{(-1)^I}{I!} \partial_x^I b_I^i. \tag{5.20}$$

□

Recall that the coordinates $(t^1, \ldots, t^n)$ of $\mathcal{Q}$ developed in Theorem 1.2 are quasi-homogeneous polynomials in the coordinates $(z^1, \ldots, z^n)$. Thus, we get the following corollary by construction.

Corollary 5.3 *Proposition 5.2 is valid when we replace $z^i(x)$ by $t^i(x)$.*

5.2 Further reduction

In this section, we reduce Drinfeld–Sokolov bihamiltonian structure to $\mathcal{N}$ and analyze the leading term using the coordinates $(t^1, \ldots, t^n)$ obtained by Theorem 1.2.

Proposition 5.4 *The reduced bihamiltonian structure on $\mathcal{Q}$ is exact with Liouville vector field ∂_{t^r}. The Poisson bracket with t^1 preserve the relations defining classical W-algebra, i.e.,*

$$\begin{aligned} \{t^1(x), t^1(y)\}_2^{\mathcal{Q}} &= c\delta'''(x-y) + 2t^1(x)\delta'(x-y) + t_x^1\delta(x-y),\\ \{t^1(x), t^i(y)\}_2^{\mathcal{Q}} &= (\eta_i+1)t^i(x)\delta'(x-y) + \eta_i t_x^i \delta(x-y),\ i=2,\ldots,n. \end{aligned} \tag{5.21}$$

for some nonzero constant c.

Proof We take $t^1(z), \ldots, t^n(z)$ as generators for the invariant ring R. By Corollary 5.3, $t^r(x)$ is the only invariant which depends on $b_0^r(x)$. This implies that the invariant $t^r(x)$ appears in the expression of $\{t^i(x), t^j(y)\}_2^{\mathcal{Q}}$ only if, when using the Leibniz rule (5.11), we encounter terms of $\{.,.\}_2$ depend explicitly on $b_0^r(x)$. Thus, $\{t^i(x), t^j(y)\}_2^{\mathcal{Q}}$ is at most linear on $z^r(x)$ and its derivatives. But the bihamiltonian structure on $\mathfrak{L}(\mathfrak{g})$ is exact and $\{.,.\}_1$ is obtained from $\{.,.\}_2$ by the shift along b_0^r. Hence, $\{t^i(x), t^j(y)\}_1^{\mathcal{Q}}$ is obtained by the shift of $\{t^i(x), t^j(y)\}_2^{\mathcal{Q}}$ along $t^r(x)$, i.e., substituting $t^r(x)$ by $t^r(x)+\epsilon$ and evaluate $\frac{d}{d\epsilon}|_{\epsilon=0}$. Therefore, $\{.,.\}_1^{\mathcal{Q}}$ does not depend on $t^r(x)$ or its derivatives. From the work in [33], the reduced Poisson bracket $\{.,.\}_2^{\mathcal{Q}}$ is a classical W-algebra in the coordinates $(z^1, \ldots, z^n)$, i.e., it satisfies the identities 2.4. Then, the argument for identities (5.21) will be similar to the one given in the proof of proposition 6.2 below.

□

Then, Theorem 1.4 gives compatible local Poisson brackets $\{.,.\}_\alpha^{\mathcal{N}}$, $\alpha = 1, 2$ on the loop space $\mathcal{N} = \mathfrak{L}(N)$ of the space of common equilibrium points N. The proof is as follows.

Proof of Theorem 1.4 From Theorem 1.3, the leading terms of $\{.,.\}_k^{\mathcal{Q}}$, $k=1,2$ have the form:

$$\{t^i(x),t^j(y)\}_k^{[-1]} = F_k^{ij}(t(x))\delta(x-y), \tag{5.22}$$

where $F_1^{i\alpha}(t)=0$ and N is defined by $F_2^{i\alpha}(t)=0$, $1\le i\le r$ and $r+1\le \alpha\le n$. Thus, $\{.,.\}_\lambda^{\mathcal{Q}}$ satisfies the hypothesis of proposition 2.5 with the coordinates $(t^1,\dots,t^r)$ on N. Using Corollary 2.6, the reduced local Poisson bracket $\{.,.\}_\lambda^{\mathcal{N}}$ on $\mathcal{N}$ is obtained by setting $\{t^i(x),t^j(y)\}_\lambda^{\mathcal{N}}$ equals $\{t^i(x),t^j(y)\}_\lambda^{\mathcal{Q}}$ and substitute the variables t^i, $i>r$ by solutions of the polynomial equations $F_2^{i\alpha}=0$ defining N. In particular, $\{t^i(x),t^j(y)\}_\lambda^{\mathcal{N}}$ is an algebraic local Poisson bracket and it is linear in λ. This leads to compatible local Poisson brackets $\{.,.\}_2^{\mathcal{N}}$ and $\{.,.\}_1^{\mathcal{N}}$ on N where the former still satisfies the identities (5.21) defining classical W-algebras. From Theorem 1.3 again, they both admit a dispersionless limit. Note that the defining equation $F_2^{i\alpha}=0$ of N do not depends on t^r. Thus, from proposition 5.4, the reduced Poisson brackets form an exact Poisson pencil. □

