Letters in Mathematical Physics (2021) 111:71
https://doi.org/10.1007/s11005-021-01417-x

Check for updates

The $\mathbb{F}_p$-Selberg integral of type A_n

Richárd Rimányi[1] **· Alexander Varchenko**[1,2,3]

Received: 22 December 2020 / Revised: 1 May 2021 / Accepted: 11 May 2021 / Published online: 22 May 2021

Abstract

We present an $\mathbb{F}_p$-Selberg integral formula of type A_n, in which the $\mathbb{F}_p$-Selberg integral is an element of the finite field $\mathbb{F}_p$, where p is an odd prime. The formula is motivated by analogy between multidimensional hypergeometric solutions of the KZ equations and polynomial solutions of the same equations reduced modulo p. The A_1-type formula was proved in a previous paper by the authors. The A_2-type formula is proved in this paper. We also sketch the proof of the A_n-type formula for $n > 2$.

Keywords KZ and dynamical equations · Reduction modulo p · Selberg integrals · $\mathbb{F}_p$-integrals

Mathematics Subject Classification 13A35 · 33C60 · 32G20

Contents

R. Rimányi: supported in part by Simons Foundation grant 523882. A. Varchenko: supported in part by NSF grant DMS-1954266.

✉ Alexander Varchenko
anv@email.unc.edu

Richárd Rimányi
rimanyi@email.unc.edu

1 Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

2 Faculty of Mathematics and Mechanics, Lomonosov Moscow State University, Leninskiye Gory 1, Moscow GSP-1, Russia 119991

3 Moscow Center of Fundamental and Applied Mathematics, Leninskiye Gory 1, Moscow GSP-1, Russia 119991

1 Introduction

In 1944, Atle Selberg proved the following integral formula:

$$\int_0^1 \dots \int_0^1 \prod_{1\le i<j\le k} |x_i - x_j|^{2\gamma} \prod_{i=1}^{k} x_i^{\alpha-1}(1-x_i)^{\beta-1}\, dx_1 \dots dx_k$$

$$= \prod_{j=1}^{k} \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma(\alpha+(j-1)\gamma)\,\Gamma(\beta+(j-1)\gamma)}{\Gamma(\alpha+\beta+(k+j-2)\gamma)}\,, \tag{1.1}$$

where α, β, γ are complex numbers such that $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, and $\operatorname{Re}\gamma > -\min\,[(\operatorname{Re}\alpha)/(n-1)\operatorname{Re}\beta)/(n-1)]$. See [1,22]. Hundreds of papers are devoted to the generalizations of the Selberg integral formula and its applications, see for example [1,10] and references therein. There are q-analysis versions of the formula, the generalizations associated with Lie algebras, elliptic versions, finite field versions, see some references in [1,2,4,5,7,8,10–12,17,18,24,27,28,30,38,39]. In the finite field versions, one considers additive and multiplicative characters of a finite field, which map the field to the field of complex numbers, and forms an analog of equation (1.1), in which both sides are complex numbers. The simplest of such formulas is the classical relation between Jacobi and Gauss sums, see [1,2,7].

In [21], we suggested another version of the Selberg integral formula, in which the $\mathbb{F}_p$-Selberg integral is an element of the finite field $\mathbb{F}_p$ with an odd prime number p of elements.

Our motivation in [21] came from the theory of Knizhnik–Zamolodchikov (KZ) equations, see [6,13]. These are the systems of linear differential equations, satisfied by conformal blocks on the sphere in the WZW model of conformal field theory. The KZ equations were solved in multidimensional hypergeometric integrals in [25], see also [31,32]. The following general principle was formulated in [16]: if an example of the KZ-type equations has a one-dimensional space of solutions, then the corresponding multidimensional hypergeometric integral can be evaluated explicitly. As an illustration of that principle in [16], an example of the $\mathfrak{sl}_2$ differential KZ equations with a one-dimensional space of solutions was considered, the corresponding multidimensional hypergeometric integral was reduced to the Selberg integral and then evaluated by formula (1.1). See other illustrations in [8,9,20,27,28,30,33].

Recently in [26], the KZ equations were considered modulo a prime number p and polynomial solutions of the reduced equations were constructed, see also [23, 33–37]. The construction is analogous to the construction of the multidimensional hypergeometric solutions, and the constructed polynomial solutions were called the $\mathbb{F}_p$-hypergeometric solutions.

In [21], we considered the reduction modulo p of the same example of the $\mathfrak{sl}_2$ differential KZ equations, that led in [16] to the Selberg integral. We evaluated the corresponding $\mathbb{F}_p$-hypergeometric solution by analogy with the evaluation of the Selberg integral and obtained the $\mathbb{F}_p$-Selberg integral formula in [21, Theorem 4.1].

In [30, Theorem 3.3], the Selberg integral formula of type A_2 was proposed and proved,

$$\begin{aligned}
&\int_{C^{k_1,k_2}[0,1]} \prod_{i=1}^{k_1} t_i^{\alpha-1}(1-t_i)^{\beta_1-1} \prod_{j=1}^{k_2}(1-s_j)^{\beta_2-1} \prod_{i=1}^{k_1}\prod_{j=1}^{k_2} |s_j-t_i|^{-\gamma} \\
&\quad\times \prod_{1\le i<i'\le k_1} |t_i-t_{i'}|^{2\gamma} \prod_{1\le j<j'\le k_2} |s_j-s_{j'}|^{2\gamma}\, dt_1\dots dt_{k_1}\, ds_1\dots ds_{k_2} \\
&= \prod_{i=1}^{k_1-k_2} \frac{\Gamma(\beta_1+(i-1)\gamma)}{\Gamma(\alpha+\beta_1+(i+k_1-2)\gamma)} \\
&\quad\times \prod_{i=1}^{k_2} \frac{\Gamma(\beta_2+(i-1)\gamma)}{\Gamma(1+\beta_2+(i+k_2-k_1-2)\gamma)}\, \frac{\Gamma(\beta_1+\beta_2+(i-2)\gamma)}{\Gamma(\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma)} \\
&\quad\times \prod_{i=1}^{k_2} \frac{\Gamma(1+(i-k_1-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{i=1}^{k_1} \frac{\Gamma(\alpha+(i-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\,. \qquad (1.2)
\end{aligned}$$

Here, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta_1 > 0$, $\operatorname{Re}\beta_2 > 0$, $\operatorname{Re}\gamma < 0$ and $|\operatorname{Re}\gamma|$ is sufficiently small. The integration cycle $C^{k_1,k_2}[0,1]$ is defined in [30, Sect. 3], also see its definition in [10,38,39].

The starting point of this formula was an example of the joint system of the $\mathfrak{sl}_3$ trigonometric differential KZ equations and associated dynamical difference equations, an example in which the space of solutions is one-dimensional. The A_n-type Selberg integral formula for arbitrary n was obtained in [38,39], see also [10].

In this paper, we consider the reduction modulo p of the same example of the joint system of the $\mathfrak{sl}_{n+1}$ trigonometric differential KZ equations and associated dynamical difference equations, which led in [30,38] to the A_n-type Selberg integral formula. Using the reduction modulo p of these differential and difference equations, we obtain our A_n-type $\mathbb{F}_p$-Selberg integral formula for $n \ge 1$, see (3.11). For $n = 1$, the formula is proved in [21, Theorem 4.1]. For $n = 2$, the formula is proved in Theorem 3.4 below. We sketch the proof of the formula for $n > 2$ in Sect. 5.4. The details of that sketch will appear elsewhere.

The paper is organized as follows. In Sect. 2, we collect useful facts. In Sect. 3, we introduce the notion of $\mathbb{F}_p$-integral and discuss the integral formula for the $\mathbb{F}_p$-beta integral. In Sect. 3, we define the A_n-type $\mathbb{F}_p$-Selberg integral and present its evaluation formula. Theorem 3.4 states that the formula holds for $n = 2$. In Section 3, we also prove Theorem 3.7, which is used in the transition from the A_{n-1}-type formula to the A_n-type formula, in particular, in the transition from the known A_1-type formula to the new A_2-type formula. In Sect. 4, we sketch the proof of formula (1.2) following [30]. In Sect. 5, we adapt this proof to prove Theorem 3.4.

The authors thank I. Cherednik, P. Etingof, E. Rains for useful discussions.

2 Preliminary remarks

In this paper, p is an odd prime number.

2.1 Cancellation of factorials

Lemma 2.1 *If a, b are non-negative integers and $a + b = p - 1$, then in $\mathbb{F}_p$ we have*

$$a!\,b! \;=\; (-1)^{a+1}\,. \tag{2.1}$$

Proof We have $a! = (-1)^a(p-1)\dots(p-a)$ and $p - a = b + 1$. Hence, $a!\,b! = (-1)^a(p-1)! = (-1)^{a+1}$ by Wilson's Theorem. □

2.2 Dyson's formula

We shall use Dyson's formula

$$\text{C.T.} \prod_{1\le i<j\le k} (1 - x_i/x_j)^c (1 - x_j/x_i)^c = \frac{(kc)!}{(c!)^k}\,, \tag{2.2}$$

where C.T. denotes the constant term. See the formula in [1, Sect. 8.8].

2.3 $\mathbb{F}_p$-Integrals

Let M be an $\mathbb{F}_p$-module. Let $P(x_1,\dots,x_k)$ be a polynomial with coefficients in M,

$$P(x_1,\dots,x_k) = \sum_d c_d\, x_1^{d_1}\dots x_k^{d_k}\,. \tag{2.3}$$

Let $l = (l_1,\dots,l_k) \in \mathbb{Z}^k_{>0}$. The coefficient $c_{l_1p-1,\dots,l_kp-1}$ is called the $\mathbb{F}_p$*-integral over the p-cycle* $[l_1,\dots,l_k]_p$ and is denoted by $\int_{[l_1,\dots,l_k]_p} P(x_1,\dots,x_k)\,dx_1\dots dx_k$.

