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Bannai–Ito algebras and the universal R-matrix of $\mathfrak{osp}(1|2)$

Nicolas Crampé[1] · **Luc Vinet**[2] · **Meri Zaimi**[2]

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Abstract

The Bannai–Ito algebra $BI(n)$ is viewed as the centralizer of the action of $\mathfrak{osp}(1|2)$ in the n-fold tensor product of the universal algebra of this Lie superalgebra. The generators of this centralizer are constructed with the help of the universal R-matrix of $\mathfrak{osp}(1|2)$. The specific structure of the $\mathfrak{osp}(1|2)$ embeddings to which the centralizing elements are attached as Casimir elements is explained. With the generators defined, the structure relations of $BI(n)$ are derived from those of $BI(3)$ by repeated action of the coproduct and using properties of the R-matrix and of the generators of the symmetric group $\mathfrak{S}_n$.

Keywords Centralizer · Super-algebra · Bannai-Ito algebra · Universal R-matrix

Mathematics Subject Classification 15A72 · 16T05

1 Introduction

This paper explains the essential role that the universal R-matrix of $\mathfrak{osp}(1|2)$ plays in the algebraic underpinnings of the Bannai–Ito algebra $BI(3)$ and its higher-rank generalization $BI(n)$.

The universal Bannai–Ito algebra $BI(3)$ is generated by the central elements $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_{123}$ and three generators $\mathcal{C}_{12}$, $\mathcal{C}_{23}$ and $\mathcal{C}_{13}$ satisfying the defining relations [4]

$$\{\mathcal{C}_{12}, \mathcal{C}_{23}\} = 2(-\mathcal{C}_{13} + \mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_2\mathcal{C}_{123}), \tag{1a}$$

$$\{\mathcal{C}_{12}, \mathcal{C}_{13}\} = 2(-\mathcal{C}_{23} + \mathcal{C}_2\mathcal{C}_3 + \mathcal{C}_1\mathcal{C}_{123}), \tag{1b}$$

✉ Nicolas Crampé
crampe1977@gmail.com

1 Institut Denis-Poisson CNRS/UMR 7013, Université de Tours - Université d'Orléans, Parc de Grammont, 37200 Tours, France

2 Centre de recherches mathématiques, Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal, Québec H3C 3J7, Canada

$$\{\mathcal{C}_{23}, \mathcal{C}_{13}\} = 2(-\mathcal{C}_{12} + \mathcal{C}_1\mathcal{C}_2 + \mathcal{C}_3\mathcal{C}_{123}), \quad (1c)$$

where $\{X, Y\} = XY + YX$.

The algebra $BI(3)$ was first introduced in [15] as an encoding of the bispectral properties of the eponym orthogonal polynomials [1]. In this context, the generators $\mathcal{C}_{12}$ and $\mathcal{C}_{23}$ are realized by the Dunkl shift operators of which the Bannai–Ito polynomials are eigenfunctions and the operator multiplication by the argument of those polynomials. In this representation, the central terms $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{123})$ become constants related to the four parameters of the polynomials.

The centrally extended $BI(3)$ was subsequently defined in [9] following the observation that the Bannai–Ito polynomials are essentially the Racah coefficients of the Lie superalgebra $\mathfrak{osp}(1|2)$. This casts $BI(3)$ as the centralizer of the action of $\mathfrak{osp}(1|2)$ in the threefold tensor product $U(\mathfrak{osp}(1|2))^{\otimes 3}$ where $U(\mathfrak{osp}(1|2))$ stands for the universal enveloping algebra of $\mathfrak{osp}(1|2)$. The generators $\mathcal{C}_{12}, \mathcal{C}_{23}, \mathcal{C}_{13}$ are then mapped to the Casimir elements attached to embeddings of $\mathfrak{osp}(1|2)$ into $\mathfrak{osp}(1|2)^{\otimes 3}$ which are indexed by the 2-element subsets of $\{1, 2, 3\}$. This paved the way to the construction of the extension $BI(n)$ of arbitrary rank as the centralizer of the action of $\mathfrak{osp}(1|2)$ in the n-fold product $U(\mathfrak{osp}(1|2))^{\otimes n}$ with the generators identified as the Casimir elements associated with $\mathfrak{osp}(1|2)$ embeddings now labelled by subsets A of $[n] = \{1, 2, \ldots, n\}$. This was actually achieved using models of $\mathfrak{osp}(1|2)$ given in terms of Dirac–Dunkl operators [6,7]. For reviews of these algebras and some of their applications, see [4,5].

