



# Quadratic open quantum harmonic oscillator

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## Abstract

We study the quantum open system evolution described by a Gorini–Kossakowski–Sudarshan–Lindblad generator with creation and annihilation operators arising in Fock representations of the  $\mathfrak{sl}_2$  Lie algebra. We show that any initial density matrix evolves to a fully supported density matrix and converges towards a unique equilibrium state. We show that the convergence is exponentially fast and we exactly compute the rate for a wide range of parameters. We also discuss the connection with the two-photon absorption and emission process.

**Keywords** Quantum harmonic oscillator · Quantum Markov semigroup · Fock representations of the  $\mathfrak{sl}_2$  algebra · Spectral gap

**Mathematics Subject Classification** 46L55 · 82C10 · 60J27

## 1 Introduction

Models of quantum harmonic oscillators are usually based on commutation relations. The Heisenberg–Weyl algebra commutation relations  $[H, A] = -A$ ,  $[H, A^+] = A^+$ ,  $[A, A^+] = 1$ , or in terms of position  $Q = (A^+ + A)/\sqrt{2}$  and momentum  $P = i(A^+ - A)/\sqrt{2}$ ,  $[H, Q] = -iP$ ,  $[H, P] = iQ$ ,  $[Q, P] = i$  are the foundation at the best known one. This model arises, for instance, replacing time derivatives

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in the classical equation  $q'' = -q$  by commutators with the Hamiltonian operator  $H = (P^2 + Q^2)/2$  so that we can write it as  $[H, [H, Q]] = Q$ . If we fix  $H$  and define  $P = i[H, Q]$ , then the double commutator equation reads as  $[H, P] = iQ$  and if, moreover, we want  $H, P, Q$  to be elements of a Lie algebra, the Jacobi identity  $[H, [P, Q]] + [P, [Q, H]] + [Q, [H, P]] = 0$  implies that  $[P, Q]$  commutes with  $H$ . The most natural choice as  $-i\mathbb{1}$  corresponds to the commutation relations of the Heisenberg–Weyl algebra.

Other choices lead to different models of quantum oscillators (see, for instance, [6] and the references therein), and for some of them, it is possible to develop a complete theory describing explicitly spectra of observables, eigenvectors, time evolution, etc. The choice  $[P, Q] = -2iH$  corresponds to the commutation relations of the  $\mathfrak{sl}_2$  Lie algebra.

This is a three-dimensional simple  $\ast$ -Lie algebra with basis  $\{B^+, B, M\}$ , commutation relations  $[B, B^+] = M, [M, B^+] = 2B^+, [M, B] = -2B$  and involution  $B^\ast = B^+, M^\ast = M$ . The construction of Fock representations of  $\mathfrak{sl}_2$  Lie algebra and of the current algebra associated to its central extension motivated a large number of papers extending it in different directions: see Ref. [26] for the case of free white noise; Ref. [4] for the connection with quantum Lévy processes; Refs. [1,2,10,11] for the construction of the quadratic Fock functor.

The weak coupling limit (see [5]) of an harmonic oscillator coupled with a reservoir in equilibrium with inverse temperature  $\beta > 0$  gives rise to a fundamental model of an open quantum system with a lot of deep properties and quantities that can be computed explicitly called in the literature the *open quantum harmonic oscillator* (see, e.g. Ref. [21] and the references therein). If we consider, instead, the formal Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) generator arising in the weak coupling limit of an oscillator based on the Fock representations of  $\mathfrak{sl}_2$  commutation relations, we find

$$\begin{aligned} \mathcal{L}(x) = & -\frac{\lambda^2}{2} (BB^+x - 2BxB^+ + xBB^+) \\ & -\frac{\mu^2}{2} (B^+Bx - 2B^+xB + xB^+B) + i[\zeta^+BB^+ + \zeta^-B^+B, x] \end{aligned} \quad (1)$$

where  $\zeta^\pm, \lambda, \mu$  are real parameters and  $\lambda, \mu > 0$ . This is called the *quadratic open quantum harmonic oscillator* because the operators  $B, B^+$  are the annihilation and creation operators arising in Fock representations of  $\mathfrak{sl}_2$  and the action of  $BB^+$  and  $B^+B$  [see formulae (2)] is quadratic with respect to the level of the system while, for the open quantum harmonic oscillator, it is linear. Constants  $\lambda^2, \mu^2$  are related with the inverse temperature  $\beta$  by  $\lambda^2\mu^{-2} = e^{-s\beta}$  for some  $s > 0$ .

This is a simple and natural model; however, contrary to what happens for the open quantum harmonic oscillator, it does not admit explicit solutions except for the formula of the invariant state. As an example, if one looks at the action on the Abelian algebra of functions of the number operator, one finds a birth and death process with quadratic jump rates for which explicit representations for transition probabilities, to the best of our knowledge (see Ref. [25]), are not known.

In this paper, we first show that the formal GKSL generator with unbounded operators  $B, B^+$  generates a unique quantum Markov semigroup and we establish the existence of a unique explicit equilibrium state. Then, we study the behaviour of the evolution of states and observables for all values of parameters involved. We prove that any initial state converges towards the unique equilibrium state for the trace norm (Theorem 1). We also prove (Theorem 3) that any initial state  $\rho_0$ , in particular also a pure state, evolves to a faithful state  $\rho_t$  for all  $t > 0$ . Moreover, we show that, for some special values of a parameter  $r$  determining the Fock representation of the  $\mathfrak{sl}_2$  commutation relations this model is intimately related with the two-photon absorption and emission process studied in [8, 16]. Finally, we show that convergence towards the unique invariant state is exponentially fast (with respect to the Hilbert–Schmidt norm induced by the invariant state) and we also compute the sharp exponential rate for a wide range of parameters (Theorem 6). Our analysis shows, in particular, that the decay rate of off-diagonal terms of density matrices is smaller than the rate of convergence of the diagonal part towards the unique equilibrium state for more and more values of the parameter  $r$  as the inverse temperature  $\beta$  becomes big, i.e. the reservoir becomes cooler. In other words, at low temperatures, decoherence is slower than relaxation for  $r$  away from 0.

The paper is organized as follows. In Sect. 2, we introduce the model of the quadratic open quantum harmonic oscillator. The full characterization of invariant states and the asymptotic behaviour of the associated quantum Markov semigroup are studied in Sect. 3. The close relationship with two-photon absorption and emission process is studied in Sect. 4. In Sect. 5, we show that for all initial state the support of the state evolved at any time  $t > 0$  is full. The rate of the exponentially fast convergence towards the unique invariant state is studied in Sect. 6.

## 2 The model

Let  $\mathfrak{h}$  be the Hilbert space  $\mathfrak{h} = \ell^2(\mathbb{N}) \simeq \Gamma(\mathbb{C})$  with canonical orthonormal basis  $(e_n)_{n \geq 0}$ . We consider the operators  $B, B^+, M$  of the Fock representation of the renormalized square of the white noise Lie algebra  $B, B^+$  and  $M$  with domain

$$\text{Dom}(B) = \text{Dom}(B^+) = \text{Dom}(M) = \left\{ u = \sum_{n \geq 0} u_n e_n \mid \sum_{n \geq 0} n^2 |u_n|^2 < \infty \right\}$$

defined, on vectors of the canonical orthonormal basis, by

$$\begin{aligned} B e_n &= \omega_n^{1/2} e_{n-1}, \text{ for } n > 0, & B e_0 &= 0, \\ B^+ e_n &= \omega_{n+1}^{1/2} e_{n+1} \\ M e_n &= (2n + r) e_n, \end{aligned} \tag{2}$$

where  $r > 0$  is a real parameter (see Ref. [4] Section 3.2 p. 134 for the explanation why  $r$  must be non-negative) and

$$\omega_n = n(n + r - 1)$$

which is strictly positive for all  $n \geq 1$  and satisfies  $\omega_0 = 0$ .

Note that the domain of  $B$ ,  $B^+$  and  $M$  coincides with the domain of the number operator  $N$  defined by  $Ne_n = ne_n$  for all  $n \geq 0$ .

We consider the formal Lindblad generator (1) which is of weak coupling limit type (see Refs. [3,5]) since it arises in the weak coupling limit of a system with Hilbert space  $\ell^2(\mathbb{N})$  and Hamiltonian  $H_S$  given by a constant  $s > 0$  times the number operator  $N$  coupled to a Boson reservoir in equilibrium with inverse temperature  $\beta > 0$  and interaction operator

$$B \otimes A^+(g) + B^+ \otimes A(g).$$

Constants  $\lambda^2, \mu^2$  satisfy  $\lambda^2\mu^{-2} = e^{-s\beta}$  for some  $s > 0$ .

Moreover (see Sect. 4) for  $r = 1/2$  (resp.  $r = 3/2$ ) and a suitable choice of the real constants  $\zeta^-, \zeta^+$  we find the even (resp. odd) part of the two-photon absorption and emission generator studied in Refs. [8,16].