Lemma 2.2 *For $i = 1, \dots, k-1$, we have*

$$\int_{[l_1,\dots,l_{i+1},l_i,\dots,l_k]_p} P(x_1, \dots, x_{i+1}, x_i, \dots, x_k) dx_1 \dots dx_k$$
$$= \int_{[l_1,\dots,l_k]_p} P(x_1, \dots, x_k)\, dx_1 \dots dx_k\,. \tag{2.4}$$

□

Lemma 2.3 *For any $i = 1, \dots, k$, we have*

$$\int_{[l_1,\dots,l_k]_p} \frac{\partial P}{\partial x_i}(x_1, \dots, x_k)\, dx_1 \dots dx_k = 0\,.$$

□

Let $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{>0}$ and

$$[\boldsymbol{k}]_p := [(1)_{k_1}; (k_1)_{k_2}; \dots; (k_{n-1})_{k_n}]_p, \tag{2.5}$$

where x_y denotes the y-tuple $(x, \dots, x)$. For example for $n = 2$, $\boldsymbol{k} = (3, 2)$, we have $[\boldsymbol{k}]_p = [1, 1, 1; 3, 3]_p$.

2.4 $\mathbb{F}_p$-Beta integral

For non-negative integers, the classical beta integral formula says

$$\int_0^1 x^a (1-x)^b dx = \frac{a!\, b!}{(a+b+1)!}\,. \tag{2.6}$$

Theorem 2.4 *([37]) Let $a < p$, $b < p$, $p - 1 \le a + b$. Then, in $\mathbb{F}_p$ we have*

$$\int_{[1]_p} x^a (1-x)^b dx \;=\; -\frac{a!\, b!}{(a+b+1-p)!}\,. \tag{2.7}$$

If $a + b < p - 1$, then

$$\int_{[1]_p} x^a (1-x)^b dx \;= 0\,. \tag{2.8}$$

3 $\mathbb{F}_p$-Selberg integral of type A_n

3.1 Admissible parameters

Let $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{>0}$ and $k_i > k_{i+1}$, $i = 1, \dots, n-1$. Set $k_0 = k_{n+1} = 0$.

Let $a, b_1, \ldots, b_n, c \in \mathbb{Z}_{>0}$. Denote $b = (b_1, \ldots, b_n)$ and

$$R_{\boldsymbol{k}}(a,b,c) = \\ = \prod_{1\le s\le r\le n} \prod_{i=1}^{k_r-k_{r+1}} \frac{(r-s+b_s+\cdots+b_r+(i+s-r-1)c)!}{(r-s+1+a_s+b_s+\cdots+b_r+(i+s-r+k_s-k_{s-1}-2)c-\delta_{s,1}p)!} \\ \times (-1)^{\sum_{i=1}^n k_i}\Big(\prod_{i=1}^{k_1}(a_1+(i-1)c)!\Big)\Big(\prod_{r=1}^{n}\prod_{i=1}^{k_r}\frac{(ic)!}{c!}\Big)\Big(\prod_{r=2}^{n}\prod_{i=1}^{k_r}(p+(i-k_{r-1}-1)c)!\Big), \quad (3.1)$$

where $a_1 = a,\ a_2 = \cdots = a_n = 0;\ \delta_{s,1}$ is 1 if $s = 1$ and is zero otherwise.

We say that $a, b_1, \ldots, b_n, c \in \mathbb{Z}_{>0}$ are *admissible* if $a + (k_1 - 1)c < p - 1$ and for any factorial $x!$ on the right-hand side of (3.1) we have $0 \le x < p$. The set of all admissible (a, b, c) is denoted by $\mathcal{A}_{\boldsymbol{k}}$.

Lemma 3.1 *The set $\mathcal{A}_{\boldsymbol{k}}$ is defined in $\mathbb{Z}^{n+2}_{>0}$ by the following system of inequalities:*

$$0 \le r-s+b_s+\cdots+b_r+(s-r)c, \\ r-s+b_s+\cdots+b_r+(k_r-k_{r+1}+s-r-1)c \le p-1, \quad (3.2)$$

for $1 \le s \le r \le n$*;*

$$0 \le r-s+1+b_s+\cdots+b_r+(s-r+k_s-k_{s-1}-1)c, \\ r-s+1+b_s+\cdots+b_r+(s-r+k_r-k_{r+1}+k_s-k_{s-1}-2)c \le p-1, \quad (3.3)$$

for $2 \le s \le r \le n$*;*

$$p \le r+a+b_1+\cdots+b_r+(k_1-r)c, \\ r+a+b_1+\cdots+b_r+(k_r-k_{r+1}+k_1-r-1)c < 2p, \quad (3.4)$$

for $1 \le r \le n$*;*

$$a+(k_1-1)c < p-1, \quad b_1 \ge p-1-(a+(k_1-1)c), \quad 0 < k_1c < p\,. \quad (3.5)$$

□

Lemma 3.2 *Assume that $(a, b, c) \in \mathcal{A}_{\boldsymbol{k}}$. Then,*

$$b_1 \ge p-1-(a+(k_1-1)c), \qquad b_s \ge (k_{s-1}-k_s+1)c-1, \quad s = 2, \ldots, n. \quad (3.6)$$

Proof The inequality $b_s \ge (k_{s-1} - k_s + 1)c - 1$ for $s = 2, \ldots, n$ follows from the first inequality in (3.3) for $r = s$. The inequality $b_1 \ge p - 1 - (a + (k_1 - 1)c)$ follows from the first inequality in (3.4) for $r = 1$. □

Example Let $n = 1$, $\boldsymbol{k} = (k_1)$. Then,

$$R_{(k_1)}(a, b_1, c) = \prod_{i=1}^{k_1} \frac{(ic)!}{c!} \frac{(a + (i-1)c)!\,(b_1 + (i-1)c)!}{(1 + a + b_1 + (i + k_1 - 2)c - p)!} \tag{3.7}$$

and $\mathcal{A}_{(k_1)}$ consists of $a, b, c \in \mathbb{Z}_{>0}$ such that

$$\begin{aligned} &a + (k_1 - 1)c < p - 1, \qquad b_1 + (k_1 - 1)c \leq p - 1, \qquad k_1 c \leq p - 1, \\ &p - 1 \leq a + b_1 + (k_1 - 1)c, \qquad a + b_1 + (2k_1 - 2)c < 2p - 1\,. \end{aligned} \tag{3.8}$$

3.2 Main result

Given $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{>0}$ introduce $k_1 + \cdots + k_n$ variables

$$t = (t^{(1)}, \ldots, t^{(n)}), \quad \text{where} \quad t^{(i)} = (t^{(i)}_1, \ldots, t^{(i)}_{k_i}), \quad i = 1, \ldots, n. \tag{3.9}$$

Define the *master polynomial*

$$\begin{aligned} \Phi_{\boldsymbol{k}}(t; a, b, c) = \prod_{i=1}^{n} \Big(\prod_{j=1}^{k_i} (t^{(i)}_j)^{a_i} (1 - t^{(i)}_j)^{b_i} \prod_{1 \leq j < j' \leq k_i} (t^{(i)}_j - t^{(i)}_{j'})^{2c} \Big) \\ \prod_{i=1}^{n-1} \prod_{j=1}^{k_{i+1}} \prod_{j'=1}^{k_i} (t^{(i+1)}_j - t^{(i)}_{j'})^{p-c}\,. \end{aligned}$$

Denote

$$S_{\boldsymbol{k}}(a, b, c) = \int_{[\boldsymbol{k}]_p} \Phi_{\boldsymbol{k}}\, dt\,. \tag{3.10}$$

The $\mathbb{F}_p$-integral $S_{\boldsymbol{k}}(a, b, c)$ is called the *$\mathbb{F}_p$-Selberg integral of type A_n.*

Conjecture 3.3 *Let n be a positive integer. Let $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{>0}$, $k_i > k_{i+1}$, $i = 1, \ldots, n-1$. Then, for any $(a, b, c) \in \mathcal{A}_{\boldsymbol{k}}$ we have the equality in $\mathbb{F}_p$:*

$$S_{\boldsymbol{k}}(a, b, c) \;=\; R_{\boldsymbol{k}}(a, b, c)\,. \tag{3.11}$$

For $n = 1$, formula (3.11) is proved in [21, Theorem 4.1]. For $n = 2$, formula (3.11) is proved in the next theorem.

Theorem 3.4 *Let $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2_{>0}$, $k_1 > k_2$. Then, for any $(a, b, c) \in \mathcal{A}_{\boldsymbol{k}}$ we have the equality in $\mathbb{F}_p$:*

$$S_{\boldsymbol{k}}(a, b, c) \;=\; R_{\boldsymbol{k}}(a, b, c)\,. \tag{3.12}$$

Formula (3.11) for $n = 2$ is deduced from formula (3.11) for $n = 1$ in Section 5.

More generally, for any k formula (3.11) for $n = k$ can be deduced from formula (3.11) for $n = k - 1$ similarly, see the sketch of that in Sect. 5.4. Details of that deduction will appear elsewhere. Because of that formula (3.11) for any n is formulated as a conjecture and not as a theorem.

Remark Theorem 3.4 can be extended to the case of $\boldsymbol{k}$ such that $k_1 \geq k_2$, but the structure of inequalities in Lemma 3.1 will depend on the appearance of the equality $k_1 = k_2$ in $\boldsymbol{k}$, and the proof of Theorem 3.4 will split into different sub-cases. To shorten the exposition, we restrict ourselves to $\boldsymbol{k}$ such that $k_1 > k_2$.

Example Here is the simplest A_2-type $\mathbb{F}_p$-Selberg integral formula with $k_1 = k_2 = 1$.

