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Semiclassical asymptotic behavior of orthogonal polynomials

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Abstract
Our goal is to find asymptotic formulas for orthonormal polynomials $P_n(z)$ with the recurrence coefficients slowly stabilizing as $n \to \infty$. To that end, we develop scattering theory of Jacobi operators with long-range coefficients and study the corresponding second-order difference equation. We introduce the Jost solutions $f_n(z)$ of this equation by a condition for $n \to \infty$ and suggest an Ansatz for them playing the role of the semiclassical Liouville–Green Ansatz for the corresponding solutions of the Schrödinger equation. This allows us to study Jacobi operators and their eigenfunctions $P_n(z)$ by traditional methods of spectral theory developed for differential equations. In particular, we express all coefficients in asymptotic formulas for $P_n(z)$ as $\to \infty$ in terms of the Wronskian of the solutions $\{P_n(z)\}$ and $\{f_n(z)\}$.

Keywords Jacobi matrices · Long-range perturbations · Difference equations · Orthogonal polynomials · Asymptotics for large numbers

Mathematics Subject Classification 33C45 · 39A70 · 47A40 · 47B39

1 Introduction

1.1 Jacobi and orthogonal polynomials

As is well known, the theories of Jacobi operators given by three-diagonal matrices

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$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

in the space $\ell^2(\mathbb{Z}_+)$ and of differential Schrödinger operators $H = Dp(x)D + q(x)$ (with, for example, the boundary condition $u(0) = 0$) in the space $L^2(\mathbb{R}_+)$ are to a large extent similar. For Jacobi operators, $n \in \mathbb{Z}_+$ plays the role of $x \in \mathbb{R}_+$ and the coefficients a_n, b_n, play the roles of the functions $p(x), q(x)$, respectively.

In our opinion, a consistent analogy between Jacobi and Schrödinger operators sheds a new light on some aspects of the orthogonal polynomials theory. Of course, this point of view is not new; for example, it was advocated long ago by Case [6].

In this paper, the sequences $a_n > 0$ and $b_n = \bar{b}_n$ in (1.1) are assumed to be bounded, so that J is a bounded self-adjoint operator in the space $\ell^2(\mathbb{Z}_+)$. Its spectral family will be denoted $E(\lambda)$. The spectrum of J is simple with $e_0 = (1, 0, 0, \ldots)^\top$ being a generating vector. It is natural to define the spectral measure of J by the relation $d\rho(\lambda) = d(E(\lambda)e_0, e_0)$.

Orthogonal polynomials $P_n(z)$ associated with the Jacobi matrix (1.1) are defined by the recurrence relation

$$a_{n-1}P_{n-1}(z) + b_n P_n(z) + a_n P_{n+1}(z) = z P_n(z), \quad n \in \mathbb{Z}_+, \tag{1.2}$$

and the boundary conditions $P_{-1}(z) = 0$, $P_0(z) = 1$. Obviously, $P_n(z)$ is a polynomial of degree n and the vector $P(z) = \{P_n(z)\}_{n=-1}^\infty$ formally satisfies the equation $JP(z) = zP(z)$, that is, it is an "eigenvector" of the operator H. The polynomials $P_n(\lambda)$ are orthogonal and normalized in the space $L^2(\mathbb{R}; d\rho)$:

$$\int_{-\infty}^{\infty} P_n(\lambda)P_m(\lambda)d\rho(\lambda) = \delta_{n,m};$$

as usual, $\delta_{n,n} = 1$ and $\delta_{n,m} = 0$ for $n \neq m$. Alternatively, given the probability measure $d\rho(\lambda)$, the polynomials $P_0(\lambda), P_1(\lambda), \ldots, P_n(\lambda), \ldots$ can be obtained by the Gram–Schmidt orthonormalization of the monomials $1, \lambda, \ldots, \lambda^n, \ldots$ in the space $L^2(\mathbb{R}_+; d\rho)$; one also has to additionally require that $P_n(\lambda) = k_n(\lambda^n + r_n\lambda^{n-1} + \cdots)$ with $k_n > 0$. The coefficients a_n, b_n can be recovered by the formulas $a_n = k_n/k_{n+1}$, $b_n = r_n - r_{n+1}$.

The operator (1.1) with the coefficients $a_n = 1/2, b_n = 0$ is known as the "free" discrete Schrödinger operator. This operator, denoted J_0, plays the role of the differential operator D^2 in the space $L^2(\mathbb{R}_+)$. The operator J_0 can be diagonalized explicitly. Its spectrum is absolutely continuous and coincides with the interval $[-1, 1]$. The eigenfunctions of J_0 are normalized Chebyshev polynomials $P_n(\lambda)$ of the second kind, and the corresponding spectral measure $d\rho_0(\lambda) = d(E_0(\lambda)e_0, e_0)$ is given by the formula

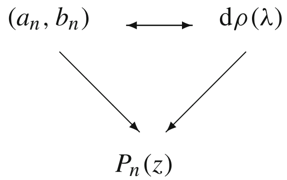
$$d\rho_0(\lambda) = 2\pi^{-1}\sqrt{1-\lambda^2}d\lambda, \quad \lambda \in (-1, 1). \tag{1.3}$$

Obviously, $J = J_0 + V$ where the "perturbation" V is given by equality (1.1) with a_n replaced by $\alpha_n = a_n - 1/2$.

We refer to the books [1,13,27] and surveys [17,28] for general information about orthogonal polynomials.

1.2 Statement of the problem

Our goal is to study an asymptotic behavior of the polynomials $P_n(z)$ as $n \to \infty$. Of course, one has to distinguish the cases of z in the spectrum $\sigma(J)$ of the Jacobi operator J and of $z \notin \sigma(J)$. Asymptotic properties of $P_n(z)$ can be deduced either from the coefficients a_n, b_n of the operator J or from its spectral measure $\mathrm{d}\rho(\lambda)$:



Asymptotic formulas were very well known (see, e.g., the book [10]) for the classical, that is, Jacobi, Laguerre and Hermite, polynomials, but the first general result is probably due to Bernstein (see his pioneering paper [2,3] or Theorem 12.1.4 in the Szegő book [27]). These results were stated in terms of the measure $\mathrm{d}\rho(\lambda)$. It was required that $\operatorname{supp} \rho \subset [-1, 1]$, the measure is absolutely continuous, $\mathrm{d}\rho(\lambda) = w(\lambda)\mathrm{d}\lambda$, and the weight $w(\lambda)$ satisfies certain regularity conditions. The assumption $\operatorname{supp} \rho \subset [-1, 1]$ accepted in [2,3,27] was later partially removed in [12,24]. In recent years, the implication $\mathrm{d}\rho(\lambda) \to P_n(z)$ has been successfully developed with a help of the Riemann–Hilbert problem method. (See the book by Deift [9].)

We suppose that the coefficients a_n, b_n are known and deduce asymptotic properties of the polynomials $P_n(z)$ from the behavior of a_n, b_n as $n \to \infty$. Apparently, this line of research $(a_n, b_n) \to P_n(z)$ was initiated by P. G. Nevai (see his book [23]) and then successfully developed in various papers, some of them are quite ingenious. His approach is briefly described in Sect. 1.5. We will not try to consistently discuss specific methods of orthogonal polynomials theory because one of our goals is to advocate a different approach based on a direct analogy with the theory of differential operators.

Here we illustrate this approach on the case

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} b_n = 0 \tag{1.4}$$

when the perturbation $V = J - J_0$ is compact. Then the essential spectrum of the operator J coincides with the interval $[-1, 1]$, and its point spectrum consists of simple eigenvalues accumulating, possibly, to the points 1 and -1. We always assume that condition (1.4) is satisfied. Our objective is to investigate the case where $\alpha_n \to 0$ and

$b_n \to 0$ as n^{-1} or slower. More precisely, we assume that

$$\sum_{n=0}^{\infty} \big(|a_{n+1} - a_n| + |b_{n+1} - b_n|\big) < \infty. \tag{1.5}$$

Thus, the sequences $\{a_n\}$ and $\{b_n\}$ are of bounded variation. In the quantum mechanical terminology, such perturbations V of the operator J_0 are called long range.

The traditional approach to scattering theory for differential operators relies on a study of the so called Jost solutions $f(x, z)$ of the Schrödinger equation

$$-\,(p(x) f'(x, z))' + q(x) f(x, z) = z f(x, z) \tag{1.6}$$

distinguished by their asymptotics as $x \to \infty$. We follow the same scheme and so start with a construction of solutions $f(z) = \{f_n(z)\}_{n=-1}^{\infty}$ of the second order Jacobi difference equation

$$a_{n-1} f_{n-1}(z) + b_n f_n(z) + a_n f_{n+1}(z) = z f_n(z), \quad n \in \mathbb{Z}_+, \tag{1.7}$$

(the number $a_{-1} \neq 0$ may be chosen at our convenience; for definiteness, we put $a_{-1} = 1/2$) satisfying a certain asymptotic condition as $n \to \infty$. We call $f(z)$ the Jost solutions and $f_{-1}(z)$ the Jost functions. The Jost function is related to the Wronskian $\{P(z), f(z)\}$ of the solutions $P(z) = \{P_n(z)\}$ and $f(z) = \{f_n(z)\}$ of Eq. (1.7) by the identity

$$-\,2^{-1} f_{-1}(z) = \{P(z), f(z)\} =: \Omega(z). \tag{1.8}$$

We express all coefficients in asymptotic formulas for $P_n(z)$ in terms of the Jost function $f_{-1}(z)$.

The asymptotics of the Jost solutions for the Schrödinger equation (1.6) with long-range coefficients is given by the famous semiclassical Liouville–Green Ansatz. So, our first goal is to find its analogue for Jacobi operators. Then we develop spectral theory of Jacobi operators J with long-range coefficients essentially along the same lines (see [33]) as in the short-range case when $\alpha_n \to 0$ and $b_n \to 0$ faster than n^{-1}. We emphasize that in the problem we consider, the semiclassical approximation applies for large n when oscillations of the coefficients α_n and b_n are not too strong.

Of course the asymptotic formulas we obtain are quite different for regular points z of J, for its eigenvalues and for $z = \lambda \in (-1, 1)$ lying on its continuous spectrum. Since $P_n(\lambda)$ are the continuous spectrum eigenfunctions of J, the last problem is natural to consider in the scattering theory framework.

1.3 Short-range perturbations

Let us describe the main ideas of our approach presented in [33] for the case of short-range perturbations $V = J - J_0$ when the condition

$$\sum_{n=0}^{\infty} \left(|a_n - 1/2| + |b_n| \right) < \infty \tag{1.9}$$

is satisfied. Then the Jost solutions of Eq. (1.7) are distinguished by their asymptotics

$$f_n(z) \sim \zeta(z)^n \tag{1.10}$$

as $n \to \infty$. Here

$$\zeta(z) = z - \sqrt{z^2 - 1} = (z + \sqrt{z^2 - 1})^{-1} \tag{1.11}$$

(we choose $\sqrt{z^2 - 1} > 0$ for $z > 1$). The sequence $f_n(z)$ rapidly tends to zero for $z \in \mathbb{C} \setminus [-1, 1]$ and oscillates for $z = \lambda \pm i0$, $\lambda \in (-1, 1)$. At a formal level, solutions $f_n(z)$ of Eq. (1.7) with asymptotics (1.10) were introduced in [6]. Under assumption (1.9) their existence was proven in [33]. It is important that $f_n(z)$ are analytic functions of $z \in \mathbb{C} \setminus [-1, 1]$ and are continuous up to the cut along $(-1, 1)$.