A notable feature of these tensorial constructs is the fact that the embeddings involved do not all correspond to the simple ones where only the factors of the n-fold product that are enumerated by the elements the sets A enter non-trivially. The proper Casimir elements are in some cases associated with modified embeddings. Sorting this out is addressed here. It will be shown that conjugations of the simple embeddings by the universal R-matrix will in general be required to ensure that the attached Casimir elements belong to the centralizer.

Throughout this paper, we shall use a presentation of $\mathfrak{osp}(1|2)$ that calls upon a grading involution P. This P is group-like under the coproduct, and when this framework is used, it enters in the formula for the universal R-matrix. In a separate study [8], expressions for the centralizing elements of $\mathfrak{osp}(1|2)$ have been provided in situations where the grade involution admits a refinement as a product of supplementary involutions. This is manifestly the case under embeddings in tensor products. The centralizing elements thus given have been shown in [8] to coincide with the Casimir's of the modified embeddings. That this should be so will be made clear in the following.

The description of the Bannai–Ito algebra in the framework of the universal R-matrix of $\mathfrak{osp}(1|2)$ has the striking benefit of allowing to fully characterize abstractly $BI(n)$ for arbitrary n (in the centralizer view) without recourse to any model. As shall be shown, the centralizing elements associated with subsets A of $[n] = \{1, 2, \ldots, n\}$ are given through repeated action of the coproduct on $\mathfrak{osp}(1|2)$ Casimir elements and conjugation by products of braided universal R-matrices. With these generators in hand, the structure relations that they verify can be inferred consistently from those of $BI(3)$ (i.e. (1a)–(1c)) by exploiting properties of the R-matrix and of the permutations of the symmetric group $\mathfrak{S}_n$. A definite picture for the generalized Bannai–Ito algebra

$BI(n)$ as the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$ is thus obtained. This approach based on the universal R-matrix has already contributed to the understanding of the Askey–Wilson algebra of rank 1 [2], and the advances presented here in the description of the Bannai–Ito algebra for $n > 3$ should show the way towards a complete picture of the higher-rank Askey–Wilson algebras.

This paper will proceed as follows. Section 2 will offer a short review of $\mathfrak{osp}(1|2)$ and will focus on the universal R-matrix of this Lie superalgebra. In Sect. 3, the centralizing elements of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$ will be given in terms of Casimir elements and the universal R-matrix will be shown to play a key role. The connection between that centralizer and $BI(3)$ will moreover be made. Section 4 will extend the results to $n > 3$ and derive the algebra homomorphism $BI(n) \to U(\mathfrak{osp}(1|2))^{\otimes n}$ making essential use of the universal R-matrix formalism. Short concluding remarks will follow in Sect. 5.

2 Properties of the Lie superalgebra $\mathfrak{osp}(1|2)$

2.1 The Lie superalgebra $\mathfrak{osp}(1|2)$

The superalgebra $\mathfrak{osp}(1|2)$ has two odd generators $F^{\pm}$ and three even generators H, $E^{\pm}$ satisfying the following (anti-)commutation relations [11]

$$[H, E^{\pm}] = \pm E^{\pm}, \quad [E^{+}, E^{-}] = 2H, \tag{2}$$

$$[H, F^{\pm}] = \pm\frac{1}{2}F^{\pm}, \quad \{F^{+}, F^{-}\} = \frac{1}{2}H, \tag{3}$$

$$[E^{\pm}, F^{\mp}] = -F^{\pm}, \quad \{F^{\pm}, F^{\pm}\} = \pm\frac{1}{2}E^{\pm}. \tag{4}$$

The $\mathbb{Z}_2$-grading of $\mathfrak{osp}(1|2)$ can be encoded by the grading involution P satisfying

$$[P, E^{\pm}] = 0, \quad [P, H] = 0, \quad \{P, F^{\pm}\} = 0 \quad \text{and} \quad P^2 = 1. \tag{5}$$

One defines the central element C of $U(\mathfrak{osp}(1|2))$ by [12,13]

$$C = 8[F^{+}, F^{-}]P + P. \tag{6}$$

The $U(\mathfrak{osp}(1|2))$ algebra is endowed with a coproduct Δ defined as the algebra homomorphism satisfying

$$\Delta(E^{\pm}) = E^{\pm} \otimes 1 + 1 \otimes E^{\pm}, \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \tag{7}$$

$$\Delta(F^{\pm}) = F^{\pm} \otimes P + 1 \otimes F^{\pm}, \quad \Delta(P) = P \otimes P. \tag{8}$$

We recall that this comultiplication is coassociative

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta\ . \tag{9}$$

2.2 The universal R-matrix of $\mathfrak{osp}(1|2)$

The universal R-matrix of $\mathfrak{osp}(1|2)$ is given by

$$\mathcal{R} = \frac{1}{2}(1 \otimes 1 + P \otimes 1 + 1 \otimes P - P \otimes P). \tag{10}$$