Theorem 3.5 *Assume that a, b_1, b_2, c are integers such that*

$$0 \leq a < p, \quad 0 < c \leq p, \quad 0 \leq b_2 - c + 1 < p,$$
$$0 \leq b_1 + b_2 - c + 1 < p, \quad p - 1 \leq a + b_1 + b_2 - c + 1 < 2p - 1.$$

Then, in $\mathbb{F}_p$ we have

$$\int_{[1;1]_p} t^a (1-t)^{b_1} (s-t)^{p-c} (1-s)^{b_2} dt\, ds$$
$$= \frac{a!\,(b_1 + b_2 - c + 1)!}{(a + b_1 + b_2 - c + 2 - p)!}\, \frac{(p-c)!\,(b_2)!}{(b_2 - c + 1)!}\,. \tag{3.13}$$

Proof Change variables $s = t + (1-t)v$, then the $\mathbb{F}_p$-integral becomes equal to

$$\int_{[1,1]_p} t^a (1-t)^{b_1 + b_2 - c + 1} v^{p-c} (1-v)^{b_2} dt\, dv.$$

Applying the $\mathbb{F}_p$-beta integral formula, we obtain the theorem. □

The simplest A_3-type $\mathbb{F}_p$-Selberg integral formula $k_1 = k_2 = k_3 = 1$ is given by the next theorem.

Theorem 3.6 *Let a, b_1, b_2, b_3, c be integers such that all factorials on the right-hand side of formula (3.14) are factorials of non-negative integers less than p. Then, in $\mathbb{F}_p$ we have*

$$\int_{[1;1;1]_p} t^a (1-t)^{b_1} (s-t)^{p-c} (1-s)^{b_2} (u-s)^{p-c} (1-u)^{b_2} dt\, ds\, du$$
$$= -\frac{a!\,(b_1 + b_2 + b_3 - 2c + 2)!}{(a + b_1 + b_2 + b_3 - 2c + 3 - p)!}\, \frac{(p-c)!\,(b_2 + b_3 - c + 1)!}{(b_2 + b_3 - 2c + 2)!}\, \frac{(p-c)!\,(b_3)!}{(b_3 - c + 1)!}\,. \tag{3.14}$$

Proof The proof is the same as the proof of the previous theorem. □

The versions of identities (3.13), (3.14) over complex numbers see in [16, Theorem 1].

3.3 Relation between the $\mathbb{F}_p$-Selberg integrals of types A_{n-1} and A_n

Theorem 3.7 *Let $n > 1$ and $\boldsymbol{k} = (k_1, \dots, k_n)$, $\boldsymbol{k}' = (k_1, \dots, k_{n-1})$, $b = (b_1, \dots, b_n)$, $b' = (b_1, \dots, b_{n-1})$. Assume that formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}']_p}(a, b', c)$ of type A_{n-1}. Also assume that $b_n = (k_{n-1} - k_n + 1)c - 1$. Then, formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}]_p}(a, b, c)$ of type A_n.*

Proof Under the assumption $b_n = (k_{n-1} - k_n + 1)c - 1$, all variables $(t_j^{(n)})$ in $\Phi_{[\boldsymbol{k}]_p}(\boldsymbol{t}; a, b, c)$ are used to reach the monomial $\prod_{j=1}^{k_n}(t_j^{(n)})^{k_{n-1}p-1}$ in the calculation of the $\mathbb{F}_p$-integral $S_{[\boldsymbol{k}]_p}(a, b, c)$. The remaining free variables $(t_j^{(i)})$ with $i < n$ all belong to the factor

$\Phi_{[\boldsymbol{k}']_p}(t^{(1)}, \dots, t^{(n-1)}; a, b', c)$ of $\Phi_{[\boldsymbol{k}]_p}(\boldsymbol{t}; a, b, c)$ and are used to calculate the coefficient of $\prod_{j=1}^{k_1}(t_j^{(1)})^{p-1}\prod_{i=2}^{n-1}\prod_{j=1}^{k_i}(t_j^{(i)})^{k_{i-1}p-1}$.

More precisely, under the assumptions of the theorem we have

$$S_{\boldsymbol{k}}(a, b, c) = (-1)^{b_n k_n + c k_n (k_n - 1)/2} \frac{(k_n c)!}{(c!)^{k_n}} S_{\boldsymbol{k}'}(a', b', c),$$

where $(-1)^{b_n k_n + c k_n (k_n - 1)/2} \frac{(k_n c)!}{(c!)^{k_n}}$ is the coefficient of $\prod_{j=1}^{k_n}(t_j^{(n)})^{k_{n-1}p-1}$ in the expansion of

$$\prod_{j=1}^{k_n}\prod_{j'=1}^{k_{n-1}}(t_j^{(n)} - t_{j'}^{(n-1)})^{p-c} \prod_{1\le j<j'\le k_n}(t_j^{(n)} - t_{j'}^{(n)})^{2c},$$

see Dyson's formula. We have $(-1)^{b_n k_n + c k_n (k_n - 1)/2} = (-1)^{(k_{n-1}k_n - k_n(k_n+1)/2)c - k_n}$. Hence,

$$\begin{aligned} S_{\boldsymbol{k}}(a, b, c) &= (-1)^{(k_{n-1}k_n - k_n(k_n+1)/2)c - k_n} \frac{(k_n c)!}{(c!)^{k_n}} S_{\boldsymbol{k}'}(a, b', c) \\ &= (-1)^{(k_{n-1}k_n - k_n(k_n+1)/2)c - k_n} \frac{(k_n c)!}{(c!)^{k_n}} R_{\boldsymbol{k}'}(a, b', c), \end{aligned}$$

where $S_{\boldsymbol{k}'}(a, b', c) = R_{\boldsymbol{k}'}(a, b', c)$ holds by assumptions. To prove the theorem, we need to show that

$$R_{[\boldsymbol{k}]_p}(a, (b', b_n), c) = (-1)^{(k_{n-1}k_n - k_n(k_n+1)/2)c - k_n} \frac{(k_n c)!}{(c!)^{k_n}} R_{\boldsymbol{k}'}(a, b', c).$$

Indeed, we have

$$\begin{aligned} R_{\boldsymbol{k}}(a, (b', b_n), c) = {} & R_{\boldsymbol{k}'}(a, b', c) \prod_{i=1}^{k_n} \frac{(ic)!}{c!} \prod_{i=1}^{k_n}(p + (i - k_{n-1} - 1)c)! \\ & \times \prod_{1\le s\le n}\prod_{i=1}^{k_n} \frac{(n - s + b_s + \dots + b_n + (i + s - n - 1)c)!}{(n - s + 1 + a_s + b_s + \dots + b_n + (i + s - n + k_s - k_{s-1} - 2)c - \delta_{s,1}p)!} \end{aligned}$$

$$\times \prod_{1\le s\le n-1}\prod_{i=1}^{k_n} \frac{(n-s+a_s+b_s+\dots+b_{n-1}+(i+s-n+k_s-k_{s-1}-3)c-\delta_{s,1}p)!}{(n-1-s+b_s+\dots+b_{n-1}+(i+s-r-1)c)!}$$

$$= R_{\boldsymbol{k}'}(a,b',c)\prod_{i=1}^{k_n}\frac{(ic)!}{c!}\prod_{i=1}^{k_n}\frac{(b_n+(i-1)c)!\,(p+(i-k_{n-1}-1)c)!}{(1+b_n+(i+k_n-k_{n-1}-2)c)!}\,.$$

$$= R_{\boldsymbol{k}'}(a,b',c)\prod_{i=1}^{k_n}\frac{(ic)!}{c!}\prod_{i=1}^{k_n}\frac{((i+k_{n-1}-k_n)c-1)!\,(p+(i-k_{n-1}-1)c)!}{((i-1)c))!}\,.$$

$$= R_{\boldsymbol{k}'}(a,b',c)\,(-1)^{(k_{n-1}k_n-k_n(k_n+1)/2)c-k_n}\,\frac{(k_nc)!}{(c!)^{k_n}}\,,$$

where in the last step we use the cancellation Lemma 2.1. The theorem is proved. □

Corollary 3.8 *Let $n>1$, $\boldsymbol{k}=(k_1,\dots,k_n)$, and $(a,b_1,c)\in\mathcal{A}_{(k_1)}$. Let $b=(b_1,\dots,b_n)$, where $b_i=(k_{i-1}-k_i+1)c-1$ for $i=2,\dots,n$. Then, formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}]_p}(a,b,c)$ of type A_n.*

Proof Formula (3.11) for the $\mathbb{F}_p$-Selberg integrals of type A_1 is proved in [21]. Hence, the corollary follows from Theorem 3.7 by induction on n. □

4 The A_2-type Selberg integral over $\mathbb{C}$

In this section, we formulate the A_2-type Selberg integral formula over $\mathbb{C}$, formulated and proved in [30], and sketch the proof of the formula, following [30]. In Sect. 5, we adapt this proof to prove the A_2-type $\mathbb{F}_p$-Selberg integral formula, that is, formula (3.11) for $n=2$.

4.1 The A_2-formula over $\mathbb{C}$

For $k_1\ge k_2\ge 0$, let $t=(t_1,\dots,t_{k_1})$, $s=(s_1,\dots,s_{k_2})$. Define the *master function*

$$\Phi(t;s)=\prod_{i=1}^{k_1}t_i^{\alpha-1}(1-t_i)^{\beta_1-1}\prod_{j=1}^{k_2}(1-s_j)^{\beta_2-1}\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}|s_j-t_i|^{-\gamma}$$
$$\times\prod_{1\le i<i'\le k_1}|t_i-t_{i'}|^{2\gamma}\prod_{1\le j<j'\le k_2}|s_j-s_{j'}|^{2\gamma} \qquad (4.1)$$

and the integral

$$\tilde S(\alpha,\beta_1,\beta_2,\gamma)=\int_{C^{k_1,k_2}[0,1]}\Phi(t;s)\,dt\,ds\,, \qquad (4.2)$$

where the integration cycle $C^{k_1,k_2}[0,1]$ is defined in [30, Section 3]. The explicit description of this cycle is of no importance in this paper.