Once these results are established, spectral and scattering theories for the operator J can be developed quite in the same way as for differential operators. Since

$$\begin{aligned} P_n(\lambda) &= \frac{f_{-1}(\lambda + i0) f_n(\lambda - i0) - f_{-1}(\lambda - i0) f_n(\lambda + i0)}{2i\sqrt{1 - \lambda^2}}, \quad \lambda \in (-1, 1), \\ & n = 0, 1, 2, \ldots, \end{aligned} \tag{1.12}$$

we see that $f_{-1}(\lambda + i0) = \overline{f_{-1}(\lambda - i0)} \neq 0$. It easily follows that the spectrum of the operator J is absolutely continuous on the interval $(-1, 1)$ and $\mathrm{d}\rho(\lambda) = w(\lambda)\mathrm{d}\lambda$ with a continuous and positive weight

$$w(\lambda) = \frac{2}{\pi} \sqrt{1 - \lambda^2} |f_{-1}(\lambda + i0)|^{-2}. \tag{1.13}$$

The operator J can also have infinite number of simple eigenvalues accumulating, possibly, to the points 1 and -1; it is not excluded that these points are eigenvalues of J.

Relations (1.10) and (1.12) lead to the asymptotic formula as $n \to \infty$ for orthogonal polynomials:

$$\begin{aligned} P_n(\lambda) &= (2/\pi)^{1/2} w(\lambda)^{-1/2} (1 - \lambda^2)^{-1/4} \sin((n + 1) \arccos \lambda \\ &+ \pi \xi(\lambda)) + o(1), \quad \lambda \in (-1, 1). \end{aligned} \tag{1.14}$$

The function $\xi(\lambda)$ has different names: (a) In scattering theory, it is known as the scattering phase or phase shift. (b) It can be also identified with the Kreĭn spectral shift function for the pair J_0, J, that is

$$\xi(\lambda) = \pi^{-1} \lim_{\varepsilon\to+0} \arg \operatorname{Det}\big(I + V(J_0 - \lambda - i\varepsilon)^{-1}\big).$$

(c) In orthogonal polynomial theory, it is constructed as the Hilbert transform of the function $\ln w(\lambda)$ and is known as the Szegő function. It is shown in [33] that

$$\operatorname{Det}\big(I + V(J_0 - z)^{-1}\big) = A\zeta(z) f_{-1}(z) \quad \text{where} \quad A = \prod_{k=0}^{\infty}(2a_k),$$

so that in view (1.13), $\ln w(\lambda)$ and $\xi(\lambda)$ are (up to insignificant factors) harmonic conjugate functions. Thus, the spectral shift and Szegő functions are different definitions of the same object.

Formula (1.14) is of course the same as in the classical works [2,3,27], but our definition of the phase shift $\xi(\lambda)$ and our assumptions on J are quite different from [2,3,27]. We emphasize that the inclusion $\operatorname{supp}\rho \subset [-1, 1]$ is irrelevant for our approach. Note that for the operator J_0, the asymptotic formula (1.14) reduces to the exact expression for the normalized Chebyshev polynomials of the second kind.

Formula (1.14) is very natural from the scattering theory viewpoint. Indeed, the solutions $f_n(z)$ of the Jacobi equation (1.7) with asymptotics (1.10) are discrete analogues of the Jost solutions $f(x, z)$ of the Schrödinger equation (1.6)

where $p(x) > 0$, $p(x) \to p_0 > 0$ as $x \to \infty$ and $p(x) - p_0$, $q(x)$ are in L^1. These conditions correspond to (1.9). The Jost solution is distinguished by its asymptotics

$$f(x, z) \sim e^{-x\sqrt{-z/p_0}}, \quad x \to \infty, \tag{1.15}$$

where $\operatorname{Re}\sqrt{-z/p_0} \geq 0$, while the regular solution $\varphi(x, z)$ of (1.6) is fixed by the conditions $\varphi(0, z) = 0$, $\varphi'(0, z) = 1$. The regular solution plays the role of the polynomial solution $P(z) = \{P_n(z)\}$ of Eq. (1.7). Since $\varphi(x, \lambda)$ is a linear combination of the Jost solutions $f_n(\lambda \pm i0)$ [cf. (1.12)], it can be standardly (see, e.g., §4.1 and §4.2 of [32]) deduced from (1.15) for $z = \lambda \pm i0$ that

$$\begin{aligned}\varphi(x, \lambda) &= (2/\pi)^{1/2} w(\lambda)^{-1/2}\lambda^{-1/4} \sin(x\sqrt{\lambda/p_0} - \pi\xi(\lambda)) + o(1),\\ &\lambda > 0, \quad x \to \infty.\end{aligned} \tag{1.16}$$

Here $w(\lambda)$ is the derivative of the spectral measure of the operator $H = Dp(x)D + q(x)$, and $\xi(\lambda)$ is the spectral shift function for the pair of the operators $H_0 = p_0 D^2$, H. Obviously, formulas (1.14) and (1.16) are quite analogous with n and x playing similar roles.

1.4 Scheme of the approach

Our main goal is to study Jacobi operators $J = J_0 + V$ with the coefficients α_n and b_n decaying so slowly that the condition (1.9) is not satisfied. Instead we require a weaker assumption (1.5). We follow here closely the well developed semiclassical approach in the theory of differential equations reviewed recently in [34]. The starting point of this approach is to find a suitable modification of the Jost solutions for the long-range case. The paper [34] relied on a simplified Liouville–Green Ansatz given by the formula

$$f(x,z) \sim \exp\Big(-\int_0^x \sqrt{\frac{q(y)-z}{p(y)}}\mathrm{d}y\Big), \quad x\to\infty, \tag{1.17}$$

for solutions of Eq. (1.6).

In this paper, we combine the methods of [33] where short-range perturbations of Jacobi operators were considered and of [34] where differential operators with long-range coefficients were studied. Let us discuss the main steps of our approach in more details:

a. We show in Sect. 2.2 that an analogue of (1.17) for solutions of the difference equation (1.7) is given by the formula

$$f_n(z) \sim \zeta\big(\frac{z-b_0}{2a_0}\big)\zeta\big(\frac{z-b_1}{2a_1}\big)\cdots\zeta\big(\frac{z-b_{n-1}}{2a_{n-1}}\big) =: q_n(z), \quad n\to\infty, \tag{1.18}$$

where the function $\zeta(z)$ is defined by (1.11). This means that the relative remainder

$$r_n(z) := q_n(z)^{-1}\big(a_{n-1}q_{n-1}(z) + (b_n - z)q_n(z) + a_n q_{n+1}(z)\big), \quad n\in\mathbb{Z}_+, \tag{1.19}$$

belongs to $\ell^1(\mathbb{Z}_+)$. Note that, unlike (1.10), asymptotic formula (1.18) takes into account decay properties of the coefficients α_n and b_n. Since $|\zeta(z)| < 1$ for $z\in\mathbb{C}\setminus[-1,1]$ and $2a_n \to 1, b_n \to 0$, we see that the Jost solutions $f_n(z) = O(\varepsilon^n)$ for some $\varepsilon = \varepsilon(z) < 1$ as $n\to\infty$. It is also easy to see that $f_n(\lambda \pm i0)$ oscillate as $n\to\infty$ if $\lambda\in(-1,1)$.

b. Then, we make in Sect. 2.3 a multiplicative change of variables

$$f_n(z) = q_n(z)u_n(z) \tag{1.20}$$

and reduce the difference equation (1.7) to a Volterra "integral" equation for the sequence $u_n(z)$ satisfying the condition $u_n(z)\to 1$ as $n\to\infty$.

c. This equation is standardly solved by iterations in Sect. 2.4 which allows us to prove that $u_n(z)$ are analytic functions of $z\in\mathbb{C}\setminus[-1,1]$ and are continuous up to the cut along $(-1,1)$. According to (1.20) the same is true for the functions $f_n(z)$ (see Theorem 3.1). All standard statements about spectral properties of the Jacobi operator J and asymptotic formulas for the polynomials $P_n(z)$ are easy consequences of this analytic result.

d. If $z \in \mathbb{C} \setminus [-1, 1]$ but is not an eigenvalue of the operator J, we prove in Theorem 4.4 that

$$\lim_{n \to \infty} \Big(q_n(z) P_n(z) \Big) = - \frac{\{P(z), f(z)\}}{\sqrt{z^2 - 1}}. \tag{1.21}$$

To that end, we construct an exponentially growing solution $g_n(z)$ of Eq. (1.7) by the formula

$$g_n(z) = f_n(z) \sum_{m=0}^{n} (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \in \mathbb{Z}_+.$$

Perhaps this formula is new.

e. Since formula (1.12) remains true in the long-range case, we immediately obtain the asymptotics

$$\begin{aligned} P_n(\lambda) &= (1 - \lambda^2)^{-1/2} |f_{-1}(\lambda + i0)| \sin \big(n \arccos \lambda \\ &\quad + \Phi_n(\lambda) \big) + o(1), \quad \lambda \in (-1, 1), \end{aligned} \tag{1.22}$$

of the orthogonal polynomials. The phase $\Phi_n(\lambda)$ is expressed in terms of the phase of the Jost function $f_{-1}(\lambda + i0)$. It depends explicitly on the coefficients a_n, b_n and satisfies the condition $\Phi_n(\lambda) = o(n)$ as $n \to \infty$. The precise definition of $\Phi_n(\lambda)$ is given in Theorem 3.5; see also formula (3.11). In view of (1.13) the amplitude factors in (1.14) and (1.22) are the same.

f. Our results on the Jost solution $f_n(z)$ directly imply that, under assumptions (1.4), (1.5), the spectrum of the Jacobi operator J is absolutely continuous on $(-1, 1)$ and the corresponding weight $w(\lambda)$ is continuous and strictly positive and can be constructed by formula (1.13). This result is stated in Theorem 5.6. At the same time, we obtain the limiting absorption principle for the operator J stating that matrix elements of its resolvent $R(z) = (J - z)^{-1}$, $\operatorname{Im} z \neq 0$, are continuous functions of z up to the cut along $(-1, 1)$.

We emphasize that in contrast to differential operators, short-range perturbations obeying (1.9) are included in the class of long-range perturbations satisfying (1.5). For example, condition (1.5) is satisfied if

$$a_n = 1/2 + \alpha n^{-r_1} + \tilde{\alpha}_n, \quad b_n = b n^{-r_2} + \tilde{b}_n \tag{1.23}$$

where $\alpha, b \in \mathbb{R}$, $r_1, r_2 \in (0, 1]$ and $\tilde{\alpha}_n, \tilde{b}_n \in \ell^1(\mathbb{Z}_+)$. This reduces to (1.9) if $\alpha = b = 0$. For Pollaczek polynomials, relations (1.23) are true with $r_1 = r_2 = 1$ and $\tilde{\alpha}_n = O(n^{-2})$, $\tilde{b}_n = O(n^{-2})$. In this case the phase in formula (1.14) is essentially changed (see the book [27, Section 5 in the Appendix]). This resembles the modification of the phase function for the Schrödinger operator with the Coulomb potential (see, e.g., formula (36,23) in the book [16]).

Condition (1.5) is very precise. Indeed, as shown in [22] (see also the preceding paper [25]), there exist coefficients b_n decaying only slightly worse than n^{-1} and oscillating as $n \to \infty$ such that the point spectrum of the corresponding Jacobi operator J with $a_n = 1/2$ is dense in $[-1, 1]$. In this case, the limiting absorption principle for the operator J does not of course hold.

To emphasize the analogy between differential and difference operators, we often use "continuous" terminology (Volterra integral equations, integration by parts, etc.) for sequences labelled by the discrete variable n. Below C, sometimes with indices, and c are different positive constants whose precise values are of no importance.

1.5 Related research

Now we are in a position to compare our approach with the paper [19] that is well known in the orthogonal polynomial literature. The results of this paper (see also the survey [29]) are close to some of our results but not quite coincide with them, and the methods are essentially different.

The paper [19] relies on specific methods of orthogonal polynomials theory. Apparently, the initial point of [19] is the relation

$$\lim_{n\to\infty} \frac{P_{n-1}(z)}{P_n(z)} = \zeta(z), \quad \operatorname{Im} z \neq 0,$$

established earlier by Nevai [23, Theorem 4.1.13], and improving one of Poincaré's theorems.

We proceed from spectral theory of Jacobi operators. One of important differences compared to [19] is the introduction and consistent use of Jost solutions $f_n(z)$ of Eq. (1.7) distinguished by their asymptotics (1.18) as $n \to \infty$. Then relation (1.12) yields asymptotic formula (1.22). Our proof of asymptotic relation (1.21) is also motivated by results on differential equations.