For $x \in U(\mathfrak{osp}(1|2))$, it satisfies

$$\Delta(x)\mathcal{R} = \mathcal{R}\Delta^{op}(x), \tag{11}$$

where the opposite comultiplication $\Delta^{op}(x) = x^{(2)} \otimes x^{(1)}$ if $\Delta(x) = x^{(1)} \otimes x^{(2)}$ in the Sweedler's notation. Let us note that

$$\mathcal{R}^2 = 1 \otimes 1, \quad \mathcal{R}_{21} = \mathcal{R}. \tag{12}$$

The universal R-matrix (10) satisfies

$$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13} \quad \text{and} \quad (\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{23}\mathcal{R}_{13}. \tag{13}$$

It also satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{14}$$

We remark that in the case of $\mathfrak{osp}(1|2)$, the universal R-matrix satisfies $[\mathcal{R}_{12}, \mathcal{R}_{13}] = 0$. However, we shall not use this property in the following so as to keep the computations performed in this paper more generic and applicable to situations involving algebras other than $\mathfrak{osp}(1|2)$.

3 The Bannai–Ito algebra as the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$

3.1 Centralizer of the diagonal action of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$

To identify $BI(3)$ as the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$, it is appropriate to first look for the centralizing elements $X \in U(\mathfrak{osp}(1|2))^{\otimes 3}$ such that

$$[(\Delta \otimes \text{id})\Delta(x), X] = 0 \quad \text{for } x \in \mathfrak{osp}(1|2). \tag{15}$$

It is straightforward to observe that the elements

$$C_1 = C \otimes 1 \otimes 1, \quad C_2 = 1 \otimes C \otimes 1, \quad C_3 = 1 \otimes 1 \otimes C \tag{16}$$

$$C_{12} = \Delta(C) \otimes 1, \quad C_{23} = 1 \otimes \Delta(C), \quad C_{123} = (\Delta \otimes \text{id})\Delta(C). \tag{17}$$

will be centralizing. Now let $\Delta(C) = C^{(1)} \otimes C^{(2)}$ in the Sweedler's notation and write

$$\overline{C}_{13} = C^{(1)} \otimes 1 \otimes C^{(2)}. \tag{18}$$

At first glance, one might think that $\overline{C}_{13}$ also belongs to the centralizer. It is the Casimir element corresponding to the simple homomorphism

$$\begin{aligned} \mathfrak{osp}(1|2) &\to \mathfrak{osp}(1|2)^{\otimes 3} \\ x &\mapsto x^{(1)} \otimes 1 \otimes x^{(2)} \end{aligned}$$

with $\Delta(x) = x^{(1)} \otimes x^{(2)}$. This, however, is not true and is where the universal R-matrix comes in.

Proposition 3.1 *The element*[1]

$$C_{13} = \mathcal{R}_{32}^{-1} \overline{C}_{13} \mathcal{R}_{32} \tag{19}$$

belongs to the centralizer of $\mathfrak{osp}(1|2)$ *in* $U(\mathfrak{osp}(1|2))^{\otimes 3}$.

Proof Since the Casimir element is central, we have for $x \in \mathfrak{osp}(1|2)$,

$$[(\Delta \otimes \mathrm{id})\Delta(x), \Delta(C) \otimes 1] = 0. \tag{20}$$

Using the coassociativity of the comultiplication (9) and conjugating by $\mathcal{R}_{23}$ transforms the previous relation into

$$[(\mathrm{id} \otimes \Delta^{op})\Delta(x), \mathcal{R}_{23}^{-1}(\Delta(C) \otimes 1)\mathcal{R}_{23}] = 0. \tag{21}$$

Finally, exchanging the spaces 2 and 3, one gets that C_{13} is in the centralizer

$$[(\mathrm{id} \otimes \Delta)\Delta(x), \mathcal{R}_{32}^{-1} \overline{C}_{13} \mathcal{R}_{32}] = [(\mathrm{id} \otimes \Delta)\Delta(x), C_{13}] = 0. \tag{22}$$

□

Let us emphasize that C_{13} is in the centralizer, whereas $\overline{C}_{13}$ is not. In particular, for $x = C$ in the previous relation, we get

$$[C_{123}, C_{13}] = 0. \tag{23}$$

There is the following alternative formula for C_{13}.

Proposition 3.2 *The element* C_{13} *is also given by*

$$C_{13} = \mathcal{R}_{12} \overline{C}_{13} \mathcal{R}_{12}^{-1}. \tag{24}$$

[1] In what follows we shall keep using the inverse of $\mathcal{R}$ even though $\mathcal{R}^{-1} = \mathcal{R}$ (for $\mathfrak{osp}(1|2)$) to make clear that conjugations are involved.