Theorem 4.1 *([30, Theorem 3.3]) Let* $\alpha, \beta_1, \beta_2, \gamma$ *be complex numbers such that* $\operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta_1 > 0,\ \operatorname{Re}\beta_2 > 0,\ \operatorname{Re}\gamma < 0$ *and* $|\operatorname{Re}\gamma|$ *sufficiently small. Then,*

$$\tilde{S}(\alpha, \beta_1, \beta_2, \gamma) = \prod_{i=1}^{k_1-k_2} \frac{\Gamma(\beta_1 + (i-1)\gamma)}{\Gamma(\alpha + \beta_1 + (i + k_1 - 2)\gamma)}$$
$$\times \prod_{i=1}^{k_2} \frac{\Gamma(\beta_2 + (i-1)\gamma)}{\Gamma(1 + \beta_2 + (i + k_2 - k_1 - 2)\gamma)} \frac{\Gamma(\beta_1 + \beta_2 + (i-2)\gamma)}{\Gamma(\alpha + \beta_1 + \beta_2 + (i + k_2 - 3)\gamma)}$$
$$\times \prod_{i=1}^{k_2} \frac{\Gamma(1 + (i - k_1 - 1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{i=1}^{k_1} \frac{\Gamma(\alpha + (i-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\,. \tag{4.3}$$

In the next Sect. 4.2, 4.3, we sketch the proof of formula (4.3) following [30].

4.2 Weight functions

To evaluate $\tilde{S}(\alpha, \beta_1, \beta_2, \gamma)$, we introduce a collection of new integrals $J_{l_1,l_2,m}(\alpha, \beta_1, \beta_2, \gamma)$, which also can be evaluated explicitly, see [30].

For a function $f(t_1, \dots, t_k)$ set

$$\operatorname{Sym}_{t_1,\dots,t_k} f(t_1, \dots, t_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f(t_{\sigma_1}, \dots, t_{\sigma_k})\,.$$

Given $k_1 \geq k_2 \geq 0$, we say that a triple of non-negative integers (l_1, l_2, m) is *allowable* if $l_1 \leq k_1 - k_2 + l_2,\ l_2 \leq k_2$ and $m \leq \min(l_1, l_2)$. For any allowable triple (l_1, l_2, m) define the weight function

$$\begin{aligned} &W_{l_1,l_2,m}(t_1, \dots, t_{k_1}; s_1, \dots, s_{k_2}) = \\ &\quad = \operatorname{Sym}_{t_1,\dots,t_{k_1}} \operatorname{Sym}_{s_1,\dots,s_{k_2}} \\ &\qquad \times \Big(\prod_{a=1}^{l_1} t_a \prod_{a=l_1+1}^{k_1} (1 - t_a) \prod_{b=1}^{m} \frac{1 - s_b}{s_b - t_b} \prod_{b=l_2+1}^{k_2} \frac{1 - s_b}{s_b - t_{b+k_1-k_2}} \Big) \end{aligned}$$

and the integral

$$J_{l_1,l_2,m}(\alpha, \beta_1, \beta_2, \gamma) = \int_{C^{k_1,k_2}[0,1]} \Phi(t; s)\, W_{l_1,l_2,m}(t; s)\, dt\, ds\,.$$

In particular,

$$J_{0,k_2,0}(\alpha, \beta_1, \beta_2, \gamma) = \tilde{S}(\alpha, \beta_1 + 1, \beta_2, \gamma). \tag{4.4}$$

4.3 Representations of $\mathfrak{sl}_3$

Consider the complex Lie algebra $\mathfrak{sl}_3$ with standard generators $f_1, f_2, e_1, e_2, h_1, h_2$, simple roots σ_1, σ_2, fundamental weights ω_1, ω_2. Let $V_{\lambda_1}, V_{\lambda_2}$ be the irreducible $\mathfrak{sl}_3$-modules with highest weights

$$\lambda_1 = -\frac{\alpha}{\gamma}\,\omega_1\,, \qquad \lambda_2 = -\frac{\beta_1}{\gamma}\,\omega_1 - \frac{\beta_2}{\gamma}\,\omega_2$$

and highest weight vectors v_1, v_2. For $k_1 \geq k_2 \geq 0$, consider the weight subspace $V_{\lambda_1} \otimes V_{\lambda_2}[\lambda_1 + \lambda_2 - k_1\sigma_1 - k_2\sigma_2]$ of the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ and the singular weight subspace Sing $V_{\lambda_1} \otimes V_{\lambda_2}[\lambda_1 + \lambda_2 - k_1\sigma_1 - k_2\sigma_2]$ consisting of the vectors $w \in V_{\lambda_1} \otimes V_{\lambda_2}[\lambda_1 + \lambda_2 - k_1\sigma_1 - k_2\sigma_2]$ such that $e_1 w = 0$, $e_2 w = 0$. A basis of $V_{\lambda_1} \otimes V_{\lambda_2}[\lambda_1 + \lambda_2 - k_1\sigma_1 - k_2\sigma_2]$ is formed by the vectors

$$v_{l_1,l_2,m} = \frac{f_1^{k_1-k_2-l_1+l_2}[f_1, f_2]^{k_2-l_2} v_1 \otimes f_1^{l_1-m}[f_1, f_2]^m f_2^{l_2-m} v_2}{(k_1-k_2-l_1+l_2)!\,(k_2-l_2)!\,(l_1-m)!\,m!\,(l_2-m)!}$$

labeled by allowable triples (l_1, l_2, m). It is known from the theory of KZ equations that the vector

$$J = \sum_{l_1,l_2,m} (-1)^{l_1} J_{l_1,l_2,m}(\alpha, \beta_1, \beta_2, \gamma)\, v_{l_1,l_2,m}$$

is a singular vector, see [14, Theorem 2.4], [15, Corollary 10.3], cf. [19].

The singular vector equations $e_1 J = 0$, $e_2 J = 0$ are calculated with the help of the formulas:

$$\begin{aligned}
&h_1 v_1 = -\frac{\alpha}{\gamma}\, v_1, \qquad h_2 v_1 = 0,\\
&h_1 v_2 = -\frac{\beta_1}{\gamma}\, v_2, \qquad h_2 v_2 = -\frac{\beta_2}{\gamma}\, v_2\,,\\
&[h_1, f_1] = -2f_1, \quad [h_1, f_2] = f_2, \quad [h_2, f_1] = f_1, \qquad [h_2, f_2] = -2f_2,\\
&[e_1, f_1] = h_1, \qquad [e_1, f_2] = [e_2, f_1] = 0, \qquad [e_2, f_2] = h_2,\\
&[h_1, [f_1, f_2]] = -[f_1, f_2], \quad [h_2, [f_1, f_2]] = -[f_1, f_2],\\
&[e_1, [f_1, f_2]] = f_2, \quad [e_2, [f_1, f_2]] = -f_1.
\end{aligned}$$

Here are some of the singular vector relations.

Theorem 4.2 *(cf. [30, Theorem 5.2]) We have*

$$J_{0,l_2,0} = (-1)^{l_2} J_{0,0,0} \prod_{i=0}^{l_2-1} \frac{(k_1 - k_2 + i + 1)\gamma}{\beta_2 + i\gamma}\,. \tag{4.5}$$

Proof We have

$$
\begin{aligned}
& e_2 \frac{f_1^{k_1-k_2+i}[f_1, f_2]^{k_2-i} v_1 \otimes f_2^i v_2}{(k_1-k_2+i)!\,(k_2-i)!\,i!} \\
& \quad = -(k_1-k_2+i+1) \frac{f_1^{k_1-k_2+i+1}[f_1, f_2]^{k_2-i-1} v_1 \otimes f_2^i v_2}{(k_1-k_2+i+1)!\,(k_2-i-1)!\,i!} \\
& \qquad + \Big(-\frac{\beta_2}{\gamma} - i + 1\Big) \frac{f_1^{k_1-k_2+i}[f_1, f_2]^{k_2-i} v_1 \otimes f_2^{i-1} v_2}{(k_1-k_2+i)!\,(k_2-i)!\,(i-1)!}\,.
\end{aligned}
$$

Calculating the coefficient of $\frac{f_1^{k_1-k_2+i+1}[f_1,f_2]^{k_2-i-1} v_1 \otimes f_2^i v_2}{(k_1-k_2+i)!\,(k_2-i)!\,i!}$ in $e_2 J = 0$, we obtain

$$
(k_1-k_2+i+1)\gamma\, J_{0,i,0} + (\beta_2 + i\gamma)\, J_{0,i+1,0} = 0. \tag{4.6}
$$

This implies the theorem. □

Hence,

$$
J_{0,k_2,0}(\alpha, \beta_1, \beta_2, \gamma) = (-1)^{k_2} J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma) \prod_{i=0}^{k_2-1} \frac{(k_1-k_2+i+1)\gamma}{\beta_2 + i\gamma}\,. \tag{4.7}
$$

Combining (4.4) and (4.7), we observe that formula (4.3) is equivalent to the formula

$$
\begin{aligned}
J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma) = & \prod_{i=1}^{k_1-k_2} \frac{\Gamma(1+\beta_1+(i-1)\gamma)}{\Gamma(1+\alpha+\beta_1+(i+k_1-2)\gamma)} \\
& \times \prod_{i=1}^{k_2} \frac{\Gamma(1+\beta_2+(i-1)\gamma)}{\Gamma(1+\beta_2+(i+k_2-k_1-2)\gamma)} \frac{\Gamma(1+\beta_1+\beta_2+(i-2)\gamma)}{\Gamma(1+\alpha+\beta_1+\beta_2+(i+k_2-3)\gamma)} \\
& \times \prod_{i=1}^{k_2} \frac{\Gamma((i-k_1-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)} \prod_{i=1}^{k_1} \frac{\Gamma(\alpha+(i-1)\gamma)\,\Gamma(i\gamma)}{\Gamma(\gamma)}\,.
\end{aligned} \tag{4.8}
$$

Denote by $R_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$ the right-hand side of (4.8).

