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# Level one Weyl modules for toroidal Lie algebras

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**Abstract**
We identify level one global Weyl modules for toroidal Lie algebras with certain twists of modules constructed by Moody–Eswara Rao–Yokonuma via vertex operators for type ADE and by Iohara–Saito–Wakimoto and Eswara Rao for general type. The twist is given by an action of $\mathrm{SL}_2(\mathbb{Z})$ on the toroidal Lie algebra. As a by-product, we obtain a formula for the character of the level one local Weyl module over the toroidal Lie algebra and that for the graded character of the level one graded local Weyl module over an affine analog of the current Lie algebra.

**Keywords** Toroidal Lie algebra · Weyl module · Character · Vertex operator

**Mathematics Subject Classification** Primary 17B67; Secondary 17B10 · 17B65 · 17B69

## 1 Introduction

### 1.1 Motivation

We study global/local Weyl modules for toroidal Lie algebras and an affine analog of current Lie algebras. The notion of Weyl modules for affine Lie algebras has been introduced by Chari–Pressley in [5] as a family of integrable highest weight modules with a universal property. Later Chari–Loktev initiated in [4] to study Weyl modules for current Lie algebras in a graded setting. The graded characters of local Weyl modules for current Lie algebras have been studied by many authors. Now they are known to coincide with Macdonald polynomials specialized at $t = 0$, a.k.a. $q$-Whittaker functions (Chari–Loktev [4], Fourier–Littelmann [10], Naoi [17], Sanderson [19], Ion [12], and Lenart–Naito–Sagaki–Schilling–Shimozono [14]).

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Toroidal Lie algebras are natural generalization of affine Lie algebras. For a finite-dimensional simple Lie algebra $\mathfrak{g}$, the corresponding toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$ is defined as the universal central extension of the double loop Lie algebra $\mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ with the degree operators. We can also consider a Lie algebra $\mathfrak{g}_{\mathrm{tor}}^{+}$ which is defined by replacing $\mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ with $\mathbb{C}[s, t^{\pm 1}]$. See Sect. 2.2 for precise definitions. We expect that the characters of Weyl modules for $\mathfrak{g}_{\mathrm{tor}}$ and $\mathfrak{g}_{\mathrm{tor}}^{+}$ produce a very interesting class of special functions. In this article, we study the first nontrivial example: the Weyl module associated with the level one dominant integral weight.

A big difference between the toroidal and the affine Lie algebra is the structure of their centers. The toroidal Lie algebra without the degree operators has an infinite-dimensional center, while the center of the affine Lie algebra is one-dimensional. The Weyl modules are examples of modules over the toroidal Lie algebra on which the action of the center does not factor a finite-dimensional quotient. We note that Chari–Le have studied in [3] local Weyl modules for a quotient of the toroidal Lie algebra. The resulting quotient is an extension of the double loop Lie algebra by a two-dimensional center with the degree operators. In particular, the Weyl modules considered in this article are possibly bigger than those studied in [3] (see 1.3 below).

### 1.2 Outline

Let us summarize contents and results of the article. In Sect. 2, we introduce the main object: the toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$. We also introduce an affine analog of the current Lie algebra which is denoted by $\mathfrak{g}_{\mathrm{tor}}^{+}$. Then, we recall their basic properties. Among other things, a certain automorphism of $\mathfrak{g}_{\mathrm{tor}}$ will play an important role. The ring $\mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ admits an $\mathrm{SL}_2(\mathbb{Z})$-action by the coordinate change. This action naturally induces automorphisms of $\mathfrak{g}_{\mathrm{tor}}$. We denote by $S$ the automorphism corresponding to the $S$-transformation.

In Sect. 3, we define the global and the local Weyl modules following [1,3–5,9]. The global Weyl module $W_{\mathrm{glob}}(\Lambda)$ for $\mathfrak{g}_{\mathrm{tor}}$ is attached to each dominant integral weight $\Lambda$ of the affine Lie algebra. We identify the endomorphism ring of $W_{\mathrm{glob}}(\Lambda)$ with a symmetric Laurent polynomial ring $A(\Lambda)$ in Proposition 3.6 and define the local Weyl module $W_{\mathrm{loc}}(\Lambda, \mathbf{a})$ for each maximal ideal $\mathbf{a}$ of $A(\Lambda)$. The argument is similar to known one for the affine and the current Lie algebras. The global/local Weyl modules $W_{\mathrm{glob}}^{+}(\Lambda)$ and $W_{\mathrm{loc}}^{+}(\Lambda, \mathbf{a})$ for $\mathfrak{g}_{\mathrm{tor}}^{+}$ are similarly defined. We prove in Proposition 3.9 a finiteness property for weight spaces of the Weyl modules. By this property, the characters of the local Weyl modules are well-defined. This result has been established for the case of the affine Lie algebra in [5] and for a quotient of the toroidal Lie algebra in [3]. We remark that we need to investigate the action of the infinite-dimensional center, which is not treated in [3]. Then, we turn to a special case where $\Lambda$ is of level one. By the diagram automorphism, we can reduce the general level one case to that for the basic level one weight $\Lambda_0$. Therefore, we only consider the case of $\Lambda_0$ in the sequel. We give an upper bound for the graded character of the level one local Weyl module $W_{\mathrm{loc}}^{+}(\Lambda_0, 0)$ over $\mathfrak{g}_{\mathrm{tor}}^{+}$ in Proposition 3.19.

In Sect. 4, we prove an isomorphism between the level one global Weyl module $W_{\mathrm{glob}}(\Lambda_0)$ over the toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$ and the twist of a module $\mathbb{V}(0)$ by the automorphism $S^{-1}$, where $\mathbb{V}(0)$ has been constructed in works of Moody–Eswara Rao–Yokonuma [16], Iohara–Saito–Wakimoto [13] and Eswara Rao [6]. This is our main theorem.

**Theorem 1.1** (Theorem 4.10) *We have an isomorphism*

$$W_{\mathrm{glob}}(\Lambda_0) \xrightarrow{\cong} (S^{-1})^*\mathbb{V}(0)$$

*of* $\mathfrak{g}_{\mathrm{tor}}$*-modules.*

As a by-product, we prove that the upper bound in Proposition 3.19 indeed gives the characters of the level one local Weyl modules (see Sect. 2.5 for the definition of $\mathrm{ch}_p$ and $\mathrm{ch}_{p,q}$).

**Corollary 1.2** (Corollary 4.11) *We have*

$$\mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda_0, a) = \mathrm{ch}_p\, W^+_{\mathrm{loc}}(\Lambda_0, a) = \mathrm{ch}_p\, L(\Lambda_0) \left( \prod_{n>0} \frac{1}{1-p^n} \right)$$

*for* $a \in \mathbb{C}^\times$ *and*

$$\mathrm{ch}_{p,q}\, W^+_{\mathrm{loc}}(\Lambda_0, 0) = \mathrm{ch}_p\, L(\Lambda_0) \left( \prod_{n>0} \frac{1}{1-p^n q} \right).$$

*Here,* $L(\Lambda_0)$ *is the level one integrable irreducible module of the affine Lie algebra with highest weight* $\Lambda_0$*.*

### 1.3 Related works

Let us give two comments regarding other works. The first one is for [3] mentioned earlier. In [3], Chari–Le have studied local Weyl modules for some quotients of $\mathfrak{g}_{\mathrm{tor}}$ and $\mathfrak{g}^+_{\mathrm{tor}}$. They have proved that the level one local Weyl modules in their setting are irreducible and are isomorphic to the evaluation modules [3, Theorem 4]. Hence, we see by our results that the level one local Weyl modules for $\mathfrak{g}_{\mathrm{tor}}$ and $\mathfrak{g}^+_{\mathrm{tor}}$ are bigger than those studied in [3]. We remark that one of our results (Proposition 3.19) gives an alternative proof of [3, Theorem 4].

The second one is for [21]. In [21, Theorem 3.8], Tsymbaliuk has proved that the level one Fock representation of Saito–Takemura–Uglov [20] and Feigin–Jimbo–Miwa–Mukhin [7] over the quantum toroidal algebra of type A is isomorphic to a twist of the vertex representation of Saito [18]. Here, the twist is given by an automorphism analogous to $S^{-1}$ which has been constructed by Miki [15]. This result motivated the present work. In the situation of [21], both the Fock and the vertex representations are known to be irreducible, and hence, it can be checked by comparing their highest weights to show the isomorphism. Thus, although the calculation of $S^{-1}$ in the quantum toroidal case is much more involved, the argument to show the isomorphism is simple.

It is an interesting problem to establish the results analogous to this article for quantum toroidal algebras and affine Yangians.

## 2 Preliminaries

### 2.1 Simple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h}$. We also fix a Borel subalgebra containing $\mathfrak{h}$. The index set of simple roots is denoted by $I$. Let $\alpha_i$ $(i \in I)$ be simple roots. We denote by $\Delta, \Delta^+, \Delta^-$ the sets of roots, positive roots, negative roots, respectively. Let $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ be the corresponding root space and put $\mathfrak{g}_0 = \mathfrak{h}$. The highest root is denoted by $\theta$.

Let $( \, , \, )$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. We denote by the same letter the bilinear form on $\mathfrak{h}^*$ induced from $( \, , \, )$ and normalize them by $(\theta, \theta) = 2$. Put $d_i = (\alpha_i, \alpha_i)/2$. We fix Chevalley generators $e_i, f_i, h_i$ $(i \in I)$ so that $(e_i, f_i) = d_i^{-1}$ and $h_i = [e_i, f_i]$. We also fix root vectors $e_\theta \in \mathfrak{g}_\theta$ and $f_\theta \in \mathfrak{g}_{-\theta}$ so that $(e_\theta, f_\theta) = 1$. We denote by $h_\alpha \in \mathfrak{h}$ the coroot corresponding to $\alpha \in \Delta$. The root lattice $Q$ is defined by $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$.

### 2.2 Toroidal Lie algebras

The universal central extension of the Lie algebra $\mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ is given by

$$\mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}] \oplus \Omega_{\mathbb{C}[s^{\pm 1}, t^{\pm 1}]}/\operatorname{Im} \mathrm{d}.$$

Here, $\Omega_A$ for a commutative $\mathbb{C}$-algebra $A$ denotes the module of differentials, and $\mathrm{d} : A \to \Omega_A$ the differential map. The Lie bracket is given by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x, y)(\mathrm{d}a)b.$$

See [16, Section 2] for details.

We put

$$c(k, l) = \begin{cases} s^k t^{l-1} \, \mathrm{d}t & \text{if } k \neq 0, \\ s^{-1} t^l \, \mathrm{d}s & \text{if } k = 0 \end{cases}$$

for $(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ and $c_s = s^{-1}\mathrm{d}s$, $c_t = t^{-1}\mathrm{d}t$. Then, $\Omega_{\mathbb{C}[s^{\pm 1}, t^{\pm 1}]}/\mathrm{Im d}$ has a $\mathbb{C}$-basis $c(k,l)$ with $(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}$, $c_s$, $c_t$. We can explicitly describe the Lie bracket as follows:

$$\left[x \otimes s^k t^l, y \otimes s^m t^n\right] = \begin{cases} [x,y] \otimes s^{k+m} t^{l+n} + (x,y) \dfrac{lm - kn}{k+m} c(k+m, l+n) & \text{if } k+m \neq 0, \\ [x,y] \otimes t^{l+n} + (x,y) k c(0, l+n) & \text{if } k+m = 0 \text{ and } l+n \neq 0, \\ [x,y] \otimes 1 + (x,y)(k c_s + l c_t) & \text{if } k+m = 0 \text{ and } l+n = 0. \end{cases} \tag{2.1}$$

We add the degree operators $d_s, d_t$ to this central extension and define the toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$ by

$$\mathfrak{g}_{\mathrm{tor}} = \mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}] \oplus \bigoplus_{(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \mathbb{C}c(k,l) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t,$$

where the additional commutation relations are as follows:

$$[d_s, x \otimes s^k t^l] = kx \otimes s^k t^l, \quad [d_t, x \otimes s^k t^l] = lx \otimes s^k t^l,$$
$$[d_s, c(k,l)] = kc(k,l), \quad [d_t, c(k,l)] = lc(k,l),$$
$$[d_s, c_s] = [d_t, c_s] = [d_s, c_t] = [d_t, c_t] = [d_s, d_t] = 0.$$

***Remark 2.1*** Note that we have

$$c(k,l) = \begin{cases} (-k/l) s^{k-1} t^l \mathrm{d}s & \text{if } k \neq 0, \\ s^{-1} t^l \mathrm{d}s & \text{if } k = 0 \end{cases}$$

for $l \neq 0$. In particular, $c(k+1, l)$ is a nonzero multiple of $s^k t^l \mathrm{d}s$ if $l \neq 0$. We will use this fact throughout the article.