Proof From property (11), one gets

$$C_{13}=\mathcal{R}_{23}^{-1}\mathcal{R}_{13}\left(C^{(2)}\otimes 1\otimes C^{(1)}\right)\mathcal{R}_{13}^{-1}\mathcal{R}_{23}. \tag{25}$$

Using the Yang–Baxter equation (14) and equality (12), this relation becomes

$$C_{13}=\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}^{-1}\mathcal{R}_{12}^{-1}\left(C^{(2)}\otimes 1\otimes C^{(1)}\right)\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{13}^{-1}\mathcal{R}_{12}^{-1}. \tag{26}$$

Now, from (13), one deduces that $[\Delta(C)\otimes 1, (\Delta\otimes\mathrm{id})(\mathcal{R})]=[\Delta(C)\otimes 1, \mathcal{R}_{23}\mathcal{R}_{13}]=0$ and that $[\left(C^{(2)}\otimes 1\otimes C^{(1)}\right), \mathcal{R}_{12}\mathcal{R}_{23}]=0$. One then obtains

$$C_{13}=\mathcal{R}_{12}\mathcal{R}_{13}\left(C^{(2)}\otimes 1\otimes C^{(1)}\right)\mathcal{R}_{13}^{-1}\mathcal{R}_{12}^{-1} \tag{27}$$

which after using (11) again yields the desired result. □

At this point, we can introduce two maps $\hat{\tau}$ and $\check{\tau}$ from $U(\mathfrak{osp}(1|2))$ to $U(\mathfrak{osp}(1|2))^{\otimes 2}$ by

$$\hat{\tau}(x)=\mathcal{R}^{-1}(1\otimes x)\mathcal{R} \quad \text{and} \quad \check{\tau}(x)=\mathcal{R}^{-1}(x\otimes 1)\mathcal{R}. \tag{28}$$

Corollary 3.1 *The following relations hold in* $U(\mathfrak{osp}(1|2))^{\otimes 3}$

$$(id\otimes\hat{\tau})\Delta(C)=C_{13} \quad \textit{and} \quad (\check{\tau}\otimes id)\Delta(C)=C_{13}. \tag{29}$$

Proof These results follow directly from Propositions 3.1 and 3.2 and the fact that $\mathcal{R}=\mathcal{R}^{-1}$. □

Using the definitions (28) and the universal R-matrix (10), one gets

$$\hat{\tau}(P)=1\otimes P,\quad \hat{\tau}(F^{\pm})=P\otimes F^{\pm},\quad \hat{\tau}(E^{\pm})=I\otimes E^{\pm},\quad \hat{\tau}(H)=I\otimes H \tag{30}$$
$$\check{\tau}(P)=P\otimes 1,\quad \check{\tau}(F^{\pm})=F^{\pm}\otimes P,\quad \check{\tau}(E^{\pm})=E^{\pm}\otimes I,\quad \check{\tau}(H)=H\otimes I. \tag{31}$$

Either more abstractly with the help of Eqs. (13) or using the formulas above, one readily observes that $\hat{\tau}$ and $\check{\tau}$ define coactions, that is verify

$$(\mathrm{id}\otimes\hat{\tau})\hat{\tau}=(\Delta\otimes\mathrm{id})\hat{\tau} \quad (\check{\tau}\otimes\mathrm{id})\check{\tau}=(\mathrm{id}\otimes\Delta)\check{\tau}. \tag{32}$$

It hence follows that $(\mathrm{id}\otimes\hat{\tau})\Delta$ and $(\check{\tau}\otimes\mathrm{id})\Delta$ define two different homomorphisms of $U(\mathfrak{osp}(1|2))$ into $U(\mathfrak{osp}(1|2))^{\otimes 3}$ which yield for C the same image, namely:

$$C_{13}=\left(8[F^{+}\otimes P\otimes P+1\otimes 1\otimes F^{+}, F^{-}\otimes P\otimes P+1\otimes 1\otimes F^{-}]+1\right) P\otimes 1\otimes P. \tag{33}$$

This can be checked directly by applying both (id $\otimes\,\hat{\tau}$) and ($\check{\tau} \otimes$ id) to

$$\Delta(C) = 8\left([F^+ \otimes P + 1 \otimes F^+, F^- \otimes P + 1 \otimes F^-] + 1\right) P \otimes P \tag{34}$$
$$= 16\left(F^- \otimes F^+ - F^+ \otimes F^-\right)\,(P \otimes 1) + 8C \otimes P + P \otimes C - P \otimes P. \tag{35}$$

Note that

$$\hat{\tau}(C) = 1 \otimes C \quad \text{and} \quad \check{\tau}(C) = C \otimes 1. \tag{36}$$