Let  $G$  be the operator defined on  $\text{Dom}(N^2)$ , the domain of the square of the number operator, by

$$G = -\frac{\lambda^2}{2}BB^+ - \frac{\mu^2}{2}B^+B - i(\zeta^+BB^+ + \zeta^-B^+B)$$

and let  $L_1, L_2$  be the operators defined on  $\text{Dom}(N)$  by

$$L_1 = \mu B, \quad L_2 = \lambda B^+.$$

Clearly  $G$  is a function of the number operator  $N$  and since

$$G = -\left(\frac{\lambda^2}{2} + i\zeta^+\right)\omega_{N+1} - \left(\frac{\mu^2}{2} + i\zeta^-\right)\omega_N$$

with negative real part, it generates a strongly continuous semigroup of contractions  $(P_t)_{t \geq 0}$  on  $\mathfrak{h}$  explicitly given by

$$P_t e_n = e^{-t\left(\left(\frac{\lambda^2}{2} + i\zeta^+\right)\omega_{n+1} + \left(\frac{\mu^2}{2} + i\zeta^-\right)\omega_n\right)} e_n.$$

For every  $x \in \mathcal{B}(\mathfrak{h})$ , the formal generator is the sesquilinear form

$$\mathcal{E}(x)[u, v] = \langle Gu, xv \rangle + \sum_{\ell=1}^2 \langle L_\ell u, xL_\ell v \rangle + \langle u, xGv \rangle, \quad (3)$$

for  $u, v \in \text{Dom}(G) = \text{Dom}(N^2)$ . One can easily check that conditions for constructing the minimal quantum dynamical semigroup (QDS) associated with the above  $G, L_1, L_2$  [(H-min) in Ref. [12]] hold and this semigroup  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  satisfies the so-called Lindblad equation

$$\langle v, T_t(x)u \rangle = \langle v, P_t^* x P_t u \rangle + \sum_{\ell=1}^2 \int_0^t \langle L_\ell P_{t-s} v, T_s(x) L_\ell P_{t-s} u \rangle ds, \tag{4}$$

for all  $u, v \in \text{Dom}(G)$ .

A straightforward computation using the CCR (it could be done considering quadratic forms on the linear manifold generated by vectors  $(e_n)_{n \geq 0}$  if one wants to cope with unboundedness of the involved operators but we prefer to simplify the notation) shows that

$$\begin{aligned} \mathcal{E}(f(N)) &= \lambda^2(N + 1)(N + r) (f(N + 1) - f(N)) \\ &\quad + \mu^2 N(N + r - 1) (f(N - 1) - f(N)). \end{aligned}$$

Taking  $f(n) = (n + 1)^2$ , for  $n \geq 1 - r$ , we easily find

$$\begin{aligned} &\lambda^2(n + 1)(n + r) (f(n + 1) - f(n)) + \mu^2 n(n + r - 1) (f(n - 1) - f(n)) \\ &= 2(\lambda^2 - \mu^2)n^3 + (\lambda^2(2r + 3) + \mu^2(2r - 3))n^2 \\ &\quad + (\lambda^2(3r + 1) + \mu^2(r - 1))n + \lambda^2 r \end{aligned}$$

and, for  $0 \leq n < 1 - r$ , i.e.  $n = 0$  we obviously find 0. Therefore, defining as  $b$  the maximum of the three constants

$$\left| \lambda^2(2r + 3) + \mu^2(2r - 3) \right|, \frac{|\lambda^2(3r + 1) + \mu^2(r - 1)|}{2}, \lambda^2|r|,$$

we have

$$\mathcal{E}((N + \mathbf{1})^2) \leq -2(\mu^2 - \lambda^2)N^3 + b(N + \mathbf{1})^2.$$

As a consequence, if  $\lambda \leq \mu$ ,  $\mathcal{E}$  satisfies a well known criterion for conservativity (Ref. [12] Theorem 3.40). Moreover, for  $\lambda > \mu$  the formal generator satisfies a simple criterion for conservativity, see Ref. [19], Example 2. Then the minimal QDS is Markov (or conservative) if and only if  $\lambda \leq \mu$ . It follows from conservativity that the minimal QDS is the unique solution of Eq. (4). Moreover an operator  $x \in \mathcal{B}(\mathfrak{h})$  belongs to the domain of the generator  $\mathcal{L}$  if and only if the sesquilinear form  $\mathcal{E}(x)$  is bounded (see Ref. [12] Prop. 3.33 p.64).

The action of  $\mathcal{L}$  on the linear manifold  $\mathcal{M} = \text{span}\{|e_j\rangle\langle e_k| : j, k \geq 0\}$  of finite range operators is given by

$$\begin{aligned} \mathcal{L}(x) &= i \sum_{j,k} (\zeta^+(\omega_{j+1} - \omega_{k+1}) + \zeta^-(\omega_j - \omega_k)) x_{jk} |e_j\rangle\langle e_k| \\ &\quad + \sum_{j,k} \left( \mu^2 \omega_k^{1/2} \omega_j^{1/2} x_{j-1k-1} - \frac{\mu^2}{2} (\omega_j + \omega_k) x_{jk} \right. \\ &\quad \left. + \lambda^2 \omega_{j+1}^{1/2} \omega_{k+1}^{1/2} x_{j+1k+1} - \frac{\lambda^2}{2} (\omega_{j+1} + \omega_{k+1}) x_{jk} \right) |e_j\rangle\langle e_k|. \end{aligned} \tag{5}$$

### 3 Invariant states and asymptotic behaviour

The behaviour of the quadratic open quantum harmonic oscillator and the structure of its invariant states depends crucially upon the parameters  $\lambda$  and  $\mu$ . We begin by considering the case where  $\mu > \lambda > 0$ . Notice that, as mentioned in the introduction, since the parameters are related with the inverse temperature  $\beta$  by  $\lambda^2/\mu^2 = e^{-\beta s}$ , this condition corresponds to  $\beta > 0$ .

**Proposition 1** *If  $v = \lambda/\mu < 1$  then the normal state*

$$\rho = (1 - v^2) \sum_{n \geq 0} v^{2n} |e_n\rangle\langle e_n| \tag{6}$$

is invariant.

**Proof** Let  $\mathcal{L}_*$  be the generator of the predual semigroup  $\mathcal{T}_* = (\mathcal{T}_{*t})_{t \geq 0}$ , acting on the Banach space of trace class operators on  $\mathfrak{h}$ . Consider the approximations  $\rho_n = (1 - v^2) \sum_{k=0}^{2n} v^{2k} |e_k\rangle\langle e_k|$ , of  $\rho$  by finite rank operators.

The operators  $\rho_n$  belong to the domain of  $\mathcal{L}_*$  and we can write  $\mathcal{L}_*(\rho_n)$  as  $(1 - v^2)$  times

$$\begin{aligned} & \sum_{k=0}^n \lambda^2 v^{2k} \omega_{k+1} (|e_{k+1}\rangle\langle e_{k+1}| - |e_k\rangle\langle e_k|) + \sum_{k=1}^n \mu^2 v^{2k} \omega_k (|e_{k-1}\rangle\langle e_{k-1}| - |e_k\rangle\langle e_k|) \\ &= \sum_{k=1}^n \left( \lambda^2 v^{2(k-1)} - \mu^2 v^{2k} \right) \omega_k |e_k\rangle\langle e_k| + \sum_{k=0}^{n-1} \left( \mu^2 v^{2(k+1)} - \lambda^2 v^{2k} \right) |e_k\rangle\langle e_k| \\ & \quad + \lambda^2 v^{2n} (\omega_{n+1} |e_{n+1}\rangle\langle e_{n+1}| - \omega_n |e_n\rangle\langle e_n|). \end{aligned}$$

Terms in the above summations vanish because  $\mu^2 v^{2(k+1)} = \lambda^2 v^{2k}$  for all  $k \geq 0$ . Moreover

$$\lim_{n \rightarrow \infty} \|\mathcal{L}_*(\rho_n)\|_1 = \lim_{n \rightarrow \infty} \lambda^2 v^{2n} (\omega_{n+1} + \omega_n) = 0$$

because  $\lambda < \mu$ . Since the operator  $\mathcal{L}_*$  is closed, it follows that  $\rho$  belongs to the domain of  $\mathcal{L}_*$  and  $\mathcal{L}_*(\rho) = 0$ . □

**Remark** It is worth noticing here that the above invariant state is the thermal state for the Hamiltonian  $d\Gamma(s)$  at inverse temperature  $\beta = s^{-1} \log(\mu^2/\lambda^2)$ . It is now clear that the condition  $\lambda < \mu$ , that corresponds to  $\beta > 0$ , means that the absorption rate is smaller than the emission rate. On the contrary, the opposite inequality  $\lambda > \mu$ , meaning that the absorption rate is greater than the emission rate, implies a continuous heating of the system, hence the absence of an invariant state.