Let $\mathfrak{g}'_{\mathrm{tor}}$ be the Lie subalgebra of $\mathfrak{g}_{\mathrm{tor}}$ without $d_s$:

$$\mathfrak{g}'_{\mathrm{tor}} = \mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}] \oplus \bigoplus_{(k,l) \in \mathbb{Z}^2 \backslash \{(0,0)\}} \mathbb{C}c(k,l) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t \oplus \mathbb{C}d_t.$$

We also consider the following Lie subalgebra $\mathfrak{g}^+_{\mathrm{tor}}$ of $\mathfrak{g}_{\mathrm{tor}}$:

$$\mathfrak{g}^+_{\mathrm{tor}} = \mathfrak{g} \otimes \mathbb{C}[s, t^{\pm 1}] \oplus \bigoplus_{\substack{k \geq 1 \\ l \in \mathbb{Z}}} \mathbb{C}c(k,l) \oplus \mathbb{C}c_t \oplus \mathbb{C}d_t.$$

The Lie algebra $\mathfrak{g}^+_{\mathrm{tor}}$ is the semidirect product of the universal central extension of $\mathfrak{g} \otimes \mathbb{C}[s, t^{\pm 1}]$ and the one-dimensional abelian Lie algebra $\mathbb{C}d_t$. It is an affine analog of the current Lie algebra $\mathfrak{g} \otimes \mathbb{C}[s]$ and has a $\mathbb{Z}_{\geq 0}$-graded Lie algebra structure by assigning

$$\deg(x \otimes s^k t^l) = k \ (x \in \mathfrak{g}), \quad \deg c(k,l) = k \ (k \geq 1, l \in \mathbb{Z}), \quad \deg c_t = \deg d_t = 0.$$

***Remark 2.2*** Later, we will study graded $\mathfrak{g}_{\mathrm{tor}}^{+}$-modules. It is equivalent to considering modules of $\mathfrak{g}_{\mathrm{tor}}^{+} \oplus \mathbb{C} d_s$.

The toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$ contains two Lie subalgebras $\mathfrak{g}_{\mathrm{aff}}^{(s)}$ and $\mathfrak{g}_{\mathrm{aff}}^{(t)}$ isomorphic to the affine Lie algebra associated with $\mathfrak{g}$:

$$\mathfrak{g}_{\mathrm{aff}}^{(s)} = \mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}] \oplus \mathbb{C} c_s \oplus \mathbb{C} d_s, \quad \mathfrak{g}_{\mathrm{aff}}^{(t)} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C} c_t \oplus \mathbb{C} d_t.$$

Note that $\mathfrak{g}_{\mathrm{tor}}^{+}$ contains $\mathfrak{g}_{\mathrm{aff}}^{(t)}$. We have

$$\begin{aligned} \mathfrak{g}_{\mathrm{tor}} &= \left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)' \otimes \mathbb{C}[s^{\pm 1}] \oplus \bigoplus_{\substack{k \in \mathbb{Z} \\ l \neq 0}} \mathbb{C} c(k,l) \oplus \mathbb{C} c_s \oplus \mathbb{C} d_s \oplus \mathbb{C} d_t, \\ \mathfrak{g}_{\mathrm{tor}}^{+} &= \left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)' \otimes \mathbb{C}[s] \oplus \bigoplus_{\substack{k \geq 1 \\ l \neq 0}} \mathbb{C} c(k,l) \oplus \mathbb{C} d_t, \end{aligned}$$

where $\left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)' = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C} c_t$. Here, the elements $c(k,0) = s^k t^{-1} \mathrm{d}t$ are regarded as $c_t \otimes s^k \in \left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)' \otimes s^k$.

***Remark 2.3*** Chari–Le [3] have studied a version of toroidal Lie algebras which is the quotient of $\mathfrak{g}_{\mathrm{tor}}$ modulo the elements $c(k,l)$ with $l \neq 0$; namely, it is equal to

$$\begin{aligned} &\mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}] \oplus \bigoplus_{k \neq 0} \mathbb{C} c(k,0) \oplus \mathbb{C} c_s \oplus \mathbb{C} c_t \oplus \mathbb{C} d_s \oplus \mathbb{C} d_t \\ &= \left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)' \otimes \mathbb{C}[s^{\pm 1}] \oplus \mathbb{C} c_s \oplus \mathbb{C} d_s \oplus \mathbb{C} d_t \end{aligned}$$

as a $\mathbb{C}$-vector space.

We introduce presentations of $\mathfrak{g}_{\mathrm{tor}}$ and $\mathfrak{g}_{\mathrm{tor}}^{+}$. Put $I_{\mathrm{aff}} = I \sqcup \{0\}$. Let $(a_{ij})_{i,j \in I_{\mathrm{aff}}}$ be the Cartan matrix of $\mathfrak{g}_{\mathrm{aff}}^{(t)}$ and set $d_0 = 1$.

**Definition 2.4** Let $\mathfrak{t}$ be the Lie algebra generated by $e_{i,k}, f_{i,k}, h_{i,k}$ $(i \in I_{\mathrm{aff}}, k \in \mathbb{Z})$, $c_s, d_s, d_t$ subject to the following defining relations:

$$\begin{aligned} c_s : \text{central}, \quad [h_{i,k}, h_{j,l}] &= d_j^{-1} a_{ij} k \delta_{k+l,0} c_s, \\ [e_{i,k}, f_{j,l}] &= \delta_{ij} \left(h_{i,k+l} + d_i^{-1} k \delta_{k+l,0} c_s\right), \\ [h_{i,k}, e_{j,l}] = a_{ij} e_{j,k+l}, \quad [h_{i,k}, f_{j,l}] &= -a_{ij} f_{j,k+l}, \end{aligned}$$

$$[e_{i,k}, e_{i,l}] = 0, \quad [f_{i,k}, f_{i,l}] = 0,$$
$$(\mathrm{ad}\, e_{i,0})^{1-a_{ij}} e_{j,k} = 0, \quad (\mathrm{ad}\, f_{i,0})^{1-a_{ij}} f_{j,k} = 0, \quad (i \neq j)$$
$$[d_s, e_{i,k}] = k e_{i,k}, \quad [d_s, f_{i,k}] = k f_{i,k}, \quad [d_s, h_{i,k}] = k h_{i,k},$$
$$[d_t, e_{i,k}] = \delta_{i,0} e_{i,k}, \quad [d_t, f_{i,k}] = -\delta_{i,0} f_{i,k}, \quad [d_t, h_{i,k}] = 0,$$
$$[d_s, d_t] = 0.$$

**Definition 2.5** Let $\mathfrak{s}$ be the Lie algebra generated by $e_{i,k}, f_{i,k}, h_{i,k}$ ($i \in I_{\mathrm{aff}}, k \in \mathbb{Z}_{\geq 0}$), $d_t$ subject to the following defining relations:

$$[h_{i,k}, h_{j,l}] = 0, \quad [e_{i,k}, f_{j,l}] = \delta_{ij} h_{i,k+l},$$
$$[h_{i,k}, e_{j,l}] = a_{ij} e_{j,k+l}, \quad [h_{i,k}, f_{j,l}] = -a_{ij} f_{j,k+l},$$
$$[e_{i,k}, e_{i,l}] = 0, \quad [f_{i,k}, f_{i,l}] = 0,$$
$$(\mathrm{ad}\, e_{i,0})^{1-a_{ij}} e_{j,k} = 0, \quad (\mathrm{ad}\, f_{i,0})^{1-a_{ij}} f_{j,k} = 0, \quad (i \neq j)$$
$$[d_t, e_{i,k}] = \delta_{i,0} e_{i,k}, \quad [d_t, f_{i,k}] = -\delta_{i,0} f_{i,k}, \quad [d_t, h_{i,k}] = 0.$$

**Theorem 2.6** ([16] Proposition 3.5, [11] Proposition 4.4) *We have an isomorphism of Lie algebras* $\mathfrak{t} \to \mathfrak{g}_{\mathrm{tor}}$ *such that*

$$e_{i,k} \mapsto \begin{cases} e_i \otimes s^k & \text{if } i \in I, \\ f_\theta \otimes s^k t & \text{if } i = 0, \end{cases} \qquad f_{i,k} \mapsto \begin{cases} f_i \otimes s^k & \text{if } i \in I, \\ e_\theta \otimes s^k t^{-1} & \text{if } i = 0, \end{cases}$$
$$h_{i,k} \mapsto \begin{cases} h_i \otimes s^k & \text{if } i \in I, \\ -h_\theta \otimes s^k + s^k t^{-1} \mathrm{d}t & \text{if } i = 0, \end{cases} \qquad c_s \mapsto c_s, \quad d_s \mapsto d_s, \quad d_t \mapsto d_t.$$

*Moreover, this restricts to an isomorphism* $\mathfrak{s} \to \mathfrak{g}_{\mathrm{tor}}^+$.

Under the isomorphism, the elements $e_{i,0}, f_{i,0}, h_{i,0}$ are in the Lie subalgebra $\mathfrak{g}_{\mathrm{aff}}^{(t)}$ and identified with its Chevalley generators. We sometimes denote them by $e_i, f_i, h_i$. Note that $e_{i,k}, f_{i,k}, h_{i,k}$ ($i \in I$, $k \in \mathbb{Z}$), $c_s, d_s$ generate the Lie subalgebra $\mathfrak{g}_{\mathrm{aff}}^{(s)}$ of $\mathfrak{t} \cong \mathfrak{g}_{\mathrm{tor}}$.

We introduce notions for the affine Lie algebra $\mathfrak{g}_{\mathrm{aff}}^{(t)}$. Let $\mathfrak{n}_{\mathrm{aff}}^{(t)}$ be the Lie subalgebra of $\mathfrak{g}_{\mathrm{aff}}^{(t)}$ generated by $e_i$ ($i \in I_{\mathrm{aff}}$), and $\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}$ that generated by $f_i$ ($i \in I_{\mathrm{aff}}$). Set

$$\mathfrak{h}_{\mathrm{aff}}^{(t)} = \mathfrak{h} \oplus \mathbb{C} c_t \oplus \mathbb{C} d_t.$$

The generator of imaginary roots is denoted by $\delta$. We put $\alpha_0 = -\theta + \delta$ so that $\alpha_i$ ($i \in I_{\mathrm{aff}}$) forms simple roots of $\mathfrak{g}_{\mathrm{aff}}^{(t)}$. We denote by $\Delta_{\mathrm{aff}}$, $\Delta_{\mathrm{aff}}^+$ the sets of roots and positive roots, respectively. Let $\left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)_\alpha$ ($\alpha \in \Delta_{\mathrm{aff}}$) be the corresponding root space. The coroot is defined by $h_{\beta + l\delta} = h_\beta + l c_t$ for $\beta \in \Delta \cup \{0\}$ and $l \in \mathbb{Z}$. We set $Q_{\mathrm{aff}} = \bigoplus_{i \in I_{\mathrm{aff}}} \mathbb{Z} \alpha_i$ and $Q_{\mathrm{aff}}^+ = \sum_{i \in I_{\mathrm{aff}}} \mathbb{Z}_{\geq 0} \alpha_i$.

We say that an element $\Lambda$ of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{aff}}^{(t)}, \mathbb{C})$ is a dominant integral weight of $\mathfrak{g}_{\mathrm{aff}}^{(t)}$ if $\langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0}$ holds for any $i \in I_{\mathrm{aff}}$. In this article, they are further assumed to satisfy $\langle d_t, \Lambda\rangle = 0$ for simplicity. Define the fundamental weights $\Lambda_i$ ($i \in I_{\mathrm{aff}}$) by $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ and $\langle d_t, \Lambda_i\rangle = 0$. We denote by $L(\Lambda)$ the irreducible $\mathfrak{g}_{\mathrm{aff}}^{(t)}$-module with highest weight $\Lambda$. We will use the symbol $L(\Lambda)^{(s)}$ for the irreducible $\mathfrak{g}_{\mathrm{aff}}^{(s)}$-module with highest weight $\Lambda$.