As in the introduction, we write the leading terms of $\{.,.\}_\alpha^{\mathcal{N}}$, $\alpha=1,2$, in the form:

$$\{t^u(x),t^v(y)\}_\alpha^{[0]} = \Omega_\alpha^{uv}(t(x))\delta'(x-y)+\Gamma_{\alpha k}^{uv}(t(x))t_x^k\delta(x-y),\ 1\le u,v\le r. \tag{5.23}$$

In the remainder of this section, we want to prove that the determinate of the matrix $\Omega_1^{uv}(t)$ is nonzero constant. For this end, we write

$$[K_1,\mathrm{ad}_f^J L_j] = \sum_t \Delta_j^{Jt}\frac{1}{T!}\mathrm{ad}_{L_1}^T\gamma_t;\quad T=\eta_t+\eta_j-J-\eta_r\ge 0 \tag{5.24}$$

where T is constrained by the Dynkin grading of $\mathfrak{g}$. Then, the values of $\{.,.\}_1$ on the coordinates of b are given by

$$\{b_I^i(x),b_J^j(y)\}_1 = \frac{1}{\Theta_J^j}\delta^{IT}\delta^{it}\Delta_j^{Jt}\delta(x-y). \tag{5.25}$$

Thus, we get the following formula for the brackets

$$\{b_I^i(x),b_J^j(y)\}_1 = \frac{\Delta_j^{Ji}}{\Theta_J^j}\delta(x-y),\quad I=\eta_i+\eta_j-J-\eta_r \tag{5.26}$$

where Δ_j^{Jt} possibly equals 0. Expanding using the Leibniz rule, we get

$$\{t^u(x),t^v(y)\}_1^{\mathcal{Q}} = \sum_{i,j}\sum_{l,h}\frac{\Delta_j^{Ji}}{\Theta_J^j}\frac{\partial t^u(x)}{\partial (b_I^i)^{(l)}}\partial_x^l\Big(\frac{\partial t^v(y)}{\partial (b_J^j)^{(h)}}\partial_y^h\delta(x-y)\Big),\ I=\eta_i+\eta_j-J-\eta_r$$

$$= \sum_{i,j} \sum_{l,h,\alpha,\beta} (-1)^h \binom{h}{\alpha} \binom{l}{\beta} \frac{\Delta_j^{Ji}}{\Theta_J^j} \frac{\partial t^u(x)}{\partial (b_I^i)^{(l)}} \left(\frac{\partial t^v(x)}{\partial (b_J^j)^{(h)}} \right)^{(\alpha+\beta)} \delta^{(h+l-\alpha-\beta)}(x-y). \tag{5.27}$$

Here we omitted the ranges of the indices since no confusion can arise. We observe that the value of Ω^{uv} is contained in the expression

$$\mathcal{F}_1^{uv} = \sum_{i,J} \sum_{h,l} (-1)^h (l+h) \frac{\Delta_j^{Ji}}{\Theta_J^j} \frac{\partial t^u(x)}{\partial (b_I^i)^{(l)}} \left(\frac{\partial t^v(x)}{\partial (b_J^j)^{(h)}} \right)^{h+l-1}, \quad I = \eta_i + \eta_j - J - \eta_r \tag{5.28}$$

Lemma 5.5 *The matrix $\Omega_1^{uv}(t)$ is lower antidiagonal with respect to* $\mathrm{E}(L_1)$*, and the antidiagonal entries are constants. In other words, $\Omega_1^{uv}(t)$ is constant if $\eta_u + \eta_v = \eta_r + 1$ and equals zero if $\eta_u + \eta_v < \eta_r + 1$.*

Proof Assume $t^u(x)$ and $t^v(x)$ are quasihomogeneous of degree $\eta_u + 1$ and $\eta_v + 1$, respectively. Then, $\mathcal{F}_1^{uv}$ is a quasihomogeneous polynomial of degree

$$\begin{aligned} &\eta_u + 1 + \eta_v + 1 - (\eta_i - I + l + 1) - (\eta_j - J + h + 1) \\ &\quad + h + l - 1 = \eta_u + \eta_v - \eta_r - 1 \end{aligned}$$

□

Recall that from the construction of the coordinates $(t^1, \ldots, t^r)$ and the second part of proposition 3.3, the entry Ω_1^{uv} in case $u + v = r + 1$ implies that $\eta_u + \eta_v = \eta_r + 1$ and $\langle \Lambda_u | \Lambda_v \rangle = \eta_r + 1$

Proposition 5.6 *The antidiagonal entries of Ω_1^{uv} with respect to the set* $\mathrm{E}(L_1)$ *equal $\eta_r + 1$ in case $u + v = r + 1$ and zero otherwise. In particular, Ω_1^{uv} is nondegenerate and its determinant equals $(\eta_r + 1)^r$.*

Proof We need only to examine the entry Ω^{uv} where t^u and t^v are quasihomogeneous of degree $\eta_u + 1$ and $\eta_r - \eta_u + 2$, respectively. The expression (5.28) yields the constrains

$$\begin{aligned} \eta_i + 1 - I \leq \eta_u + 1 &\Rightarrow J \leq \eta_u + \eta_j - \eta_r \\ \eta_j + 1 - J \leq \eta_r - \eta_u + 2 &\Rightarrow \eta_j + \eta_u - \eta_r - 1 \leq J. \end{aligned} \tag{5.29}$$