In order to show uniqueness of the invariant state (6), we begin by recalling that the support projection  $p$  of an invariant state with density matrix  $\rho$ , i.e. the orthogonal projection onto the range of  $\rho$ , satisfies  $\mathcal{T}_t(p) \geq p$  for all  $t \geq 0$  (see, e.g. [14])

Theorem II.1). Such projections, called subharmonic, are easily characterized in terms of invariant subspaces of operators  $P_t$  and  $L_1, L_2$  considered in Sect. 2. A QMS is called *irreducible* if the only subharmonic projections are the trivial ones  $0, \mathbf{1}$ . In this case, it is well-known (see Ref. [18] Lemma 1) that a faithful invariant state, if it exists, is unique because the set of fixed points for the QMS  $\mathcal{T}$  is the trivial algebra  $\mathbb{C}\mathbf{1}$ . In our framework, we can prove the following.

**Proposition 2** *The QMS  $\mathcal{T}$  is irreducible for all  $\lambda \leq \mu$ . In particular, if  $\lambda < \mu$ , the state (6) is the unique  $\mathcal{T}$ -invariant state.*

**Proof** The range of any non-trivial subharmonic projection determines an invariant subspace for the operators  $P_t$  for all  $t > 0$  (see Ref. [14] Theorem III.1). Since these operators are normal and compact, these invariant subspaces are generated by eigenvectors of  $P_t$ . Moreover, knowing the spectral decomposition of  $P_t$  (it is a function of the number operator!) we infer that they are generated by collections of vectors  $(e_n)_{n \in I}$  for some subset  $I$  of  $\mathbb{N}$ . Invariance of these subspaces for  $B$  and  $B^+$  implies then that they must coincide with the whole of  $\mathfrak{h}$ . This proves that the QMS is irreducible.

If  $\lambda < \mu$  the QMS admits the faithful invariant state (6) and so the set of fixed points for the QMS  $\mathcal{T}$  is the trivial algebra  $\mathbb{C}\mathbf{1}$ . It follows then from Lemma 1 of Ref. [18] that (6) is the unique invariant state. □

Applying the main result of Ref. [9], we can also show convergence towards the invariant state in trace norm. As a preliminary step, we prove the following result which is interesting on its own

**Proposition 3** *If  $\nu = \lambda/\mu < 1$  the decoherence free subalgebra*

$$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{B}(\mathfrak{h}) \mid \mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x), \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x^*), \forall t \geq 0\}.$$

*and the fixed point algebra  $\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathfrak{h}) \mid \mathcal{T}_t(x) = x \forall t \geq 0\}$  are trivial.*

**Proof** It is well-known that  $\mathcal{N}(\mathcal{T})$  is a von Neumann subalgebra of  $\mathcal{B}(\mathfrak{h})$  (see, e.g. Proposition 2.1 (3) of Ref. [9]). Moreover, since the invariant state  $\rho$  defined in (6) is faithful, also  $\mathcal{F}(\mathcal{T})$  is a von Neumann subalgebra of  $\mathcal{B}(\mathfrak{h})$ . Indeed, if  $x$  belongs to  $\mathcal{F}(\mathcal{T})$ , then, by 2-positivity,  $\mathcal{T}_t(x^*x) \geq \mathcal{T}_t(x^*)\mathcal{T}_t(x) = x^*x$  and  $\text{tr}(\rho(\mathcal{T}_t(x^*x) - x^*x)) = 0$  because  $\rho$  is invariant. It follows that  $\mathcal{T}_t(x^*x) = x^*x$ , i.e.  $x^*x \in \mathcal{F}(\mathcal{T})$ .

As a by-product, if  $x \in \mathcal{F}(\mathcal{T})$ , then

$$\mathcal{T}_t(x^*x) = x^*x = \mathcal{T}_t(x^*)\mathcal{T}_t(x),$$

and the same identity holds exchanging  $x$  and  $x^*$ , i.e.  $\mathcal{F}(\mathcal{T})$  is contained in  $\mathcal{N}(\mathcal{T})$ .

Thus, it suffices to prove that  $\mathcal{N}(\mathcal{T})$  is trivial. To this end, we apply Theorem 4.1 of Ref. [9] characterizing  $\mathcal{N}(\mathcal{T})$  as the generalized commutator of the set of unbounded operators

$$\mathcal{D}(\mathcal{T}) := \left\{ e^{-itH} L_\ell e^{itH}, e^{-itH} L_\ell^* e^{itH} \mid \ell \geq 1, t \geq 0 \right\}. \tag{7}$$

where  $H = \zeta^+ B B^+ + \zeta^- B^+ B$ . The additional technical domain assumptions that can be easily checked taking as  $D$  the linear manifold spanned by finite linear combinations of vectors  $e_n$  of the orthonormal basis and as operator  $C$  the number operator or  $(N + \mathbb{1})^2$ .

If  $X$  is an operator in the generalized commutator of (7), then it is, by definition of generalized commutator, bounded and, in particular, it satisfies

$$XB \subseteq BX, \quad XB^+ \subseteq B^+X$$

(meaning that  $BX$  is an ampliation of  $XB$  and  $B^+X$  is an ampliation of  $XB^+$ ). It follows that

$$XB^+B \subseteq B^+XB \subseteq B^+BX, \quad XBB^+ \subseteq BXB^+ \subseteq BB^+X,$$

and so, since the difference  $BB^+ - B^+B$  is  $2N + r\mathbf{1}$ ,  $NX$  is an ampliation of  $NX$  and

$$X(s + N) \subseteq (s + N)X$$

for all  $s > 0$ . Left and right multiplying by the resolvent  $(s + N)^{-1}$ , since the operators  $(s + N)^{-1}X$  and  $X(s + N)^{-1}$  are bounded, we find  $(s + N)^{-1}X = X(s + N)^{-1}$  for all  $s > 0$ . This shows that  $X$  commutes with every spectral projection  $|e_n\rangle\langle e_n|$  of the number operator and so it is a function  $f(N)$  of the number operator itself. However, since  $\omega_{n+1} > 0$  for all  $n \geq 0$  if  $r > 0$

$$f(N)B^+e_n - B^+f(N)e_n = \omega_{n+1}^{1/2}(f(n+1) - f(n))e_n$$

vanishes if and only if  $f$  is constant and so the generalized commutator of (7) is trivial.  $\square$

We are now in a position to prove the following.

**Theorem 1** *If  $0 < \nu = \lambda/\mu < 1$  then (6) is the unique invariant state and*

$$\lim_{t \rightarrow \infty} T_{*t}(\eta) = \rho$$

*in trace norm for all initial state  $\eta$ .*

**Proof** Since  $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbf{1}$  by Proposition 3, the conclusion is immediate from Theorem 3.3 of Ref. [9].  $\square$

We complete the study of the asymptotic behaviour by the following.

**Proposition 4** *If  $\lambda = \mu > 0$  the QMS is transient. In particular, it has no invariant state.*



**Proof** By Theorem 5 Ref. [15], it suffices to find a strictly positive bounded operator  $X$  such that  $\mathcal{T}_t(X) \leq X$  for all  $t \geq 0$ . Inspired by a result on classical birth and death processes ([22] Theorem 2 and Lemma 1), we consider the operator

$$X = \left( \sum_{k \geq n} (k + 1)^{-1} (k + r)^{-1} \right) |e_n\rangle \langle e_n|$$

which is clearly bounded since  $\sum_{k \geq 0} (k + 1)^{-1} (k + r)^{-1} < \infty$  and is a function of the number operator. A straightforward computation shows that

$$\mathcal{L}(X) = -|e_0\rangle \langle e_0| \leq 0.$$

It follows that  $X$  belongs to the domain of  $\mathcal{L}$  ([12] Prop. 3.33 p.64) and

$$\frac{d}{dt} \mathcal{T}_t(X) = -\mathcal{T}_t(|e_0\rangle \langle e_0|) \leq 0$$

so that  $\mathcal{T}_t(X) \leq X$  for all  $t \geq 0$ . Since the QMS  $\mathcal{T}$  is transient, it has no invariant state by Proposition 6 of Ref. [15]. □

In the case where there is a faithful invariant state, it is not difficult to show that the quantum detailed balance condition (in most of its quantum formulations as in [17]) holds.

### 4 Relationship with the two-photon absorption and emission process

The two-photon absorption and emission process quantum Markov semigroup is generated by

$$\begin{aligned} \mathcal{L}_{\text{tp}}(x) = & i [\xi^- a^{+2} a^2 + \xi^+ a^2 a^{+2}, x] \\ & - \frac{\lambda^2}{2} (a^2 a^{+2} x - 2a^2 x a^{+2} + x a^2 a^{+2}) \\ & - \frac{\mu^2}{2} (a^{+2} a^2 x - 2a^{+2} x a^2 + x a^{+2} a^2) \end{aligned}$$

where  $a, a^+$  are the usual annihilation and creation operators in  $\mathfrak{h}$ ,  $\lambda, \mu, \xi^+, \xi^- \in \mathbb{R}$ ,  $\lambda \leq \mu$ .