To prove (4.8), we use the following observation. The weight subspace $V_{\lambda_1}[\lambda_1 - k_1\sigma_1 - k_2\sigma_2] \subset V_{\lambda_1}$ is one-dimensional with a basis vector

$$
v_{0,0,0} = \frac{f_1^{k_1-k_2}[f_1, f_2]^{k_2} v_1 \otimes v_2}{(k_1-k_2)!\,(k_2)!}\,.
$$

By [15, Theorem 5.1], the vector-valued function $J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma) v_{0,0,0}$ satisfies the dynamical difference equations introduced in [29],

$$
J_{0,0,0}(\alpha, \beta_1 - 1, \beta_2, \gamma) v_{0,0,0} = J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)\, \mathbb{B}_1 v_{0,0,0}\,, \tag{4.9}
$$

$$
J_{0,0,0}(\alpha, \beta_1, \beta_2 - 1, \gamma) v_{0,0,0} = J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)\, \mathbb{B}_2 v_{0,0,0}\,. \tag{4.10}
$$

Here, $\mathbb{B}_1$, $\mathbb{B}_2$ are certain linear operators acting on V_{λ_1} and preserving the weight decomposition of V_{λ_1}, see formulas for these operators in the example in [15, Sect. 7.1] and in [15, Sect. 3.1], also see [29, Formula (8)].

Written explicitly equations (4.9), (4.10) give us the difference equations for the scalar function $J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$ with respect to the shift of the variables $\beta_1 \to \beta_1 - 1$ and $\beta_2 \to \beta_2 - 1$,

$$J_{0,0,0}(\alpha, \beta_1 - 1, \beta_2, \gamma) = J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma) \prod_{i=1}^{k_1-k_2} \frac{\alpha + \beta_1 + (i + k_1 - 2)\gamma}{\beta_1 + (i-1)\gamma}$$

$$\times \prod_{i=1}^{k_2} \frac{\alpha + \beta_1 + \beta_2 + (i + k_2 - 3)\gamma}{\beta_1 + \beta_2 + (i-2)\gamma}, \tag{4.11}$$

$$J_{0,0,0}(\alpha, \beta_1, \beta_2 - 1, \gamma) = J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$$

$$\times \prod_{i=1}^{k_2} \frac{\beta_2 + (i + k_2 - k_1 - 2)\gamma}{\beta_2 + (i-1)\gamma} \frac{\alpha + \beta_1 + \beta_2 + (i + k_2 - 3)\gamma}{\beta_1 + \beta_2 + (i-2)\gamma}. \tag{4.12}$$

The difference equations for $J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$ are the same as the difference equations for the function $R_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$ with respect to the shift of the variables $\beta_1 \to \beta_1 - 1$ and $\beta_2 \to \beta_2 - 1$. Therefore, the functions $J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$ and $R_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)$ are proportional up to a periodic function of β_1, β_2. The periodic function can be fixed by comparing asymptotics as $\operatorname{Re} \beta_1 \to \infty$, $\operatorname{Re} \beta_2 \to \infty$. This finishes the proof in [30] of formulas (4.8) and (4.3).

5 The A_2-type Selberg integrals over $\mathbb{F}_p$

5.1 Relations between $\mathbb{F}_p$-integrals

For $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$ and integers
$0 < a, b_1, b_2, c < p$ define the master polynomial

$$\Phi_{\boldsymbol{k}}(t; s; a, b_1, b_2, c) = \prod_{i=1}^{k_1} t_i^a (1 - t_i)^{b_1} \prod_{j=1}^{k_2} (1 - s_j)^{b_2} \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} (s_j - t_i)^{p-c}$$

$$\times \prod_{1 \le i < i' \le k_1} (t_i - t_{i'})^{2c} \prod_{1 \le j < j' \le k_2} (s_j - s_{j'})^{2c} \tag{5.1}$$

and the $\mathbb{F}_p$-integral

$$S_{\boldsymbol{k}}(a, b_1, b_2, c) = \int_{[\boldsymbol{k}]_p} \Phi_{\boldsymbol{k}}(t; s; a, b_1, b_2, c)\, dt\, ds\,, \tag{5.2}$$

where the p-cycle $[\boldsymbol{k}]_p$ is defined in (2.5). This is the A_2-type $\mathbb{F}_p$-Selberg integral, see (3.10).

For an allowable triple (l_1, l_2, m) define the $\mathbb{F}_p$-integral

$$I_{l_1,l_2,m}(a, b_1, b_2, c) = \int_{[\boldsymbol{k}]_p} \Phi_{\boldsymbol{k}}(t; s; a, b_1, b_2, c) \frac{W_{l_1,l_2,m}(t; s)}{\prod_{i=1}^{k_1} t_i(1 - t_i) \prod_{j=1}^{k_2}(1 - s_j)} \, dt \, ds \,, \quad (5.3)$$

where $W_{l_1,l_2,m}(t; s)$ is the weight function defined in Section 4.2.

Clearly, we have

$$I_{0,k_2,0}(a, b_1, b_2, c) = S_{\boldsymbol{k}}(a - 1, b_1, b_2 - 1, c). \quad (5.4)$$

Denote

$$B_0(a, b_1, b_2, c) = (-1)^{k_2} \prod_{i=0}^{k_2-1} \frac{(k_1 - k_2 + i + 1)c}{b_2 + ic} \,,$$

$$B_1(a, b_1, b_2, c) = \prod_{i=1}^{k_1-k_2} \frac{a+b_1+(i + k_1 - 2)c}{b_1 + (i - 1)c} \prod_{i=1}^{k_2} \frac{a + b_1 + b_2 + (i + k_2 - 3)c}{b_1 + b_2 + (i - 2)c} \,,$$

$$B_2(a, b_1, b_2, c) = \prod_{i=1}^{k_2} \frac{b_2 + (i + k_2 - k_1 - 2)c}{b_2 + (i - 1)c} \frac{a + b_1 + b_2 + (i + k_2 - 3)c}{b_1 + b_2 + (i - 2)c} \,. \quad (5.5)$$

Theorem 5.1 *Assume that* $k_1 < p$.

(i) Assume that every factor in B_0 *in the numerator or denominator is a nonzero element of* $\mathbb{F}_p$. *Then,*

$$I_{0,k_2,0}(a, b_1, b_2, c) = B_0(a, b_1, b_2, c) \, I_{0,0,0}(a, b_1, b_2, c) \,. \quad (5.6)$$

(ii) Assume that every factor in B_1 *in the numerator or denominator is a nonzero element of* $\mathbb{F}_p$ *and* $b_1 > 1$, *then*

$$I_{0,0,0}(a, b_1 - 1, b_2, c) \; = \; B_1(a, b_1, b_2, c) \, I_{0,0,0}(a, b_1, b_2, c). \quad (5.7)$$

(iii) Assume that every factor in B_2 *in the numerator or denominator is a nonzero element of* $\mathbb{F}_p$ *and* $b_2 > 1$, *then*

$$I_{0,0,0}(a, b_1, b_2 - 1, c) \; = \; B_2(a, b_1, b_2, c) \, I_{0,0,0}(a, b_1, b_2, c). \quad (5.8)$$

Proof Equation (5.6) is an $\mathbb{F}_p$-analog of equation (4.7), and its proof is analogous to the proof of equation (4.7).

More precisely, consider the complex Lie algebra $\mathfrak{sl}_3$ with standard generators $f_1, f_2, e_1, e_2, h_1, h_2$, simple roots σ_1, σ_2, fundamental weights ω_1, ω_2. Let $V_{\lambda_1}, V_{\lambda_2}$

be the irreducible $\mathfrak{sl}_3$-modules with highest weights

$$\lambda_1 = -\frac{a}{c+p}\,\omega_1\,, \qquad \lambda_2 = -\frac{b_1}{c+p}\,\omega_1 - \frac{b_2}{c+p}\,\omega_2$$

and highest weight vectors v_1, v_2. The module V_{λ_1} has a basis $(f_1^{r_1}[f_1, f_2]^{r_2} v_1)$ labeled by non-negative integers r_1, r_2, and the module V_{λ_2} has a basis $(f_1^{r_1}[f_1, f_2]^{r_2} f_2^{r_3} v_2)$ labeled by non-negative integers r_1, r_2, r_3. For every generator of $\mathfrak{sl}_3$, the matrix of its action on V_{λ_1} or on V_{λ_2} in these bases is a polynomial in $-\frac{a}{c+p}$ $-\frac{b_1}{c+p}$, $-\frac{b_2}{c+p}$ with integer coefficients.

Consider the Lie algebra $\mathfrak{sl}_3$ over the field $\mathbb{F}_p$. Let $V_{\lambda_1}^{\mathbb{F}_p}$ be the vector space over $\mathbb{F}_p$ with basis $(f_1^{r_1}[f_1, f_2]^{r_2} v_1)$ labeled by non-negative integers r_1, r_2 and with the action of $\mathfrak{sl}_3$ defined by the same formulas as on V_{λ_1} but reduced modulo p. Similarly, we define the $\mathfrak{sl}_3$-module $V_{\lambda_2}^{\mathbb{F}_p}$.

Recall $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$. Consider the weight subspace $V_{\lambda_1}^{\mathbb{F}_p} \otimes V_{\lambda_2}^{\mathbb{F}_p}[\lambda_1 + \lambda_2 - k_1\sigma_1 - k_2\sigma_2]$ of the tensor product $V_{\lambda_1}^{\mathbb{F}_p} \otimes V_{\lambda_2}^{\mathbb{F}_p}$. This weight subspace has a basis formed by the vectors

$$v_{l_1,l_2,m} = \frac{f_1^{k_1-k_2-l_1+l_2}[f_1, f_2]^{k_2-l_2} v_1 \otimes f_1^{l_1-m}[f_1, f_2]^m f_2^{l_2-m} v_2}{(k_1-k_2-l_1+l_2)!\,(k_2-l_2)!\,(l_1-m)!\,m!\,(l_2-m)!}$$

labeled by allowable triples (l_1, l_2, m).