Hence, J equals $\eta_u + \eta_j - \eta_r - 1$ or $\eta_u + \eta_j - \eta_r$. Consider $J = \eta_u + \eta_j - \eta_r - 1$. Then $\deg(b_J^j)^{(h)} = \eta_j - J + 1 + h = \deg t^v + h$. This forces $h = 0$ and t^v is linear in b_J^j. Therefore, from proposition 5.2, $j = v$ and $J = 0$ which leads to $\frac{\partial t^v(x)}{\partial (b_J^j)^{(h)}} = 1$. Also

$$\begin{aligned} \deg(b_I^i)^{(l)} &= \eta_i - I + h + 1 = \eta_i - (\eta_i + \eta_j - J - \eta_r) + l + 1 \\ &= \eta_u + l = \deg t^u. \end{aligned} \tag{5.30}$$

Thus, the only possible value for l is 1. Note that $I = \eta_i - \eta_u + 1$. Hence, $\deg t^u = \deg(b_I^i)'$ and t^u is linear in $(b_I^i)'$. Then, $i = u$ and $I = 1$ and from proposition 5.2, $\frac{\partial t^u(x)}{\partial (b_I^i)^{(l)}} = -1$. Therefore, the case $J = \eta_u + \eta_j - \eta_r - 1$, the expression (5.28) contributes to Ω_1^{uv} with the value $-\frac{\Delta_v^{0u}}{\Theta_0^v} = -\Delta_v^{0u}$ since $J = \eta_u + \eta_v - \eta_r - 1 = 0$. By definition,

$$-\Delta_v^{0u} = \frac{1}{\eta_u}\langle ad_f L_u | [K_1, L_v]\rangle = \eta_v \delta_{u+v,r+1}. \tag{5.31}$$

A similar analysis when $J = \eta_u + \eta_j - \eta_r$ leads to the value $\eta_u \delta_{u+v,r+1}$. By the normalization of Λ_i, it follows that the value of Ω_1^{uv} equals $\eta_u + \eta_v = \eta_r + 1$ when $u + v = r + 1$ and zero otherwise. The determinant of the matrix Ω_1^{uv} follows accordingly. □

Corollary 5.7 *The matrix $\Omega_2^{uv}(t)$ is nondegenerate on $\mathcal{N}$.*

Proof It follows from the exactness of the Poisson pencil, i.e., $\Omega_1^{uv}(t) = \partial_{t^r} \Omega_2^{uv}(t)$. □

Recall the duality of the multiset $\overline{\mathrm{E}}(L_1)$ stated in Corollary 3.7. Then, the following proposition is useful to find the rank of B_1^Q. Note that the proof depends only on the linear part of the invariants $t^i(x)$.

Proposition 5.8 *The matrix $F_1^{uv}(t)$, $u, v = 1, \ldots, n$ is a lower antidiagonal in the sense that $F_1^{uv}(t) = 0$ if $\eta_u + \eta_v < \eta_r$. In particular, if $\eta_u + \eta_v = \eta_r$ then*

$$F_1^{uv}(t) = \langle L_u | [K_1, L_v]\rangle, \tag{5.32}$$

and if $\eta_u + \eta_v = \eta_r + 1$ then $F_1^{uv}(t) = 0$

Proof Note that the value of the matrix $F_1^{uv}(t)$ is contained in the expression

$$\sum_{i,J}\sum_{h,l}(-1)^h \frac{\Delta_j^{Ji}}{\Theta_J^j} \frac{\partial t^u(x)}{\partial (b_I^i)^{(l)}} \left(\frac{\partial t^v(x)}{\partial (b_J^j)^{(h)}}\right)^{h+l}, \quad I = \eta_i + \eta_j - J - \eta_r. \tag{5.33}$$

Then, the proof will be similar to the proof of lemma 5.5 and proposition 5.6. The degree of this expression is $\eta_u + \eta_v - \eta_r$. Thus, the matrix will be lower antidiagonal as claimed. Let us assume $\eta_v + \eta_u = \eta_r$. Then, the only possible value for J is $\eta_u + \eta_j - \eta_r$. We also find h (resp. j, l and i) must equal 0 (resp. v, 0 and u). Therefore, $J = 0$ and the expression (5.33) will be $\Delta_v^{0u} = \langle L_u | [K_1, L_v]\rangle$. For the last statement, note that $F_1^{uv}(t)$ is a polynomial [19] and there is no variable of degree 1. □

6 Algebraic Frobenius manifold

In this section, we obtain the promised algebraic Frobenius structure and give examples

6.1 General construction

We consider the flat pencil of metrics on N consists of $\Omega_1^{uv}(t)$ and $\Omega_2^{uv}(t)$ which is afforded by Theorems 1.4, 2.4, Proposition 5.6 and Corollary 5.7. From the exactness of Poisson pencil on $\mathcal{N}$ and defining equations of W-algebra given in proposition 5.4, we have

$$\mathfrak{L}_{\partial_{t^r}} \Omega_2^{uv} = \Omega_1^{uv}, \quad g_2^{1u}(t) = (\eta_u + 1)t, \; \Gamma_{2k}^{1j}(t) = \eta_j \delta_k^j. \tag{6.1}$$

Recall that we assign degree t^u equals $\eta_u + 1$.