This generator has been studied in Ref. [16] for  $\xi^+ = 0$ ; however, this does not change any result of that paper. In particular, it has been proved that the even and odd projections

$$p_e = \sum_{k \geq 0} |e_{2k}\rangle \langle e_{2k}|, \quad p_o = \sum_{k \geq 0} |e_{2k+1}\rangle \langle e_{2k+1}|$$

are harmonic (i.e. invariant) for the QMS generated by  $\mathcal{L}_{\text{tp}}$ . As a consequence, we can consider the restricted semigroups  $\mathcal{T}^e$  and  $\mathcal{T}^o$  on the von Neumann subalgebras  $p_e\mathcal{B}(\mathfrak{h})p_e$  and  $p_o\mathcal{B}(\mathfrak{h})p_o$ , identified, respectively, with  $\mathcal{B}(p_e\mathfrak{h})$  and  $\mathcal{B}(p_o\mathfrak{h})$ .

Let  $U_e, U_o$  be the unitary operators

$$U_e : \mathfrak{h} \rightarrow p_e\mathfrak{h}, \quad U_e e_k = e_{2k}, \quad U_o : \mathfrak{h} \rightarrow p_o\mathfrak{h}, \quad U_o e_k = e_{2k+1}$$

A straightforward computation shows that, if  $r = 1/2$ ,

$$U_e B U_e^* e_{2k} = 2 a^2 e_{2k}, \quad U_e B^+ U_e^* e_{2k} = 2 a^{+2} e_{2k}$$

and, if  $r = 3/2$ , similarly

$$U_o B U_o^* e_{2k+1} = 2 a^2 e_{2k+1}, \quad U_o B^+ U_o^* e_{2k+1} = 2 a^{+2} e_{2k+1},$$

so that, in both cases,

$$U_e (\zeta^+ B B^+ + \zeta^- B^+ B) U_e^* = 4 (\zeta^+ a^2 a^{+2} + \zeta^- a^{+2} a^2)$$

As a consequence, the quadratic open quantum harmonic oscillator generator  $\mathcal{L}$ , for  $\xi^\pm = 4\zeta^\pm$ , satisfies

$$\begin{aligned} U_e \mathcal{L}(U_e^* x U_e) U_e^* &= 4 \mathcal{L}_{\text{tp}}(p_e x p_e) && \text{for } r = 1/2 \\ U_o \mathcal{L}(U_o^* x U_o) U_o^* &= 4 \mathcal{L}_{\text{tp}}(p_o x p_o) && \text{for } r = 3/2 \end{aligned}$$

This shows that the QMS of the quadratic open quantum harmonic oscillator is unitarily equivalent (up to the multiplicative constant 4) to the QMS obtained by restriction of the two-photon absorption and emission process to the even (resp. odd) states of the number operator for  $r = 1/2$  (resp.  $r = 3/2$ ), for a suitable choice of the parameters  $\xi^\pm, \zeta^\pm$ . This analogy will serve as an inspiration to study the exponential speed of convergence towards the equilibrium state.

## 5 Instantaneous spread of state supports

In this section, we will show that for all initial state  $\eta$  the range of the support of the state  $\mathcal{T}_{*t}(\eta)$  at any time  $t > 0$  is the whole of  $\mathfrak{h}$ .

This property follows from a recent result by Hachicha, Nasroui and Gliouez [20] Theorem 3.3 for QMSs associated with operators  $G, L_\ell$  (in our case  $L_1 = \lambda B^+, L_2 = \mu B$ ) constructed from generators.

**Theorem 2** *Suppose that  $G$  generates an analytic semigroup in a sector  $\Delta = \{z \in \mathbb{C} \mid |\arg z| < \theta\}$  with  $\theta \in ]0, \frac{\pi}{2}]$  and, moreover, that*

$$L_\ell(\text{Dom}(G^k)) \subseteq \text{Dom}(G^{k-1}) \quad (8)$$

for all  $k > 0$ . For all state  $\omega = \sum_{j \in J} \omega_j |\psi_j\rangle\langle\psi_j|$ , with  $\omega_j > 0$  for all  $j \in J$  and all  $t \geq 0$ , the support projection  $\mathcal{S}_t(\omega)$  of the state  $\mathcal{T}_{*t}(\omega)$  is the closure of linear manifold generated by vectors

$$P_t \psi_j, \delta_G^{m_1}(L_{\ell_1}) \delta_G^{m_2}(L_{\ell_2}) \dots \delta_G^{m_n}(L_{\ell_n}) P_t \psi_j \tag{9}$$

for all  $j \in J, n \geq 1, m_1, \dots, m_n \geq 0$  and  $\ell_1, \dots, \ell_n \geq 1$  where  $\delta_G^m$  denotes the  $m$ -th iterated commutator with  $G$  and  $\delta_G^0(L_\ell) = L_\ell$ .

In our framework the operator  $G$  can be written as

$$G = - \left( \frac{\lambda^2 + \mu^2}{2} + i(\zeta^+ + \zeta^-) \right) N^2 + \left( \frac{\lambda^2}{2} + i\zeta^+ \right) r - \left( \left( \frac{\lambda^2 + \mu^2}{2} + i(\zeta^+ + \zeta^-) \right) r + \left( \frac{\lambda^2 - \mu^2}{2} + i(\zeta^+ - \zeta^-) \right) \right) N.$$

Note that the self-adjoint operator  $N^2$  generates a semigroup  $(e^{zN^2})_{z \in \mathbb{C}; \Re(z) \leq 0}$  defined on the complex plane with non-positive real part which is an analytic semigroup in the half plane  $\{z \in \mathbb{C} \mid \Re(z) < 0\}$ . Thus, by a change of variable, the operator

$$G_0 = - \left( \frac{\lambda^2 + \mu^2}{2} + i(\zeta^+ + \zeta^-) \right) N^2 \tag{10}$$

generates an analytic semigroup in the sector

$$\Re \left( \left( \frac{\lambda^2 + \mu^2}{2} + i(\zeta^+ + \zeta^-) \right) z \right) > 0,$$

which is equivalent to  $\Re(z) > 0$  for  $\zeta^+ + \zeta^- = 0$  and

$$\begin{aligned} \Im(z) &< \frac{\lambda^2 + \mu^2}{2(\zeta^+ + \zeta^-)} \Re(z), & \text{if } \zeta^+ + \zeta^- > 0, \\ \Im(z) &> \frac{\lambda^2 + \mu^2}{2(\zeta^+ + \zeta^-)} \Re(z), & \text{if } \zeta^+ + \zeta^- < 0. \end{aligned}$$

In any case, the semigroup generated by the operator  $G_0$  in (10) is analytic in the sector

$$\left\{ z \in \mathbb{C} \mid |\arg(z)| < \arctan \left( (\lambda^2 + \mu^2) / (2|\zeta^+ + \zeta^-|) \right) \right\} \tag{11}$$

with the convention  $\arctan(+\infty) = \pi/2$ . Clearly  $\text{Dom}(G) = \text{Dom}(G_0) = \text{Dom}(N^2)$  and  $G$  is the sum of  $G_0$  and an operator with domain  $\text{Dom}(N) = \text{Dom}(G_0^{1/2})$  where  $G_0^{1/2}$  is defined as  $N$  times a square root of the complex number  $(\lambda^2 + \mu^2)/2 + i(\zeta^+ + \zeta^-)$ . It follows then from Corollary 2.4 p.81 of [24] that  $G$  generates an analytic semigroup in the sector (11).

We can now prove the following.

**Theorem 3** For all initial state  $\eta$ , the range of the support of the state  $\mathcal{T}_{*t}(\eta)$  at any time  $t > 0$  is the whole of  $\mathfrak{h}$ .