For the Schrödinger operator with short-range coefficients, the scheme of Sect. 1.4 [except formula (1.21)] goes back to the paper [14] by R. Jost; it is described in Sect. 4.1 of the book [32]. There exists another approach due to N. Levinson (see the book [7]) adapted to difference equations in the book [4]. Probably the methods of Jost and Levinson are essentially equivalent, but, for our purposes, it is more convenient to use the Jost method, moreover that it admits a direct quantum mechanical interpretation.

We followed here the paper [34] devoted to differential equations. Actually, the Ansatz (1.17) appeared in [34] as an attempt to adjust the famous Liouville–Green Ansatz (which compared to (1.17) contains an additional pre-exponential factor) to difference equations. This allowed us in [34] to get rid of conditions on the second derivatives of $p(x)$ and $q(x)$ required by the Liouville-Green Ansatz and to develop spectral theory of the Schrödinger operator H under the assumptions $p' \in L^1(\mathbb{R}_+)$, $q' \in L^1(\mathbb{R}_+)$.

Let us try to advocate our approach. Its specific feature is that we treat the discrete and continuous cases on equal footing. Thus we can use traditional methods well known for differential operators in the discrete case. Introduction of the Jost solutions allows us to study various problems of spectral theory in a unified way. Here are some examples:

1. Both asymptotic relations (1.21) and (1.22) were obtained in [19]. However, expressions for the coefficients in the right-hand sides were not, at least in the author's opinion, very efficient. It was conjectured in [19] that the asymptotic coefficients in (1.22) can be obtained from that in (1.21) as the limit on $(-1, 1)$ from complex values of z. This conjecture was later justified in [30]. In our approach this problem does not even arise since both coefficients are expressed in terms of the Wronskian $\{P(z), f(z)\}$ of the polynomial and Jost solutions of Eq. (1.7).
2. Introducing and consistently using the Jost solutions, we avoid many more recent methods, for example, transfer matrices, Prüfer variables, Gilbert–Pearson subordinacy theory, etc.
3. The scheme used here yields automatically an expression for the spectral measure in terms of the Jost function. This shows, in particular, that this measure is absolutely continuous and the corresponding density is a continuous positive function.

Note that quite general conditions of the absolutely continuity of the spectral measure were given in the paper [26] and the book [31, Theorem 14.25], where the subordinacy method of [11] was used.

I thank G. Świderski who kindly informed the author about the papers [19,30].

2 Volterra integral equation

Here we reduce the Jacobi difference equation (1.7) with asymptotics (1.18) to a Volterra equation whose solution can be constructed by iterations. Below conditions (1.4) and (1.5) are always assumed unless indicated otherwise.

2.1 Preliminaries

Let us consider Eq. (1.7). Note that the values of f_{N-1} and f_N for some $N \in \mathbb{Z}_+$ determine the whole sequence f_n satisfying the difference equation (1.7).

Let $f = \{f_n\}_{n=-1}^{\infty}$ and $g = \{g_n\}_{n=-1}^{\infty}$ be two solutions of Eq. (1.7). A direct calculation shows that their Wronskian

$$\{f, g\} := a_n(f_n g_{n+1} - f_{n+1} g_n) \tag{2.1}$$

does not depend on $n = -1, 0, 1, \ldots$. In particular, for $n = -1$ and $n = 0$, we have

$$\{f, g\} = 2^{-1}(f_{-1} g_0 - f_0 g_{-1}) \quad \text{and} \quad \{f, g\} = a_0(f_0 g_1 - f_1 g_0). \tag{2.2}$$

Clearly, the Wronskian $\{f, g\} = 0$ if and only if the solutions f and g are proportional.

It is convenient to introduce a notation

$$x'_n = x_{n+1} - x_n$$

for the "derivative" of a sequence x_n. We note the Abel summation formula ("integration by parts"):

$$\sum_{n=N}^{M} x_n y'_n = x_M y_{M+1} - x_{N-1} y_N - \sum_{n=N}^{M} x'_{n-1} y_n; \tag{2.3}$$

here $M \geq N \geq 0$ are arbitrary, but we have to set $x_{-1} = 0$ so that $x'_{-1} = x_0$.

Let us fix the branch of the analytic function $\sqrt{z^2-1}$ of $z \in \mathbb{C} \setminus [-1, 1]$ by the condition $\sqrt{z^2-1} > 0$ for $z > 1$. Obviously, this function is continuous up to the cut along $[-1, 1]$, it equals $\pm i\sqrt{1-\lambda^2}$ for $z = \lambda \pm i0$, $\lambda \in (-1, 1)$, and $\sqrt{z^2-1} < 0$ for $z < -1$. Define the one-to-one, onto mapping $\zeta : \mathbb{C} \setminus [-1, 1] \to \mathbb{D}$ (the unit disc) by formula (1.11). Since $2z = \zeta(z) + \zeta(z)^{-1}$, the sequence $\{\zeta(z)^n\}_{n=-1}^{\infty}$ satisfies the "free" Eq. (1.7) where $a_n = 1/2$, $b_n = 0$. For $\lambda \in [-1, 1]$, it is common to set $\lambda = \cos\theta$ with $\theta \in [0, \pi]$. Then $\zeta(\lambda \pm i0) = e^{\mp i\theta}$.

Below the values of $|z|$ are bounded, and hence, the values of $\zeta = \zeta(z)$ are separated from 0.

2.2 Ansatz

Here we show that the sequence $q_n(z)$ defined in (1.18) is an approximate solution (Ansatz) of Eq. (1.7).

Let $\Pi = \mathbb{C} \setminus \mathbb{R}$, and let $\operatorname{clos} \Pi$ be the closure of Π. For $z \in \operatorname{clos} \Pi$, we set

$$z_n = \frac{z - b_n}{2a_n} \tag{2.4}$$

and

$$\zeta_n = \zeta(z_n) = z_n - \sqrt{z_n^2 - 1} = \left(z_n + \sqrt{z_n^2 - 1}\right)^{-1}. \tag{2.5}$$

Note that

$$a_n\zeta_n + a_n\zeta_n^{-1} + b_n - z = 0. \tag{2.6}$$

Obviously, $z_n \to z$ and $\zeta_n \to \zeta(z)$ as $n \to \infty$ because $2a_n \to 1$ and $b_n \to 0$.

Now, we define a sequence $q_n = q_n(z)$ by the relations $q_0(z) = 1$ and

$$q_n(z) = \zeta_0\zeta_1 \cdots \zeta_{n-1}, \quad n \geq 1, \tag{2.7}$$

which coincides with the right-hand side of (1.18). The functions $q_n(z)$ are analytic in $z \in \Pi$ and continuous up to the real axis. Of course $\zeta(\bar{z}) = \overline{\zeta(z)}$ and $q_n(\bar{z}) = \overline{q_n(z)}$. Note also that

$$|q_n(z)| \leq 1, \tag{2.8}$$

but $q_n(z) \neq 0$ for all n and all $z \in \operatorname{clos}\Pi$. We consider $q_n(z)$ for $z \in \Pi$ (not for $z \in \mathbb{C}\setminus[-1,1]$) because the values of $\zeta_n(\lambda + i0)$ and $\zeta_n(\lambda - i0)$ may be different for $|\lambda| > 1$ if n is not too large.

We will show that the sequence $q_n(z)$ satisfies approximately Eq. (1.7). Let us introduce a (relative) remainder in this equation by the formula (1.19) (the values of $q_{-1}(z)$ and $r_0(z)$ are of course inessential). Since $q_{n-1}q_n^{-1} = \zeta_{n-1}^{-1}$ and $q_{n+1}q_n^{-1} = \zeta_n$, we can rewrite (1.19) as

$$r_n = a_{n-1}\zeta_{n-1}^{-1} + a_n\zeta_n + b_n - z, \tag{2.9}$$

or using (2.6) as

$$r_n = a_{n-1}\zeta_{n-1}^{-1} - a_n\zeta_n^{-1} = (a_{n-1} - a_n)\zeta_{n-1}^{-1} + a_n(\zeta_{n-1}^{-1} - \zeta_n^{-1}). \tag{2.10}$$

Lemma 2.1 *For* $z \in \operatorname{clos}\Pi$, *an estimate*

$$|r_n(z)| \leq C\frac{|a_n - a_{n-1}| + |b_n - b_{n-1}|}{\sqrt{z_{n-1}^2 - 1} + \sqrt{z_n^2 - 1}} \tag{2.11}$$

holds. In particular, $\{r_n(z)\}_{n=0}^{\infty} \in \ell^1(\mathbb{Z}_+)$ *if* $z \neq \pm 1$.

Proof Let us proceed from representation (2.10). By definition (2.5), we have

$$\zeta_{n-1}^{-1} - \zeta_n^{-1} = (z_{n-1} - z_n)\Big(1 + \frac{z_{n-1} + z_n}{\sqrt{z_{n-1}^2 - 1} + \sqrt{z_n^2 - 1}}\Big), \tag{2.12}$$

where according to (2.4)

$$z_{n-1} - z_n = \frac{(z - b_n)(a_n - a_{n-1}) + a_n(b_n - b_{n-1})}{2a_{n-1}a_n}$$

so that

$$|z_n - z_{n-1}| \leq C(|a_n - a_{n-1}| + |b_n - b_{n-1}|). \tag{2.13}$$

Thus equality (2.12) yields estimate (2.11). □

2.3 Multiplicative substitution

Let the sequence $q_n(z)$ be given by formulas (2.4), (2.5) and (2.7). We are looking for solutions $f_n(z)$ of the difference equation (1.7) satisfying the condition

$$f_n(z) = q_n(z)(1 + o(1)), \quad n \to \infty. \tag{2.14}$$

The uniqueness of such solutions is almost obvious.

Lemma 2.2 *Equation* (1.7) *may have only one solution* $f_n(z)$ *satisfying condition* (2.14).

Proof Let $\tilde{f}_n(z)$ be another solution of (1.7) satisfying (2.14). Then the Wronskian (2.1) of these solutions equals

$$\{f, \tilde{f}\} = a_n q_n q_{n+1} o(1).$$

This expression tends to zero because the sequences a_n and q_n are bounded according to (1.4) and (2.8). Thus $\tilde{f} = Cf$ where $C = 1$ by virtue again of condition (2.14). □

For construction of $f_n(z)$, we will reformulate the problem introducing a sequence

$$u_n(z) = q_n(z)^{-1} f_n(z), \quad n \in \mathbb{Z}_+. \tag{2.15}$$

Then (2.14) is equivalent to the condition

$$\lim_{n\to\infty} u_n(z) = 1. \tag{2.16}$$

Let us derive a difference equation for $u_n(z)$.

Lemma 2.3 *Let* $z \in \operatorname{clos} \Pi$ *and let* $r_n(z)$ *be given by formula* (1.19). *Then Eq.* (1.7) *for a sequence* $f_n(z)$ *is equivalent to the equation*

$$a_n \zeta_n (u_{n+1}(z) - u_n(z)) - a_{n-1} \zeta_{n-1}^{-1} (u_n(z) - u_{n-1}(z)) = -r_n(z) u_n(z), \quad n \in \mathbb{Z}_+, \tag{2.17}$$

for sequence (2.15).