We may hence pick the homomorphism given by $(\check{\tau} \otimes \mathrm{id})\Delta$ and identify the three embeddings labelled by the pairs (1, 2), (2, 3) and (1, 3) (see also [8]):

$$\begin{aligned} &H_{ij} = H_i + H_j, \quad E^{\pm}_{ij} = E^{\pm}_i + E^{\pm}_j, \quad i, j = 1, 2, 3,\\ &F^{\pm}_{12} = F^{\pm}_1 P_2 + F^{\pm}_2, \quad F^{\pm}_{23} = F^{\pm}_2 P_3 + F^{\pm}_3, \quad F^{\pm}_{13} = F^{\pm}_1 P_2 P_3 + F^{\pm}_3,\\ &P_{12} = P_1 P_2, \quad P_{23} = P_2 P_3, \quad P_{13} = P_1 P_3, \end{aligned} \tag{37}$$

with the subscripts denoting (as on the R-matrix) the factor in the tensor product where the element appears. The centralizing elements C_{ij} are then simply the Casimir element given by

$$C_{ij} = \left(8[F^+_{ij}, F^-_{ij}] + 1\right) P_{ij} \tag{38}$$

as is manifested in particular from (33) and we now understand the reasons for the choice of the (13) embedding. In this notation, we have

$$C_i = \left(8[F^+_i, F^-_i] + 1\right) P_i, \quad i = 1, 2, 3 \tag{39}$$
$$\text{and} \quad C_{123} = \left(8[F^+_{123}, F^-_{123}] + 1\right) P_1 P_2 P_3 \tag{40}$$
$$\text{with} \quad F^{\pm}_{123} = F^{\pm}_1 P_2 P_3 + F^{\pm}_2 P_3 + F^{\pm}_3. \tag{41}$$

3.2 The image of $BI(3)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$

We wish to identify the Bannai–Ito algebra $BI(3)$ with relations (1a)–(1c) by mapping its generators $\mathcal{C}$ with one, two and three indices onto the corresponding C. To that end, we need to obtain the relations between the Casimir elements C. Using the formulas (38), (39), (40), relation (1a) is readily verified under $\mathcal{C} \to C$.

Note that C_{13} could have been taken to be defined by (1a) assuming that the Bannai–Ito relations will be realized. (This is typically the approach.) Given that C_{12} and C_{23} are centralizing $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes 3}$, it then follows that C_{13} must also be in the centralizer. We have here adopted the view point of first identifying the centralizing elements and hence of first defining C_{13}, before obtaining the relations between the generators of the centralizers. Since the tensorial embedding is so far the only approach that has been designed to obtain the higher-rank generalization of the Bannai–Ito algebra, having these definitions of the centralizing elements proves essential in this respect.

Given the definitions of C_{12}, C_{23} and C_{13}, as already said, one directly checks that (1a) is satisfied. It is then seen, remarkably, that the remaining defining relations of the Bannai–Ito algebra are implied. One has

$$\{C_{12}, C_{23}\} = 2(-\mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1} + C_1C_3 + C_2C_{123}). \tag{42}$$

Interchanging the factors 1 and 2 yields

$$\{C_{21}, \overline{C}_{13}\} = 2(-\mathcal{R}_{21}C_{23}\mathcal{R}_{21}^{-1} + C_2C_3 + C_1C_{213}). \tag{43}$$

Mindful that $C_{21} = \Delta^{op}(C) \otimes 1$ and that $C_{213} = (\Delta^{op} \otimes 1)\Delta(C)$, upon conjugating with $\mathcal{R}_{21}^{-1} = \mathcal{R}_{12}$, we find

$$\{C_{12}, \mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1}\} = 2(-C_{23} + C_2C_3 + C_1C_{123}) \tag{44}$$

given that $\mathcal{R}\Delta^{op} = \Delta\mathcal{R}$. We thus recover (1b) from (1a). The defining relation (1c) is also obtained from (1a) in a similar fashion. In this case, one interchanges the factors 2 and 3 and makes use of the other expression for C_{13}, namely $C_{13} = \mathcal{R}_{23}^{-1}\overline{C}_{13}\mathcal{R}_{23}$.

In conclusion, given C_{12}, C_{23} and once C_{13} has been defined with the help of the universal R-matrix, it is a matter of calculation to obtain one relation between these centralizing elements and one sees thereafter that the other two defining relations of the Bannai–Ito algebra follow simply from the first one in the light of the properties of the generators and their connection to the R-matrix.