### 2.3 Triangular decomposition

Let $\mathfrak{n}_{\mathrm{tor}}$ be the Lie subalgebra of $\mathfrak{g}_{\mathrm{tor}}$ generated by $e_{i,k}$ ($i \in I_{\mathrm{aff}}, k \in \mathbb{Z}$), and $\bar{\mathfrak{n}}_{\mathrm{tor}}$ that generated by $f_{i,k}$ ($i \in I_{\mathrm{aff}}, k \in \mathbb{Z}$). Set

$$\begin{aligned}\mathfrak{a}_{\mathrm{tor}} &= \mathfrak{h} \otimes \mathbb{C}[s^{\pm 1}] \oplus \bigoplus_{k\neq 0} \mathbb{C}c(k,0) \oplus \mathbb{C}c_s \oplus \mathbb{C}c_t \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t \\ &= (\mathfrak{h} \oplus \mathbb{C}c_t) \otimes \mathbb{C}[s^{\pm 1}] \oplus \mathbb{C}c_s \oplus \mathbb{C}d_s \oplus \mathbb{C}d_t.\end{aligned}$$

**Proposition 2.7** *We have*

$$\mathfrak{n}_{\mathrm{tor}} = \mathfrak{n}_{\mathrm{aff}}^{(t)} \otimes \mathbb{C}[s^{\pm 1}] \oplus \bigoplus_{\substack{k\in\mathbb{Z}\\ l\geq 1}} \mathbb{C}c(k,l), \quad \bar{\mathfrak{n}}_{\mathrm{tor}} = \bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)} \otimes \mathbb{C}[s^{\pm 1}] \oplus \bigoplus_{\substack{k\in\mathbb{Z}\\ l\leq -1}} \mathbb{C}c(k,l).$$

***Proof*** Denote by $\mathfrak{n}'_{\mathrm{tor}}$ and $\bar{\mathfrak{n}}'_{\mathrm{tor}}$ the right-hand sides. Then, we see by the formula of the Lie bracket (2.1) that $\mathfrak{n}_{\mathrm{tor}} \supset \mathfrak{n}'_{\mathrm{tor}}$ and $\bar{\mathfrak{n}}_{\mathrm{tor}} \supset \bar{\mathfrak{n}}'_{\mathrm{tor}}$. We also see that $\bar{\mathfrak{n}}_{\mathrm{tor}} + \mathfrak{a}_{\mathrm{tor}} + \mathfrak{n}_{\mathrm{tor}} = \bar{\mathfrak{n}}_{\mathrm{tor}} \oplus \mathfrak{a}_{\mathrm{tor}} \oplus \mathfrak{n}_{\mathrm{tor}}$. Since we have $\mathfrak{g}_{\mathrm{tor}} = \bar{\mathfrak{n}}'_{\mathrm{tor}} \oplus \mathfrak{a}_{\mathrm{tor}} \oplus \mathfrak{n}'_{\mathrm{tor}}$, the assertion holds. □

In this article, we call

$$\mathfrak{g}_{\mathrm{tor}} = \bar{\mathfrak{n}}_{\mathrm{tor}} \oplus \mathfrak{a}_{\mathrm{tor}} \oplus \mathfrak{n}_{\mathrm{tor}}$$

the triangular decomposition of $\mathfrak{g}_{\mathrm{tor}}$.

In $\mathfrak{g}_{\mathrm{tor}}^{+}$, the elements $e_{i,k}$ ($i \in I_{\mathrm{aff}}, k \in \mathbb{Z}_{\geq 0}$) generate

$$\mathfrak{n}_{\mathrm{tor}} \cap \mathfrak{g}_{\mathrm{tor}}^{+} = \mathfrak{n}_{\mathrm{aff}}^{(t)} \otimes \mathbb{C}[s] \oplus \bigoplus_{\substack{k\geq 1\\ l\geq 1}} \mathbb{C}c(k,l),$$

and $f_{i,k}$ ($i \in I_{\mathrm{aff}}, k \in \mathbb{Z}_{\geq 0}$) generate

$$\bar{\mathfrak{n}}_{\mathrm{tor}} \cap \mathfrak{g}_{\mathrm{tor}}^{+} = \bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)} \otimes \mathbb{C}[s] \oplus \bigoplus_{\substack{k\geq 1\\ l\leq -1}} \mathbb{C}c(k,l).$$

Further set

$$\mathfrak{a}'_{\mathrm{tor}} = \mathfrak{a}_{\mathrm{tor}} \cap \mathfrak{g}'_{\mathrm{tor}} = (\mathfrak{h} \oplus \mathbb{C}c_t) \otimes \mathbb{C}[s^{\pm 1}] \oplus \mathbb{C}c_s \oplus \mathbb{C}d_t.$$

### 2.4 Automorphisms

Let $S$ be the ring automorphism of $\mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ defined by $s \mapsto t, t \mapsto s^{-1}$. It naturally induces a Lie algebra automorphism of $\mathfrak{g}_{\mathrm{tor}}$ which is denoted by the same letter $S$. Later, we will rather use its inverse $S^{-1}$. It corresponds to the assignment $s \mapsto t^{-1}$, $t \mapsto s$. In particular, we have

$$S^{-1}(c(k,l)) = \begin{cases} (k/l)c(l,-k) & \text{if } k,l \neq 0, \\ -c(l,0) & \text{if } k = 0, \\ c(0,-k) & \text{if } l = 0, \end{cases} \qquad S^{-1}(c_s) = -c_t, \quad S^{-1}(c_t) = c_s.$$

We introduce Lie algebra automorphisms $T_0$ and $T_\theta$ of $\mathfrak{g}_{\mathrm{tor}}$ by

$$T_0 = \exp \operatorname{ad} e_0 \circ \exp \operatorname{ad}(-f_0) \circ \exp \operatorname{ad} e_0,$$
$$T_\theta = \exp \operatorname{ad} e_\theta \circ \exp \operatorname{ad}(-f_\theta) \circ \exp \operatorname{ad} e_\theta.$$

We can regard them as automorphisms of $\mathfrak{g}_{\mathrm{tor}}^+$ by restriction.

**Lemma 2.8** *We have* $e_\theta \otimes s^k t^l = T_0 T_\theta(e_\theta \otimes s^k t^{l+2})$.

***Proof*** By a direct calculation. We use the following:

$$T_\theta(e_\theta \otimes s^k t^{l+2}) = -f_\theta \otimes s^k t^{l+2},$$
$$\exp \operatorname{ad} e_0(f_\theta \otimes s^k t^{l+2}) = f_\theta \otimes s^k t^{l+2},$$
$$\exp \operatorname{ad}(-f_0)(f_\theta \otimes s^k t^{l+2}) = f_\theta \otimes s^k t^{l+2} - (h_\theta \otimes s^k t^{l+1} - s^k t^l \mathrm{d}t) - e_\theta \otimes s^k t^l,$$
$$\exp \operatorname{ad} e_0(h_\theta \otimes s^k t^{l+1}) = h_\theta \otimes s^k t^{l+1} + 2 f_\theta \otimes s^k t^{l+2},$$
$$\exp \operatorname{ad} e_0(e_\theta \otimes s^k t^l) = e_\theta \otimes s^k t^l - h_\theta \otimes s^k t^{l+1} + s^k t^l \mathrm{d}t - f_\theta \otimes s^k t^{l+2}.$$

□

Let $M$ be a module of $\mathcal{A} = \mathfrak{g}_{\mathrm{tor}}, \mathfrak{g}'_{\mathrm{tor}}$, or $\mathfrak{g}_{\mathrm{tor}}^+$ and assume that $M$ is integrable as a $\mathfrak{g}_{\mathrm{aff}}^{(t)}$-module. Then, $T_0, T_\theta \in \operatorname{Aut} M$ are similarly defined. Moreover, they satisfy

$$T_0(xv) = T_0(x)T_0(v), \quad T_\theta(xv) = T_\theta(x)T_\theta(v)$$

for $x \in \mathcal{A}$ and $v \in M$.

The Lie algebra automorphism $\tau_a$ $(a \in \mathbb{C})$ of $\mathfrak{g}_{\mathrm{tor}}^+$ is induced from the map $s \mapsto s+a$.

### 2.5 Characters

Let $M$ be a module of $\mathcal{A} = \mathfrak{g}_{\mathrm{tor}}, \mathfrak{g}'_{\mathrm{tor}}$, or $\mathfrak{g}_{\mathrm{tor}}^+$ and regard it as a $\mathfrak{g}_{\mathrm{aff}}^{(t)}$-module by restriction. For $\lambda \in \mathfrak{h}^*$ and $m \in \mathbb{C}$, let $M_{\lambda - m\delta}$ be the corresponding weight space. In this article, we always assume that any $\mathfrak{g}_{\mathrm{aff}}^{(t)}$-module $M$ has the weight space decomposition and $M_{\lambda - m\delta} = 0$ unless $m \in \mathbb{Z}$.

We define the $p$-character $\operatorname{ch}_p M$ of $M$ by

$$\operatorname{ch}_p M = \sum_{\substack{\lambda \in \mathfrak{h}^* \\ m \in \mathbb{Z}}} (\dim M_{\lambda - m\delta}) e^{\lambda} p^m$$

if it is well-defined. This is nothing but the ordinary $\mathfrak{g}_{\mathrm{aff}}^{(t)}$-character with $p = e^{-\delta}$. Let $M$ be a graded $\mathfrak{g}_{\mathrm{tor}}^{+}$-module and $M_{\lambda - m\delta} = \bigoplus_{n \in \mathbb{Z}} M_{\lambda - m\delta}[n]$ the decomposition of the weight space into graded pieces. We define the $(p, q)$-character $\operatorname{ch}_{p,q} M$ of $M$ by

$$\operatorname{ch}_{p,q} M = \sum_{\substack{\lambda \in \mathfrak{h}^* \\ m,n \in \mathbb{Z}}} (\dim M_{\lambda - m\delta}[n]) e^{\lambda} p^m q^n$$

if it is well-defined. For two formal sums

$$f = \sum_{\substack{\lambda \in \mathfrak{h}^* \\ m \in \mathbb{Z}}} f_{\lambda,m} e^{\lambda} p^m, \quad g = \sum_{\substack{\lambda \in \mathfrak{h}^* \\ m \in \mathbb{Z}}} g_{\lambda,m} e^{\lambda} p^m \quad (f_{\lambda,m}, g_{\lambda,m} \in \mathbb{Z}),$$

we say $f \leq g$ if $f_{\lambda,m} \leq g_{\lambda,m}$ holds for all $\lambda$ and $m$. We define an inequality $\leq$ for

$$f = \sum_{\substack{\lambda \in \mathfrak{h}^* \\ m,n \in \mathbb{Z}}} f_{\lambda,m,n} e^{\lambda} p^m q^n, \quad g = \sum_{\substack{\lambda \in \mathfrak{h}^* \\ m,n \in \mathbb{Z}}} g_{\lambda,m,n} e^{\lambda} p^m q^n \quad (f_{\lambda,m,n}, g_{\lambda,m,n} \in \mathbb{Z})$$

similarly.