Proof Substituting expression $f_n = q_n u_n$ into (1.7), we see that

$$\begin{aligned} q_n^{-1}\Big(a_{n-1} f_{n-1} + (b_n - z) f_n + a_n f_{n+1}\Big) &= a_{n-1}\zeta_{n-1}^{-1} u_{n-1} + (b_n - z) u_n + a_n \zeta_n u_{n+1} \\ &= a_{n-1}\zeta_{n-1}^{-1}(u_{n-1} - u_n) \\ &\quad + r_n u_n + a_n \zeta_n (u_{n+1} - u_n). \end{aligned}$$

by virtue of (2.9). Thus Eqs. (1.7) and (2.17) coincide. □

According to Lemma 2.1 the sequence $r_n(z)$ of the coefficients of $u_n(z)$ in the right-hand side of (2.17) belongs to $\ell^1(\mathbb{Z}_+)$. This allows us to reduce the difference equation (2.17) with condition (2.16) to a "Volterra integral" equation

$$u_n(z) = 1 - \sum_{m=n+1}^{\infty} G_{n,m}(z) r_m(z) u_m(z) \tag{2.18}$$

with kernel

$$G_{n,m}(z) = q_m(z)^2 \sum_{p=n}^{m-1} (a_p \zeta_p)^{-1} q_p(z)^{-2}, \quad n, m \in \mathbb{Z}_+, \quad m \geq n+1. \quad (2.19)$$

Note that $G_{n,m}(\bar{z}) = \overline{G_{n,m}(z)}$. The functions $G_{n,m}(z)$ are analytic in $z \in \Pi$ and are continuous up to the real axis.

2.4 Integral equation

Our plan is now the following. We first prove that, for all $z \in \operatorname{clos} \Pi \setminus \{-1, 1\}$, a solution $u_n(z)$ of the integral equation (2.18) exists for sufficiently large n and tends to 1 as $n \to \infty$. Then we show that, for such n, the sequence $u_n(z)$ satisfies also the difference equation (2.17). To all $n \geq -1$, the sequence $u_n(z)$ is extended as a solution of Eq. (2.17).

The following assertion plays the crucial role in our analysis of Eq. (2.18), in particular, for z lying on the cut along $[-1, 1]$. It shows that the sequence (2.19) is bounded uniformly in n and m provided the points ± 1 are excluded.

Lemma 2.4 *There exist constants $C(z)$ and $N(z)$ such that an estimate*

$$|G_{n,m}(z)| \leq C(z) < \infty, \quad m - 1 \geq n \geq N(z), \quad (2.20)$$

is true for all $z \in \operatorname{clos} \Pi \setminus \{-1, 1\}$. The constants $C(z)$ and $N(z)$ are common for z in compact subsets of $\operatorname{clos} \Pi \setminus \{-1, 1\}$, that is, for all $z \in \operatorname{clos} \Pi$ such that $|z^2 - 1| \geq \epsilon$ and $|z| \leq R$ where $\epsilon > 0$ and $R < \infty$ are some fixed numbers.

Proof According to definition (2.7), we have

$$(q_p^{-2})' = (\zeta_p^{-2} - 1) q_p^{-2}.$$

Set

$$\eta_p = a_p^{-1} \zeta_p (1 - \zeta_p^2)^{-1}.$$

According to (1.4) $\zeta_p \to \zeta$ where $\zeta^2 \neq 1$ as $p \to \infty$, so that $|\eta_p| \leq C < \infty$ for $p \geq N$ if N is sufficiently large. Therefore, it follows [cf. (2.12) and (2.13)] from condition (1.5) that

$$\sum_{p=N}^{\infty} |\eta_p'| \leq C \sum_{p=N}^{\infty} (|a_p'| + |\zeta_p'|) \leq C_1 \sum_{p=N}^{\infty} (|a_p'| + |b_p'|) < \infty. \quad (2.21)$$

Integrating by parts, that is, using identity (2.3), we find that

$$\sum_{p=n}^{m-1} (a_p \zeta_p)^{-1} q_p^{-2} = \sum_{p=n}^{m-1} \eta_p (q_p^{-2})' = \eta_m q_{m+1}^{-2} - \eta_{n-1} q_n^{-2} - \sum_{p=n}^{m-1} \eta'_{p-1} q_p^{-2}.$$

Substituting this expression into (2.19) and using the estimates $|q_m q_p^{-1}| \leq 1$ for $m \geq p$ and (2.21), we see that

$$|G_{n,m}| \leq C\big(1 + \sum_{p=n}^{m-1} |\eta'_{p-1}||q_m^2 q_p^{-2}|\big) \leq C_1 < \infty.$$

This proves (2.20). □

Lemmas 2.1 and 2.4 allow us to solve the Volterra equation (2.18) by iterations.

Lemma 2.5 *Let $z \in \operatorname{clos} \Pi \setminus \{-1, 1\}$. Set $u_n^{(0)}(z) = 1$ and*

$$u_n^{(k+1)}(z) = - \sum_{m=n+1}^{\infty} G_{n,m}(z) r_m(z) u_m^{(k)}(z), \quad k \geq 0, \tag{2.22}$$

for all $n \in \mathbb{Z}_+$. Then estimates

$$|u_n^{(k)}(z)| \leq \frac{C(z)^k}{k!} \left(\sum_{m=n+1}^{\infty} |r_m(z)| \right)^k, \quad \forall k \in \mathbb{Z}_+, \tag{2.23}$$

are true for all sufficiently large n with the same constant $C(z)$ as in Lemma 2.4.

Proof Suppose that (2.23) is satisfied for some $k \in \mathbb{Z}_+$. We have to check the same estimate (with k replaced by $k+1$ in the right-hand side) for $u_n^{(k+1)}$. Set

$$\mathcal{R}_m = \sum_{p=m+1}^{\infty} |r_p|.$$

According to definition (2.22), it follows from (2.20) and (2.23) that

$$|u_n^{(k+1)}| \leq \frac{C^{k+1}}{k!} \sum_{m=n+1}^{\infty} |r_m| \mathcal{R}_m^k. \tag{2.24}$$

Observe that

$$\mathcal{R}_m^{k+1} + (k+1)|r_m| \mathcal{R}_m^k \leq \mathcal{R}_{m-1}^{k+1},$$

and hence, for all $N \in \mathbb{Z}_+$,

$$(k+1)\sum_{m=n+1}^{N} |r_m|\mathcal{R}_m^k \le \sum_{m=n+1}^{N} (\mathcal{R}_{m-1}^{k+1} - \mathcal{R}_m^{k+1}) = \mathcal{R}_n^{k+1} - \mathcal{R}_N^{k+1} \le \mathcal{R}_n^{k+1}.$$

Substituting this bound into (2.24), we obtain estimate (2.23) for $u_n^{(k+1)}$. □

Now we can conclude our study of the "integral" equation (2.18).

Theorem 2.6 *Let assumptions* (1.4) *and* (1.5) *be satisfied. For* $z \in \operatorname{clos}\Pi\setminus\{-1, 1\}$*, Eq.* (2.18) *has a* (*unique*) *bounded solution* $\{u_n(z)\}_{n=0}^{\infty}$*. This sequence obeys an estimate*

$$|u_n(z) - 1| \le C\varepsilon_n \tag{2.25}$$

where the constant C *is common for* z *in compact subsets of* $\operatorname{clos}\Pi\setminus\{-1, 1\}$ *and*

$$\varepsilon_n := \sum_{m=n}^{\infty} (|\alpha'_m| + |b'_m|). \tag{2.26}$$

For all $n \in \mathbb{Z}_+$*, the functions* $u_n(z)$ *are analytic in* $z \in \Pi$ *and are continuous up to the cut along* $\mathbb{R}$ *with possible exception of the points* $z = -1$ *and* $z = 1$.

Proof Set

$$u_n = \sum_{k=0}^{\infty} u_n^{(k)} \tag{2.27}$$

where $u_n^{(k)}$ are defined by recurrence relations (2.22). Estimate (2.23) shows that this series is absolutely convergent. Using the Fubini theorem to interchange the order of summations in m and p, we see that

$$\sum_{m=n+1}^{\infty} G_{n,m} r_m u_m = \sum_{k=0}^{\infty}\sum_{m=n+1}^{\infty} G_{n,m} r_m u_m^{(k)} = -\sum_{k=0}^{\infty} u_n^{(k+1)} = 1 - \sum_{k=0}^{\infty} u_n^{(k)} = 1 - u_n.$$

This is Eq. (2.18) for sequence (2.27). It also follows from (2.23) that

$$|u_n(z) - 1| \le C \sum_{m=n+1}^{\infty} |r_m(z)|$$

which in view of (2.11) implies (2.25). Since every function $u_n^{(k)}(z)$ is analytic in $z \in \Pi$ and is continuous up to the cut $\mathbb{R}$ (away from the points ± 1), estimate (2.23) guarantees that function (2.27) possesses the same properties. □

Let us come back to the difference equation (2.17).

Lemma 2.7 *For $z \in \operatorname{clos}\Pi\backslash\{-1, 1\}$, a solution $u_n(z)$ of integral equation (2.18) satisfies also difference equation (2.17).*

Proof Below we consistently take into account that $q_{n+1} = \zeta_n q_n$. It follows from (2.18) that

$$u_{n+1} - u_n = -\sum_{m=n+2}^{\infty} (G_{n+1,m} - G_{n,m}) r_m u_m + G_{n,n+1} r_{n+1} u_{n+1}. \tag{2.28}$$

Since according to (2.19)

$$G_{n+1,m} - G_{n,m} = -(a_n\zeta_n)^{-1} q_n^{-2} q_m^2 \quad \text{and} \quad G_{n,n+1} = \zeta_n a_n^{-1},$$

equality (2.28) can be rewritten as

$$a_n(u_{n+1} - u_n) = \zeta_n^{-1} q_n^{-2} \sum_{m=n+1}^{\infty} q_m^2 r_m u_m. \tag{2.29}$$

Putting together this equality with the same equality where n is replaced by $n-1$, we arrive to Eq. (2.17). □

Remark 2.8 Let $z \in \Pi$ or $z = \lambda \pm i0$ where $|\lambda| > 1$. Since $|\zeta| < 1$ and $\zeta_n \to \zeta$ as $n \to \infty$, we see that $|\zeta_m| \le \varepsilon$ for some $\varepsilon < 1$ and sufficiently large m whence

$$\big|q_n/q_m\big| = |\zeta_m \cdots \zeta_{n-1}| \le \varepsilon^{n-m}, \quad n \ge m. \tag{2.30}$$

Note also that $\{r_m u_m\} \in \ell^1(\mathbb{Z}_+)$. Thus, it follows from representation (2.29) that $\{u_n'\} \in \ell^1(\mathbb{Z}_+)$.

3 Modified Jost solutions

3.1 Construction

Let us put together the results obtained in the previous section. According to Theorem 2.6 for every $z \in \operatorname{clos}\Pi\backslash\{-1, 1\}$ there exists a solution $u_n(z)$, $n \in \mathbb{Z}_+$, of the integral equation (2.18). By Lemma 2.7 it satisfies also the difference equation (2.17). Then Lemma 2.3 implies that the function $f_n(z) := q_n(z)u_n(z)$ satisfies Eq. (1.7). Estimate (2.25) for $u_n(z)$ is obviously equivalent to the asymptotics

$$f_n(z) = q_n(z)\big(1 + O(\varepsilon_n)\big), \quad n \to \infty, \tag{3.1}$$

for $f_n(z)$.

Thus we arrive at the following result.

Theorem 3.1 *Let assumptions* (1.4) *and* (1.5) *be satisfied, and let* $z \in \operatorname{clos}\Pi\setminus\{-1,1\}$. *Denote by* $u_n(z)$ *the sequence constructed in Theorem* 2.6. *Then the sequence* $f_n(z)$ *defined by equality* (1.20) *satisfies Eq.* (1.7), *and it has asymptotics* (3.1). *For all* $n \in \mathbb{Z}_+$, *the functions* $f_n(z)$ *are analytic in* $z \in \Pi$ *and are continuous up to the cut along* $\mathbb{R}$ *with possible exception of the points* $z = -1$ *and* $z = 1$. *Asymptotics* (3.1) *is uniform in z from compact subsets of the set* $\operatorname{clos}\Pi\setminus\{-1,1\}$.