Proposition 6.1 *Each entry $\Omega_2^{uv}(t)$ is quasihomogeneous of degree $\eta_u + \eta_v$, while $\Gamma_{2k}^{uv}(t)$ is quasihomogeneous of degree $\eta_u + \eta_v - (\eta_k + 1)$.*

Proof First part follows from the proof of lemma 5.5. Analyzing the coefficient of $\delta(x - y)$ is the expression (5.27) leads to the degree of $\Gamma_{2k}^{uv}(t)$. □

Proposition 6.2 *There exist a quasihomogeneous polynomial change of coordinates of the form*

$$s^i = t^i + \text{non linear terms} \tag{6.2}$$

such that the matrix $\Omega_1^{uv}(s) = (\eta_r + 1)\delta^{u+v,r+1}$. Furthermore, in these coordinates the metric $\Omega_2^{uv}(s)$ and its Christoffel symbols preserve the identities

$$\Omega_2^{1,v}(s) = (\eta_v + 1)s^v, \; \Gamma_{2k}^{1v}(t) = \eta_v \delta_k^v. \tag{6.3}$$

Proof A local flat coordinates of the metric $\Omega_1^{uv}(s)$ exist at each point of N and can be found by solving the system [23]

$$\Omega_1^{uv} \partial_{t^u} \partial_{t^k} s + \Gamma_{1k}^{uv} \partial_{t^v} s = 0, \; u, k = 1, \ldots, r. \tag{6.4}$$

First, we search for a quasihomogeneous change of coordinates in the form $s^i = s^i(t^1, \ldots, t^r)$ with $\deg s^i = \deg t^i$ such that the matrix $\Omega_1^{uv}(s)$ is constant antidiagonal with respect to the set $\mathrm{E}(L_1)$. The proof of its existence can be obtained by following the proof of a similar statement in ([23], Corollary 2.4). Note that we can write s^i in the form (6.2) using eliminations. But then, after reordering, we can apply proposition 5.6 to get $\Omega_1^{uv}(s) = (\eta_r + 1)\delta^{u+v,r+1}$. For the second part of the statement, we need only to show that

$$\Omega_2^{1,i}(s) = (\eta_i + 1)s^i, \; \Gamma_{2k}^{1j}(s) = \eta_j \delta_k^j. \tag{6.5}$$

Let us introduce the Euler vector field

$$E' := \sum_i (\eta_i + 1)t^i \partial_{t^i}. \tag{6.6}$$

Then, the formula for change of coordinates gives

$$\Omega_2^{1j}(s) = \partial_{t^a} s^1 \partial_{t^b} s^j \ \Omega_2^{ab}(t) = E'(s^j) = (\eta_j + 1)s^j. \tag{6.7}$$

Here, the last equality comes from quasihomogeneity of the coordinates s^i. For $\Gamma_{2k}^{1j}(t)$, the change of coordinates has the following formula:

$$\Gamma_{2k}^{ij}(s)ds^k = \Big(\partial_{t^a} s^i \partial_{t^c} \partial_{t^b} s^j \Omega_2^{ab}(t) + \partial_{t^a} s^i \partial_{t^b} s^j \Gamma_c^{ab}(t)\Big)ds^c. \tag{6.8}$$

But then we get

$$\begin{aligned}\Gamma_{2k}^{1j} ds^k &= \Big(E'(\partial_{t^c} s^j) + \partial_{t^b} s^j \Gamma_{2c}^{1b}\Big)dt^c \\ &= \Big((\eta_j - \eta_c)\partial_{t^c} s^j + \eta_c \partial_{t^c} s^j\Big)dt^c = \eta_j \partial_{t^c} s^j dt^c = \eta_j ds^j. \end{aligned} \tag{6.9}$$

□

From proposition 6.2, we can assume without loss of generality that the coordinates t^i are the flat coordinates for Ω_1^{ij}. Then, we get a regular quasihomogeneous flat pencil of metrics of degree $\frac{\eta_r - 1}{\eta_r + 1}$ formed by Ω_1^{ij} and Ω_2^{ij} on N as Theorem 1.5 states.

Proof of Theorem 1.5 In the notation of equations (2.10), we set $\tau := \frac{1}{\eta_r + 1} t^1$. Then,

$$\begin{aligned} E &:= \Omega_2^{ij} \partial_{t^j} \tau \ \partial_{t^i} = \frac{1}{\eta_r + 1} \sum_i (\eta_i + 1) t^i \partial_{t^i}, \\ e &:= \Omega_1^{ij} \partial_{t^j} \tau \ \partial_{t^i} = \partial_{t^r}. \end{aligned} \tag{6.10}$$

The identities $[e, E] = e$, $\mathfrak{L}_{\partial_{t^r}} \Omega_2^{uv} = \Omega_1^{uv}$ and $\mathfrak{L}_{\partial_{t^r}} \Omega_1^{uv} = 0$ are fulfilled. We also obtain from proposition 6.1 that

$$\mathfrak{L}_E \Omega^{ij} = E(\Omega_2^{ij}) - \frac{\eta_i + 1}{\eta_r + 1}\Omega_2^{ij} - \frac{\eta_j + 1}{\eta_r + 1}\Omega_2^{ij} = \frac{-2}{\eta_r + 1}\Omega_2^{ij} = (d - 1)\Omega^{ij}. \tag{6.11}$$

We also have the regularity condition since the (1,1)-tensor R_i^j has the entries

$$R_i^j = \frac{d-1}{2}\delta_i^j + \nabla_{1i} E^j = \frac{\eta_i}{\eta_r + 1}\delta_i^j. \tag{6.12}$$

□

Now we can prove the main result, Theorem 1.1.