Lemma 5.2 *The vector*

$$I = \sum_{l_1,l_2,m} (-1)^{l_1} I_{l_1,l_2,l_m}(a, b_1, b_2, c)\, v_{l_1,l_2,m}$$

is a singular vector of $V_{\lambda_1}^{\mathbb{F}_p} \otimes V_{\lambda_2}^{\mathbb{F}_p}$*, that is,* $e_1 I = 0,\ e_2 I = 0$.

Proof Equations $e_1 I = 0,\ e_2 I = 0$ are $\mathbb{F}_p$-analogs of equations $e_1 J = 0,\ \ e_2 J = 0$ over $\mathbb{C}$.

For $i = 1, 2$, the vector $e_i J$ is the integral of a certain differential $k_1 + k_2$-form μ_i. It is shown in [26, Theorems 6.16.2], [14, Theorem 2.4] that $\mu_i = d\nu_i$, where ν_i is some explicitly written differential $k_1 + k_2 - 1$-form. This implies $e_i J = 0$ by Stokes' theorem.

The vector $e_i I$ is the $\mathbb{F}_p$-integral of the same μ_i reduced modulo p. It is explained in [26, Section 4] that the differential form ν_i also can be reduced modulo p and this implies that the $\mathbb{F}_p$-integral $e_i I$ is zero by Lemma 2.3. Cf. the proof of [26, Theorem 2.4].

Lemma 5.2 implies the equations

$$(k_1 - k_2 + i + 1)c\, I_{0,i,0} + (b_2 + ic)\, I_{0,i+1,0} = 0 \tag{5.9}$$

for $i = 0, \ldots, k_2 - 1$, similarly to the proof of equations (4.6). The iterated application of equation (5.9) implies equation (5.6).

The proof of equations (5.7), (5.8) is parallel to the proof of equations (4.11), (4.12). We prove (5.7). The proof of (5.8) is similar.

Equation (4.11) follows from equation (4.9):

$$J_{0,0,0}(\alpha, \beta_1 - 1, \beta_2, \gamma) v_{0,0,0} - J_{0,0,0}(\alpha, \beta_1, \beta_2, \gamma)\, \mathbb{B}_1 v_{0,0,0} = 0 \tag{5.10}$$

and equation $\mathbb{B}_1 v_{0,0,0} = B_1 v_{0,0,0}$ in V_{λ_1}. The explicit formulas for $\mathbb{B}_1$ show that under the assumptions of Theorem 5.1 the action of $\mathbb{B}_1$ on $v_{0,0,0}$ is well-defined modulo p and gives the same result $\mathbb{B}_1 v_{0,0,0} = B_1 v_{0,0,0}$ but in $V_{\lambda_1}^{\mathbb{F}_p}$.

The proof of (5.10) in [15] goes as follows. The left-hand side of (5.10) is a vector-valued integral of a suitable differential $k_1 + k_2$-form μ. It is shown in [14, Theorem 5.1] that $\mu = d\nu$, where ν is some explicitly written differential $k_1 + k_2 - 1$-form. This implies (5.10) by Stokes' theorem.

The p-analog of the left-hand side of (5.10) is the element

$$I_{0,0,0}(a, b_1 - 1, b_2, c) v_{0,0,0} - I_{0,0,0}(a, b_1, b_2, c)\, \mathbb{B}_1 v_{0,0,0} \in V_{\lambda_1}^{\mathbb{F}_p}. \tag{5.11}$$

This element is the $\mathbb{F}_p$-integral of the same μ reduced modulo p. It is explained in [26, Section 4] that the differential form ν also can be reduced modulo p and this implies that the $\mathbb{F}_p$-integral in the left-hand side of (5.11) equals zero by Lemma 2.3. Hence, equation (5.7) is proved and Theorem 5.1 is proved. □

5.2 Proof of Theorem 3.4

Recall the set of admissible parameters $\mathcal{A}_{\boldsymbol{k}}$ introduced in Section 3.1 for $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$.

Lemma 5.3 *Assume that* $(a, b_1 - 1, b_2, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$. *Then,*

$$\begin{aligned} S_{\boldsymbol{k}}(a, b_1 - 1, b_2, c) = S_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_1 - k_2} \frac{1 + a + b_1 + (i + k_1 - 2)c - p}{b_1 + (i-1)c} \\ \times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \end{aligned} \tag{5.12}$$

Assume that $(a, b_1, b_2 - 1, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$. *Then,*

$$\begin{aligned} S_{\boldsymbol{k}}(a, b_1, b_2 - 1, c) = S_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_2} \frac{1 + b_2 + (i + k_2 - k_1 - 2)c}{b_2 + (i-1)c} \\ \times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \end{aligned} \tag{5.13}$$

Proof The lemma follows from formulas (5.4) and (5.6) and Theorem 5.1. □

For $n = 2$, formula (3.1) takes the form:

$$R_{\boldsymbol{k}}(a, b_1, b_2, c) = (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(b_1 + (i-1)c)!}{(1 + a + b_1 + (i + k_1 - 2)c - p)!}$$
$$\times \prod_{i=1}^{k_2} \frac{(b_2 + (i-1)c)!}{(1 + b_2 + (i + k_2 - k_1 - 2)c)!} \frac{(1 + b_1 + b_2 + (i-2)c)!}{(2 + a + b_1 + b_2 + (i + k_1 - 3)c - p)!}$$
$$\times \prod_{i=1}^{k_1}(a + (i-1)c)! \prod_{i=1}^{k_2}(p + (i - k_1 - 1)c)! \prod_{r=1}^{2}\prod_{i=1}^{k_r} \frac{(ic)!}{c!}. \tag{5.14}$$

Lemma 5.4 *Assume that* $(a, b_1 - 1, b_2, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$. *Then,*

$$R_{\boldsymbol{k}}(a, b_1 - 1, b_2, c) = R_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_1-k_2} \frac{1 + a + b_1 + (i + k_1 - 2)c - p}{b_1 + (i-1)c}$$
$$\times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \tag{5.15}$$

Assume that $(a, b_1, b_2 - 1, c), (a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}(a, b_1, b_2, c)$. *Then,*

$$R_{\boldsymbol{k}}(a, b_1, b_2 - 1, c) = R_{\boldsymbol{k}}(a, b_1, b_2, c) \prod_{i=1}^{k_2} \frac{1 + b_2 + (i + k_2 - k_1 - 2)c}{b_2 + (i-1)c}$$
$$\times \prod_{i=1}^{k_2} \frac{2 + a + b_1 + b_2 + (i + k_1 - 3)c - p}{1 + b_1 + b_2 + (i-2)c}. \tag{5.16}$$

□

By Lemmas 5.3 and 5.4, the functions $S_{\boldsymbol{k}}(a, b_1, b_2, c)$ and $R_{\boldsymbol{k}}(a, b_1, b_2, c)$ defined on $\mathcal{A}_{\boldsymbol{k}}$ satisfy the same difference equations with respects to the shifts of variables $b_1 \to b_1 - 1$ and $b_2 \to b_2 - 1$.

Lemma 5.5 *Assume that* a, c *are positive integers such that* $0 < k_1 c \le p - 1$, $a + (k_1 - 1)c < p - 1$. *Then, the point*

$$(a, b_1, b_2, c) = (a, p - 1 - (a + (k_1 - 1)c), (k_1 - k_2 + 1)c - 1, c) \tag{5.17}$$

lies in $\mathcal{A}_{\boldsymbol{k}}$.

Proof If (a, b_1, b_2, c) is given by (5.17), then

$$R_{\boldsymbol{k}} = (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(p - 1 - (a + (k_1 - i)c))!}{((i-1)c)!}$$

$$\times \prod_{i=1}^{k_2} \frac{((k_1-k_2+i)c-1)!}{((i-1)c)!} \frac{(p-1-(a+(k_2-i)c))!}{((k_1-k_2+i-1)c)!}$$
$$\times \prod_{i=1}^{k_1}(a+(i-1)c)! \prod_{i=1}^{k_2}(p+(i-k_1-1)c)! \prod_{r=1}^{2}\prod_{i=1}^{k_r} \frac{(ic)!}{c!}\,.$$

This proves the lemma. □

Lemma 5.6 *Assume that $\tilde a, \tilde c$ are non-negative integers such that $0 < k_1\tilde c \le p-1$, $\tilde a + (k_1-1)\tilde c \le p-1$. Denote by $\mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ the set of all $(a, b_1, b_2, c) \in \mathcal{A}_{\boldsymbol{k}}$ such that $a = \tilde a$, $c = \tilde c$. Then, $\mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ consists of the pairs (b_1, b_2) of non-negative integers satisfying the inequalities*

$$p-1-(\tilde a+(k_1-1)\tilde c) \le b_1, \qquad (k_1-k_2+1)\tilde c - 1 \le b_2 \tag{5.18}$$

and some other inequalities of the form

$$b_1 \le A_1, \qquad b_2 \le A_2 \qquad b_1+b_2 \le A_{12}, \tag{5.19}$$

where $A_1,\ A_2,\ A_{12}$ are some integers such that

$$A_1 \ge p-1-(\tilde a+(k_1-1)\tilde c), \quad A_2 \ge (k_1-k_2+1)\tilde c - 1,$$
$$A_{12} \ge p-1-(\tilde a+(k_1-1)\tilde c) + (k_1-k_2+1)\tilde c - 1\,.$$

Proof The lemma follows from Lemmas 3.1 and 3.2. □

Corollary 5.7 *Any point $(\tilde a, b_1, b_2, \tilde c) \in \mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ can be connected with the point $(\tilde a,\ p-1-(\tilde a+(k_1-1)\tilde c),\ (k_1-k_2+1)\tilde c-1,\ \tilde c) \in \mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ by a piece-wise linear path in $\mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$ consisting of the vectors $(0,-1,0,0)$ or $(0,0,-1,0)$.* □

Proof of Theorem 3.4. For $n=1$, $\boldsymbol{k}=(k_1)$ and the point $(\tilde a,\ p-1-(\tilde a+(k_1-1)\tilde c),\ \tilde c)$ formula (3.11) holds by [21, Theorem 4.1].