# 3 Weyl modules

## 3.1 Definitions of global/local Weyl modules

**Definition 3.1** Let $\Lambda$ be a dominant integral weight of $\mathfrak{g}_{\mathrm{aff}}^{(t)}$. The global Weyl module $W_{\mathrm{glob}}(\Lambda)$ for $\mathfrak{g}_{\mathrm{tor}}$ with highest weight $\Lambda$ is the $\mathfrak{g}_{\mathrm{tor}}$-module generated by $v_{\Lambda}$ subject to the following defining relations:

$$\begin{aligned}
&e_{i,k} v_{\Lambda} = 0 \ (i \in I_{\mathrm{aff}}, k \in \mathbb{Z}), \quad h v_{\Lambda} = \langle h, \Lambda \rangle v_{\Lambda} \ \left(h \in \mathfrak{h}_{\mathrm{aff}}^{(t)}\right), \\
&f_i^{\langle h_i, \Lambda \rangle + 1} v_{\Lambda} = 0 \ (i \in I_{\mathrm{aff}}), \\
&c_s v_{\Lambda} = d_s v_{\Lambda} = 0.
\end{aligned}$$

The global Weyl module $W_{\mathrm{glob}}^{+}(\Lambda)$ for $\mathfrak{g}_{\mathrm{tor}}^{+}$ with highest weight $\Lambda$ is the $\mathfrak{g}_{\mathrm{tor}}^{+}$-module generated by $v_{\Lambda}^{+}$ subject to the following defining relations:

$$e_{i,k} v_{\Lambda}^{+} = 0 \ (i \in I_{\mathrm{aff}}, k \in \mathbb{Z}_{\geq 0}), \quad h v_{\Lambda}^{+} = \langle h, \Lambda \rangle v_{\Lambda}^{+} \ \left(h \in \mathfrak{h}_{\mathrm{aff}}^{(t)}\right),$$

$$f_i^{\langle h_i,\Lambda\rangle+1} v_\Lambda^+ = 0 \ (i \in I_{\mathrm{aff}}).$$

We describe the endomorphism rings of $W_{\mathrm{glob}}(\Lambda)$ and $W_{\mathrm{glob}}^+(\Lambda)$. The following argument is the same as in the case of the affine and the current Lie algebras. We give some details for completeness.

**Lemma 3.2** *We have an action $\varphi$ of $U(\mathfrak{a}'_{\mathrm{tor}})$ on each weight space $W_{\mathrm{glob}}(\Lambda)_{\Lambda-\beta}$ ($\beta \in Q_{\mathrm{aff}}^+$) defined by*

$$\varphi(a)(Xv_\Lambda) = X(av_\Lambda)$$

*for $a \in U(\mathfrak{a}'_{\mathrm{tor}})$ and $X \in U(\mathfrak{g}'_{\mathrm{tor}})$.*

***Proof*** To see that the action is well-defined, we need to check that $Xv_\Lambda = 0$ implies $X(av_\Lambda) = 0$. By the same argument as [1, 3.4], we can show that if $v$ satisfies the relations

$$e_{i,k}v = 0 \ (i \in I_{\mathrm{aff}}, k \in \mathbb{Z}),\ hv = \langle h, \Lambda\rangle v \ \left(h \in \mathfrak{h}_{\mathrm{aff}}^{(t)}\right),$$
$$f_i^{\langle h_i,\Lambda\rangle+1} v = 0 \ (i \in I_{\mathrm{aff}}),\ c_s v = 0,$$

then so does $av$. This completes the proof. □

Let Ann $v_\Lambda$ be the annihilator ideal of $U(\mathfrak{a}'_{\mathrm{tor}})$ and set

$$\tilde{A}(\Lambda) = U(\mathfrak{a}'_{\mathrm{tor}})/\operatorname{Ann} v_\Lambda.$$

Since the action $\varphi$ of $\mathfrak{a}'_{\mathrm{tor}}$ factors through an abelian Lie algebra $\mathfrak{a}'_{\mathrm{tor}}/\mathbb{C}c_s \oplus \mathbb{C}d_t$, $\tilde{A}(\Lambda)$ is a commutative algebra.

**Lemma 3.3** *The action map*

$$\tilde{A}(\Lambda) \to W_{\mathrm{glob}}(\Lambda)_\Lambda, \quad a \mapsto av_\Lambda$$

*gives an isomorphism of $\mathbb{C}$-vector spaces.*

***Proof*** The well-definedness and the injectivity immediately follow from the definition of $\tilde{A}(\Lambda)$. The surjectivity holds since we have $W_{\mathrm{glob}}(\Lambda)_\Lambda = U(\mathfrak{a}'_{\mathrm{tor}})v_\Lambda$. □

**Lemma 3.4** *The natural map*

$$\tilde{A}(\Lambda) \to \operatorname{End}_{\mathfrak{g}'_{\mathrm{tor}}} W_{\mathrm{glob}}(\Lambda), \quad a \mapsto \varphi(a)$$

*gives an isomorphism of $\mathbb{C}$-algebras.*

***Proof*** By the definition of $\tilde{A}(\Lambda)$, we have a natural injective algebra homomorphism

$$\tilde{A}(\Lambda) \to \mathrm{End}_{\mathfrak{g}'_{\mathrm{tor}}} W_{\mathrm{glob}}(\Lambda), \quad a \mapsto \varphi(a).$$

We also have a natural $\mathbb{C}$-linear map

$$\mathrm{End}_{\mathfrak{g}'_{\mathrm{tor}}} W_{\mathrm{glob}}(\Lambda) \to W_{\mathrm{glob}}(\Lambda)_{\Lambda}, \quad f \mapsto f(v_{\Lambda})$$

and this is injective since $W_{\mathrm{glob}}(\Lambda)$ is generated by $v_{\Lambda}$. The composite of the maps

$$\tilde{A}(\Lambda) \hookrightarrow \mathrm{End}_{\mathfrak{g}'_{\mathrm{tor}}} W_{\mathrm{glob}}(\Lambda) \hookrightarrow W_{\mathrm{glob}}(\Lambda)_{\Lambda}$$

is given by $a \mapsto av_{\Lambda}$. Since this map is bijective by Lemma 3.3, the two injective maps are bijective. □

Write $\Lambda = \sum_{i \in I_{\mathrm{aff}}} m_i \Lambda_i$ with the fundamental weights $\Lambda_i$ and $m_i \in \mathbb{Z}_{\geq 0}$. We define $A(\Lambda)$ by

$$A(\Lambda) = \bigotimes_{i \in I_{\mathrm{aff}}} \mathbb{C}\left[z_{i,1}^{\pm 1}, \ldots, z_{i,m_i}^{\pm 1}\right]^{\mathfrak{S}_{m_i}},$$

the symmetric Laurent polynomial algebra associated with $\Lambda$.

**Proposition 3.5** *The assignment*

$$\sum_{m=1}^{m_i} z_{i,m}^k \mapsto h_{i,k}$$

*gives an isomorphism $A(\Lambda) \cong \tilde{A}(\Lambda)$ of $\mathbb{C}$-algebras.*

***Proof*** The well-definedness and the surjectivity of the map are proved in the same way as [5, Proposition 1.1 (i), (iv), (v)].

We follow the argument in [2, 5.6] to show the injectivity. Take a nonzero element $a$ of $A(\Lambda)$ and fix a maximal ideal $\mathfrak{m}$ which does not contain $a$. Assume that $W_{\mathrm{glob}}(\Lambda) \otimes_{A(\Lambda)} A(\Lambda)/\mathfrak{m}$ is nonzero. Then, the image of $a$ in $A(\Lambda)/\mathfrak{m}$ acts on $W_{\mathrm{glob}}(\Lambda) \otimes_{A(\Lambda)} A(\Lambda)/\mathfrak{m}$ by a nonzero scaler. Hence, we conclude that $a$ acts on $W_{\mathrm{glob}}(\Lambda)$ nontrivially and the map $A(\Lambda) \to \tilde{A}(\Lambda) \cong \mathrm{End}_{\mathfrak{g}'_{\mathrm{tor}}} W_{\mathrm{glob}}(\Lambda)$ is shown to be injective.

Thus, it is enough to show that $W_{\mathrm{glob}}(\Lambda) \otimes_{A(\Lambda)} A(\Lambda)/\mathfrak{m}$ is nonzero. We denote by $\bar{p}_k^{(i)}$ ($i \in I_{\mathrm{aff}}, k \in \mathbb{Z}$) the image of the power some function $p_k^{(i)} = \sum_{m=1}^{m_i} z_{i,m}^k$ in $A(\Lambda)/\mathfrak{m}$. We can choose a set of nonzero complex numbers $\{a_{i,m}\}$ satisfying

$$\sum_{m=1}^{m_i} a_{i,m}^k = \bar{p}_k^{(i)}$$

under an identification $A(\Lambda)/\mathfrak{m} \cong \mathbb{C}$. For each $a \in \mathbb{C}^\times$, we have the evaluation map

$$\mathrm{ev}_a \colon \mathfrak{g}'_{\mathrm{tor}} \to \mathfrak{g}_{\mathrm{aff}}^{(t)}$$

defined as the composite of

$$\mathfrak{g}'_{\mathrm{tor}} \to \mathfrak{g}'_{\mathrm{tor}} / \bigoplus_{\substack{k \in \mathbb{Z} \\ l \neq 0}} \mathbb{C}c(k,l) \oplus \mathbb{C}c_s \cong \left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)' \otimes \mathbb{C}[s^{\pm 1}] \oplus \mathbb{C}d_t$$

and the evaluation at $s = a$. Then, we have a nonzero $\mathfrak{g}'_{\mathrm{tor}}$-module homomorphism

$$W_{\mathrm{glob}}(\Lambda) \otimes_{A(\Lambda)} A(\Lambda)/\mathfrak{m} \to \bigotimes_{i \in I_{\mathrm{aff}}} \bigotimes_{m=1}^{m_i} \mathrm{ev}^*_{a_{i,m}} L(\Lambda_i)$$

assigning $v_\Lambda \otimes 1$ to the tensor product of highest weight vectors. This proves the assertion. □

We have a completely analogous story for the global Weyl module $W^+_{\mathrm{glob}}(\Lambda)$ over $\mathfrak{g}^+_{\mathrm{tor}}$ if we replace $A(\Lambda)$ with

$$A^+(\Lambda) = \bigotimes_{i \in I_{\mathrm{aff}}} \mathbb{C}[z_{i,1}, \ldots, z_{i,m_i}]^{\mathfrak{S}_{m_i}}.$$

We can summarize the discussion so far as follows.

**Proposition 3.6** *We have* $\mathrm{End}_{\mathfrak{g}'_{\mathrm{tor}}} W_{\mathrm{glob}}(\Lambda) \cong A(\Lambda)$ *and* $\mathrm{End}_{\mathfrak{g}^+_{\mathrm{tor}}} W^+_{\mathrm{glob}}(\Lambda) \cong A^+(\Lambda)$.

For a maximal ideal $\mathbf{a}$ of $A = A(\Lambda)$ or $A^+(\Lambda)$, we denote by $\mathbb{C}_{\mathbf{a}}$ the corresponding one-dimensional module $A/\mathbf{a}$.

**Definition 3.7** We call

$$W_{\mathrm{loc}}(\Lambda, \mathbf{a}) = W_{\mathrm{glob}}(\Lambda) \otimes_{A(\Lambda)} \mathbb{C}_{\mathbf{a}}, \quad W^+_{\mathrm{loc}}(\Lambda, \mathbf{a}) = W^+_{\mathrm{glob}}(\Lambda) \otimes_{A^+(\Lambda)} \mathbb{C}_{\mathbf{a}}$$

the local Weyl modules for $\mathfrak{g}'_{\mathrm{tor}}$ and $\mathfrak{g}^+_{\mathrm{tor}}$, respectively.

We denote the images of $v_\Lambda$ and $v^+_\Lambda$ in the local Weyl modules by $v_{\Lambda,\mathbf{a}}$ and $v^+_{\Lambda,\mathbf{a}}$.

***Remark 3.8*** The global/local Weyl modules for $\mathfrak{g}_{\mathrm{tor}}$ and $\mathfrak{g}^+_{\mathrm{tor}}$ can be regarded as a sort of highest weight modules with respect to their triangular decompositions:

$$\mathfrak{g}_{\mathrm{tor}} = \bar{\mathfrak{n}}_{\mathrm{tor}} \oplus \mathfrak{a}_{\mathrm{tor}} \oplus \mathfrak{n}_{\mathrm{tor}}, \quad \mathfrak{g}^+_{\mathrm{tor}} = \left(\bar{\mathfrak{n}}_{\mathrm{tor}} \cap \mathfrak{g}^+_{\mathrm{tor}}\right) \oplus \left(\mathfrak{a}_{\mathrm{tor}} \cap \mathfrak{g}^+_{\mathrm{tor}}\right) \oplus \left(\mathfrak{n}_{\mathrm{tor}} \cap \mathfrak{g}^+_{\mathrm{tor}}\right).$$

### 3.2 Finiteness of weight spaces

The goal of this subsection is to prove the following.