Proof of Theorem 1.1 It follows from Theorems 1.5 and 2.8 that N has a natural Frobenius structure of charge $\frac{\eta_r - 1}{\eta_r + 1}$. This Frobenius structure is algebraic since the potential $\mathbb{F}$ is constructed using equations (2.13) and from Theorem 1.4 the matrix Ω_2^{uv} may contain variables $t^k, k > r$ which are solution of the polynomial equations (1.10) defining N. The Euler vector field is given by the formula (6.10). By construction, different choices of a representative L_1 or transverse subspace other than Slodowy slice will lead to the same Frobenius structure. □

6.2 Examples

6.2.1 Regular nilpotent orbits

Suppose L_1 is a regular nilpotent element in $\mathfrak{g}$. Then, the multisets $\mathrm{E}(L_1)$ and $\mathrm{E}(\mathfrak{g})$ coincide. In this case, we get the standard Drinfeld–Sokolov reduction [22] on Slodowy slice Q and the local bihamiltonian structure admits a dispersionless limit. Hence, the space of common equilibrium points N equals Q. The algebraic Frobenius manifold is polynomial. It coincides [30] with the polynomial Frobenius manifold constructed by Dubrovin on the orbit spaces of the underlined Weyl group [23]. The construction using the methods of this article was also obtained in [15].

6.2.2 Subregular nilpotent orbits

A nilpotent elements is called subregular if $\dim \mathfrak{g}_0 = r + 2$. The set of all subregular nilpotent elements form one nilpotent orbit which exists in any complex simple Lie algebra. However, not all subregular nilpotent elements of simple Lie algebras are of semisimple type, which was wrongly assumed in the article [18]. Only the subregular nilpotent elements of type $D_4(a_1)$, $F_4(a_1)$, $E_6(a_1)$, $E_7(a_1)$ and $E_8(a_1)$ are of semisimple type. Hence, all statements in [18] are valid only when considering those cases. Let L_1 be a subregular nilpotent element of semisimple type. Then, Slodowy slice Q is of dimension $r + 2$. In [18], the set of common equilibrium points N was defined in terms of the invariant polynomials $P_1, \ldots, P_r$ using the normalization of the transverse Lie-Poisson bracket $\{.,.\}_2^Q$ obtained in [9]. Moreover, the article [18] contains in detail the construction of the potential of the algebraic Frobenius manifold associated to $D_4(a_1)$. So we are not keen to repeat writing this example here. We also constructed the potential associated with $E_8(a_1)$, but it results in a huge polynomial in 8 variables (consist of 303 monomials) with vast numbers and by all means unpublishable [16]. A simpler formula for this potential appears in [17].

6.2.3 Nilpotent element of type $F_4(a_2)$

We use minimal representation of F_4 which is given by square matrices of size 27. The following computations can be verified using any computer algebra systems. Below $\epsilon_{i,j}$ denote the standard basis of the set of square matrices of size 27. To simplify the notation we use $E_{c_1c_2c_3c_4}$ to denote the root vector corresponding to the root $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ while $F_{c_1c_2c_3c_4}$ for the root vector corresponding to the negative root. We always set $F_{c_1c_2c_3c_4}$ equals the transpose of the matrix $E_{c_1c_2c_3c_4}$. Then, the simple root vectors are

$$E_{0001} := -\epsilon_{4,5} + \epsilon_{7,8} + \epsilon_{9,11} + \epsilon_{20,22} + \epsilon_{21,6} + \epsilon_{23,24},$$

$$\begin{aligned} E_{0010} &:= -\epsilon_{3,4} + \epsilon_{8,10} + \epsilon_{11,13} + \epsilon_{18,20} + \epsilon_{19,21} + \epsilon_{24,25}, \\ E_{0100} &:= -\epsilon_{2,3} - \epsilon_{4,7} + \epsilon_{5,8} + \epsilon_{6,24} + \epsilon_{10,12} + \epsilon_{13,15} + \epsilon_{13,16} + \epsilon_{15,18} \\ &\quad + \epsilon_{16,18} + \epsilon_{17,19} + \epsilon_{21,23} + \epsilon_{25,26}, \\ E_{1000} &:= -\epsilon_{1,2} - \epsilon_{7,9} - \epsilon_{8,11} - \epsilon_{10,13} + \epsilon_{12,14} - \epsilon_{12,15} - \epsilon_{14,17} \\ &\quad + \epsilon_{15,17} + \epsilon_{18,19} + \epsilon_{20,21} + \epsilon_{22,6} + \epsilon_{26,27}. \end{aligned} \tag{6.13}$$

We construct the remaining root vectors by setting

$E_{0011} = [E_{0001}, E_{0010}]$	$E_{0110} = [E_{0010}, E_{0100}]$	$E_{1100} = [E_{0100}, E_{1000}]$
$E_{0111} = [E_{0011}, E_{0100}]$	$E_{0210} = [E_{0100}, E_{0110}]$	$E_{1110} = [E_{1000}, E_{0110}]$
$E_{0211} = [E_{0111}, E_{0100}]$	$E_{1111} = [E_{1110}, E_{0001}]$	$E_{1210} = [E_{1110}, E_{0100}]$
$E_{0221} = [E_{0211}, E_{0010}]$	$E_{1211} = [E_{1111}, E_{0100}]$	$E_{2210} = [E_{1210}, E_{1000}]$
$E_{1221} = [E_{0221}, E_{1000}]$	$E_{2211} = [E_{1211}, E_{1000}]$	$E_{1321} = [E_{1221}, E_{0100}]$
$E_{2221} = [E_{2211}, E_{0010}]$	$E_{2321} = [E_{2221}, E_{0100}]$	$E_{2421} = [E_{2321}, E_{0100}]$
$E_{2431} = [E_{2421}, E_{0010}]$	$E_{2432} = [E_{2431}, E_{0001}]$	