**Proof** Assumption (8) obviously holds because  $\text{Dom}(G^k) = \text{Dom}(N^{2k})$  and  $B, B^+$  map  $\text{Dom}(N^h)$  in  $\text{Dom}(N^{h-1})$  for all  $h$  so that

$$B^\pm(\text{Dom}(G^k)) \subseteq \text{Dom}(N^{2k-1}) \subseteq \text{Dom}(N^{2k-2}) = \text{Dom}(G^{k-1}).$$

For all state  $\omega$ , we can write its spectral decomposition  $\omega = \sum_{j \in J} \omega_j |\psi_j\rangle\langle\psi_j|$ , for a collection  $(\psi_j)_{j \in J}$  of orthonormal vectors and  $\omega_j > 0$  for all  $j \in J$  and  $J$  non-empty. Since  $L_1 = \lambda B^+, L_2 = \mu B^-, \delta_G^0(B^\pm) = B^\pm$  and  $[B, B^+] - r \mathbb{1} = 2N$ , by Theorem 2, all vectors

$$N^n P_t \psi_j = 2^{-n} ([B, B^+] - r \mathbb{1})^n P_t \psi_j \tag{12}$$

belong to the range of the support of  $\mathcal{T}_{*t}(\omega)$ . Write  $P_t \psi_j = \sum_{k \geq 0} z_{jk}(t) e_k$  and let  $m_j$  be the minimum  $k$  for which  $z_{jk}(t) \neq 0$ . Since the function  $z \rightarrow e^{-zN}$  is analytic for  $\Re z > 0$ , and  $P_t \psi_j$  belongs to the range of the support of  $\mathcal{T}_{*t}(\omega)$ , for all real number  $s > 0$

$$\begin{aligned} \sum_{n \geq 0} \frac{(-s)^n}{n!} (N - m_j \mathbb{1})^n P_t \psi_j &= e^{-s(N - m_j \mathbb{1})} P_t \psi_j \\ &= z_{jm_j}(t) e_{m_j} + \sum_{k > m_j} z_{jk}(t) e^{-s(k - m_j)} e_k \end{aligned}$$

belongs to range of the support of  $\mathcal{T}_{*t}(\omega)$  by (12) for all  $s > 0$ . Taking the limit as  $s$  tends to infinity, we conclude that  $e_{m_j}$  belongs to the range of the support of  $\mathcal{T}_{*t}(\eta)$ . In the same way, starting from either

$$B^k P_t \psi_j = (\delta_G^0(B))^k P_t \psi_j \quad \text{or} \quad B^{+k} P_t \psi_j = (\delta_G^0(B^+))^k P_t \psi_j,$$

we can conclude that

$$B^k e_{m_j} \quad \text{and} \quad B^{+k} e_{m_j}$$

belong to the range of the support of  $\mathcal{T}_{*t}(\omega)$  for  $t > 0$  for all  $k$ . As a consequence, the range of the support of  $\mathcal{T}_{*t}(\omega)$  is the whole of  $\mathfrak{h}$  for all  $t > 0$ . □

### 6 Spectral gap

In this section, we discuss the spectral gap of the generator of the semigroup of the quadratic harmonic oscillator. For the purpose, we will follow the methods developed in Refs. [7,8].

### 6.1 Dirichlet form and spectral gap

Recall the invariant state  $\rho$  in (6). Let  $L_2(\mathfrak{h})$  be the space of Hilbert–Schmidt operators on  $\mathfrak{h}$  with inner product  $\langle x, y \rangle = \text{tr}(x^*y)$ . Consider the embedding

$$\iota : \mathcal{B}(\mathfrak{h}) \rightarrow L_2(\mathfrak{h}), \quad \iota(x) = \rho^{1/4}x\rho^{1/4}.$$

Let  $T = (T_t)_{t \geq 0}$  be the strongly continuous contraction semigroup on  $L_2(\mathfrak{h})$  defined by

$$T_t(\iota(x)) = \iota(\mathcal{T}_t(x)).$$

Let  $L$  be the generator of the semigroup  $(T_t)_{t \geq 0}$ . We can check that

$$L(\rho^{1/4}x\rho^{1/4}) = \rho^{1/4}\mathcal{L}(x)\rho^{1/4}, \quad \text{for } x \in D(\mathcal{L}).$$

**Lemma 1** For  $x = \sum_{j,k} x_{jk}|e_j\rangle\langle e_k| \in \mathcal{M}$ ,

$$\begin{aligned} L(x) = & i \sum_{j,k \geq 0} (\zeta^+(\omega_{j+1} - \omega_{k+1}) + \zeta^-(\omega_j - \omega_k))x_{jk}|e_j\rangle\langle e_k| \\ & + \sum_{j,k \geq 0} \left( \mu\lambda(\omega_k\omega_j)^{1/2}x_{j-1k-1} - \frac{\mu^2}{2}(\omega_j + \omega_k)x_{jk} \right. \\ & \left. + \mu\lambda(\omega_{j+1}\omega_{k+1})^{1/2}x_{j+1k+1} - \frac{\lambda^2}{2}(\omega_{j+1} + \omega_{k+1})x_{jk} \right)|e_j\rangle\langle e_k|. \end{aligned} \quad (13)$$

with the convention  $x_{j-1k-1} = 0$  if  $j = 0$  or  $k = 0$ .

The Dirichlet form, defined for  $\xi \in D(L)$ , is the quadratic form  $\mathcal{E}$

$$\mathcal{E}(\xi) = -\text{Re}\langle \xi, L(\xi) \rangle.$$

The spectral gap of the operator  $L$  is the non-negative number

$$\text{gap } L := \inf\{\mathcal{E}(\xi) : \|\xi\| = 1, \xi \in (\text{Ker } L)^\perp\}.$$

**Lemma 2** For  $\xi \in D(L)$  with  $\xi = \sum_{j,k} \xi_{jk}|e_j\rangle\langle e_k|$

$$\begin{aligned} \mathcal{E}(\xi) = & \frac{1}{2}\mu^2 \left( \sum_{k \geq 1} \omega_k |\xi_{0k}|^2 + \sum_{j \geq 1} \omega_j |\xi_{j0}|^2 \right) \\ & + \frac{1}{2} \sum_{j,k \geq 0} \left( \left| \mu\omega_{j+1}^{1/2}\xi_{j+1k+1} - \lambda\omega_{k+1}^{1/2}\xi_{jk} \right|^2 + \left| \mu\omega_{k+1}^{1/2}\xi_{j+1k+1} - \lambda\omega_{j+1}^{1/2}\xi_{jk} \right|^2 \right). \end{aligned}$$

In particular,  $\mathcal{E}(\xi) = \mathcal{E}(\xi^*)$ , where  $\xi^*$  is the adjoint operator of  $\xi$ .

**Proof** By Lemma 1, we get

$$\begin{aligned} \mathcal{E}(\xi) &= -\operatorname{Re} \langle \xi, L(\xi) \rangle \\ &= -\operatorname{Re} \sum_{j,k} \left[ \mu\lambda(\omega_j\omega_k)^{1/2} \overline{\xi_{jk}} \xi_{j-1k-1} - \frac{1}{2} \mu^2(\omega_j + \omega_k) |\xi_{jk}|^2 \right. \\ &\quad \left. + \mu\lambda(\omega_{j+1}\omega_{k+1})^{1/2} \overline{\xi_{jk}} \xi_{j+1k+1} - \frac{1}{2} \lambda^2(\omega_{j+1} + \omega_{k+1}) |\xi_{jk}|^2 \right] \end{aligned}$$

Rearranging the terms, we get the desired expression. □

**Proposition 5** If  $W = \{\xi \in D(L) : \mathcal{E}(\xi) = 0\}$  then

$$\operatorname{span}\{\rho^{1/2}\} = \iota(\operatorname{Ker} \mathcal{L}) = \operatorname{Ker} L = W.$$

**Proof** Since  $\mathbb{1} \in \operatorname{Ker} \mathcal{L}$ ,  $\rho^{1/2} = \iota(\mathbb{1}) \in \operatorname{Ker} L$ . It is obvious that  $\operatorname{Ker} L \subset W$ . Suppose  $\xi = \sum_{j,k} \xi_{jk} |e_j\rangle\langle e_k| \in W$ . By Lemma 2

$$\xi_{j+1k+1} = v \sqrt{\frac{\omega_{k+1}}{\omega_{j+1}}} \xi_{jk} = v \sqrt{\frac{\omega_{j+1}}{\omega_{k+1}}} \xi_{jk} \tag{14}$$

and

$$\xi_{0k} = \xi_{k0} = 0, \quad \text{for } k \geq 1.$$

Thus  $\xi$  must be diagonal and by (14),  $\xi_{k+1k+1} = v\xi_{kk}$ . Hence for  $k \geq 0$ ,  $\xi_{kk} = v^k \xi_{00}$ , i.e.  $\xi = \operatorname{const} \cdot \rho^{1/2}$ . This completes the proof. □

Like the model of two-photon absorption and emission process discussed in [8], there are invariant subspaces for the process of quadratic open quantum harmonic oscillator. For  $m \in \mathbb{Z}$ , define

$$\mathcal{G}_m := \overline{\operatorname{span}\{|e_k\rangle\langle e_{k+m}| : k \geq \max\{0, -m\}\}}.$$

One can easily check the following properties.

- (1)  $\xi \in \mathcal{G}_m \Leftrightarrow \xi^* \in \mathcal{G}_{-m}$ .
- (2) The linear spaces  $\mathcal{G}_m$  are orthogonal in  $L_2(\mathfrak{h})$ , and

$$L_2(\mathfrak{h}) = \oplus\{\mathcal{G}_m : m \in \mathbb{Z}\}.$$

- (3) Each  $\mathcal{G}_m$  is invariant for the generator  $L$  and so also for the semigroup  $T$ .
- (4) Each  $\mathcal{G}_m$  is isometrically isomorphic to the space  $l^2(\mathbb{N})$  of square summable sequences.

By mimicking the proof of [8, Proposition 4], we can show the following.