**Proposition 3.9** (i) *Every weight space $W_{\mathrm{glob}}(\Lambda)_{\Lambda-\beta}$ is finitely generated over $A(\Lambda)$. Every weight space $W_{\mathrm{loc}}(\Lambda, \mathbf{a})_{\Lambda-\beta}$ is finite-dimensional.*

(ii) *Every weight space $W^{+}_{\mathrm{glob}}(\Lambda)_{\Lambda-\beta}$ is finitely generated over $A^{+}(\Lambda)$. Every weight space $W^{+}_{\mathrm{loc}}(\Lambda, \mathbf{a})_{\Lambda-\beta}$ is finite-dimensional.*

(iii) *We have $W_{\mathrm{loc}}(\Lambda, \mathbf{a}) = U(\mathfrak{g}^{+}_{\mathrm{tor}})v_{\Lambda,\mathbf{a}}$.*

We start to prove the following lemma.

**Lemma 3.10** *Let $\Lambda$ be a dominant integral weight of $\mathfrak{g}^{(t)}_{\mathrm{aff}}$.*

(i) *For each positive root $\beta \in \Delta^{+}_{\mathrm{aff}}$, there exists a nonnegative integer $N(\beta)$ satisfying the following: we have*

$$(X_{-\beta} \otimes s^{k})v_{\Lambda} \in \sum_{m=0}^{N(\beta)} (X_{-\beta} \otimes s^{m})A(\Lambda)v_{\Lambda}$$

*for any root vector $X_{-\beta}$ of $\bar{\mathfrak{n}}^{(t)}_{\mathrm{aff}}$ corresponding to a negative root $-\beta$ and any $k$.*

(ii) *For each positive integer $l > 0$, there exists a nonnegative integer $N_l$ satisfying the following: we have*

$$c(k, -l)v_{\Lambda} \in \sum_{m=1}^{N_l} c(m, -l)A(\Lambda)v_{\Lambda} + \sum_{m=0}^{N_l} \left( \left(\mathfrak{g}^{(t)}_{\mathrm{aff}}\right)_{-l\delta} \otimes s^{m} \right) A(\Lambda)v_{\Lambda}$$

*for any $k$.*

***Proof*** The assertion (i) is proved in the same way as [3, Proposition 3.2 and Corollary 3.1].

We prove (ii). Take an arbitrary element $\alpha$ of $\Delta^{+}$ and fix root vectors $x_{\alpha} \in \mathfrak{g}_{\alpha}$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ satisfying $(x_{\alpha}, x_{-\alpha}) = 1$. Then, we have

$$\begin{aligned} \left(s^{k}t^{-l}\mathrm{d}s\right) v_{\Lambda} &= \left(\left[x_{\alpha} \otimes s, x_{-\alpha} \otimes s^{k}t^{-l}\right] - h_{\alpha} \otimes s^{k+1}t^{-l}\right) v_{\Lambda} \\ &= (x_{\alpha} \otimes s) \left(x_{-\alpha} \otimes s^{k}t^{-l}\right) v_{\Lambda} - \left(h_{\alpha} \otimes s^{k+1}t^{-l}\right) v_{\Lambda}. \end{aligned}$$

We have

$$(x_{\alpha} \otimes s) \left(x_{-\alpha} \otimes s^{k}t^{-l}\right) v_{\Lambda} \in (x_{\alpha} \otimes s) \sum_{m=0}^{N(\alpha+l\delta)} (x_{-\alpha} \otimes s^{m}t^{-l})A(\Lambda)v_{\Lambda}$$

by (i). The right-hand side is equal to

$$\sum_{m=0}^{N(\alpha+l\delta)} \left(h_{\alpha} \otimes s^{m+1}t^{-l} + s^{m}t^{-l}\mathrm{d}s\right) A(\Lambda)v_{\Lambda}$$

$$= \sum_{m=1}^{N(\alpha+l\delta)+1} \left(h_\alpha \otimes s^m t^{-l} + c(m,-l)\right) A(\Lambda) v_\Lambda.$$

We have

$$(h_\alpha \otimes s^{k+1} t^{-l}) v_\Lambda \in \sum_{m=0}^{N(l\delta)} (h_\alpha \otimes s^m t^{-l}) A(\Lambda) v_\Lambda$$

again by (i). Hence, we conclude that

$$(s^k t^{-l} \mathrm{d}s) v_\Lambda \in \sum_{m=1}^{N_l} c(m,-l) A(\Lambda) v_\Lambda + \sum_{m=0}^{N_l} \left( \left(\mathfrak{g}_{\mathrm{aff}}^{(t)}\right)_{-l\delta} \otimes s^m \right) A(\Lambda) v_\Lambda$$

if we put $N_l = \max(N(l\delta), N(\alpha + l\delta) + 1)$. □

The following proposition is an analog of [5, Proposition 1.2] for the case of the affine Lie algebra and of [3, Proposition 3.2 and Corollary 3.1] for the quotient of $\mathfrak{g}_{\mathrm{tor}}$ modulo the elements $c(k,l)$ with $l \neq 0$ (cf. Remark 2.3).

**Proposition 3.11** *For each positive root $\beta_j \in \Delta_{\mathrm{aff}}^+$ and each positive integer $l > 0$, there exist nonnegative integers $N(\beta_j)$ and $N_l$ such that the weight space $W_{\mathrm{glob}}(\Lambda)_{\Lambda-\beta}$ for $\beta \in Q_{\mathrm{aff}}^+$ is spanned by elements of the form*

$$(X_{-\beta_1} \otimes s^{k_1}) \cdots (X_{-\beta_a} \otimes s^{k_a}) \left( \prod_{j=1}^{b} c(m_j, -l_j) \right) A(\Lambda) v_\Lambda, \tag{3.1}$$

*where each $X_{-\beta_j}$ is a root vector of $\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}$ corresponding to a negative root $-\beta_j$ and each $l_j > 0$ is a positive integer satisfying $\beta = \sum_{j=1}^{a} \beta_j + \left(\sum_{j=1}^{b} l_j\right) \delta$ and $0 \le k_j \le N(\beta_j)$, $1 \le m_j \le N_{l_j}$. A similar statement also holds for $W_{\mathrm{glob}}^+(\Lambda)_{\Lambda-\beta}$.*

***Proof*** By the PBW theorem, we see that $W_{\mathrm{glob}}(\Lambda)_{\Lambda-\beta}$ is spanned by elements of the form as (3.1) without any conditions on $k_j$ and $m_j$. Then, we use Lemma 3.10 to show the assertion by the induction on $a + b$. □

Thus, we establish Proposition 3.9 from Proposition 3.11. We also have the following.

**Proposition 3.12** *Let* **a** *be a maximal ideal of $A(\Lambda)$ and regard it also as a maximal ideal of $A^+(\Lambda)$. Then we have $\mathrm{ch}_p\, W_{\mathrm{loc}}^+(\Lambda, \mathbf{a}) \ge \mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda, \mathbf{a})$.*

***Proof*** We have a $\mathfrak{g}_{\mathrm{tor}}^+$-homomorphism $W_{\mathrm{loc}}^+(\Lambda, \mathbf{a}) \to \mathrm{Res}\, W_{\mathrm{loc}}(\Lambda, \mathbf{a})$ assigning $v_{\Lambda,\mathbf{a}}^+ \mapsto v_{\Lambda,\mathbf{a}}$. It is surjective by Proposition 3.9 (iii). □

### 3.3 Upper bound for the level one Weyl module

In this subsection, we consider the case $\Lambda = \Lambda_0$. The ring $A(\Lambda_0)$ is identified with $\mathbb{C}[z^{\pm 1}]$ and the action on $W_{\mathrm{glob}}(\Lambda_0)$ is given by

$$z^k (X v_{\Lambda_0}) = X(h_{0,k} v_{\Lambda_0})$$

for $X \in U(\mathfrak{g}'_{\mathrm{tor}})$. This identification induces $A^+(\Lambda_0) = \mathbb{C}[z]$.

**Lemma 3.13** *We have $h_{i,k} v_{\Lambda_0} = 0$ for $i \in I$ and $k \in \mathbb{Z}$.*

***Proof*** The defining relations $e_{i,k} v_{\Lambda_0} = 0$ and $f_i v_{\Lambda_0} = 0$ for $i \in I$ imply the assertion. □

Recall that $\sum_{i \in I_{\mathrm{aff}}} h_{i,k} = s^k t^{-1} \mathrm{d}t$. By Lemma 3.13, we see that the action of $A(\Lambda_0)$ on $W_{\mathrm{glob}}(\Lambda_0)$ is given by $z^k \mapsto s^k t^{-1} \mathrm{d}t$. In particular, $z$ acts by $c(1, 0) = s t^{-1} \mathrm{d}t$.

We have defined the local Weyl modules $W_{\mathrm{loc}}(\Lambda_0, a)$ for $a \in \mathbb{C}^\times$ and $W^+_{\mathrm{loc}}(\Lambda_0, a)$ for $a \in \mathbb{C}$ by

$$W_{\mathrm{loc}}(\Lambda_0, a) = W_{\mathrm{glob}}(\Lambda_0) \otimes_{A(\Lambda_0)} \mathbb{C}_a, \quad W^+_{\mathrm{loc}}(\Lambda_0, a) = W^+_{\mathrm{glob}}(\Lambda_0) \otimes_{A^+(\Lambda_0)} \mathbb{C}_a.$$

**Proposition 3.14** *The p-character $\mathrm{ch}_p\, W^+_{\mathrm{loc}}(\Lambda_0, a)$ is independent of $a \in \mathbb{C}$.*

***Proof*** The defining relations of $W^+_{\mathrm{loc}}(\Lambda_0, a)$ are given by

$$(\mathfrak{n}_{\mathrm{tor}} \cap \mathfrak{g}^+_{\mathrm{tor}}) v^+_{\Lambda_0,a} = 0, \quad h_{i,k} v^+_{\Lambda_0,a} = \delta_{i,0} a^k v^+_{\Lambda_0,a} \ (i \in I_{\mathrm{aff}}, k \geq 0), \quad d_t v^+_{\Lambda_0,a} = 0,$$
$$f_0^2 v^+_{\Lambda_0,a} = 0, \quad f_i v^+_{\Lambda_0,a} = 0 \ (i \in I).$$

Hence, we have $\tau_a^* W^+_{\mathrm{loc}}(\Lambda_0, 0) \cong W^+_{\mathrm{loc}}(\Lambda_0, a)$, where $\tau_a$ is the automorphism of $\mathfrak{g}^+_{\mathrm{tor}}$ defined in Sect. 2.4. This proves the assertion. □

We put

$$W(\Lambda_0) = W^+_{\mathrm{loc}}(\Lambda_0, 0) = W^+_{\mathrm{glob}}(\Lambda_0) \otimes_{A^+(\Lambda_0)} \mathbb{C}_0$$

and denote its highest weight vector $v^+_{\Lambda_0,0}$ by $v_0$. This $W(\Lambda_0)$ is regarded as a graded $\mathfrak{g}^+_{\mathrm{tor}}$-module by setting $\deg v_0 = 0$.

**Lemma 3.15** *We have $f_{i,k} v_0 = 0$ for any $i \in I_{\mathrm{aff}}$ and $k \geq 1$.*

***Proof*** The assertion for $i \in I$ follows from $f_i v_0 = 0$ and $h_{i,k} v_0 = 0$. The assertion for $i = 0$ follows from

$$0 = e_{0,k} f_0^2 v_0 = \left[e_{0,k}, f_0^2\right] v_0 = (-2 f_{0,k} + 2 f_0 h_{0,k}) v_0$$

and $h_{0,k} v_0 = 0$ for $k \geq 1$. □

**Lemma 3.16** *Let $k \geq 1$. We have*

(i)

$$(e_\theta \otimes s^k t^{-l})v_0 = \begin{cases} 0 & \text{if } l \leq k, \\ \sum_{m=1}^{l-k} c(k, -l+m)(e_\theta \otimes t^{-m})v_0 & \text{if } l > k, \end{cases}$$

(ii)

$$(s^k t^{-l} \mathrm{d}s)v_0 = \begin{cases} 0 & \text{if } l \leq k, \\ \sum_{m=1}^{l-k} c(k, -l+m)(t^{-m} \mathrm{d}s)v_0 & \text{if } l > k. \end{cases}$$

***Proof*** We prove the assertions (i) and (ii) by induction on $l$.

For $l \leq 0$, $e_\theta \otimes s^k t^{-l}$ is an element of $\mathfrak{n}_{\mathrm{tor}} \cap \mathfrak{g}_{\mathrm{tor}}^+$, hence it kills $v_0$. For $l = 1$, $e_\theta \otimes s^k t^{-1} = f_{0,k}$ kills $v_0$ by Lemma 3.15. Then, we have

$$(s^k t^{-l} \mathrm{d}s)v_0 = \left([f_\theta \otimes s, e_\theta \otimes s^k t^{-l}] - [f_\theta, e_\theta \otimes s^{k+1} t^{-l}]\right) v_0 = 0$$

for $l \leq 1$. We thus have proved (i) and (ii) for $l \leq 1$.