We fix the following sl_2-triple, where the nilpotent element L_1 is of type $F_4(a_2)$

$$\begin{aligned} L_1 &= E_{0010} + E_{0011} + E_{0110} + E_{0111} + E_{0210} + E_{0211} + E_{1000} + E_{1100}, \\ f &= 3F_{0010} + 3F_{0011} + F_{0110} + F_{0111} + \frac{5}{4}F_{0210} + \frac{5}{4}F_{0211} + 6F_{1000} + 2F_{1100}, \\ h &= 5[E_{0001}, F_{0001}] + 10[E_{0010}, F_{0010}] + 7[E_{0100}, F_{0100}] + 4[E_{1000}, F_{1000}], \end{aligned} \tag{6.14}$$

The following vectors form a complete set of maximum weight vectors of the irreducible sl_2-submodules. They are of eigenvalues 1,5,5,4,3,2,1, respectively, under ad_h.

$$\begin{aligned} L_2 &= \frac{20}{13}E_{0010} - \frac{28}{13}E_{0011} - \frac{76}{13}E_{0110} - \frac{28}{13}E_{0111} + \frac{38}{13}E_{0210} \\ &\quad + \frac{2}{13}E_{0211} + \frac{32}{13}E_{1000} - \frac{88}{13}E_{1100}, \\ L_3 &= \frac{39}{20}E_{2431}, \quad L_4 = \frac{39}{20}E_{2431} + \frac{9}{4}E_{2432}, \\ L_5 &= E_{2321} + E_{2421}, \quad L_6 = 2E_{1221} + 6E_{1321} + E_{2210} - 5E_{2211}, \\ L_7 &= -4E_{221} + E_{1110} - 5E_{1111} - E_{1210} + 5E_{1211}, \\ L_8 &= \frac{2}{5}E_{0010} + 2E_{0011} - \frac{6}{5}E_{0110} - \frac{14}{5}E_{0111} \\ &\quad - \frac{1}{5}E_{0210} + E_{0211} - \frac{4}{5}E_{1100}. \end{aligned} \tag{6.15}$$

Then setting

$$
\begin{aligned}
K_1 &= F_{2432}, \quad K_2 = \frac{15}{13}F_{2431} - F_{2432}\\
K_3 &= \frac{39}{20}F_{0010} - \frac{39}{20}F_{0011} - \frac{39}{8}F_{0110} - \frac{273}{40}F_{111} + \frac{39}{10}F_{0211} - \frac{39}{10}F_{1100}\\
K_4 &= \frac{39}{20}F_{0010} + \frac{204}{5}F_{0011} + 3F_{0110} - \frac{51}{5}F_{0111}\\
&\quad + \frac{9}{2}F_{0210} + \frac{129}{10}F_{0211} + \frac{9}{4}F_{1000} + \frac{48}{5}F_{1100}
\end{aligned} \tag{6.16}
$$

The vectors $\Lambda_i = L_i + K_i$ are basis of the opposite Cartan subalgebra $\mathfrak{h}'$. The normalized bilinear form is given by $\langle g_1|g_2\rangle = \frac{1}{216}\mathrm{Tr}(g_1.g_2)$. Then, one can check that $\langle \Lambda_i|\Lambda_j\rangle = 6\delta_{ij}$. The basis $\gamma_i \in \mathfrak{g}^f$ such that $\langle \gamma_i|L_j\rangle = \delta_{ij}$ are given by the formula

$$
\begin{aligned}
\gamma_1 &= f,\\
\gamma_2 &= \frac{1677}{1120}F_{0010} - \frac{1833}{1120}F_{0011} - \frac{923}{1120}F_{0110} + \frac{247}{1120}F_{0111} + \frac{403}{2240}F_{0210}\\
&\quad + \frac{247}{2240}F_{0211} + \frac{39}{35}F_{1000} - \frac{143}{140}F_{1100},\\
\gamma_3 &= \frac{15}{13}F_{2431} - F_{2432}, \quad \gamma_4 = F_{2432},\\
\gamma_5 &= \frac{27}{10}F_{2321} + \frac{9}{10}F_{2421}, \quad \gamma_6 = \frac{5}{16}F_{1221} + \frac{5}{16}F_{1321} - \frac{3}{8}F_{2210} - \frac{7}{8}F_{2211},\\
\gamma_7 &= -\frac{15}{28}F_{0221} - \frac{27}{28}F_{1110} - \frac{9}{4}F_{1111} + \frac{9}{28}F_{1210} + \frac{3}{4}F_{1211},\\
\gamma_8 &= -\frac{405}{112}F_{0010} + \frac{135}{16}F_{0011} + \frac{75}{112}F_{0110} - \frac{375}{112}F_{0111}\\
&\quad + \frac{45}{224}F_{0210} + \frac{15}{32}F_{0211} - \frac{15}{14}F_{1100}.
\end{aligned} \tag{6.17}
$$