**Proposition 6**

$$\text{gap } L = \inf_{m \geq 0} A_m,$$

$$\text{where } A_m = \begin{cases} \inf\{\mathcal{E}(\xi) : \|\xi\| = 1, \xi \in \mathcal{G}_m\}, & \text{for } m \neq 0 \\ \inf\{\mathcal{E}(\xi) : \|\xi\| = 1, \xi \in \mathcal{G}_0, \xi \perp \rho^{1/2}\}, & \text{for } m = 0 \end{cases}$$

We will now study separately off-diagonal and diagonal minima.

**6.2 Off-diagonal minimum problems**

Fix  $m > 0$ . For  $\xi \in \mathcal{G}_m$ , we can write for some sequence  $y = (y_j)_{j \geq 0}$  in  $l^2(\mathbb{N})$ ,

$$\xi = \sum_j y_j |e_j\rangle \langle e_{j+m}|,$$

where  $y_j = \xi_{j, j+m}$ . Then

$$\begin{aligned} \mathcal{E}(\xi) = & \frac{1}{2} \mu^2 \omega_m |y_0|^2 + \frac{1}{2} \sum_j \left( |\mu \sqrt{\omega_{j+1}} y_{j+1} - \lambda \sqrt{\omega_{j+m+1}} y_j|^2 \right. \\ & \left. + |\mu \sqrt{\omega_{j+m+1}} y_{j+1} - \lambda \sqrt{\omega_{j+1}} y_j|^2 \right). \end{aligned} \tag{15}$$

**Proposition 7** For any  $m \geq 1$ ,

$$A_m \geq \frac{\mu^2 m}{2} \left( (m+r-1) + v^2(m-r+1) \right) \tag{16}$$

The lower bound is attained by  $\mathcal{E}(\xi)$  for

$$y_j^2 = v^{2(j+1)} \binom{j+m}{m} \frac{\Gamma(m+r)\Gamma(j+r)}{\Gamma(r)\Gamma(j+m+r)} y_0^2$$

and  $y_0 \geq 0$  such that  $\|\xi\|^2 = 1$ .

**Proof** We fix  $m \geq 1$  and  $\xi \in \mathcal{G}_m$ . From formula (15), without loss of generality we may assume  $y_j \geq 0$  for all  $j \geq 0$  because  $|a - b| \geq ||a| - |b||$  for any complex numbers  $a$  and  $b$ . Then we can rewrite

$$\begin{aligned} \mathcal{E}(\xi) = & \frac{1}{2} \mu^2 \omega_m y_0^2 + \frac{1}{2} \sum_{j \geq 0} \left( (\omega_{j+1} + \omega_{j+m+1}) (\mu^2 y_{j+1}^2 + \lambda^2 y_j^2) \right. \\ & \left. - 4\mu\lambda \sqrt{\omega_{j+1}\omega_{j+m+1}} y_{j+1} y_j \right). \end{aligned} \tag{17}$$

Since

$$2\lambda\mu y_{j+1}y_j \leq \mu^2\theta_j y_{j+1}^2 + \frac{\lambda^2}{\theta_j} y_j^2 \quad (18)$$

where  $\theta_j > 0$  is such that

$$\omega_{j+m+1} + \omega_{j+1} - 2\theta_j \sqrt{\omega_{j+m+1}\omega_{j+1}} = \omega_m$$

i.e.

$$\theta_j = \frac{\omega_{j+m+1} + \omega_{j+1} - \omega_m}{2\sqrt{\omega_{j+m+1}\omega_{j+1}}}.$$

In this way, we get the inequality

$$\begin{aligned} \mathcal{E}(\xi) &\geq \frac{\mu^2}{2} \omega_m y_0^2 + \frac{\mu^2}{2} \sum_{j \geq 0} \omega_m y_{j+1}^2 \\ &\quad + \frac{\lambda^2}{2} \sum_{j \geq 0} \left( \omega_{j+m+1} + \omega_{j+1} - \frac{2\sqrt{\omega_{j+m+1}\omega_{j+1}}}{\theta_j} \right) y_j^2 \\ &= \frac{\mu^2}{2} \omega_m \|y\|^2 + \frac{\lambda^2}{2} \sum_{j \geq 0} \left( \omega_{j+m+1} + \omega_{j+1} - \frac{4\omega_{j+m+1}\omega_{j+1}}{\omega_{j+m+1} + \omega_{j+1} - \omega_m} \right) y_j^2 \\ &= \frac{\mu^2}{2} \omega_m \|y\|^2 + \frac{\lambda^2}{2} \sum_{j \geq 0} \frac{m(m-r+1)(j+1)(j+m+r)}{(1+j)(j+m+r)} y_j^2 \\ &= \frac{\mu^2}{2} \omega_m \|y\|^2 + \frac{\lambda^2}{2} \sum_{j \geq 0} m(m-r+1) y_j^2. \end{aligned}$$

It follows that

$$A_m \geq \frac{\mu^2 \omega_m}{2} + \frac{\lambda^2 m(m-r+1)}{2} = \frac{\mu^2 m}{2} \left( (m+r-1) + \nu^2(m-r+1) \right)$$

The above lower bound for the Dirichlet form is attained if and only if the Schwarz inequalities (18) are equalities namely  $\mu\theta_j^{1/2} y_{j+1} = \lambda\theta_j^{-1/2} y_j$  so that

$$y_{j+1} = \frac{\nu}{\theta_j} y_j$$

for all  $j \geq 1$ . Iterating we find

$$y_{j+1} = \frac{\nu^{j+1}}{\theta_j \dots \theta_0} y_0 = \nu^{j+1} y_0 \prod_{i=0}^j \frac{2\sqrt{\omega_{i+m+1}\omega_{i+1}}}{\omega_{i+m+1} + \omega_{i+1} - \omega_m}.$$



Since

$$\lim_{j \rightarrow \infty} \frac{y_{j+1}^2}{y_j^2} = v^2 \lim_{j \rightarrow \infty} \frac{4\omega_{j+m+1}\omega_{j+1}}{(\omega_{j+m+1} + \omega_{j+1} - \omega_m)^2} = v^2 < 1,$$

we find  $\sum_{j \geq 0} y_j^2 < \infty$  and so the lower bound is a minimum. Minimizers can be written in an explicit form. First note that

$$y_{j+1}^2 = v^2 \frac{4(j+m+1)(j+m+r)(j+1)(j+r)}{4(j+1)^2(j+m+r)^2} y_j^2 = v^2 \frac{(j+m+1)(j+r)}{(j+m+r)(j+1)} y_j^2.$$

Iterating

$$\begin{aligned} y_{j+1}^2 &= v^{2(j+1)} \frac{(j+m+1) \dots (m+1)(j+r) \dots r}{(j+m+r) \dots (m+r)(j+1)!} y_0^2 \\ &= v^{2(j+1)} \frac{(j+m+1)!}{m!(j+1)!} \frac{(j+r) \dots r}{(j+m+r) \dots (j+1+r)} y_0^2 \\ &= v^{2(j+1)} \binom{j+m+1}{m} \frac{\Gamma(m+r)\Gamma(j+r+1)}{\Gamma(r)\Gamma(j+m+r+1)} y_0^2. \end{aligned}$$

□

The function  $m \rightarrow (m+r-1) + v^2(m-r+1)$  is a positive and increasing function of  $m$  for  $m \geq 1$  and so we have

**Theorem 4** *The off-diagonal gap is*

$$\inf_{m \geq 1} \frac{\mu^2 m}{2} \left( (m+r-1) + v^2(m-r+1) \right) = \frac{\mu^2}{2} \left( 2v^2 + (1-v^2)r \right).$$

### 6.3 Diagonal minima

For any  $f$  in  $l^2(\mathbb{N})$ , let us denote by  $M_f \in \mathcal{B}(\mathfrak{h})$  the multiplication operator by  $f$ ,  $M_f = \sum_j f_j |e_j\rangle\langle e_j|$ . Then we get

$$\mathcal{L}(M_f) = M_{\mathcal{A}f} \text{ for } f \in D(\mathcal{A}) := \{f \in l^\infty : M_f \in D(\mathcal{L})\}.$$

The (formal) explicit expression for  $\mathcal{A}$  is given by

$$\mathcal{A}f = \sum_j \left( \lambda^2 \omega_{j+1}(f_{j+1} - f_j) + \mu^2 \omega_j(f_{j-1} - f_j) \right) e_j.$$

The invariant measure for this classical birth and death process is

$$\pi = (\pi_u)_{u \geq 0} \text{ with } \pi_u = (1 - v^2)v^{2u}.$$

From now on, whenever there is no confusion we write simply  $f$  for  $M_f \in \mathcal{B}(\mathfrak{h})$ . By Lemma 2, we can compute

$$\mathcal{E}(\pi^{1/2} f) = \sum_j \lambda^2 \omega_{j+1} |f_{j+1} - f_j|^2 \pi_j.$$

**Proposition 8** *For any positive sequence  $(a_n)_n$ , define the (strictly positive) constant*

$$B(v) := \sup_{u \geq 0} \frac{(\mu^2(1 - v^2))^{-1}}{a_u \omega_{u+1}} \left\{ \sum_{v \leq u} (1 - v^{2(v+1)}) a_v + (v^{-2(u+1)} - 1) \sum_{v > u} v^{2(v+1)} a_v \right\}.$$

Then  $\text{gap}(\mathcal{A}) \geq B(v)^{-1}$ .