Let $l \geq 2$. We assume the assertions (i) and (ii) for all $l' < l$. By Lemma 2.8, we have

$$\begin{aligned}(e_\theta \otimes s^k t^{-l})v_0 &= T_0 T_\theta \left((e_\theta \otimes s^k t^{-l+2}) T_\theta^{-1} T_0^{-1} v_0\right) \\ &= T_0 T_\theta \left((e_\theta \otimes s^k t^{-l+2}) T_\theta^{-1} (f_0 v_0)\right) \\ &= T_0 T_\theta \left((e_\theta \otimes s^k t^{-l+2}) T_\theta^{-1} (f_0) v_0\right) \\ &= T_0 T_\theta \left(T_\theta^{-1}(f_0)(e_\theta \otimes s^k t^{-l+2})v_0 + [e_\theta \otimes s^k t^{-l+2}, T_\theta^{-1}(f_0)]v_0\right).\end{aligned} \tag{3.2}$$

We have

$$\begin{aligned}[e_\theta \otimes s^k t^{-l+2}, T_\theta^{-1}(f_0)] &= [e_\theta \otimes s^k t^{-l+2}, -f_\theta \otimes t^{-1}] \\ &= -\left([e_\theta \otimes s^k t^{-l+1}, f_\theta] + c(k, -l+1)\right) \\ &= [f_\theta, e_\theta \otimes s^k t^{-l+1}] - c(k, -l+1).\end{aligned}$$

Put

$$A = T_\theta^{-1}(f_0)(e_\theta \otimes s^k t^{-l+2})v_0, \quad B = f_\theta(e_\theta \otimes s^k t^{-l+1})v_0.$$

Then, (3.2) is equal to $T_0 T_\theta(A + B - c(k, -l+1)v_0)$. By the induction assumption, we have

$$\begin{aligned} A &= T_\theta^{-1}(f_0) \sum_{m=1}^{l-2-k} c(k, -l+2+m)(e_\theta \otimes t^{-m})v_0, \\ B &= f_\theta \sum_{m=1}^{l-1-k} c(k, -l+1+m)(e_\theta \otimes t^{-m})v_0 \\ &= f_\theta \sum_{m=0}^{l-2-k} c(k, -l+2+m)(e_\theta \otimes t^{-m-1})v_0. \end{aligned}$$

Then, (3.2) is equal to

$$\begin{aligned} &T_0 T_\theta \Bigg( \sum_{m=1}^{l-2-k} c(k, -l+2+m) \Big( T_\theta^{-1}(f_0)(e_\theta \otimes t^{-m}) + f_\theta(e_\theta \otimes t^{-m-1}) \Big) v_0 \\ &\quad + c(k, -l+2) f_\theta (e_\theta \otimes t^{-1}) v_0 - c(k, -l+1)v_0 \Bigg) \end{aligned} \tag{3.3}$$

if $l \geq k+2$ and to $T_0 T_\theta(-c(k, -l+1)v_0)$ if $l \leq k+1$.

We prove (i) for $l$. First, consider the case $l \leq k$. In this case, we have

$$\left(e_\theta \otimes s^k t^{-l}\right) v_0 = T_0 T_\theta(-c(k, -l+1)v_0) = \frac{k}{-l+1} T_0 T_\theta((s^{k-1} t^{-(l-1)} \mathrm{d}s)v_0) = 0$$

by the induction assumption. Hence, (i) holds for $l$. Next, consider the case $l = k+1$. In this case, we have

$$\left(e_\theta \otimes s^k t^{-l}\right) v_0 = T_0 T_\theta(-c(k, -l+1)v_0) = -c(k, -l+1) T_0 T_\theta(v_0).$$

Since we have $T_0 T_\theta(v_0) = -f_0 v = -(e_\theta \otimes t^{-1})v_0$, (i) holds for $l = k+1$. Finally, consider the case $l \geq k+2$. The equality (3.2) is valid even for $k = 0$, and hence, we have

$$\left(e_\theta \otimes t^{-m-2}\right) v_0 = T_0 T_\theta \Big( \Big( T_\theta^{-1}(f_0)(e_\theta \otimes t^{-m}) + f_\theta(e_\theta \otimes t^{-m-1}) \Big) v_0 \Big)$$

for each $m$. This implies that (3.3) is equal to

$$\begin{aligned} &\sum_{m=1}^{l-2-k} c(k, -l+2+m) \left(e_\theta \otimes t^{-m-2}\right) v_0 \\ &\quad + c(k, -l+2) T_0 T_\theta (f_\theta (e_\theta \otimes t^{-1}) v_0) + c(k, -l+1)(e_\theta \otimes t^{-1}) v_0. \end{aligned}$$

Since we can easily show $T_0 T_\theta(f_\theta(e_\theta \otimes t^{-1})v_0) = (e_\theta \otimes t^{-2})v_0$, (i) is proved for $l$.

We prove (ii) for $l$. By (i), we have

$$\begin{aligned}(s^k t^{-l} \mathrm{d}s)v_0 &= \Big([f_\theta \otimes s, e_\theta \otimes s^k t^{-l}] - [f_\theta, e_\theta \otimes s^{k+1} t^{-l}]\Big)\, v_0 \\ &= (f_\theta \otimes s) \sum_{m=1}^{l-k} c(k, -l+m)(e_\theta \otimes t^{-m})v_0 \\ &\quad - f_\theta \sum_{n=1}^{l-(k+1)} c(k+1, -l+n)(e_\theta \otimes t^{-n})v_0\end{aligned}$$

if $l > k$ and $(s^k t^{-l} \mathrm{d}s)v_0 = 0$ otherwise. Therefore, we may assume $l > k$. We have

$$\begin{aligned}(f_\theta \otimes s)(e_\theta \otimes t^{-m})v_0 &= [f_\theta \otimes s, e_\theta \otimes t^{-m}]v_0 \\ &= \big([f_\theta, e_\theta \otimes s t^{-m}] + t^{-m}\mathrm{d}s\big)\, v_0 \\ &= f_\theta(e_\theta \otimes s t^{-m})v_0 + (t^{-m}\mathrm{d}s)v_0 \\ &= f_\theta \sum_{n=1}^{m-1} c(1, -m+n)(e_\theta \otimes t^{-n})v_0 + (t^{-m}\mathrm{d}s)v_0.\end{aligned}$$

We claim that

$$\begin{aligned}&\sum_{m=1}^{l-k} c(k, -l+m) \sum_{n=1}^{m-1} c(1, -m+n)(e_\theta \otimes t^{-n})v_0 \\ &= \sum_{n=1}^{l-(k+1)} c(k+1, -l+n)(e_\theta \otimes t^{-n})v_0\end{aligned}$$

holds. Indeed, this equality is obtained by applying $h_\theta \otimes s$ to both sides of (i). Hence, we conclude

$$\begin{aligned}(s^k t^{-l} \mathrm{d}s)v_0 &= \sum_{m=1}^{l-k} c(k, -l+m)\left( f_\theta \sum_{n=1}^{m-1} c(1, -m+n)(e_\theta \otimes t^{-n})v_0 + (t^{-m}\mathrm{d}s)v_0\right) \\ &\quad - f_\theta \sum_{n=1}^{l-(k+1)} c(k+1, -l+n)(e_\theta \otimes t^{-n})v_0 \\ &= \sum_{m=1}^{l-k} c(k, -l+m)(t^{-m}\mathrm{d}s)v_0.\end{aligned}$$

□

We define the subalgebra $\bar{C}$ of $U(\mathfrak{g}_{\mathrm{tor}}^+)$ to be generated by $c(k, -l)$ $(k \geq 1, l \geq 1)$. Let $\bar{C}_1$ be the subalgebra of $\bar{C}$ generated by $c(1, -l)$ $(l \geq 1)$.

**Lemma 3.17** *We have $\bar{C} v_0 = \bar{C}_1 v_0$.*

***Proof*** Suppose $k \geq 1$ and $l \geq 1$. We rewrite Lemma 3.16 (ii) as

$$(s^k t^{-l} \mathrm{d}s) v_0 = \begin{cases} 0 & \text{if } l \leq k, \\ \sum_{m=1}^{l-k} \dfrac{k}{l-m} (s^{k-1} t^{-l+m} \mathrm{d}s)(t^{-m} \mathrm{d}s) v_0 & \text{if } l > k. \end{cases}$$

This implies that the action of $c(k+1, -l) = ((k+1)/l) s^k t^{-l} \mathrm{d}s$ on $v_0$ is written in terms of a polynomial in $c(1, -m) = (1/m) t^{-m} \mathrm{d}s$ with $m \geq 1$. □

**Lemma 3.18** *We have*

$$\left( \bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)} \otimes s\mathbb{C}[s] \right) v_0 \subset \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_0.$$

***Proof*** Note that we have

$$\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)} \otimes s^k = \bigoplus_{\substack{\alpha \in \Delta^+ \cup \{0\} \\ l \geq 1}} \mathfrak{g}_\alpha \otimes s^k t^{-l} \oplus \bigoplus_{\substack{\alpha \in \Delta^- \\ l \geq 0}} \mathfrak{g}_\alpha \otimes s^k t^{-l}.$$

Suppose $k \geq 1$. We show

$$(x \otimes s^k t^{-l}) v_0 \in \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_0 \tag{3.4}$$

for

- $x \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta^+ \cup \{0\})$ and $l \geq 1$;
- $x \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta^-)$ and $l \geq 0$.

Lemma 3.16 (i) and 3.17 imply (3.4) for $x = e_\theta$ and $l \geq 1$. Then, we obtain (3.4) for $x \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta^+)$ and $l \geq 1$ by successively applying $f_i$'s $(i \in I)$ to $(e_\theta \otimes s^k t^{-l}) v_0$. We obtain (3.4) for $x = h_i$ $(i \in I)$ and $l \geq 1$ by applying $f_i$ to $(e_i \otimes s^k t^{-l}) v_0$. We show (3.4) for $x \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta^-)$ and $l \geq 0$. The case $l = 0$ is immediate from Lemma 3.15. Assume $l \geq 1$. We use $[h_\alpha \otimes s^k t^{-l}, x] = 2x \otimes s^k t^{-l}$ and $x v_0 = 0$ to deduce

$$(x \otimes s^k t^{-l}) v_0 = -\frac{1}{2} x (h_\alpha \otimes s^k t^{-l}) v_0 \in x \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_0 \subset \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_0.$$

□

**Proposition 3.19** *We have*

$$W(\Lambda_0) = \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_0.$$

*In particular, we have an inequality*

$$\mathrm{ch}_{p,q} W(\Lambda_0) \leq \mathrm{ch}_p L(\Lambda_0) \prod_{n>0} \frac{1}{1 - p^n q}.$$

***Proof*** Let $N$ be the $\mathbb{C}$-span of monomials in $\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)} \otimes s\mathbb{C}[s]$. Then, the PBW theorem and Lemma 3.17 imply

$$W(\Lambda_0) = U(\bar{\mathfrak{n}}_{\mathrm{tor}} \cap \mathfrak{g}_{\mathrm{tor}}^{+})v_0 = \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) N v_0.$$

Since $\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)} \otimes s\mathbb{C}[s]$ is ad $\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}$-invariant modulo central elements, we prove the assertion by Lemmas 3.17 and 3.18. □

***Remark 3.20*** We will show in Corollay 4.11 that the equality

$$\mathrm{ch}_{p,q}\, W(\Lambda_0) = \mathrm{ch}_p\, L(\Lambda_0) \prod_{n>0} \frac{1}{1-p^n q}$$

holds.