Recall that the polynomials $P_n(z)$ are solutions of Eq. (1.7) satisfying the conditions $P_{-1}(z) = 0$, $P_0(z) = 1$. Put $P(z) = \{P_n(z)\}_{n=-1}^{\infty}$, $f(z) = \{f_n(z)\}_{n=-1}^{\infty}$. Then the first formula (2.2) yields relation (1.8) for the Wronskian $\Omega(z)$ of the solutions $P(z)$ and $f(z)$. By analogy with the continuous case, the sequence $\{f_n(z)\}_{n=-1}^{\infty}$ will be called the (modified) Jost solution of Eq. (1.7) and $f_{-1}(z)$ will be called the (modified) Jost function. For the operator H_0, the Jost solution is $\{\zeta(z)^n\}_{n=-1}^{\infty}$ and the corresponding Wronskian

$$\Omega_0(z) = -(2\zeta(z))^{-1}. \tag{3.2}$$

The following result is a direct consequence of Theorem 3.1.

Corollary 3.2 *The Wronskian* $\Omega(z)$ *depends analytically on* $z \in \Pi$, *and it is continuous in z up to the cut along* $\mathbb{R}$ *except, possibly, the points* ± 1.

Note that

$$f_n(\bar{z}) = \overline{f_n(z)}$$

and, in particular,

$$f_n(\lambda - i0) = \overline{f_n(\lambda + i0)}, \quad \lambda \in \mathbb{R}\setminus\{-1, 1\}. \tag{3.3}$$

Remark 3.3 (i) In contrast to the short-range case, the functions $f_n(z)$ are not analytic on the whole set $\mathbb{C}\setminus[-1,1]$ because, in general, $f_n(\lambda - i0) \neq f_n(\lambda + i0)$ even for $|\lambda| > 1$. This circumstance is, however, inessential—see Remark 5.2 below.

(ii) The definitions of the function $q_n(z)$ and of the Jost solution $f_n(z)$ are not intrinsic. One can multiply them by a factor depending on z but not on n. For example, we can set

$$\tilde{f}_n(z) = q_{N_0}(z)^{-1} f_n(z)$$

where N_0 is some fixed number. Then $\tilde{f}_n(z)$ satisfies Eq. (1.7) and $\tilde{f}_n(z) = \tilde{q}_n(z)(1 + O(\varepsilon_n))$ where $\tilde{q}_n(z) = q_{N_0}(z)^{-1} q_n(z)$ as $n \to \infty$.

Let us find asymptotics of the sequence (2.7) as $n \to \infty$.

Lemma 3.4 *Let assumption* (1.4) *be satisfied, let* $z \in \operatorname{clos}\Pi\setminus\{-1,1\}$, *and let* $\zeta = \zeta(z)$ *be given be equality* (1.11). *Then*

$$q_n(z) = e^{n(\ln\zeta + o(1))}, \quad n \to \infty. \tag{3.4}$$

Proof Let the sequences z_n and ζ_n be defined by formulas (2.4) and (2.5). It follows from (1.4) that $z_n = z + o(1)$ and hence $\zeta_n = \zeta(1+\epsilon_n)$ where $\epsilon_n \to 0$ as $n \to \infty$. For the sequence (2.7), this yields

$$\ln q_n = n \ln \zeta + \sum_{m=0}^{n-1} \ln(1+\epsilon_m) = n \ln \zeta + o(n)$$

which is equivalent to (3.4). □

Note that $|\zeta(z)| < 1$ for $z \in \Pi$ and for $z = \lambda \pm i0$ if $|\lambda| > 1$. Therefore for such z according to (3.1) and (3.4), $f_n(z) = O(\delta^n)$ with some $\delta < 1$ as $n \to \infty$ whence $f(z) \in \ell^2(\mathbb{Z}_+)$.

Observe that $\Omega(z) \neq 0$ for $z \in \Pi$. Indeed, if $\Omega(z) = 0$, then, by definition (1.8), $P_n(z) = c f_n(z)$ for some $c \in \mathbb{C}$. Since $f(z) \in \ell^2(\mathbb{Z}_+)$, it follows that the complex number z is an eigenvalue of the self-adjoint operator J which is impossible. This argument also shows that a number $\lambda \in (-\infty, -1) \cup (1, \infty)$ is an eigenvalue of J if and only if $\Omega(\lambda \pm i0) = 0$. By (3.3), these equalities for the signs "$+''$ and "$-''$ are equivalent to each other.

3.2 On the cut

Suppose now that $z = \lambda \pm i0$ where $\lambda \in (-1, 1)$ so that $\lambda = \cos\theta$, $\theta \in (0, \pi)$. Set

$$\lambda_n := \frac{\lambda - b_n}{2a_n}. \tag{3.5}$$

For $|\lambda_n| < 1$, we have

$$\zeta(\lambda_n \pm i0) = \lambda_n \mp i\sqrt{1-\lambda_n^2} = e^{\mp i\theta_n(\lambda)}$$

where

$$\theta_n(\lambda) = \arccos \lambda_n \in (0, \pi). \tag{3.6}$$

Set $\theta_n(\lambda) = 0$ for $\lambda_n \geq 1$ when $\zeta(\lambda_n) > 0$ and, similarly, $\theta_n(\lambda) = \pi$ for $\lambda_n \leq -1$, when $\zeta(\lambda_n) < 0$. Then formula (2.7) reads as

$$q_n(\lambda \pm i0) = e^{\mp i\varphi_n(\lambda)} \prod_{m=0}^{n-1} |\zeta(\lambda_m \pm i0)|$$

where

$$\varphi_n(\lambda) = \sum_{m=0}^{n-1} \theta_m(\lambda). \tag{3.7}$$

Since $|\zeta(\lambda_m \pm i0)| = 1$ for sufficiently large m, the product

$$k(\lambda) := \prod_{m=0}^{\infty} |\zeta(\lambda_m \pm i0)|$$

consists of a finite number of terms. Obviously, $k(\lambda)$ is a continuous function of $\lambda \in (-1, 1)$ and $k(\lambda) \neq 0$.

As mentioned in Remark 3.3 (ii), the definitions of the Jost solution $f_n(z)$ and hence of the Wronskian $\Omega(z)$ are not unique. For $\lambda \in (-1, 1)$, it is convenient to normalize them dividing by the factor $k(\lambda)$:

$$\begin{aligned} \mathbf{f}_n(\lambda \pm i0) &= k(\lambda)^{-1} f_n(\lambda \pm i0), \quad \boldsymbol{\Omega}(\lambda \pm i0) = \{P(\lambda), \mathbf{f}(\lambda \pm i0)\} \\ &= k(\lambda)^{-1} \Omega(\lambda \pm i0). \end{aligned} \tag{3.8}$$

This simplifies formulas below.

The following direct consequence of Theorem 3.1 is stated in terms of the normalized Jost solutions.

Theorem 3.5 *Let assumptions* (1.4) *and* (1.5) *be satisfied. For* $\lambda \in (-1, 1)$*, define the phases* $\varphi_n(\lambda)$ *by formulas* (3.5)–(3.7). *Then*

$$\mathbf{f}_n(\lambda \pm i0) = e^{\mp i\varphi_n(\lambda)}(1 + O(\varepsilon_n)), \quad n \to \infty, \tag{3.9}$$

where ε_n *is given by* (2.26).

By definitions (3.5), (3.6), we have

$$\lambda_n = \lambda + o(1) \quad \text{and} \quad \theta_n = \theta + o(1) \tag{3.10}$$

where $\theta = \arccos \lambda$. It follows that

$$\varphi_n(\lambda) = n \arccos \lambda + \Phi_n(\lambda)$$

where

$$\Phi_n(\lambda) = \sum_{m=0}^{n-1} \big(\theta_m(\lambda) - \theta\big) = o(n). \tag{3.11}$$

In particular, we see that asymptotics (3.9) of $\mathbf{f}_n(\lambda \pm i0)$ as $n \to \infty$ is oscillating.

4 Orthogonal polynomials

4.1 Uniform estimates

Let us start with an estimate on orthogonal polynomials $P_n(z)$ for large n. This estimate will be uniform in z from compact subsets of $\mathbb{C} \setminus \{-1, 1\}$. We recall that $P_n(z)$ satisfy the difference equation (1.2) and the conditions $P_{-1}(z) = 0$, $P_0(z) = 1$. As before, we define the sequence $q_n(z)$ by formula (2.7) but instead of (2.15) we make a substitution

$$u_n(z) = q_n(z)P_n(z). \tag{4.1}$$

Observe that $q_n(z)^{-1}$ is an approximate solution of Eq. (1.7), that is

$$r_n(z) := q_n(z)\big(a_{n-1}q_{n-1}(z)^{-1} + (b_n - z)q_n(z)^{-1} + a_n q_{n+1}(z)^{-1}\big), \quad n \in \mathbb{Z}_+,$$

where the remainder $r_n(z)$ is given [cf. (1.19), (2.9), (2.10)] by the formula

$$r_n = a_{n-1}\zeta_{n-1} + a_n\zeta_n^{-1} + b_n - z = a_{n-1}\zeta_{n-1} - a_n\zeta_n.$$

Similarly to Lemma 2.1, it is easy to show that $\{r_n\} \in \ell^1(\mathbb{Z}_+)$ and, similarly to Lemma 2.3, it is easy to obtain an equation

$$a_n\zeta_n^{-1}(u_{n+1}(z) - u_n(z)) - a_{n-1}\zeta_{n-1}(u_n(z) - u_{n-1}(z)) = -r_n(z)u_n(z), \quad n \in \mathbb{Z}_+, \tag{4.2}$$

for sequence (4.1). This difference equation can be reduced to a Volterra integral equation.

Lemma 4.1 *The sequence* (4.1) *obeys an equation*

$$\mathsf{u}_{n+1}(z) = \mathsf{u}_n^{(0)}(z) - \sum_{m=1}^{n} K_{n,m}(z)\mathsf{r}_m(z)\mathsf{u}_m(z), \quad n \geq 1, \tag{4.3}$$

where $\mathsf{u}_1(z) = q_1 P_1(z) = 2z_0\zeta_0$,

$$\mathsf{u}_n^{(0)}(z) = \mathsf{u}_1(z) + a_0\zeta_0 \sum_{p=1}^{n} a_p^{-1}\zeta_p q_p(z)^2$$

and

$$K_{n,m}(z) = q_m(z)^{-2} \sum_{p=m}^{n} a_p^{-1}\zeta_p q_p(z)^2.$$

Proof Set

$$v_n = a_n(\mathsf{u}_{n+1} - \mathsf{u}_n) \quad \text{and} \quad \varrho_n = r_n u_n. \tag{4.4}$$

Then the second-order difference equation (4.2) for u_n yields a first-order difference equation

$$\zeta_n^{-1}v_n - \zeta_{n-1}v_{n-1} = -\varrho_n \tag{4.5}$$

for the sequence v_n. Note that $\zeta_n q_n^2$ is the solution of the corresponding homogeneous equation because $q_n = \zeta_{n-1} q_{n-1}$, by (2.7). Thus, setting $v_n = \zeta_n q_n^2 w_n$, we rewrite (4.5) as an equation

$$w_n - w_{n-1} = -q_n^{-2} \varrho_n$$

for w_n whence

$$w_n = w_0 - \sum_{m=1}^{n} q_m^{-2} \varrho_m.$$

It follows that the solution of Eq. (4.5) is given by the formula

$$v_n = v_0 q_n^2 \zeta_n \zeta_0^{-1} - q_n^2 \zeta_n \sum_{m=1}^{n} q_m^{-2} \varrho_m, \quad n \geq 1.$$

Using now (4.4), we find that

$$u_{n+1} - u_1 = \sum_{p=1}^{n} a_p^{-1} v_p = v_0 \zeta_0^{-1} \sum_{p=1}^{n} a_p^{-1} q_p^2 \zeta_p - \sum_{p=1}^{n} a_p^{-1} q_p^2 \zeta_p \sum_{m=1}^{p} q_m^{-2} r_m u_m.$$

Interchanging the summations over p and m here and observing that $v_0 \zeta_0^{-1} = a_0 \zeta_0$, we obtain Eq. (4.3). □

Similarly to Lemma 2.4, integration by parts shows that the functions $\mathsf{u}_n^{(0)}(z)$ and $K_{n,m}(z)$ are uniformly bounded. Therefore solving Eq. (4.3) by iterations (cf. Lemma 2.5), we see that its solution $\mathsf{u}_n(z)$ is also bounded. Coming back to relation (4.1), we can state the following result.