For $n=2$, $\boldsymbol{k}=(k_1,k_2)$ and the point $(\tilde a,\ p-1-(\tilde a+(k_1-1)\tilde c),\ (k_1-k_2+1)\tilde c-1,\ \tilde c)$, formula (3.11) holds by Lemma 5.5 and Theorem 3.7.

For $n=2$, $\boldsymbol{k}=(k_1,k_2)$ and arbitrary $(\tilde a, b_1, b_2, \tilde c) \in \mathcal{A}_{\boldsymbol{k}}(\tilde a, \tilde c)$, formula (3.11) holds by Lemmas 5.3, 5.4 and Corollary 5.7. Theorem 3.4 for $n=2$ is proved. □

Corollary 5.8 *Let $n > 2$, $\boldsymbol{k} = (k_1, \dots, k_n)$, and $(a, (b_1, b_2), c) \in \mathcal{A}_{(k_1,k_2)}$. Let $b = (b_1, \dots, b_n)$, where $b_i = (k_{i-1}-k_i+1)c-1$ for $i = 3, \dots, n$. Then, formula (3.11) holds for the $\mathbb{F}_p$-Selberg integral $S_{[\boldsymbol{k}]_p}(a, b, c)$ of type A_n.*

Proof Formula (3.11) for the $\mathbb{F}_p$-Selberg integrals of type A_2 is proved in Theorem 3.4. Hence, the corollary follows from Theorem 3.7 by induction on n. □

5.3 Evaluation of $I_{0,0,0}(a, b_1, b_2, c)$

In this section, we evaluate $I_{0,0,0}(a, b_1, b_2, c)$ without using the evaluation of $S_{\boldsymbol{k}}(a, b_1, b_2, c)$.

Theorem 5.9 *Let $\boldsymbol{k} = (k_1, k_2)$, $k_1 > k_2 > 0$ and $a, b_1, b_2, c \in \mathbb{Z}_{>0}$. Assume that $a + (k_1 - 1)c < p$ and all factorials on the right-hand side of the next formula are factorials of the non-negative integers less than p. Then,*

$$I_{0,0,0}(a, b_1, b_2, c) = (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(b_1 + (i-1)c)!}{(a + b_1 + (i + k_1 - 2)c - p)!}$$
$$\times \prod_{i=1}^{k_2} \frac{(b_2 + (i-1)c)!}{(b_2 + (i + k_2 - k_1 - 2)c)!} \frac{(b_1 + b_2 + (i-2)c)!}{(a + b_1 + b_2 + (i + k_1 - 3)c - p)!}$$
$$\times \prod_{i=1}^{k_1} (a + (i-1)c - 1)! \prod_{i=1}^{k_2} (p + (i - k_1 - 1)c - 1)! \prod_{r=1}^{2} \prod_{i=1}^{k_i} \frac{(ic)!}{c!}. \quad (5.20)$$

Proof The proof is parallel to the proof of Theorem 3.4 for $n = 2$.

Denote by $\mathcal{A}^I_{\boldsymbol{k}}$ the set of all $a, b_1, b_2, c \in \mathbb{Z}_{>0}$ satisfying the assumptions of Theorem 5.9. Notice that if $(a, b_1, b_2, c) \in \mathcal{A}^I_{\boldsymbol{k}}$, then

$$b_1 \geq p - (a + (k_1 - 1)c), \qquad b_2 \geq (k_1 - k_2 + 1)c. \quad (5.21)$$

□

Lemma 5.10 *Formula (5.20) holds if $b_2 = (k_1 - k_2 + 1)c$ and $(a, b_1, (k_1 - k_2 + 1)c, c) \in \mathcal{A}^I_{\boldsymbol{k}}$.*

Proof If $b_2 = (k_1 - k_2 + 1)c$, then all variables (s_j) in the integrand of $I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c)$ are used to reach the monomial $\prod_{j=1}^{k_n} s_j^{k_1 p - 1}$ in the calculation of the $\mathbb{F}_p$-integral $I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c)$. The remaining free variables (t_i) all belong to the factor

$$\Phi_{(k_1)}(t_1, \ldots, t_{k_1}, a - 1, b_1, c) = \prod_{1 \leq i < i' \leq k_1} (t_i - t_{i'})^{2c} \prod_{i=1}^{k_1} t_i^{a-1} (1 - t_i)^{b_1}$$

of the integrand and are used to calculate the coefficient of the monomial $\prod_{j=1}^{k_1} t_i^{p-1}$.

More precisely, under the assumptions of the theorem we have

$$I_{0,0,0}(a, b_1, (k_1 - k_2 + 1)c, c) = (-1)^{b_2 k_2 + c k_2 (k_2 - 1)/2} \frac{(k_2 c)!}{(c!)^{k_2}} S_{(k_1)}(a - 1, b_1, c),$$

cf. the proof of Theorem 3.7. We have $(-1)^{b_2 k_2 + c k_2 (k_2 - 1)/2} = (-1)^{(k_1 k_2 - k_2 (k_2 + 1)/2)c - k_2}$.

By [21, Theorem 4.1] we have $S_{(k_1)}(a-1,b_1,c) = R_{(k_1)}(a-1,b_1,c)$. Hence,

$$I_{0,0,0}(a,b_1,(k_1-k_2+1)c,c) = (-1)^{(k_1k_2-k_2(k_2+1)/2)c}(-1)^{k_1+k_2} \times \frac{(k_2c)!}{(c!)^{k_2}} \prod_{j=1}^{k_1} \frac{(jc)!}{c!} \frac{(a+(j-1)c-1)!\,(b+(j-1)c)!}{(a+b+(k_1+j-2)c-p)!}. \tag{5.22}$$

Denote by $R^I_{\boldsymbol{k}}(a,b_1,b_2,c)$ the right-hand side in (5.20). We have

$$\begin{aligned} R^I_{\boldsymbol{k}}(a,b_1,(k_1-k_2+1)c,c) &= (-1)^{k_1+k_2} \prod_{i=1}^{k_1-k_2} \frac{(b_1+(i-1)c)!}{(a+b_1+(i+k_1-2)c-p)!} \\ &\times \prod_{i=1}^{k_2} \frac{((k_1-k_2+1)c+(i-1)c)!}{((k_1-k_2+1)c+(i+k_2-k_1-2)c)!} \\ &\times \prod_{i=1}^{k_2} \frac{(b_1+(k_1-k_2+1)c+(i-2)c)!}{(a+b_1+(k_1-k_2+1)c+(i+k_1-3)c-p)!} \\ &\times \prod_{i=1}^{k_1}(a+(i-1)c-1)! \prod_{i=1}^{k_2}(p+(i-k_1-1)c-1)! \prod_{r=1}^{2}\prod_{i=1}^{k_i} \frac{(ic)!}{c!}. \\ &= (-1)^{k_1+k_2}(-1)^{(k_1k_2-k_2(k_2+1)/2)c} \\ &\frac{(k_2c)!}{(c!)^{k_2}} \prod_{j=1}^{k_1} \frac{(ic)!}{c!} \frac{(a+(j-1)c-1)!\,(b+(j-1)c)!}{(a+b+(k_1+j-2)c-p)!}, \end{aligned} \tag{5.23}$$

where we used the cancellation Lemma 2.1 in the last step. Hence,

$I_{0,0,0}(a,b_1,(k_1-k_2+1)c,c) = R^I_{\boldsymbol{k}}(a,b_1,(k_1-k_2+1)c,c)$ and Lemma 5.10 is proved. □

Comparing equations (5.7), (5.8) and the formula for $R^I_{\boldsymbol{k}}(a,b_1,b_2,c)$, we conclude that the functions $I_{0,0,0}(a,b_1,b_2,c)$ and $R^I_{\boldsymbol{k}}(a,b_1,b_2,c)$ on $\mathcal{A}^I_{\boldsymbol{k}}$ satisfy the same difference equations with respect to the shifts of variables $b_1 \to b_1-1$ and $b_2 \to b_2-1$ and are equal if b_2 takes its minimal value $(k_1-k_2+1)c$. This implies Theorem 5.9, cf. Lemmas 5.5, 5.6 and Corollary 5.7.

5.4 Sketch of the proof of formula (3.11) for $n > 2$

The proof is parallel to the proof of Theorem 3.4.

Analogously to the proof of Theorem 5.1, consider the Lie algebra $\mathfrak{sl}_{n+1}$ and its representations $V^{\mathbb{F}_p}_{\lambda_1}$ and $V^{\mathbb{F}_p}_{\lambda_2}$ over $\mathbb{F}_p$ with highest weights

$$\lambda_1 = -\frac{a}{c+p}\,\omega_1\,, \qquad \lambda_2 = -\frac{b_1}{c+p}\,\omega_1 - \cdots - \frac{b_n}{c+p}\,\omega_n$$

and highest weight vectors v_1, v_2. Consider the PBW basis $\mathcal{B} = (u)$ of the weight subspace $V^{\mathbb{F}_p}_{\lambda_1} \otimes V^{\mathbb{F}_p}_{\lambda_2}[\lambda_1 + \lambda_2 - \sum_{i=1}^n k_i\sigma_i]$ like in the proof of Theorem 5.1. We distinguish two elements of that basis:

$$u_1 = \frac{f_1^{k_1-k_2}[f_1, f_2]^{k_2-k_3} \dots [f_1, [f_2, \dots, [f_{n-1}, f_n]\dots]]^{k_n} v_1 \otimes v_2}{(k_1-k_2)!\,(k_2-k_3)!\,\dots\,(k_n)!},$$

$$u_2 = \frac{f_1^{k_1} v_1 \otimes f_2^{k_2} \dots f_n^{k_n} v_2}{(k_1)!\,(k_2)!\,\dots\,(k_n)!}.$$

For $n = 2$, these vectors are the vectors $v_{0,0,0}$ and $v_{0,k_2,0}$ in the proof of Theorem 5.1.