***Remark 3.21*** By Propositions 3.12, 3.14 and 3.19, we have an inequality

$$\mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda_0, a) \leq \mathrm{ch}_p\, L(\Lambda_0) \prod_{n>0} \frac{1}{1-p^n}.$$

We will show in Corollary 4.11 that the equality holds. In fact, we can directly prove this inequality for $\mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda_0, a)$ by a similar calculation for $W_{\mathrm{loc}}(\Lambda_0, a)$ instead of $W(\Lambda_0)$. More precisely, we can show $W_{\mathrm{loc}}(\Lambda_0, a) = \bar{C}_1 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_{\Lambda_0, a}$. Moreover, we can show that

$$W_{\mathrm{loc}}(\Lambda_0, a) = \bar{C}_0 U(\bar{\mathfrak{n}}_{\mathrm{aff}}^{(t)}) v_{\Lambda_0, a}$$

also holds, where $\bar{C}_0$ is the subalgebra of $U(\mathfrak{g}_{\mathrm{tor}}')$ generated by $c(0, -l)$ ($l \geq 1$).

Here, we gave the calculation for $W(\Lambda_0)$ by two reasons:

(i) we are interested in the $(p, q)$-characters of the graded local Weyl modules for $\mathfrak{g}_{\mathrm{tor}}^{+}$;
(ii) the calculation for $W(\Lambda_0)$ is easier than that for $W_{\mathrm{loc}}(\Lambda_0, a)$.

## 4 Vertex operator construction and Weyl modules

### 4.1 Heisenberg Lie algebras

We assume that $\mathfrak{g}$ is of type ADE in Sects. 4.1 and 4.2. Recall that $Q_{\mathrm{aff}} = \bigoplus_{i \in I_{\mathrm{aff}}} \mathbb{Z}\alpha_i$ is the root lattice of $\mathfrak{g}_{\mathrm{aff}}^{(t)}$. We fix a bimultiplicative 2-cocycle $\varepsilon \colon Q_{\mathrm{aff}} \times Q_{\mathrm{aff}} \to \{\pm 1\}$ satisfying

$$\varepsilon(\alpha, \alpha) = (-1)^{(\alpha,\alpha)/2}, \quad \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)}, \quad \varepsilon(\alpha, \delta) = 1$$

as in [16, Section 4]. Let $\mathbb{C}[Q_{\mathrm{aff}}]$ be the group algebra of $Q_{\mathrm{aff}}$ with a $\mathbb{C}$-basis denoted by $e^{\alpha}$ ($\alpha \in Q_{\mathrm{aff}}$). We make $\mathbb{C}[Q_{\mathrm{aff}}]$ into a $\mathbb{C}[Q_{\mathrm{aff}}]$-module via $\varepsilon$, that is, we define

$e^\alpha \cdot e^\beta = \varepsilon(\alpha, \beta)e^{\alpha+\beta}$. We denote by $\mathbb{C}_\varepsilon[Q_{\mathrm{aff}}]$ this module. We define an action of $h \in \mathfrak{h}_{\mathrm{aff}}^{(t)}$ on $\mathbb{C}_\varepsilon[Q_{\mathrm{aff}}]$ by $h \cdot e^\alpha = \langle h, \alpha\rangle e^\alpha$.

The toroidal Lie algebra $\mathfrak{g}_{\mathrm{tor}}$ contains a Heisenberg Lie algebra

$$\mathcal{H} = \bigoplus_{\substack{i \in I_{\mathrm{aff}} \\ k \neq 0}} \mathbb{C}h_{i,k} \oplus \mathbb{C}c_s.$$

Define the Fock representation $\mathcal{F}_{\mathrm{aff}}$ of $\mathcal{H}$ by

$$\mathcal{F}_{\mathrm{aff}} = U(\mathcal{H}) / \sum_{\substack{i \in I_{\mathrm{aff}} \\ k > 0}} U(\mathcal{H})h_{i,k} + U(\mathcal{H})(c_s - 1).$$

We set

$$\mathbb{V}(0) = \mathcal{F}_{\mathrm{aff}} \otimes \mathbb{C}_\varepsilon[Q_{\mathrm{aff}}].$$

Define the degree on $\mathbb{V}(0)$ by $\deg h_{i,k} = k$ and $\deg e^\alpha = (\alpha, \alpha)/2$. Then, we regard $\mathbb{V}(0)$ as a module of $\mathfrak{a}_{\mathrm{tor}} = \mathcal{H} \oplus \mathfrak{h}_{\mathrm{aff}}^{(t)} \oplus \mathbb{C}d_s$ via the actions of $\mathcal{H}$ and $\mathfrak{h}_{\mathrm{aff}}^{(t)}$ on $\mathcal{F}_{\mathrm{aff}}$ and $\mathbb{C}_\varepsilon[Q_{\mathrm{aff}}]$, respectively, and so that $d_s$ counts the degree.

Similarly, we define $\mathcal{F}$ to be the Fock representation for a Heisenberg Lie subalgebra

$$\bigoplus_{\substack{i \in I \\ k \neq 0}} \mathbb{C}h_{i,k} \oplus \mathbb{C}c_s$$

of $\mathfrak{g}_{\mathrm{aff}}^{(s)}$.

### 4.2 Vertex representations

For each $\alpha \in \Delta_{\mathrm{aff}}$, we set

$$X(\alpha, u) = u^{(\alpha,\alpha)/2} \left(e^\alpha u^{h_\alpha}\right) \exp\left(\sum_{k>0} \frac{h_\alpha \otimes s^{-k}}{k} u^k\right) \exp\left(-\sum_{k>0} \frac{h_\alpha \otimes s^{k}}{k} u^{-k}\right)$$

as an element of $(\mathrm{End}_{\mathbb{C}}\, \mathbb{V}(0))[[u^{\pm 1}]]$. Here, $u^{h_\alpha}$ acts by

$$u^{h_\alpha} \cdot e^\beta = u^{(\alpha,\beta)} e^\beta.$$

Define $X_k(\alpha)$ by the expansion

$$X(\alpha, u) = \sum_{k \in \mathbb{Z}} X_k(\alpha) u^{-k}.$$

**Theorem 4.1** ([16] Proposition 4.3) *We can extend the action of* $\mathfrak{a}_{\text{tor}} = \mathcal{H} \oplus \mathfrak{h}_{\text{aff}}^{(t)} \oplus \mathbb{C}d_s$ *to* $\mathfrak{g}_{\text{tor}}$ *on* $\mathbb{V}(0)$ *by*

$$e_{i,k} \mapsto X_k(\alpha_i), \quad f_{i,k} \mapsto X_k(-\alpha_i).$$

We denote by $\tau$ the action of $c(0, 1)$ on $\mathbb{V}(0)$. Then, by [16, (4.1) and Proposition 5.3 (ii)], the action of $c(0, k)$ for $k \neq 0$ is given by $\tau^k$. The subalgebra of $\operatorname{End}_{\mathbb{C}} \mathbb{V}(0)$ generated by $\tau^k$ $(k \in \mathbb{Z})$ is isomorphic to the Laurent polynomial algebra $\mathbb{C}[\tau^{\pm 1}]$.

We denote by $\delta(k)$ the action of $c(k, 0)$ on $\mathbb{V}(0)$ for $k < 0$. They freely generate a polynomial subalgebra of $\operatorname{End}_{\mathbb{C}} \mathbb{V}(0)$ and we denote it by $D$. We have an isomorphism of $\mathbb{C}$-vector spaces

$$\mathcal{F}_{\text{aff}} \cong \mathcal{F} \otimes D.$$

**Proposition 4.2** ([16] Lemma 5.6) *The multiplication map gives an isomorphism*

$$\mathbb{V}(0) \cong \mathcal{F} \otimes \mathbb{C}_\varepsilon[Q] \otimes D \otimes \mathbb{C}[\tau^{\pm 1}]$$

*of* $\mathbb{C}$*-vector spaces. In particular,* $\mathbb{V}(0)$ *is free over* $\mathbb{C}[\tau^{\pm 1}]$.

The $\mathfrak{g}_{\text{aff}}^{(s)}$-submodule $\mathcal{F} \otimes \mathbb{C}_\varepsilon[Q]$ is known to be isomorphic to the level one integrable irreducible $\mathfrak{g}_{\text{aff}}^{(s)}$-module $L(\Lambda_0)^{(s)}$ with highest weight $\Lambda_0$ by Frenkel–Kac [8]. Hence, it has the following defining relations:

$$(f_\theta \otimes s)(1 \otimes e^0) = 0, \quad e_i(1 \otimes e^0) = 0 \ (i \in I), \tag{4.1}$$
$$c_s(1 \otimes e^0) = 1 \otimes e^0, \quad h_i(1 \otimes e^0) = 0 \ (i \in I), \quad d_s(1 \otimes e^0) = 0, \tag{4.2}$$
$$(e_\theta \otimes s^{-1})^2(1 \otimes e^0) = 0, \quad f_i(1 \otimes e^0) = 0 \ (i \in I). \tag{4.3}$$

We will determine the defining relations of $\mathbb{V}(0)$ as a $\mathfrak{g}_{\text{tor}}$-module as a main result of this article.

### 4.3 General construction

We review the construction of $\mathfrak{g}_{\text{tor}}$-modules given by Iohara–Saito–Wakimoto [13] and Eswara Rao [6]. Assume that $\mathfrak{g}$ is an arbitrary simple Lie algebra. Let $D$ be the polynomial algebra generated by the elements $\delta(k)$ $(k < 0)$. For a given smooth $\mathfrak{g}_{\text{aff}}^{(s)}$-module $M$, we will define a $\mathfrak{g}_{\text{tor}}$-module structure on

$$M \otimes D \otimes \mathbb{C}[\tau^{\pm 1}]$$

as follows. For an element $x$ of $\mathfrak{g}$, we put $x(u) = \sum_{k \in \mathbb{Z}} (x \otimes s^k) u^{-k}$. Define a formal series $\Delta_l(u)$ for each $l \in \mathbb{Z}$ by

$$\Delta_l(u) = \exp\left(\sum_{k>0} \frac{l\delta(-k)}{k} u^k\right).$$

We make $D$ into a graded algebra by $\deg \delta(k) = k$ and let $d^{(D)}$ be the operator which counts the degree on $D$. We make $\mathbb{C}[\tau^{\pm 1}]$ into a graded algebra by $\deg \tau = 1$ and let $d^{(\tau)}$ be the operator which counts the degree on $\mathbb{C}[\tau^{\pm 1}]$.

**Theorem 4.3** ([13] Lemma 2.1, [6] Theorem 4.1) *Let $M$ be a smooth $\mathfrak{g}_{\mathrm{aff}}^{(s)}$-module. The assignment*

$$\sum_{k\in\mathbb{Z}} (x \otimes s^k t^l) u^{-k} \mapsto x(u) \otimes \Delta_l(u) \otimes \tau^l$$

*for $x \in \mathfrak{g}$,*

$$\sum_{k\in\mathbb{Z}} (s^{k-1} t^l \mathrm{d}s) u^{-k} \mapsto c_s \otimes \Delta_l(u) \otimes \tau^l, \quad s^k t^{-1} \mathrm{d}t \mapsto \begin{cases} \mathrm{id} \otimes \delta(k) \otimes \mathrm{id} & \text{if } k < 0, \\ 0 & \text{if } k \geq 0, \end{cases}$$
$$d_s \mapsto d_s \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes d^{(D)} \otimes \mathrm{id}, \quad d_t \mapsto \mathrm{id} \otimes \mathrm{id} \otimes d^{(\tau)}$$

*gives a $\mathfrak{g}_{\mathrm{tor}}$-module structure on $M \otimes D \otimes \mathbb{C}[\tau^{\pm 1}]$.*

***Remark 4.4*** Let us give a remark on the results of [6,13] stated above. In [13], the authors consider a Lie algebra bigger than $\mathfrak{g}_{\mathrm{tor}}$ and the module they construct is bigger than $M \otimes D \otimes \mathbb{C}[\tau^{\pm 1}]$. If one restricts the action to $\mathfrak{g}_{\mathrm{tor}}$, we can take $M \otimes D \otimes \mathbb{C}[\tau^{\pm 1}]$ as a $\mathfrak{g}_{\mathrm{tor}}$-submodule. Moreover, although they assume that $\mathfrak{g}$ is of type ADE in [13], the construction does not need the assumption. Later this construction of $\mathfrak{g}_{\mathrm{tor}}$-modules has been generalized in [6] to some Lie superalgebras.