Theorem 4.2 *Let assumptions* (1.4) *and* (1.5) *be satisfied, and let* $z \in \mathbb{C} \setminus \{-1, 1\}$. *Define the sequence* $q_n(z)$ *by formula* (2.7). *Then the orthogonal polynomials* $P_n(z)$ *obey an estimate*

$$|P_n(z)| \leq C |q_n(z)|^{-1}, \quad \forall n \in \mathbb{Z}_+,$$

where C *does not depend on* z *in compact subsets of* $\mathbb{C} \setminus \{-1, 1\}$.

Observe that (4.3) is different from the standard (see, e.g., equation (3.2) in [33]) equation for orthogonal polynomials relying on the perturbation theory, that is, on a comparison of $P_n(z)$ with the Chebyshev polynomials of the second kind. We emphasize that Theorem 4.2 is very close to Lemma 4 in [30], and we state it mainly for the completeness of our presentation.

4.2 Asymptotics in the complex plane

Here we find asymptotics of the polynomials $P_n(z)$ for $z \notin [-1, 1]$. We follow the scheme exposed in [33] for Jacobi operators in the short-range case and in [34] for differential operators with long-range coefficients. In this subsection we use Theorem 3.1 for a fixed $z \notin [-1, 1]$ only.

Let the sequence $q_n(z)$ be defined by formula (2.7), and let $f_n(z)$ be the Jost solution of Eq. (1.7). If $z = \lambda \in \mathbb{R}\setminus[-1, 1]$, we can choose the sequences $q_n(\lambda \pm i0)$ and $f_n(\lambda \pm i0)$ for any of the signs "$\pm$". We start by introducing a solution $g_n(z)$ of Eq. (1.7) exponentially growing as $n \to \infty$. Perhaps this construction is of interest in its own sake. In view of asymptotics (3.1), we can choose $n_0 = n_0(z)$ such that $f_n(z) \neq 0$ for all $n \geq n_0 - 1$. Note that, for $\operatorname{Im} z \neq 0$, one can set $n_0 = 0$ because the equality $f_{n_0-1}(z) = 0$ implies that the Jacobi operator $J^{(n_0)}$ with the matrix elements $a_n^{(n_0)} = a_{n+n_0}$, $b_n^{(n_0)} = b_{n+n_0}$ has eigenvalue z. Put

$$G_n(z) = \sum_{m=n_0}^{n} (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \in \mathbb{Z}_+. \tag{4.6}$$

Theorem 4.3 *Let $z \in \mathbb{C}\setminus[-1, 1]$. Under assumptions (1.4) and (1.5) the sequence $g_n(z)$ defined by*

$$g_n(z) = f_n(z) G_n(z)$$

satisfies Eq. (1.7) and

$$\lim_{n\to\infty} q_n(z) g_n(z) = \frac{1}{\sqrt{z^2 - 1}}. \tag{4.7}$$

Proof First, we check Eq. (1.7) for g_n. According to definition (4.6), we have

$$\begin{aligned} &a_{n-1} f_{n-1} G_{n-1} + (b_n - z) f_n G_n + a_n f_{n+1} G_{n+1} \\ &\quad = \big(a_{n-1} f_{n-1} + (b_n - z) f_n + a_n f_{n+1}\big) G_n + a_{n-1} f_{n-1} (G_{n-1} - G_n) \\ &\qquad + a_n f_{n+1} (G_{n+1} - G_n). \end{aligned}$$

The first term here is zero because Eq. (1.7) is true for the sequence f_n Since

$$G_{n+1} = G_n + (a_n f_n f_{n+1})^{-1},$$

the second and third terms equal $-f_n^{-1}$ and f_n^{-1}, respectively.

Next, we prove asymptotics (4.7). Recall that $f_n = q_n u_n$ where q_n is defined by formulas (2.5), (2.7) and u_n is constructed in Theorem 2.6. Let us set

$$v_m = \zeta_{m-1} \zeta_m (1 - \zeta_{m-1} \zeta_m)^{-1} (a_{m-1} u_{m-1} u_m)^{-1}, \quad t_m = (q_{m-1} q_m)^{-1}. \tag{4.8}$$

Then

$$t'_m = \big((\zeta_{m-1}\zeta_m)^{-1} - 1\big)t_m$$

and

$$(a_{m-1}f_{m-1}f_m)^{-1} = (a_{m-1}u_{m-1}u_m)^{-1}t_m = v_m t'_m.$$

Integrating by parts [see formula (2.3)] in (4.6), we find that

$$G_n = \sum_{m=n_0}^{n} v_m t'_m = v_n t_{n+1} - v_{n_0-1}t_{n_0} - \sum_{m=n_0}^{n} v'_{m-1}t_m. \tag{4.9}$$

Let us multiply this expression by $q_n f_n = q_n^2 u_n$ and pass to the limit $n \to \infty$. Since $v_n \to 2\zeta^2(1-\zeta^2)^{-1}$, we see that

$$q_n^2 u_n v_n t_{n+1} = \zeta_n^{-1} u_n v_n \to 2\zeta(1-\zeta^2)^{-1} = \frac{1}{\sqrt{z^2-1}} \tag{4.10}$$

as $n \to \infty$. It follows from (2.30) that $q_n^2 u_n v_{n_0-1} t_{n_0} = O(r^{2n})$. The contribution

$$q_n^2 u_n \sum_{m=n_0}^{n} v'_{m-1}t_m = u_n \sum_{m=n_0}^{n} v'_{m-1}\frac{q_n^2}{q_{m-1}q_m}$$

of the third term in the right-hand side (4.9) can be estimated by

$$\sum_{m=n_0}^{n} r^{2(n-m)}|v'_{m-1}| \le r^n \sum_{n_0\le m<[n/2]} |v'_{m-1}| + \sum_{[n/2]\le m\le n} |v'_{m-1}|. \tag{4.11}$$

Recall that $\{u'_n\} \in \ell^1(\mathbb{Z}_+)$ according to Remark 2.8 and $\{\zeta'_n\} \in \ell^1(\mathbb{Z}_+)$ according to definitions (2.4) and (2.5). It follows from (4.8) that $\{v'_n\} \in \ell^1(\mathbb{Z}_+)$, and therefore, expression (4.11) tends to zero as $n \to \infty$. Hence (4.10) implies (4.7). □

By definition (4.6), the Wronskian (2.1) of $f(z) = \{f_n(z)\}$ and $g(z) = \{f_n(z)G_n(z)\}$ equals

$$\{f(z), g(z)\} = a_n f_n(z) f_{n+1}(z)(G_{n+1}(z) - G_n(z)) = 1,$$

whence solutions $f(z)$ and $g(z)$ are linearly independent. It follows that

$$P_n(z) = d_+(z) f_n(z) + d_-(z) g_n(z)$$

where

$$d_+(z) = \{P(z), g(z)\} \quad \text{and} \quad d_-(z) = -\{P(z), f(z)\} = -\Omega(z)$$

according to (1.8). Obviously, $d_+(z) \neq 0$ if $d_-(z) = 0$. Therefore, Theorems 3.1 and 4.3 imply the following result.

Theorem 4.4 *Under assumptions* (1.4) *and* (1.5) *the relation* (1.21) *is true for all* $z \in \mathbb{C}\setminus[-1, 1]$ *with convergence uniform on compact subsets of* $z \in \mathbb{C}\setminus[-1, 1]$. *Moreover, if* $\Omega(z) = 0$, *then*

$$\lim_{n\to\infty} q_n(z)^{-1} P_n(z) = \{P(z), g(z)\} \neq 0. \tag{4.12}$$

Note that Theorem 4.2 does not follow from Theorem 4.4 because asymptotics (1.21) is not uniform as z approaches the cut along $(-1, 1)$.

The existence of the limit in (1.21) is the classical result of the Szegő theory. It is stated as Theorem 12.1.2 in the book [27] where the assumptions are imposed on the measure $d\rho(\lambda)$; in particular, it is assumed that supp $\rho \subset [-1, 1]$. Under short-range assumption (1.9) asymptotic relations (1.21), (4.12) were established in [15] and, by a different method, in [33].

4.3 Asymptotics on the continuous spectrum

To find asymptotic behavior of the polynomials $P_n(\lambda)$ for $\lambda \in (-1, 1)$, that is, on the continuous spectrum of the Jacobi operator J, we have to consider two (normalized) Jost solutions $\mathbf{f}(\lambda \pm i0) = \{\mathbf{f}_n(\lambda \pm i0)\}_{n=-1}^{\infty}$ for $\lambda = \cos\theta \in (-1, 1)$. Of course, these two solutions are complex conjugate to each other. Calculating the Wronskian (2.1) of $\mathbf{f}(\lambda + i0)$ and $\mathbf{f}(\lambda - i0)$ for $n \to \infty$ and using (3.7), (3.9), (3.10), we see that

$$\begin{aligned}\{\mathbf{f}(\lambda + i0), \mathbf{f}(\lambda - i0)\} &= a_n\big(e^{-i\varphi_n(\lambda)}e^{i\varphi_{n+1}(\lambda)} - e^{-i\varphi_{n+1}(\lambda)}e^{i\varphi_n(\lambda)}\big) + o(1)\\ &= i\sin\theta_n(\lambda) + o(1) = i\sin\theta(\lambda) = i\sqrt{1-\lambda^2} \neq 0,\end{aligned}$$

and hence, these solutions are linearly independent. It follows that

$$P_n(\lambda) = \overline{c(\lambda)}\mathbf{f}_n(\lambda + i0) + c(\lambda)\mathbf{f}_n(\lambda - i0) \tag{4.13}$$

for some complex constant $c(\lambda)$. Taking the Wronskian of this equation with $\mathbf{f}(\lambda+i0)$ and using notation (3.8), we find that

$$-c(\lambda)\{\mathbf{f}(\lambda + i0), \mathbf{f}(\lambda - i0)\} = \{P(\lambda), \mathbf{f}(\lambda + i0)\} = \boldsymbol{\Omega}(\lambda + i0). \tag{4.14}$$

Thus (4.13) leads to the same formula as (1.14) in the short-range case.

Lemma 4.5 *For* $\lambda \in (-1, 1)$, *the representation*

$$\begin{aligned}P_n(\lambda) \;&=\; \frac{\boldsymbol{\Omega}(\lambda - i0)\mathbf{f}_n(\lambda + i0) - \boldsymbol{\Omega}(\lambda + i0)\mathbf{f}_n(\lambda - i0)}{i\sqrt{1-\lambda^2}},\\ &\lambda \in (-1, 1), \quad n = 0, 1, 2, \ldots,\end{aligned} \tag{4.15}$$

holds true.

Properties of the Wronskian (4.14) are summarized in the following statement.

Theorem 4.6 *The Wronskians* $\boldsymbol{\Omega}(\lambda + i0)$ *and* $\boldsymbol{\Omega}(\lambda - i0) = \overline{\boldsymbol{\Omega}(\lambda + i0)}$ *are continuous functions of* $\lambda \in (-1, 1)$ *and*

$$\boldsymbol{\Omega}(\lambda \pm i0) \neq 0, \quad \lambda \in (-1, 1). \tag{4.16}$$

Proof The functions $\boldsymbol{\Omega}(\lambda \pm i0)$ are continuous by Corollary 3.2. If $\boldsymbol{\Omega}(\lambda \pm i0) = 0$, then according to (4.15) $P_n(\lambda) = 0$ for all $n \in \mathbb{Z}_+$. However, $P_0(\lambda) = 1$ for all λ. □

Let us set

$$\varkappa(\lambda) = 2|\boldsymbol{\Omega}(\lambda + i0)|, \quad -2\boldsymbol{\Omega}(\lambda \pm i0) = \varkappa(\lambda)e^{\pm i\eta(\lambda)}, \quad \varkappa(\lambda) > 0. \tag{4.17}$$

In the theory of short-range perturbations of the Schrödinger operator, the functions $\varkappa(\lambda)$ and $\eta(\lambda)$ are known as the limit amplitude and the limit phase, respectively; the function $\eta(\lambda)$ is also called the scattering phase or the phase shift. Definition (4.17) fixes $\eta(\lambda)$ only up to a term $2\pi k$ where $k \in \mathbb{Z}$.