4 The higher-rank Bannai–Ito algebras

In this section, we shall take n be any positive integer and $[n] = \{1, 2, \ldots, n\}$. The higher-rank universal Bannai–Ito algebra $BI(n)$ is generated by $\mathcal{C}_A$ for $A \subset [n]$ (by convention $\mathcal{C}_\emptyset = 1$) and the following defining relations [7], for $A, B \subset [n]$,

$$\{\mathcal{C}_A, \mathcal{C}_B\} = 2(-\mathcal{C}_{(A\cup B)\setminus(A\cap B)} + \mathcal{C}_{A\setminus(A\cap B)}\mathcal{C}_{B\setminus(A\cap B)} + \mathcal{C}_{A\cap B}\mathcal{C}_{A\cup B})\,. \tag{45}$$

Let us remark that there is a factor (-2) between the generators used here and the ones of [7] which explains the apparent discrepancy between the defining relations. We shall give an image of $BI(n)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$. For that, we follow the same logic as before and study the centralizer of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$.

We define by induction $\Delta^{(k)} = (\text{id} \otimes \Delta^{(k-1)})\Delta$ with $\Delta^{(0)} = \text{id}$ which allows to define, for $1 \le k \le \ell \le n$,

$$C_{k,k+1,\ldots\ell} = 1^{\otimes(k-1)} \otimes \Delta^{(\ell-k)}(C) \otimes 1^{\otimes(n-\ell)}. \tag{46}$$

These elements commute with $\Delta^{(n-1)}(x)$ for $x \in \mathfrak{osp}(1|2)$. We thus obtain elements of the centralizer associated with each subset $K \subset [n]$ with successive integers. We want to find centralizing elements associated with each subset $A \subset [n]$ without restriction. Let $\mathfrak{S}_n$ be the permutation group of n objects generated by the transpositions $s_1, s_2, \dots, s_{n-1}$. For $s = s_{i_1}s_{i_2}\dots s_{i_p}$ some permutation of $\mathfrak{S}_n$ (we recall that any permutation can be written as a product of transpositions), we define the action γ_s on $X \in U(\mathfrak{osp}(1|2))^{\otimes n}$ by

$$\gamma_s(X) = \check{\mathcal{R}}_{i_1}\check{\mathcal{R}}_{i_2}\dots\check{\mathcal{R}}_{i_p}X(\check{\mathcal{R}}_{i_1}\check{\mathcal{R}}_{i_2}\dots\check{\mathcal{R}}_{i_p})^{-1}, \tag{47}$$

where

$$\check{\mathcal{R}}_i = \mathcal{R}_{i,i+1}\sigma_{i,i+1} \tag{48}$$

and $\sigma_{i,i+1}(x_1 \otimes \dots \otimes x_i \otimes x_{i+1} \otimes \dots \otimes x_n) = (x_1 \otimes \dots \otimes x_{i+1} \otimes x_i \otimes \dots \otimes x_n)\sigma_{i,i+1}$. Such a $\check{\mathcal{R}}_i$ is called braided universal R-matrix. It satisfies

$$\Delta(x)\check{\mathcal{R}} = \check{\mathcal{R}}\Delta(x), \tag{49}$$

and the braided Yang–Baxter equation

$$\check{\mathcal{R}}_i\check{\mathcal{R}}_{i+1}\check{\mathcal{R}}_i = \check{\mathcal{R}}_{i+1}\check{\mathcal{R}}_i\check{\mathcal{R}}_{i+1}. \tag{50}$$

Let us emphasize that the definition of γ_s does not depend on the choice of the decomposition of the permutation s in terms of the transpositions since the $\check{R}_i$ and the s_i satisfy the same algebra.

We define the intermediate Casimir element associated with any subset $A \subset [n]$ as follows

$$C_A = C_{s(K)} = \gamma_s(C_K) \tag{51}$$

where $K \subset [n]$ with successive integers, C_K is defined by (46), and $s \in \mathfrak{S}_n$ is chosen such that

$$s(K) = s(\{K_1, K_2, \dots, K_k\}) = \{s(K_1), s(K_2), \dots, s(K_k)\} = A. \tag{52}$$

We remark that if the permutation s leaves the subset K invariant one gets $\gamma_s(C_K) = C_K$. It is easy to show using (49) that C_A is in the centralizer given that C_K is in the centralizer as already proved.

The following example shows that there are different ways to compute C_A depending on the set K we start with.