We write elements of Slodowy slice in the form $Q = L_1 + \sum_{i=1}^{8} z_i\gamma_i$. The restriction P_i^Q of the invariant polynomials P_i of degree $\nu_i + 1$ is obtained from taking the trace of the matrix Q^{ν_i+1}. We can take $P_1^Q = z_1$. The expression corresponding to the invariant of maximal degree P_4^Q is omitted since it is very large. We give instead $\partial_{z_4}P_4^Q$.

$$
\begin{aligned}
P_2^Q &= 744192z_1^3 + \frac{44928}{7}z_2z_1^2 - \frac{518400}{7}z_8z_1^2\\
&\quad - \frac{866970}{49}z_2^2z_1 + \frac{923400}{49}z_8^2z_1 - 5760z_6z_1\\
&\quad - \frac{1600560}{49}z_2z_8z_1 + \frac{228002463}{137200}z_2^3 - \frac{9871875}{686}z_8^3\\
&\quad + \frac{150984}{49}z_7^2 - \frac{37986975}{1372}z_2z_8^2 + \frac{165888}{13}z_3
\end{aligned}
$$

$$
\begin{aligned}
&-3456z_4-\frac{6786}{7}z_2z_6\\
&-\frac{45734949}{2744}z_2^2z_8+\frac{78300}{7}z_6z_8, \qquad (6.18)\\
P_3^Q &= 40799232z_1^4+958464z_2z_1^3-11059200z_8z_1^3-\frac{80016768}{35}z_2^2z_1^2\\
&+\frac{24883200}{7}z_8^2z_1^2-860160z_6z_1^2\\
&-\frac{31000320}{7}z_2z_8z_1^2+\frac{209079702}{1225}z_2^3z_1-\frac{89910000}{49}z_8^3z_1\\
&+\frac{2287872}{7}z_7^2z_1-\frac{134573400}{49}z_2z_8^2z_1\\
&+\frac{24772608}{13}z_3z_1-516096z_4z_1-109824z_2z_6z_1-\frac{84159972}{49}z_2^2z_8z_1\\
&+1267200z_6z_8z_1+\frac{9587156553}{686000}z_2^4\\
&-\frac{29615625}{343}z_8^4-\frac{87267375}{343}z_2z_8^3+25920z_6^2\\
&-\frac{112320}{49}z_2z_7^2-\frac{29362905}{343}z_2^2z_8^2\\
&+\frac{621000}{7}z_6z_8^2+207360z_2z_3-149760z_2z_4+\frac{534378}{35}z_2^2z_6\\
&+311040z_5z_7+\frac{537489459}{6860}z_2^3z_8\\
&+\frac{1296000}{49}z_7^2z_8-\frac{3456000}{13}z_3z_8+\frac{60840}{7}z_2z_6z_8, \qquad (6.19)\\
\partial_{z_4}P_4^Q &= -4505960448z_1^3-\frac{18242205696}{7}z_2z_1^2\\
&+\frac{1094860800}{7}z_8z_1^2+\frac{2043055872}{245}z_2^2z_1\\
&-\frac{410572800}{7}z_8^2z_1+12165120z_6z_1+\frac{5782233600}{49}z_2z_8z_1\\
&+\frac{20251269324}{1225}z_2^3+\frac{801900000}{343}z_8^3\\
&+\frac{87588864}{49}z_7^2-\frac{1209265200}{49}z_2z_8^2\\
&-\frac{76972032}{13}z_3-5308416z_4\\
&+\frac{41019264}{7}z_2z_6+\frac{7000116552}{343}z_2^2z_8+5702400z_6z_8. \qquad (6.20)
\end{aligned}
$$

Our special coordinates $(t_1,\dots,t_8)$ are given by

$$
t_1=z_1,\ t_2=-\frac{1}{149760}\partial_{z_4}P_3^Q-\frac{224}{65}z_1=z_2,\ t_i=z_i,\ i=5,6,7,8
$$

$$t_3 = -\frac{13}{331776000}\partial_{z_4} P_4^Q + \frac{13}{216000} P_3^Q = z_3 + \text{ nonlinear terms},$$
$$t_4 = -\frac{1}{6912000}\partial_{z_4} P_4^Q - \frac{29}{432000} P_3^Q = z_4 + \text{ nonlinear terms}. \tag{6.21}$$

Writing the restriction of the invariant polynomials in these coordinates, the space N of common equilibrium points is defined as the zero set of the following polynomials:

$$\begin{aligned}
\partial_{t_5} P_3^Q &= 311040 t_7,\\
\partial_{t_6} P_3^Q &= -\frac{1478412}{35} t_2^2 - \frac{2779920 t_8}{7} t_2 + \frac{1458000}{7} t_8^2 + 51840 t_6 - 622080 t_1 t_8,\\
\partial_{t_7} P_3^Q &= 311040 t_5 - \frac{1866240}{7} t_1 t_7 - \frac{9401184}{49} t_2 t_7 + \frac{5598720}{49} t_7 t_8,\\
\partial_{t_8} P_3^Q &= 58844160 t_1^3 + 27248832 t_2 t_1^2 - 4147200 t_8 t_1^2 - \frac{16116516}{35} t_2^2 t_1 + \frac{25758000}{7} t_8^2 t_1\\
&\quad -622080 t_6 t_1 + \frac{84240}{7} t_2 t_8 t_1 + \frac{31300659}{980} t_2^3 - \frac{66825000}{49} t_8^3 + \frac{2799360}{49} t_7^2\\
&\quad -\frac{21718125}{49} t_2 t_8^2 - \frac{3456000}{13} t_3 - \frac{2779920}{7} t_2 t_6\\
&\quad +\frac{38534535}{49} t_2^2 t_8 + \frac{2916000}{7} t_6 t_8.
\end{aligned} \tag{6.22}$$