To any basis vector $u \in \mathcal{B}$, we assign the weight function $W_u(t)$ defined in [19, Section 6.1], here t is the collection of variables defined in (3.9). Then, we consider the $\mathbb{F}_p$-integrals

$$I_u(a, b, c) = \int_{[\boldsymbol{k}]_p} \Phi(t, a, b, c) W_u(t)\, dt.$$

It follows from the formulas for the weight functions that

$$I_{u_2}(a, b, c) = S_{\boldsymbol{k}}(a-1, b_1, b_2-1, \dots, b_n-1, c),$$

cf. (5.4). It is known from the theory of KZ equations that the vector

$$I(a, b, c) = \sum_{u \in \mathcal{B}} I_u(a, b, c)\, u$$

is a singular vector in $V^{\mathbb{F}_p}_{\lambda_1} \otimes V^{\mathbb{F}_p}_{\lambda_2}[\lambda_1 + \lambda_2 - \sum_{i=1}^n k_i\sigma_i]$. From the singular vector condition, it follows that

$$I_{u_2}(a, b, c) = B_0(a, b, c)\, I_{u_1}(a, b, c), \tag{5.24}$$

where $B_0(a, b, c)$ is an explicit expression like in (5.6).

The vector $I_{u_1}(a, b, c)u_1$ is a generator of the one-dimensional weight subspace $V^{\mathbb{F}_p}_{\lambda_1}[\lambda_1 + \lambda_2 - \sum_{i=1}^n k_i\sigma_i]$. That vector satisfies the dynamical equations defined in [29]. The dynamical equations take the form

$$I_{u_1}(a, b_1, \dots, b_i - 1, \dots, b_n, c) = B_i(a, b, c) I_{u_1}(a, b, c), \qquad i = 1, \dots, n, \tag{5.25}$$

where $B_i(a, b, c)$ are explicit products like in (5.7) and (5.8).

Equation (5.24) and difference equations (5.25) imply that the two functions $S_{\boldsymbol{k}}(a, b, c)$ and $R_{\boldsymbol{k}}(a, b, c)$, defined on the set $\mathcal{A}_{\boldsymbol{k}}$, satisfy the same difference equations with respect to the shift of variables $b_i \to b_i - 1$ for $i = 1, \dots, n$. By Corollary

3.8, we also know that the two functions are equal at the distinguished point

$$(a, p-1-(a+(k_1-1)c), (k_1-k_2+1)c-1, \ldots, (k_{n-1}-k_n+1)c-1, c) \in \mathcal{A}_{\boldsymbol{k}} .$$

This implies that the two functions are equal (cf. Corollary 5.7) and formula (3.11) holds for any n. The details of this sketch will be published elsewhere.

References

1. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)
2. Anderson, G..W.: The evaluation of Selberg sums. C. R. Acad. Sci. Paris Sér. I Math **311**(8), 469–472 (1990)
3. Aomoto, K.: Jacobi polynomials associated with Selberg integral. Siam J. Math **18**(2), 545–549 (1987)
4. Askey, R.: Some basic hypergeometric extensions of integrals of Selberg and Andrews. SIAM J. Math. Anal. **11**, 938–951 (1980)
5. Cherednik, I.: *From double Hecke algebra to analysis*, Doc.Math.J.DMV, Extra Volume ICM. II, 527–531 (1998)
6. Etingof, P., Frenkel, I., Kirillov, A.: *Lectures on representation theory and Knizhnik-Zamokodchikov equations*, Mathematical Surveys and Monographs, 58, AMS, Providence, RI, (1998). xiv+198 pp. ISBN: 0-8218-0496-0
7. Evans, R.J.: The evaluation of Selberg character sums. L'Enseign. Math. **37**, 235–248 (1991)
8. Felder, G., Stevens, L., Varchenko, A.: Elliptic Selberg integrals and conformal blocks. Math. Res. Lett. **10**(5–6), 671–684 (2003)
9. Felder, G., Varchenko, A.: Integral representation of solutions of the elliptic Knizhnik-Zamolodchikov-Bernard equations. Int. Math. Res. Notices **5**, 221–233 (1995)
10. Forrester, P.. J., Warnaar, S.. O.: The importance of the Selberg integral. Bull. Amer. Math. Soc. (N.S.) **45**, 489–534 (2008)
11. Habsieger, L.: Une q-intégrale de Selberg et Askey. SIAM J. Math. Anal. **19**, 1475–1489 (1988)
12. Kaneko, J.: q-Selberg integrals and Macdonald polynomials. Ann. Sci. Ecole Norm. Sup. **29**, 583–637 (1996)
13. Knizhnik, V., Zamolodchikov, A.: Current algebra and the Wess-Zumino model in two dimensions. Nucl. Phys. B **247**, 83–103 (1984)
14. Matsuo, A.: An application of Aomoto-Gelfand Hypergeometric functions to the $SU(n)$ Kniznik-Zamolodchikov Equation. Comm. Math. Phys. **134**(1990), 65–77 (1990)
15. Markov, Y., Varchenko, A.: Hypergeometric solutions of Trigonometric KZ Equations satisfy Dynamical Difference equations. Adv. Math. **166**(1), 100–147 (2002)
16. Mukhin, E., Varchenko, A.: Remarks on critical points of phase functions and norms of Bethe vectors. Adv. Stud. Pure Math. **27**, 239–246 (2000)
17. Opdam, E.M.: Some applications of hypergeometric shift operators. Invent. Math. **98**, 1–18 (1989)
18. Rains, E.: *Multivariate Quadratic Transformations and the Interpolation Kernel*, Contribution to the Special Issue on Elliptic Hypergeometric Functions and Their Applications, SIGMA **14** (2018), 019, 69 pages, https://doi.org/10.3842/SIGMA.2018.019
19. Rimányi, R., Stevens, L., Varchenko, A.: Combinatorics of rational functions and Poincare-Birkhoff-Witt expansions of the canonical $U(\mathfrak{n}_-)$-valued differential form. Ann. Comb. **9**(1), 57–74 (2005)
20. Rimányi, R., Tarasov, V., Varchenko, A., Zinn-Justin, P.: Extended Joseph polynomials, quantized conformal blocks, and a q-Selberg type integral. J. Geometry Phys. **62**, 2188–2207 (2012)
21. Rimányi, R., Varchenko, A.: The $\mathbb{F}_p$-Selberg integral, arXiv:2011.14248, 1–19
22. Selberg, A.: Bemerkninger om et multipelt integral. Norsk Mat. Tidsskr. **26**, 71–78 (1944)
23. Slinkin, A., Varchenko, A.: Hypergeometric Integrals Modulo p and Hasse–Witt Matrices, arXiv:2001.06869, 1–36
24. Spiridonov, V.: *On the elliptic beta function*, (Russian) Uspekhi Mat. Nauk 5 **6** (2001), no. 1, 181–182; translation in Russian Math. Surveys **56** (2001), no. 1, 185–186

25. Schechtman, V., Varchenko, A.: Arrangements of Hyperplanes and Lie Algebra Homology. Invent. Math. **106**, 139–194 (1991)
26. Schechtman, V., Varchenko, A.: Solutions of KZ differential equations modulo p. Ramanujan J. **48**(3), 655–683 (2019). https://doi.org/10.1007/s11139-018-0068-x
27. Tarasov, V., Varchenko, A.: Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras. Invent. Math. **128**, 501–588 (1997)
28. Tarasov, V., Varchenko, A.: Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups. Asterisque **246**, 1–135 (1997)
29. Tarasov, V., Varchenko, A.: Difference Equations compatible with Trigonometric KZ Differential Equations. IMRN **15**, 801–829 (2000)
30. Tarasov, V., Varchenko, A.: Selberg-type integrals associated with $\mathfrak{sl}_3$. Lett. Math. Phys. **65**, 173–185 (2003)
31. Varchenko, A.: *The Euler beta-function, the Vandermonde determinant, the Legendre equation, and critical values of linear functions on a configuration of hyperplanes*, I. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. **6**, 1206–1235, 1337; translation in Math. USSR-Izv. **35** (1990), no. 3, 543–571
32. Varchenko, A.: *Special functions, KZ type equations, and Representation theory*, CBMS, Regional Conference Series in Math., n. 98, AMS (2003)
33. Varchenko, A.: Remarks on the Gaudin model modulo p. J. Singularities. 18: 486–499 (2018). arXiv:1708.06264
34. Varchenko, A.: Solutions modulo p of Gauss-Manin differential equations for multidimensional hypergeometric integrals and associated Bethe ansatz, Mathematics (2017), 5(4), 52 arXiv:1709.06189 https://doi.org/10.3390/math5040052, 1–18
35. Varchenko, A.: Hyperelliptic integrals modulo p and Cartier-Manin matrices, arXiv:1806.03289, 1–16
36. Varchenko, A.: An invariant subbundle of the KZ connection mod p and reducibility of $\widehat{\mathfrak{sl}_2}$ Verma modules mod p, arXiv:2002.05834 , 1–14
37. Varchenko, A.: Determinant of $\mathbb{F}_p$-hypergeometric solutions under ample reduction, arXiv:2010.11275, 1–22
38. Ole Warnaar, S.: A Selberg integral for the Lie algebra A_n, Acta Math. 203(2): 269–304 (2009)
39. Ole Warnaar, S.: The $\mathfrak{sl}_3$ Selberg integral. Adv. Math **224**(2), 499–524 (2010)