Example 4.1 From $s_1(\{2,3\}) = \{1,3\}$ or $s_2(\{1,2\}) = \{1,3\}$, the definition (51) gives for C_{13}

$$C_{13} = \gamma_{s_1}(C_{23}) = \check{\mathcal{R}}_1 C_{23}\check{\mathcal{R}}_1^{-1} = \mathcal{R}_{12}\sigma_{12}C_{23}\sigma_{12}^{-1}\mathcal{R}_{12}^{-1} = \mathcal{R}_{12}\overline{C}_{13}\mathcal{R}_{12}^{-1} \tag{53}$$

$$= \gamma_{s_2}(C_{12}) = \check{\mathcal{R}}_2 C_{12}\check{\mathcal{R}}_2^{-1} = \mathcal{R}_{23}\sigma_{23}C_{12}\sigma_{23}^{-1}\mathcal{R}_{23}^{-1} = \mathcal{R}_{23}\overline{C}_{13}\mathcal{R}_{23}^{-1}. \tag{54}$$

We recover the equivalent expressions (19) or (24) of C_{13} given in the previous section (we recall that $\mathcal{R}_{12} = \mathcal{R}_{21} = \mathcal{R}_{12}^{-1}$).

To have a well-posed definition of C_A, such different paths must lead to the same result. To confirm that, we must prove that for two subsets $K, L \subset [n]$ of successive integers defined by (46) the following relation holds

$$C_K = \gamma_s(C_L) \tag{55}$$

where $s(L) = K$. It is sufficient to prove (55) for the sets $L = \{1, 2, \ldots, \ell\}$ and $K = \{k+1, \ldots, k+\ell\}$ to prove it in general. The following permutation

$$s = (s_k s_{k+1} \ldots s_{k+\ell-1}) \ldots (s_2 s_3 \ldots s_{\ell+1})(s_1 s_2 \ldots s_\ell) \tag{56}$$

satisfies $s(L) = K$. Then, from definition (47), one gets

$$\begin{aligned}\gamma_s(C_L) &= (\check{\mathcal{R}}_k \check{\mathcal{R}}_{k+1} \ldots \check{\mathcal{R}}_{k+\ell-1}) \ldots (\check{\mathcal{R}}_1 \check{\mathcal{R}}_2 \\ &\quad \ldots \check{\mathcal{R}}_\ell) C_L (\check{\mathcal{R}}_\ell \ldots \check{\mathcal{R}}_2 \check{\mathcal{R}}_1) \ldots (\check{\mathcal{R}}_{k+\ell-1} \ldots \check{\mathcal{R}}_{k+1} \check{\mathcal{R}}_k) & (57)\\ &= (\mathcal{R}_{k,k+1} \mathcal{R}_{k,k+2} \ldots \mathcal{R}_{k,k+\ell}) \ldots (\mathcal{R}_{1,k+1} \mathcal{R}_{1,k+2} \ldots \mathcal{R}_{1,k+\ell}) \\ &\quad C_K (\mathcal{R}_{1,k+\ell} \ldots \mathcal{R}_{1,k+2} \mathcal{R}_{1,k+1}) \ldots (\mathcal{R}_{k,k+\ell} \ldots \mathcal{R}_{k,k+2} \mathcal{R}_{k,k+1}). & (58)\end{aligned}$$

The last relation has been obtained using the definition of $\check{\mathcal{R}}$ and the properties of $\sigma_{i,i+1}$. Then, noticing that from relation (13) one gets $(\text{id}^{\otimes k} \otimes \Delta^{(\ell-1)})(\mathcal{R}_{i,k+1}) = \mathcal{R}_{i,k+1} \mathcal{R}_{i,k+2} \ldots \mathcal{R}_{i,k+\ell}$ (for $1 \leq i \leq k$) and $(\text{id}^{\otimes k} \otimes \Delta^{(\ell-1)})(C_{k+1}) = C_K$, one obtains $[\mathcal{R}_{i,k+1} \mathcal{R}_{i,k+2} \ldots \mathcal{R}_{i,k+\ell}, C_K] = 0$ which proves (55) in view of (58).

We are now ready to present the main result of this section.

Proposition 4.1 *The map*

$$\begin{aligned} BI(n) &\rightarrow U(\mathfrak{osp}(1|2))^{\otimes n} \\ \mathcal{C}_A &\mapsto C_A \end{aligned} \tag{59}$$

is an algebra homomorphism.

Proof We must prove that the centralizing elements C_A satisfy the relations (45). We know from the previous section that one has

$$\{C_{12}, C_{23}\} = 2(-\mathcal{R}_{23} \overline{C}_{13} \mathcal{R}_{23}^{-1} + C_1 C_3 + C_2 C_{123}) \tag{60}$$

which can be transformed as

$$\{C_{12}, C_{23}\} = 2(-\gamma_s(C_{12}) + C_1 C_3 + C_2 C_{123}) \quad \text{with} \quad s(\{1,2\}) = \{1,3\}. \tag{61}$$