Combined together relations (3.9) and (4.15) yield the asymptotics of Bernstein–Szegő type for the polynomials $P_n(\lambda)$. Recall that ε_n are defined by (2.26).

Theorem 4.7 *Let assumptions* (1.4) *and* (1.5) *be satisfied, let* $\lambda \in (-1, 1)$ *and let the phase* $\varphi_n(\lambda)$ *be defined by formulas* (3.5)–(3.7). *Then the polynomials* $P_n(\lambda)$ *have asymptotics*

$$P_n(\lambda) = \varkappa(\lambda)(1 - \lambda^2)^{-1/2} \sin(\varphi_n(\lambda) + \eta(\lambda)) + O(\varepsilon_n) \tag{4.18}$$

as $n \to \infty$. *Relation* (4.18) *is uniform in* λ *on compact subintervals of* $(-1, 1)$.

The phase shifts $\eta(\lambda)$ in (4.18) and $\pi\xi(\lambda)$ in the short-range formula (1.14) play the same roles. They depend on the precise values of the coefficients a_n and b_n for all n and hence cannot be found from their asymptotic behavior as $n \to \infty$. Under additional assumptions, the growing part $\varphi_n(\lambda)$ of the phase in (4.18) can be made more explicit.

4.4 Hilbert–Schmidt perturbations

In addition to (1.5), assume now that condition

$$\sum_{n=0}^{\infty} (\alpha_n^2 + b_n^2) < \infty, \tag{4.19}$$

is satisfied, that is, $V = J - J_0$ is a Hilbert–Schmidt operator. Then asymptotic formulas of Theorems 4.4 and 4.7 can be made more explicit. We proceed from the following elementary assertion.

Lemma 4.8 *Let $z \neq \pm 1$. Under assumption (4.19), there exists a finite limit*

$$\lim_{n\to\infty}\left(\zeta(z)^{-n}\exp\left(-\frac{1}{\sqrt{z^2-1}}\sum_{m=0}^{n-1}(2z\alpha_m+b_m)\right)q_n(z)\right)\neq 0. \tag{4.20}$$

Proof It follows [cf. (2.12)] from (2.4), (2.5) that

$$\zeta_m-\zeta=(z_m-z)\Big(1-\frac{z_m+z}{\sqrt{z_m^2-1}+\sqrt{z^2-1}}\Big). \tag{4.21}$$

Since

$$z_m-z=-2z\alpha_m-b_m+O(\alpha_m^2+b_m^2)$$

and

$$1-\frac{z_m+z}{\sqrt{z_m^2-1}+\sqrt{z^2-1}}=-\frac{\zeta}{\sqrt{z^2-1}}+O(\sqrt{\alpha_m^2+b_m^2}),$$

equality (4.21) implies that

$$\frac{\zeta_m}{\zeta}=1+\frac{2z\alpha_m+b_m}{\sqrt{z^2-1}}+O(\alpha_m^2+b_m^2)=\exp\Big(\frac{2z\alpha_m+b_m}{\sqrt{z^2-1}}\Big)\big(1+O(\alpha_m^2+b_m^2)\big).$$

Taking the product over $m=0,1,\ldots,n-1$ and using condition (4.19), we arrive at (4.20). □

Now the following statement is a direct consequence of Theorem 4.4.

Theorem 4.9 *Let assumptions (1.5) and (4.19) be satisfied, and let $z\in\mathbb{C}\setminus[-1,1]$. Then there exist finite limits*

$$\lim_{n\to\infty}\left(\zeta(z)^{n}\exp\left(\frac{1}{\sqrt{z^2-1}}\sum_{m=0}^{n-1}(2z\alpha_m+b_m)\right)P_n(z)\right)\neq 0 \tag{4.22}$$

if z is not an eigenvalue of the operator J and

$$\lim_{n\to\infty}\left(\zeta(z)^{-n}\exp\left(-\frac{1}{\sqrt{z^2-1}}\sum_{m=0}^{n-1}(2z\alpha_m+b_m)\right)P_n(z)\right)\neq 0 \tag{4.23}$$

if z is an eigenvalue of J.

Corollary 4.10 *Suppose additionally that the conditions*

$$\sum_{n=0}^{\infty}\alpha_n<\infty \quad \textit{and} \quad \sum_{n=0}^{\infty}b_n<\infty \tag{4.24}$$

(*these series should be convergent but perhaps not absolutely*) *are satisfied. Then the exponential factors in* (4.22) *and* (4.23) *may be omitted, that is, the limits in* (1.21) *and* (4.12) *exist.*

We do not know whether relations (4.22) and (4.23) remain true under the only assumption (4.19).

In some cases, the exponential factors in (4.22) and (4.23) can be simplified.

Example 4.11 Let conditions (1.23) be satisfied with some $r_1, r_2 \in (1/2, 1)$. Then

$$\sum_{m=0}^{n}(2z\,\alpha_m + b_m) = 2z\alpha(1-r_1)^{-1}n^{1-r_1} + b(1-r_2)^{-1}n^{1-r_2} + 2z\alpha\boldsymbol{\gamma}_{r_1} + b\boldsymbol{\gamma}_{r_2} + \sum_{m=0}^{\infty}(2z\tilde{\alpha}_m + \tilde{b}_m) + o(1) \tag{4.25}$$

where $\boldsymbol{\gamma}_r - (1-r)^{-1}$ is the Euler–Mascheroni constant. With a natural modification, expression (4.25) remains true if $r_j = 1$ for one or both j. In this case, $(1-r_j)^{-1}n^{1-r_j}$ should be replaced by $\ln n$ and $\boldsymbol{\gamma}_1$ is the Euler–Mascheroni constant.

Let us now discuss relation (4.18). Similarly to Lemma 4.8, we have

Lemma 4.12 *Under assumption* (4.19)*, there exists a finite limit*

$$\lim_{n\to\infty}\Big(\varphi_n(\lambda) - n\theta - (\sin\theta)^{-1}\sum_{m=0}^{n-1}(2\cos\theta\,\alpha_m + b_m)\Big) =: \gamma(\lambda). \tag{4.26}$$

Proof It follows from (3.5), (3.6) that

$$\theta_m = \arccos\frac{\lambda - b_m}{2a_m} = \theta + (\sin\theta)^{-1}(2\cos\theta\,\alpha_m + b_m) + O(\alpha_m^2 + b_m^2).$$

Taking the sum over $m = 0, 1, \dots, n-1$ and using condition (4.19), we arrive at (4.26). □

Now the following statement is a direct consequence of Theorem 4.7.

Theorem 4.13 *Let assumptions* (1.5) *and* (4.19) *be satisfied. Then for* $\lambda \in (-1, 1)$*, the asymptotic formula*

$$P_n(\lambda) = \varkappa(\lambda)(1-\lambda^2)^{-1/2}\sin\Big(n\theta + (\sin\theta)^{-1}\sum_{m=0}^{n-1}(2\cos\theta\,\alpha_m + b_m) + \gamma(\lambda) + \eta(\lambda)\Big) + o(1) \tag{4.27}$$

holds as $n \to \infty$*. Relation* (4.27) *is uniform in* λ *on compact subintervals of* $(-1, 1)$*.*

Under assumption (1.23), the phase in (4.27) can be simplified if one takes relation (4.25) (where z is replaced by $\cos\theta$) into account.

Of course, formulas (4.22) and (4.27) are consistent with asymptotic formulas for Pollaczek polynomials in Appendix in the book [27].

Without any additional assumptions, Hilbert–Schmidt perturbations V of the operator J_0 were investigated in the deep papers [8,15]. In [15], necessary and sufficient conditions in terms of the spectral measure $d\rho(\lambda)$ of the operator $J = J_0 + V$ were found for V to be in the Hilbert–Schmidt class. Asymptotic behavior of the corresponding polynomials $P_n(z)$ was studied in [8]. It was proved in Theorem 5.1 that the limit of $\zeta(z)^n P_n(z)$ as $n \to \infty$ exists if and only if conditions (4.19) and (4.24) are satisfied.

As shown in Theorem 8.1 of [8], assumptions (4.19), (4.24) are sufficient also for the validity of formula (1.16) but only in some *averaged* sense. Such a regularization seems to be necessary since under these assumptions the structure of the essential spectrum of the operator J can be quite wild.

Condition (1.5) accepted in this paper is different in nature from (4.19), (4.24). On the one hand, it excludes too strong oscillations of the coefficients α_n, b_n but, on the other hand, it permits their arbitrary slow decay as $n \to \infty$.

5 Spectral theory of Jacobi operators

Here we show that the spectrum of the Jacobi operator J on the interval $(-1, 1)$ is absolutely continuous and the corresponding weight $w(\lambda)$ is expressed via the Jost function by the formula (1.13). It follows that $w(\lambda)$ is a continuous strictly positive function of $\lambda \in (-1, 1)$.

5.1 Resolvent

First, we construct the resolvent $R(z) = (J - z)^{-1}$ of the operator J. Recall that $P(z) = \{P_n(z)\}$ and $f(z) = \{f_n(z)\}$ are, respectively, the polynomial and the Jost solutions of Eq. (1.7) and $\Omega(z)$ is their Wronskian (1.8). We denote by e_n, $n \in \mathbb{Z}_+$, the canonical basis in $\ell^2(\mathbb{Z}_+)$. The proof below is almost the same as in the short-range case (cf. Lemma 2.6 in [33]).

Lemma 5.1 *For all $n, m \in \mathbb{Z}_+$, we have*

$$(R(z)e_n, e_m) = \Omega(z)^{-1} P_n(z) f_m(z), \quad \operatorname{Im} z \neq 0, \tag{5.1}$$

if $n \leq m$ and $(R(z)e_n, e_m) = (R(z)e_m, e_n)$.

Proof We will show that the operator $R(z)$ defined by relation (5.1) is the resolvent of J. We have

$$\Omega(z)(R(z)u)_n = f_n(z)A_n(z) + P_n(z)B_n(z) \tag{5.2}$$

where

$$A_n(z) = \sum_{m=0}^{n} P_m(z)u_m, \quad B_n(z) = \sum_{m=n+1}^{\infty} f_m(z)u_m, \tag{5.3}$$

at least for all sequences $u = \{u_n\}$ with a finite number of non-zero components u_n. In this case $R(z)u \in \ell^2(\mathbb{Z}_+)$ because $f_n(z) \in \ell^2(\mathbb{Z}_+)$ if $\operatorname{Im} z \neq 0$.