**Proof** We follow the proof of [8, Proposition 7] with a change  $(2u + 1)(2u + 2)$  by  $\omega_{u+1}$ , which amounts to consider the birth rate  $b_j = \lambda^2 \omega_{j+1}$  instead of  $b_j = \lambda^2(2j + 1)(2j + 2)$ . □

The following was proven in [8, Lemma 8].

**Lemma 3** *Take a positive summable sequence  $(a_n)_{n \geq 0}$  and define the positive decreasing tail sequence  $(A_k)_{k \geq 0}$  by  $A_k = \sum_{n \geq k} a_n$ . Then*

$$B(v) = \sup_{u \geq 0} \frac{1}{\mu^2 a_u \omega_{u+1}} \sum_{k \geq 0} v^{2k} (A_k - A_{k+u+1}).$$

Thus the computation of the spectral gap relies on how we choose the sequence  $(a_n)_n$ . Here we propose the following choice. (cf. [8, Lemma 9])

**Lemma 4** *Let  $a_n = \frac{1}{(n+r+1)(n+r)}$ . Then the following properties hold.*

- (i)  $A_k = \sum_{n \geq k} a_n = \frac{1}{k+r}$ .
- (ii)  $\sum_{k \geq 0} v^{2k} a_k = 2v^{-2(r+1)} \int_0^v \frac{s^{2r-1}}{1-s^2} (v^2 - s^2) ds$ .
- (iii)  $\sum_{k \geq 0} v^{2k} A_k = 2v^{-2r} \int_0^v \frac{s^{2r-1}}{1-s^2} ds$ .

**Proof** The item (i) is trivial. To prove (ii), we write

$$\sum_{k \geq 0} v^{2k} a_k = \sum_{k \geq 0} v^{2k} \left( \frac{1}{k+r} - \frac{1}{k+r+1} \right).$$

Let

$$a(v) := \sum_{k \geq 0} v^{2k} \frac{1}{k+r}.$$

Then,  $b(v) := a(v)v^{2r}$  satisfies

$$b'(v) = 2 \frac{v^{2r-1}}{1 - v^2}.$$

Since  $b(0) = 0$ , we get

$$b(v) = 2 \int_0^v \frac{s^{2r-1}}{1 - s^2} ds.$$

From this, we easily get the desired expression. For (iii), we see by (i) that the value we are looking for is  $a(v)$ , which we computed in the above.  $\square$

We can now find a lower bound for the diagonal minimum of the Dirichlet form on vectors  $\xi$  orthogonal to  $\rho^{1/2}$ .

**Theorem 5**

$$\text{gap}(\mathcal{A}) \geq \mu^2 \left( \sum_{k \geq 0} v^{2k} \frac{1}{k+r} \right)^{-1} = \mu^2 v^{2r} \left( 2 \int_0^v \frac{s^{2r-1}}{1 - s^2} ds \right)^{-1}.$$

**Proof** We choose the sequence  $(a_n)_n$  as in Lemma 4. Let us define a function  $V : \mathbb{N} \rightarrow \mathbb{R}$  by

$$\begin{aligned} V(u) &:= \frac{1}{a_u \omega_{u+1}} \sum_{k \geq 0} v^{2k} (A_k - A_{k+u+1}) \\ &= \frac{u+r+1}{u+1} \sum_{k \geq 0} v^{2k} \left( \frac{1}{k+r} - \frac{1}{k+u+1+r} \right). \end{aligned}$$

We can see that the function  $V(u)$  is increasing. In fact, regarding  $V(u)$  as a function on the interval  $[0, \infty)$ , we differentiate it. With a little computation we see that

$$V'(u) = \sum_{k \geq 0} v^{2k} \frac{k}{(k+u+1+r)^2(k+r)} > 0.$$

Therefore we get by using Lemma 4 (iii)

$$\mu^2 B(v) = \sup_{u \geq 0} V(u) = \sum_{k \geq 0} v^{2k} \frac{1}{k+r} = 2v^{-2r} \int_0^v \frac{s^{2r-1}}{1 - s^2} ds.$$

The result now follows by Proposition 8.  $\square$

**Remark** The lower bound of theorem 5 can be written in a closed form by introducing the Lerch function  $\Phi : \{z \in \mathbb{C} \mid |z| < 1\} \times \mathbb{N} \times ]0, +\infty[ \rightarrow \mathbb{C}$

$$\Phi(z, n, r) = \sum_{k \geq 0} \frac{z^k}{(k+r)^n}.$$

By Theorem 5, we have  $\text{gap}(\mathcal{A}) \geq \mu^2 (\Phi(v^2, 1, r))^{-1}$ . Moreover, by comparing the off-diagonal explicit minimum and the diagonal lower bound we find the following.

**Theorem 6** For all  $v, r$  such that  $(2v^2 + r(1 - v^2)) \Phi(v^2, 1, r) \leq 2$  we have

$$\text{gap } L = \frac{\mu^2}{2} (2v^2 + r(1 - v^2)). \tag{19}$$

In particular, if  $r \geq 2v^2/(1 - v^2)$  the above identity holds.

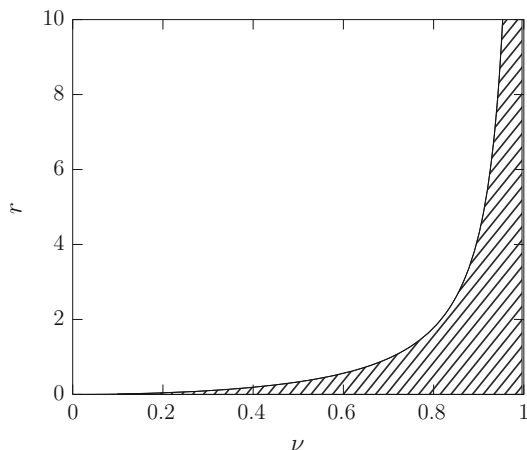
**Proof** The first claim follows immediately by comparing the diagonal lower bound of Theorem 5 and the off-diagonal minimum of Theorem 4. By the elementary inequality,

$$\sum_{k=0}^{\infty} \frac{v^{2k}}{k+r} = \frac{1}{r} \sum_{k=0}^{\infty} \frac{rv^{2k}}{k+r} \leq \frac{1}{r} \sum_{k=0}^{\infty} v^{2k} = (r(1 - v^2))^{-1}$$

we have  $\Phi(v^2, 1, r) \leq (r(1 - v^2))^{-1}$  and so identity (19) holds, in particular, if  $(r(1 - v^2))^{-1} (2v^2 + r(1 - v^2)) \leq 2$ , i.e.  $r \geq 2v^2/(1 - v^2)$ . □

The graph in Fig. 1 shows the values of  $v, r$  for which the identity  $(2v^2 + r(1 - v^2)) \Phi(v^2, 1, r) = 2$  holds. Clearly, for pairs  $(v, r)$  lying above (or on) the graph the spectral gap is given by (19).

**Fig. 1**  $\text{gap}(L) = \mu^2(2v^2 + r(1 - v^2))/2$ , off-diagonal minimum, for  $(v, r)$  above the curve



The exact value of the spectral gap, for pairs  $(v, r)$  lying below the graph, could be also the diagonal minimum whose exact value is not known and we are unable to compute. This will be clear studying upper bounds for the diagonal spectral gap.

### 6.4 Upper bound

In this section, we discuss the upper bound of the diagonal spectral gap. By definition, any value  $\mathcal{E}(\xi)$  with  $\|\xi\| = 1$ , with  $\xi$  orthogonal to  $\rho^{1/2}$  is an upper bound for the gap. Thus to get a good upper bound we need to choose a vector  $\xi$  cleverly.

Looking at the explicit form of the off-diagonal minimizer that we get for  $m = 1$ , we consider the first order polynomial  $f_j = j - c$  where  $c$  is a constant chosen in such a way that  $j \rightarrow \pi_j^{1/2} f_j$  is orthogonal to  $\pi_j^{1/2}$ , i.e.  $\sum_j \pi_j (j - c) = 0$ . It is worth noticing here that this choice yields the desired minimizer of the Dirichlet form on vectors orthogonal to  $\rho^{1/2}$  for the usual harmonic oscillator (see [7]). Using the identities

$$(1 - v^2) \sum_{k \geq 0} k v^{2k} = \frac{v^2}{1 - v^2}, \quad (1 - v^2) \sum_{k \geq 0} k^2 v^{2k} = \frac{v^2(1 + v^2)}{(1 - v^2)^2},$$

one computes  $c = v^2/(1 - v^2)$ . Then, considering  $\xi_j = \pi_j^{1/2}(j - c)$ , one finds

$$\|\xi\|^2 = (1 - v^2) \sum_{k \geq 0} (k - c)^2 v^{2k} = \frac{v^2}{(1 - v^2)^2}$$

$$\mathcal{E}(\xi) = \lambda^2 \sum_{j \geq 0} (f_{j+1} - f_j)^2 (j + 1)(j + r) \pi_j = \lambda^2 \frac{2v^2 + r(1 - v^2)}{(1 - v^2)^2}$$

and so

$$\frac{\mathcal{E}(\xi)}{\|\xi\|^2} = \mu^2(2v^2 + r(1 - v^2)).$$

As a consequence, one has the explicit upper bound

$$\text{gap}(L) \leq \mu^2(2v^2 + r(1 - v^2))$$

which is twice the spectral gap in good cases by Theorem 6.