By acting with the coproduct on the second space in relation (60), one gets

$$\{C_{123}, C_{234}\} = 2(-\mathcal{R}_{34} \mathcal{R}_{24} \overline{C}_{14} \mathcal{R}_{24}^{-1} \mathcal{R}_{34}^{-1} + C_1 C_4 + C_{23} C_{1234}) \tag{62}$$

which becomes

$$\{C_{123}, C_{234}\} = 2(-\gamma_s(C_{12}) + C_1C_4 + C_{23}C_{1234}) \quad \text{with} \quad s(\{1,2\}) = \{1,4\}. \tag{63}$$

Similarly, by acting with the coproduct successively on the second space in relation (60), one gets, for $L = \{2, \dots, \ell+1\}$,

$$\{C_{1,L}, C_{L,\ell+2}\} = 2(-\mathcal{R}_{\ell+1,\ell+2}\dots\mathcal{R}_{2,\ell+2}\overline{C}_{1,\ell+2}\mathcal{R}^{-1}_{2,\ell+2}\dots\mathcal{R}^{-1}_{\ell+1,\ell+2} + C_1C_{\ell+2} + C_LC_{1,L,\ell+2}) \tag{64}$$

which becomes

$$\{C_{1,L}, C_{L,\ell+2}\} = 2(-\gamma_s(C_{12}) + C_1C_{\ell+2} + C_LC_{1,L,\ell+2}) \quad \text{with} \quad s(\{1,2\}) = \{1, \ell+2\}. \tag{65}$$

Finally, by acting with the coproduct successively on the first and third spaces in relation (60), one can prove

$$\{C_{KL}, C_{LM}\} = 2(-\gamma_s(C_{K,k+1,k+2,\dots,k+m}) + C_KC_M + C_LC_{KLM}) \tag{66}$$

where $s(K, k+1, k+2, \dots, k+m) = KM$ and $K = \{1, \dots k\}$, $L = \{k+1, \dots k+\ell\}$ and $M = \{k+\ell+1, \dots k+\ell+m\}$. This proves that for the sets K, L and M given above, the $BI(n)$ relations (45) are satisfied by the corresponding centralizing elements. We can similarly see relation (45) to hold when K, L or M are chosen empty. Let $s \in \mathfrak{S}_n$. Using the definition (47), one gets

$$\gamma_s(XX') = \gamma_s(X)\gamma_s(X'). \tag{67}$$

Then, we have

$$\{C_{s(KL)}, C_{s(LM)}\} = 2(-C_{s(KM)} + C_{s(K)}C_{s(M)} + C_{s(L)}C_{s(KLM)}). \tag{68}$$

We conclude the proof by remarking that $s(KM) = (s(KL) \cup s(LM)) \backslash (s(KL) \cap s(LM))$, $s(K) = s(KL) \backslash (s(KL) \cap s(LM))$, $s(M) = s(LM) \backslash (s(KL) \cap s(LM))$, $s(L) = s(KL) \cap s(LM)$ and $s(KLM) = s(KL) \cup s(LM)$ and by noting that there exist K, L and M and s such that $s(KL) = A$ and $s(LM) = B$ for any $A, B \subset [n]$. □

5 Conclusions

This paper has offered a complete description of the Bannai–Ito algebras as centralizers of the diagonal action of $\mathfrak{osp}(1|2)$ in $U(\mathfrak{osp}(1|2))^{\otimes n}$ by bringing the universal R-matrix to bear on the topic. This has proved most appropriate. In addition to the elegance it confers to the presentation, this approach gave answers to questions that had so far been unsettled. It provided an intrinsic algebraic definition of all centralizing elements independently of the defining relations. It also shed light on the specific form of the intermediate embeddings of $\mathfrak{osp}(1|2)$ in $\mathfrak{osp}(1|2)^{\otimes n}$ that yield the generators through the associated Casimir elements. Importantly, it has entailed a simple constructive derivation of the structure relations of $BI(n)$ satisfied by these generators through bootstrapping from the relations of $BI(3)$. Another possible merit is that casting Bannai–Ito algebras in this framework might bring experts familiar with universal R-matrices to contribute further to the field and its applications.

This universal R-matrix approach has already been applied to the study of the Askey–Wilson algebra $AW(3)$ [10] as the centralizer of the diagonal action of $U_q(\mathfrak{sl}(2))$ into its threefold product and has also been seen to hold promises for advancing the understanding of the higher-rank $AW(n)$ where one is looking at the centralizer of $U_q(\mathfrak{sl}(2))$ in $U_q(\mathfrak{sl}(2))^{\otimes n}$ [2]. While advances have been made on this last front [3,14], a complete description of $AW(n)$ is still lacking. We trust that the treatment given here of the Bannai–Ito algebra $BI(n)$ using the universal R-matrix might hold the clues towards bringing this quest to a satisfactory conclusion. We hope to report on this in the near future.

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