The local bihamiltonian structure is polynomial in t_1, t_2, t_3, t_4 and t_8, where t_8 is a solution of a cubic equation. The potential of the Frobenius structure in the flat coordinates (s_1, s_2, s_3, s_4) is

$$\begin{aligned}
\mathbb{F} = T^2 &\left(\frac{664832691 s_1^5}{43750} + \frac{393797781 s_2 s_1^4}{8750} + \frac{117925163577 s_2^2 s_1^3}{2240000} + \frac{31524548679 s_2^3 s_1^2}{1280000}\right.\\
&-\frac{177147 s_3 s_1^2}{1820}\\
&\left.+\frac{1411599235293 s_2^4 s_1}{286720000} - \frac{59049 s_2 s_3 s_1}{1120} + \frac{8090133251733 s_2^5}{22937600000} - \frac{255879 s_2^2 s_3}{35840}\right)\\
&+T\left(\frac{81990638748 s_1^6}{546875} + \frac{157687224903 s_2 s_1^5}{546875} + \frac{252845042697 s_2^2 s_1^4}{875000}\right.\\
&+\frac{5680343128707 s_2^3 s_1^3}{28000000}\\
&-\frac{405324 s_3 s_1^3}{2275} + \frac{41422089388329 s_2^4 s_1^2}{448000000} - \frac{150903}{350} s_2 s_3 s_1^2 + \frac{349410443449509 s_2^5 s_1}{17920000000}\\
&\left.-\frac{2075463 s_2^2 s_3 s_1}{5600} + \frac{118472583689109 s_2^6}{81920000000} + \frac{675 s_3^2}{1183} - \frac{5051241 s_2^3 s_3}{89600}\right)\\
&+\frac{2446443495072 s_1^7}{13671875} + \frac{8512750428624 s_2 s_1^6}{13671875}\\
&+\frac{1593096854076 s_2^2 s_1^5}{1953125} + \frac{87566456228121 s_2^3 s_1^4}{175000000}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{391896144 s_3 s_1^4}{284375}+\frac{1357381494479907 s_2^4 s_1^3}{5600000000}\\
&-\frac{700488 s_2 s_3 s_1^3}{21875}+\frac{21326967621723933 s_2^5 s_1^2}{224000000000}\\
&-\frac{95335461 s_2^2 s_3 s_1^2}{175000}+\frac{87348137456366631 s_2^6 s_1}{3584000000000}+\frac{16}{169} s_3^2 s_1\\
&+\frac{1}{2} s_4^2 s_1-\frac{505028277 s_2^3 s_3 s_1}{1400000}\\
&+\frac{120333341133594693 s_2^7}{57344000000000}-\frac{7}{13} s_2 s_3^2\\
&-\frac{10700732367 s_2^4 s_3}{89600000}+s_2 s_3 s_4
\end{aligned}
\tag{6.23}
$$

where T is a solution of the following cubic equation

$$
\begin{aligned}
0=T^3-\frac{15552}{625} T s_1^2-\frac{4563}{2500} T s_2^2-\frac{8424}{625} T s_1 s_2-\frac{213504}{15625} s_1^3-\frac{270231}{62500} s_2^3\\
-\frac{444132 s_1}{15625} s_2^2-\frac{516672 s_1^2}{15625} s_2+\frac{256}{2925} s_3 .
\end{aligned}
\tag{6.24}
$$

Then, the quasihomogeneity condition reads

$$
\frac{1}{3} \partial_{s_1} \mathbb{F}+\frac{1}{3} \partial_{s_2} \mathbb{F}+\partial_{s_3} \mathbb{F}+\partial_{s_4} \mathbb{F}=(3-\frac{2}{3}) \mathbb{F} .
\tag{6.25}
$$

7 Conclusions and remarks

Consider a nilpotent element not of semisimple type and the associated Drinfeld–Sokolov bihamiltonian structure. Then, the space of common equilibrium points is still well defined and probably possesses a local bihamiltonian structure which admits a dispersionless limit. However, examples show that its leading term does not define a flat pencil of metrics.

It is known that for each conjugacy class in the Weyl group one can construct Drinfeld–Sokolov hierarchy [10] and, under some restrictions, an accompanied bihamiltonian structure [4]. This bihamiltonian structure agrees with the one used in this article if the conjugacy class is regular [12].

In the case of a regular primitive conjugacy classes, we obtain a new local algebraic bihamiltonian structure on the space of common equilibrium points. Since it defines an exact Poisson pencil, its central invariants are constants [32]. It will be interesting to calculate them and find if they are equal. In this case the bihamiltonian structure will be related to the topological hierarchy associated with the algebraic Frobenius structure [29]. This topological hierarchy seems to be a reduction of the Drinfeld–Sokolov hierarchy (see [19] for details on Dirac reduction of Hamiltonian equations).

In future work, we will analyze the bihamiltonian structure associated with Drinfeld–Sokolov hierarchy for a primitive non-regular conjugacy class. Hoping, this will lead to algebraic Frobenius structure not covered in this article.

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