Our goal is to check that $(J - z)R(z)u = u$. It follows from definition (1.1) of the Jacobi operator J and formula (5.2) that

$$\begin{aligned}\Omega((J - z)Ru)_n &= a_{n-1}\big(f_{n-1}A_{n-1} + P_{n-1}B_{n-1}\big)\\ &+ (b_n - z)\big(f_n A_n + P_n B_n\big) + a_n\big(f_{n+1}A_{n+1} + P_{n+1}B_{n+1}\big).\end{aligned} \tag{5.4}$$

According to (5.3) we have

$$f_{n-1}A_{n-1} + P_{n-1}B_{n-1} = f_{n-1}(A_n - P_n u_n) + P_{n-1}(B_n + f_n u_n)$$

and

$$f_{n+1}A_{n+1} + P_{n+1}B_{n+1} = f_{n+1}A_n + P_{n+1}B_n.$$

Let us substitute these expressions into the right-hand side of (5.4) and observe that the coefficients at A_n and B_n equal zero by virtue of Eq. (1.7) for $\{f_n\}$ and $\{P_n\}$, respectively. It follows that

$$((J - z)Ru)_n = \Omega^{-1}a_{n-1}(-P_n f_{n-1} + f_n P_{n-1})u_n = u_n$$

whence $R(z) = (J - z)^{-1}$. In particular, the operator $R(z)$ defined by (5.1) is bounded in the space $\ell^2(\mathbb{Z}_+)$. □

In view of Theorem 3.1, $f_n(z)$, $n = -1, 0, 1, \ldots$, and $\Omega(z)$ are analytic functions of $z \in \mathbb{C}\setminus\mathbb{R}$, and they are continuous up to the cut along $\mathbb{R}$ with possible exception of the points $z = \pm 1$. The values $f_m(\lambda \pm i0)$ and $\Omega(\lambda \pm i0)$ on the upper and lower edges of the cuts along $(-\infty, -1)$ and $(1, \infty)$ are, in general, different. Note however the following obvious

Remark 5.2 If $\lambda \in (-\infty, -1) \cup (1, \infty)$ and $\Omega(\lambda \pm i0) \neq 0$, then

$$\frac{f_m(\lambda + i0)}{\Omega(\lambda + i0)} = \frac{f_m(\lambda - i0)}{\Omega(\lambda - i0)}. \tag{5.5}$$

Indeed, consider the sequence

$$\Delta_m(\lambda) = f_m(\lambda + i0)\Omega(\lambda - i0) - f_m(\lambda - i0)\Omega(\lambda + i0).$$

It satisfies Eq. (1.7) where $z = \lambda$, belongs to $\ell^2(\mathbb{Z}_+)$ because $\lambda \notin [-1, 1]$ and $\Delta_{-1}(\lambda) = 0$ by definition (1.8) of $\Omega(\lambda \pm i0)$. Since λ is not an eigenvalues of J, we see that $\Delta_m(\lambda) = 0$ for all m which is equivalent to (5.5). It follows from (5.5) that the function $f_m(z)/\Omega(z)$ is analytic in $\mathbb{C}\setminus[-1, 1]$ with poles at eigenvalues of the operator J. This is consistent with formula (5.1) since the matrix elements $(R(z)e_n, e_m)$ are analytic functions of $z \in \mathbb{C}\setminus[-1, 1]$.

Taking also (4.16) into account, we obtain the following result.

Theorem 5.3 *Let assumptions* (1.4) *and* (1.5) *hold. Then*

(i) *The resolvent* $R(z) = (J - z)^{-1}$ *of the Jacobi operator* J *is an integral operator with matrix elements* (5.1). *For all* $n, m \in \mathbb{Z}_+$, *it is an analytic function of* $z \in \mathbb{C}\setminus[-1, 1]$ *with simple poles at eigenvalues of the operator* J. *A point* $z \in \mathbb{C}\setminus[-1, 1]$ *is an eigenvalue of* J *if and only if* $\Omega(z) = 0$.
(ii) *For all* $n, m \in \mathbb{Z}_+$, *the functions* $(R(z)e_n, e_m)$ *are continuous in* z *up to the cut* $[-1, 1]$ *except, possibly, the points* ± 1.
(iii) *Estimates*

$$|(R(z)e_n, e_m)| \le C|\Omega(z)|^{-1}\big|q_m(z)/q_n(z)\big| \le C_1 < \infty, \quad n \le m,$$

are true with some positive constants that do not depend on n, m *and on* z *in compact subsets of the set* $\operatorname{clos}(\mathbb{C}\setminus[-1, 1])$ *as long as they are away from the points* ± 1.

The statement (ii) is known as the limiting absorption principle. It implies

Corollary 5.4 *The spectrum of the operator* J *on the interval* $(-1, 1)$ *is absolutely continuous.*

Let us now consider the spectral projector $E(\lambda)$ of the operator J. By the Cauchy–Stieltjes formula, its matrix elements satisfy the identity

$$2\pi i \frac{d(E(\lambda)e_n, e_m)}{d\lambda} = (R(\lambda + i0)e_n, e_m) - (R(\lambda - i0)e_n, e_m). \tag{5.6}$$

The following assertion is a direct consequence of Theorem 5.3, part (ii).

Corollary 5.5 *For all* $n, m \in \mathbb{Z}_+$, *the functions* $(E(\lambda)e_n, e_m)$ *are continuously differentiable in* $\lambda \in (-1, 1)$.

We emphasize that the points 1 and -1 may be eigenvalues of J; see Example 4.15 in [33].

Theorem 5.3 can also be obtained by the Mourre method [21]. It was applied to Jacobi operators in [5]; to be precise, the problem in the space $\ell^2(\mathbb{Z})$ was considered in [5], but this is of no importance. However, the Mourre method does not exclude eigenvalues of J embedded in its continuous spectrum although it shows that these eigenvalues may accumulate to the points 1 and -1 only. Note also that very general conditions of the absolute continuity of spectrum were obtained in [26] by the subordinacy method of [11].

5.2 Spectral measure

Now we are in a position to calculate the spectral family $dE(\lambda)$ of the operator J. Let us proceed from the identity (5.6). Using notation (3.8) and (4.14), we can rewrite formula (5.1) as

$$(R(\lambda \pm i0)e_n, e_m) = \mathbf{\Omega}(\lambda \pm i0)^{-1} P_n(\lambda)\mathbf{f}_m(\lambda \pm i0), \quad n \le m, \tag{5.7}$$

where

$$\mathbf{f}_m(\lambda - i0) = \overline{\mathbf{f}_m(\lambda + i0)} \quad \text{and} \quad \mathbf{\Omega}(\lambda - i0) = \overline{\mathbf{\Omega}(\lambda + i0)}.$$

Substituting expression (5.7) into (5.6), we find that

$$2\pi i \frac{\mathrm{d}(E(\lambda)e_n, e_m)}{\mathrm{d}\lambda} = P_n(\lambda) \frac{\mathbf{\Omega}(\lambda - i0)\mathbf{f}_m(\lambda + i0) - \mathbf{\Omega}(\lambda + i0)\mathbf{f}_m(\lambda - i0)}{|\mathbf{\Omega}(\lambda + i0)|^2}.$$

Combining this representation with formula (4.15) for $P_m(\lambda)$, we obtain the following result.

Theorem 5.6 *Let assumptions* (1.4) *and* (1.5) *hold. Then, for all* $n, m \in \mathbb{Z}_+$ *and* $\lambda \in (-1, 1)$, *we have the representation*

$$\frac{\mathrm{d}(E(\lambda)e_n, e_m)}{\mathrm{d}\lambda} = (2\pi)^{-1}\sqrt{1-\lambda^2}|\mathbf{\Omega}(\lambda \pm i0)|^{-2} P_n(\lambda) P_m(\lambda).$$

In particular, the spectral measure of the operator J *equals*

$$\mathrm{d}\rho(\lambda) := \mathrm{d}(E(\lambda)e_0, e_0) = w(\lambda)\mathrm{d}\lambda, \quad \lambda \in (-1, 1), \tag{5.8}$$

where the weight $w(\lambda)$ *is given by the formula*

$$w(\lambda) = (2\pi)^{-1}\sqrt{1-\lambda^2}\,|\mathbf{\Omega}(\lambda \pm i0)|^{-2} \tag{5.9}$$

(*the right-hand sides here do not depend on the sign*).

According to (5.9), the amplitude factors in (1.14) and (4.18) are the same.
Putting together Theorem 4.6 and formula (5.9), we arrive at the next result.

Theorem 5.7 *Under assumptions* (1.4) *and* (1.5), *the weight* $w(\lambda)$ *is a continuous strictly positive function of* $\lambda \in (-1, 1)$.

Note that this result was earlier obtained in [18] by specific methods of the orthogonal polynomials theory.

According to (3.2) for the operator J_0, we have

$$\mathbf{\Omega}_0(\lambda \pm i0) = \Omega_0(\lambda \pm i0) = -2^{-1}(\lambda \pm i\sqrt{1-\lambda^2}),$$

and hence expressions (5.8), (5.9) reduce to (1.3).

Observe that Theorem 3.1 does not give any information on the behavior of the Jost function $\Omega(z)$ as $z \to \pm 1$. However, relation (5.9) implies that the normalized function (3.8) obeys an estimate

$$\int_{-1}^{1} \sqrt{1-\lambda^2} |\mathbf{\Omega}(\lambda \pm i0)|^{-2} d\lambda = 2\pi \int_{-1}^{1} w(\lambda) d\lambda \le 2\pi,$$

and hence $\mathbf{\Omega}(\lambda \pm i0)$ cannot vanish too rapidly as $\lambda \to 1-0$ and $\lambda \to -1+0$ (even if 1 or -1 are eigenvalues of J). For example, the behavior $\mathbf{\Omega}(\lambda + i0) \sim c_{\pm}(\lambda \mp 1)$ with $c_{\pm} \neq 0$ is excluded.

In view of (4.17), (5.9), the amplitude in (4.18) can be written as

$$\varkappa(\lambda)(1-\lambda^2)^{-1/2} = (2/\pi)^{1/2}(1-\lambda^2)^{-1/4} w(\lambda)^{-1/2}. \quad (5.10)$$

Substituting this expression into (4.18), we can reformulate Theorem 4.7 in a form more common for the orthogonal polynomials literature.

Theorem 5.8 *Let assumptions* (1.4) *and* (1.5) *be satisfied, let* $\lambda \in (-1, 1)$ *and let the phases* $\varphi_n(\lambda)$ *and* $\eta(\lambda)$ *be defined by formulas* (3.7) *and* (4.17), *respectively. Then the polynomials* $P_n(\lambda)$ *have asymptotics*

$$P_n(\lambda) = (2/\pi)^{1/2}(1-\lambda^2)^{-1/4} w(\lambda)^{-1/2} \sin(\varphi_n(\lambda) + \eta(\lambda)) + O(\varepsilon_n) \quad (5.11)$$

as $n \to \infty$. *Relation* (5.11) *is uniform in* λ *on compact subintervals of* $(-1, 1)$.

Formula (5.11) is a generalization of (1.14) and reduces to it if the short-range condition (1.9) is satisfied.

Of course, we can also replace the amplitude factor in (4.27) by its expression (5.10).

5.3 Singular weights

As was mentioned in Sect. 1.2, results on orthogonal polynomials $P_n(z)$ can be deduced from some conditions on either the recurrent coefficients a_n, b_n or on the corresponding spectral measure $d\rho(\lambda)$. We proceeded from conditions on a_n, b_n, but our approach gives automatically also some results about the regularity of $d\rho(\lambda) = w(\lambda)d\lambda$ (see Theorem 5.7).

Weights $w(\lambda)$ with singularities or zeros inside $(-1, 1)$ change both the asymptotic behavior of the recurrent coefficients a_n, b_n and of the orthogonal polynomials $P_n(z)$. For example, even weights $w(\lambda)$ behaving like $\kappa|\lambda|^{\gamma}$ where $\gamma > -1$, $\kappa > 0$ as $\lambda \to 0$ were investigated in the paper [23]. Such weights are either singular at the point $\lambda = 0$ if $\gamma < 0$ or $w(0) = 0$ if $\gamma > 0$. It was shown in [23, Theorem 7.4], that the corresponding Jacobi coefficients a_n satisfy the asymptotic relation

$$a_n = 1/2 + (-1)^n \gamma/(4n) + o(1/n)$$

(the coefficients $b_n = 0$ if the weight $w(\lambda)$ is even). Since $|a'_n| \sim |\gamma|/(2n)$, the condition (1.5) is now violated. For such weights, the asymptotic behavior of the polynomials $P_n(\lambda)$ in a neighborhood of the point $\lambda = 0$ differs from (1.22) and from (1.14), in particular. The results of [20] for weights with a jump singularity are morally similar to those in [23]. Thus, the long-range condition (1.5) is close to necessary even for our results on the weight $w(\lambda)$.

In our case, the weight $w(\lambda)$ is a continuous positive function. Nevertheless, the classical asymptotics (1.14) breaks down since an additional phase shift $\Phi_n(\lambda)$ appears in (1.22). This is quite similar to long-range scattering for the Schrödinger equation. We emphasize, however, that asymptotics (1.22) obtained under assumption (1.5) is, in some sense, more regular than the asymptotics of $P_n(\lambda)$ in [20,23].

Compliance with ethical standards

Conflict of interest The author states that there is no conflict of interest.

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