In order to show that the spectral gap converges to 0 as  $r \rightarrow 0$ , we find another upper bound computing the value of the Dirichlet form for another vector  $\xi$  suggested by our choice of the sequence  $(A_k)_k$  in Lemma 4.

#### Theorem 7

$$\text{gap } L \leq \lambda^2 \frac{\sum_j \frac{j+1}{(j+r+1)^2(j+r)} v^{2j}}{\sum_j \left(\frac{1}{j+r} - c(v, r)\right)^2 v^{2j}},$$

where  $c(v, r) = (1 - v^2) \sum_{j=0}^{\infty} \frac{v^{2j}}{j+r} = (1 - v^2)\Phi(v^2, 1, r)$ . In particular, for all  $v > 0$  fixed, gap  $L$  tends to 0 as  $r \rightarrow 0$ .

**Proof** We consider  $A_k = (k + r)^{-1}$  as in Lemma 4, and define  $f_k = A_k - c = (k + r)^{-1} - c(v, r)$ , where the constant  $c$  is such that  $\sum_j \pi_j f_j = 0$ . We compute

$$\begin{aligned} \mathcal{E}(\pi^{1/2}h) &= \sum_{j \geq 0} \lambda^2 \omega_{j+1} |f_{j+1} - f_j|^2 \pi_j \\ &= (1 - v^2) \lambda^2 \sum_{j \geq 0} \frac{j+1}{(j+r+1)^2(j+r)} v^{2j}. \end{aligned}$$

On the other hand,  $\|f\|_{\pi}^2 = (1 - v^2) \sum_{j \geq 0} ((j+r)^{-1} - c(v, r))^2 v^{2j}$ . Therefore,

$$\mathcal{E}(\xi)/\|\xi\|^2 \leq \lambda^2 \frac{\sum_j \frac{j+1}{(j+r+1)^2(j+r)} v^{2j}}{\sum_j \left(\frac{1}{j+r} - c\right)^2 v^{2j}} \quad (20)$$

We now check that the above upper bound tends to 0 as  $r \rightarrow 0$ . To this end, note that we can write the denominator as

$$\begin{aligned} &\sum_{j \geq 0} \frac{v^{2j}}{(j+r)^2} - (1 - v^2) \left( \sum_{j \geq 0} \frac{v^{2j}}{j+r} \right)^2 \\ &= \frac{1}{r^2} \left( 1 + \sum_{j \geq 1} \frac{r^2 v^{2j}}{(j+r)^2} \right) + (1 - v^2) \left( \frac{1}{r} + \sum_{j \geq 1} \frac{v^{2j}}{j+r} \right)^2 \end{aligned}$$

It follows that, if we multiply the denominator by  $r^2$  in the limit as  $r \rightarrow 0$  we get  $1 - (1 - v^2) = v^2$ . On the other hand, if we multiply also the numerator by  $r^2$  we get

$$\lim_{r \rightarrow 0} \sum_{j \geq 0} \frac{r^2(j+1)v^{2j}}{(j+r+1)^2(j+r)} \leq \lim_{r \rightarrow 0} \sum_{j \geq 0} \frac{r v^{2j}}{(j+r+1)} \leq \lim_{r \rightarrow 0} r \sum_{j \geq 0} \frac{v^{2j}}{(j+1)} = 0.$$

This completes the proof.  $\square$

## 7 Conclusion

Theorem 7 shows that, for  $r$  near 0, gap  $L$  is the diagonal gap whose exact value is not known. Moreover, it tends to 0 as  $r \rightarrow 0$ . The exact result of Theorem 6 holds in the white region above the graph in Fig. 1. It is worth noticing here that the range of values of  $r$  for which our exact result holds becomes closer and closer to the half-line  $(0, +\infty)$  as  $v^2 = e^{-s\beta}$  goes to 0, namely the inverse temperature  $\beta$  goes to  $+\infty$ . This

confirms the intuition that quantum (off-diagonal) effects prevail over the classical (diagonal) ones when the temperature is small.

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## References

1. Accardi, L., Dhahri, A.: Quadratic exponential vectors. *J. Math. Phys.* **50**, 122103 (2009)
2. Accardi, L., Dhahri, A.: The quadratic Fock functor. *J. Math. Phys.* **51**, 022105 (2010)
3. Accardi, L., Fagnola, F., Quezada, R.: On three new principles in non-equilibrium statistical mechanics and Markov semigroups of weak coupling limit type. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **19**, 1650009 (2016)
4. Carbone, R., Franz, U., Skeide, M.: Renormalized squares of white noise and other non-Gaussian noises as Lévy processes on real Lie algebras. *Commun. Math. Phys.* **228**, 123–150 (2002)
5. Accardi, L., Lu, Y.G., Volovich, I.: *Quantum Theory and Its Stochastic Limit*. Springer, Berlin (2002)
6. Atakishiyev, M.N., Atakishiyev, N.M.: On  $su_q(1, 1)$  models of quantum oscillator. *J. Math. Phys.* **47**, 093502 (2006)
7. Carbone, R., Fagnola, F.: Exponential  $L_2$ -convergence of quantum Markov semigroup on  $\mathcal{B}(h)$ . *Math. Notes* **68**(4), 452–463 (2000)
8. Carbone, R., Fagnola, F., García, J.C., Quezada, R.: Spectral properties of the two-photon absorption and emission process. *J. Math. Phys.* **49**(3), 32106 (2008)
9. Dhahri, A., Fagnola, F., Rebolledo, R.: The decoherence-free subalgebra of a quantum Markov semigroup with unbounded generator. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13**(3), 413–433 (2010)
10. Dhahri, A.: On the quadratic Fock functor. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **15**(2), 1250012 (2012)
11. Dhahri, A.: Self-adjointness and boundedness in quadratic quantization. *J. Math. Phys.* **55**, 052103 (2014)
12. Fagnola, F.: Quantum Markov semigroups and quantum Markov flows. *Proyecciones* **18**(3), 1–144 (1999)
13. Fagnola, F., Rebolledo, R.: Lectures on the qualitative analysis of quantum Markov semigroups. In: *Quantum Interacting Particle Systems*, pp. 197–239. Trento (2000). QP–PQ: Quantum Probability and White Noise Analysis, vol. 14. World Scientific Publishing, River Edge, NJ (2002)
14. Fagnola, F., Rebolledo, R.: Subharmonic projections for a quantum Markov semigroup. *J. Math. Phys.* **43**(2), 1074–1082 (2002)
15. Fagnola, F., Rebolledo, R.: Transience and recurrence of quantum Markov semigroups. *Probab. Theory Relat. Fields* **126**(2), 289–306 (2003)
16. Fagnola, F., Quezada, R.: Two-photon absorption and emission process. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8**(4), 573–591 (2005)
17. Fagnola, F., Umanità, V.: Generators of KMS symmetric Markov semigroups on  $\mathcal{B}(h)$  symmetry and quantum detailed balance. *Commun. Math. Phys.* **298**, 523–547 (2010)
18. Frigerio, A.: Quantum dynamical semigroups and approach to equilibrium. *Lett. Math. Phys.* **2**, 79–87 (1977)
19. García, J.C., Quezada, R.: Hille–Yosida estimate and nonconservativity criteria for quantum dynamical semigroups. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7**(3), 383–394 (2004)
20. Hachicha, S., Nasroui, I., Gliouez, S.: The support projection of state of QMS with unbounded generator and a quantum extension of the classical Lévy–Austin–Ornstein theorem. *Open. Syst. Inf. Dyn.*
21. Isar, A., Scheid, W.: Uncertainty functions of the open quantum harmonic oscillator in the Lindblad theory. *Phys. Rev. A* **66**, 042117 (2002)
22. Karlin, S., McGregor, J.: The classification of birth and death processes. *Trans. Am. Math. Soc.* **86**(2), 366–400 (1957)

23. Liggett, T.: Exponential  $L_2$  convergence of attractive reversible nearest particle systems. *Ann. Prob.* **17**(2), 403–432 (1989)
24. Pazy, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New-York (1983)
25. Simon, M., Valent, G.: Spectral analysis of birth-and-death processes with and without killing via perturbation. *Methods Appl. Anal.* **16**(1), 055–068 (2009)
26. Śniady, P.: Quadratic bosonic and free white noises. *Commun. Math. Phys.* **211**(3), 615–628 (2000)

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