Take $M$ as the level one integrable irreducible $\mathfrak{g}_{\mathrm{aff}}^{(s)}$-module $L(\Lambda_0)^{(s)}$ with highest weight $\Lambda_0$ and set

$$\mathbb{V}(0) = L(\Lambda_0)^{(s)} \otimes D \otimes \mathbb{C}[\tau^{\pm 1}].$$

This definition is compatible with the construction given in Sects. 4.1 and 4.2 if $\mathfrak{g}$ is of type ADE. Indeed, the definition of the vertex operator $X(\alpha, u)$ implies that

$$X(\beta + l\delta, u) = \begin{cases} X(\beta, u) \otimes \Delta_l(u) \otimes \tau^l & \text{if } \beta \in \Delta, \\ \mathrm{id} \otimes \Delta_l(u) \otimes \tau^l & \text{if } \beta = 0, \end{cases}$$

when we write $\alpha \in \Delta_{\mathrm{aff}}$ as $\alpha = \beta + l\delta$ with $\beta \in \Delta \cup \{0\}$ and $l \in \mathbb{Z}$.

Let $v^{(s)}$ be a highest weight vector of $L(\Lambda_0)^{(s)}$. We generalize the relations given in (4.1), (4.2), (4.3).

**Lemma 4.5** *We have*

$$(f_\theta \otimes s)(v^{(s)} \otimes 1 \otimes 1) = 0, \quad e_i(v^{(s)} \otimes 1 \otimes 1) = 0 \ (i \in I), \tag{4.4}$$
$$c_s(v^{(s)} \otimes 1 \otimes 1) = v^{(s)} \otimes 1 \otimes 1,$$
$$h_i(v^{(s)} \otimes 1 \otimes 1) = 0 \ (i \in I), \quad d_s(v^{(s)} \otimes 1 \otimes 1) = 0, \tag{4.5}$$
$$(e_\theta \otimes s^{-1})^2(v^{(s)} \otimes 1 \otimes 1) = 0, \quad f_i(v^{(s)} \otimes 1 \otimes 1) = 0 \ (i \in I). \tag{4.6}$$

***Proof*** These are direct consequences of the definition of the action and the relations in $L(\Lambda_0)^{(s)}$. □

**Lemma 4.6** *We have* $\mathfrak{g}_{\mathrm{aff}}^{(t)}(v^{(s)} \otimes 1 \otimes 1) = 0$.

***Proof*** We have $\mathfrak{g}(v^{(s)} \otimes 1 \otimes 1) = (\mathfrak{g}v^{(s)}) \otimes 1 \otimes 1 = 0$. To see the action of $e_0 = f_\theta \otimes t$, consider the assignment

$$\sum_{k\in\mathbb{Z}} (f_\theta \otimes s^k t) u^{-k} \mapsto f_\theta(u) \otimes \Delta_1(u) \otimes \tau.$$

Expand $\Delta_1(u) = \sum_{k\geq 0} \Delta_1^{(-k)} u^k$. Then, the action of $e_0 = f_\theta \otimes t$ is given by $\sum_{k\geq 0}(f_\theta \otimes s^k) \otimes \Delta_1^{(-k)} \otimes \tau$. Since we have $(f_\theta \otimes s^k)v^{(s)} = 0$ for $k \geq 0$, we have $e_0(v^{(s)} \otimes 1 \otimes 1) = 0$. Similarly, the action of $f_0 = e_\theta \otimes t^{-1}$ is given by $\sum_{k\geq 0}(e_\theta \otimes s^k) \otimes \Delta_{-1}^{(-k)} \otimes \tau^{-1}$, hence it acts on $v^{(s)} \otimes 1 \otimes 1$ by 0. We have $c_t(v^{(s)} \otimes 1 \otimes 1) = 0$ and $d_t(v^{(s)} \otimes 1 \otimes 1) = 0$ by the definition of the action of $c_t$ and $d_t$. □

### 4.4 Isomorphisms

We define a $\mathfrak{g}_{\mathrm{tor}}$-module $\mathbb{V}$ by the pull-back of $\mathbb{V}(0)$ via the automorphism $S^{-1}$, that is, $\mathbb{V} = (S^{-1})^* \mathbb{V}(0)$. Denote the vector of $\mathbb{V}$ corresponding to $v^{(s)} \otimes 1 \otimes 1 \in \mathbb{V}(0)$ by $\mathbf{v}$.

The action of $c(1,0)$ on $\mathbb{V}$ corresponds to $\tau^{-1}$ on $\mathbb{V}(0)$ via $S^{-1}$ since $S^{-1}(c(1,0)) = c(0,-1)$. We regard $\mathbb{V}$ as a module over $A(\Lambda_0) = \mathbb{C}[z^{\pm 1}]$ via $z \mapsto c(1,0)$, and then, $\mathbb{V}$ becomes a free $A(\Lambda_0)$-module by Proposition 4.2. We put $\mathbb{V}_a = \mathbb{V} \otimes_{A(\Lambda_0)} \mathbb{C}_a$ for $a \in \mathbb{C}^\times$. This $\mathbb{V}_a$ is a $\mathfrak{g}'_{\mathrm{tor}}$-module. The character of $\mathbb{V}_a$ is given as follows.

**Proposition 4.7** *We have* $\mathrm{ch}_p\, \mathbb{V}_a = \mathrm{ch}_p\, L(\Lambda_0) \prod_{n>0} \dfrac{1}{1-p^n}$.

***Proof*** The assertion obviously follows from the construction of the action of $\mathfrak{g}_{\mathrm{tor}}$ on $\mathbb{V}(0) = L(\Lambda_0)^{(s)} \otimes D \otimes \mathbb{C}[\tau^{\pm 1}]$. □

Let us study relation between the level one global Weyl module $W_{\mathrm{glob}}(\Lambda_0)$ and $\mathbb{V}$.

**Lemma 4.8** *We have*

$$h_{i,k}\mathbf{v} = \begin{cases} 0 & \text{if } i \in I, \\ z^k \mathbf{v} & \text{if } i = 0 \end{cases}$$

*for any* $k \in \mathbb{Z}$.

***Proof*** We have

$$S^{-1}(h_{i,k}) = \begin{cases} h_i \otimes t^{-k} & \text{if } i \in I, \\ s^{-1} t^{-k} \mathrm{d}s - h_\theta \otimes t^{-k} & \text{if } i = 0. \end{cases}$$

By Lemma 4.6, we have $(h_i \otimes t^{-k})(v^{(s)} \otimes 1 \otimes 1) = (h_\theta \otimes t^{-k})(v^{(s)} \otimes 1 \otimes 1) = 0$. Since we have $(s^{-1}t^{-k}\mathrm{d}s)(v^{(s)} \otimes 1 \otimes 1) = \tau^{-k}(v^{(s)} \otimes 1 \otimes 1)$ and $\tau^{-1}$ corresponds to $z$, the assertion is proved. □

**Lemma 4.9** *We have a surjective homomorphism $W_{\mathrm{glob}}(\Lambda_0) \to \mathbb{V}$ of modules over both $\mathfrak{g}_{\mathrm{tor}}$ and $A(\Lambda_0)$.*

***Proof*** The equalities (4.4), (4.5), (4.6) are equivalent to

$$\begin{aligned} &e_i \mathbf{v} = 0 \ (i \in I_{\mathrm{aff}}), \\ &c_t \mathbf{v} = \mathbf{v}, \quad h_i \mathbf{v} = 0 \ (i \in I), \quad d_t \mathbf{v} = 0, \\ &f_0^2 \mathbf{v} = 0, \quad f_i \mathbf{v} = 0 \ (i \in I). \end{aligned}$$

Moreover, we have

$$\begin{aligned} c_s \mathbf{v} &= S^{-1}(c_s)(v^{(s)} \otimes 1 \otimes 1) = -c_t(v^{(s)} \otimes 1 \otimes 1) = 0, \\ d_s \mathbf{v} &= S^{-1}(d_s)(v^{(s)} \otimes 1 \otimes 1) = -d_t(v^{(s)} \otimes 1 \otimes 1) = 0 \end{aligned}$$

by Lemma 4.6. We need to check $e_{i,k}\mathbf{v} = 0$ for $i \in I_{\mathrm{aff}}$ and $k \in \mathbb{Z}$. This follows from $e_i \mathbf{v} = 0$ and Lemma 4.8. □

By Lemma 4.9, we have a surjective $\mathfrak{g}'_{\mathrm{tor}}$-homomorphism $W_{\mathrm{loc}}(\Lambda_0, a) \to \mathbb{V}_a$ for every $a \in \mathbb{C}^\times$. Hence, we have inequalities of the characters

$$\mathrm{ch}_p\, W^+_{\mathrm{loc}}(\Lambda_0, a) \geq \mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda_0, a) \geq \mathrm{ch}_p\, \mathbb{V}_a \tag{4.7}$$

by Proposition 3.12.

**Theorem 4.10** *We have isomorphisms*

$$W_{\mathrm{glob}}(\Lambda_0) \xrightarrow{\cong} \mathbb{V}, \quad W_{\mathrm{loc}}(\Lambda_0, a) \xrightarrow{\cong} \mathbb{V}_a$$

*of modules over $\mathfrak{g}_{\mathrm{tor}}$ and $\mathfrak{g}'_{\mathrm{tor}}$, respectively.*

***Proof*** First, we prove the isomorphism $W_{\mathrm{loc}}(\Lambda_0, a) \cong \mathbb{V}_a$. We have

$$\mathrm{ch}_p\, W^+_{\mathrm{loc}}(\Lambda_0, a) = \mathrm{ch}_p\, W(\Lambda_0) \leq \mathrm{ch}_p\, L(\Lambda_0) \prod_{n>0} \frac{1}{1-p^n} = \mathrm{ch}_p\, \mathbb{V}_a \tag{4.8}$$

by Propositions 3.14, 3.19, 4.7. Then the inequalities (4.7) and (4.8) imply $\mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda_0, a) = \mathrm{ch}_p\, \mathbb{V}_a$. This shows that the surjective homomorphism $W_{\mathrm{loc}}(\Lambda_0, a) \to \mathbb{V}_a$ is an isomorphism for every $a \in \mathbb{C}^\times$. Next, we prove the isomorphism $W_{\mathrm{glob}}(\Lambda_0) \cong \mathbb{V}$. Since $\mathbb{V}$ is a free $A(\Lambda_0)$-module, we can take a splitting of the exact sequence

$$0 \to \mathrm{Ker} \to W_{\mathrm{glob}}(\Lambda_0) \to \mathbb{V} \to 0$$

of $A(\Lambda_0)$-modules. The isomorphism $W_{\mathrm{loc}}(\Lambda_0, a) \cong \mathbb{V}_a$ implies $\mathrm{Ker} \otimes_{A(\Lambda_0)} \mathbb{C}_a = 0$ for every $a \in \mathbb{C}^\times$. Then, by Nakayama's lemma, we see that Ker $= 0$ and obtain the isomorphism $W_{\mathrm{glob}}(\Lambda_0) \cong \mathbb{V}$. □

**Corollary 4.11** *We have*

$$\mathrm{ch}_p\, W_{\mathrm{loc}}(\Lambda_0, a) = \mathrm{ch}_p\, W^+_{\mathrm{loc}}(\Lambda_0, a) = \mathrm{ch}_p\, L(\Lambda_0) \left( \prod_{n>0} \frac{1}{1-p^n} \right)$$

*for* $a \in \mathbb{C}^\times$ *and*

$$\mathrm{ch}_{p,q}\, W(\Lambda_0) = \mathrm{ch}_p\, L(\Lambda_0) \left( \prod_{n>0} \frac{1}{1-p^n q} \right).$$

***Proof*** The equalities for the $p$-characters are verified in the proof of Theorem 4.10. The equality for the $(p, q)$-character follows from that for the $p$-character and Proposition 3.19. □

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