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On tau-functions for the Toda lattice hierarchy

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Dedicated to the memory of Boris Anatol'evich Dubrovin, with gratitude and admiration.

Abstract

We extend a recent result of Dubrovin et al. in On tau-functions for the KdV hierarchy, arXiv:1812.08488 to the Toda lattice hierarchy. Namely, for an arbitrary solution to the Toda lattice hierarchy, we define a pair of wave functions and use them to give explicit formulae for the generating series of k-point correlation functions of the solution. Applications to computing GUE correlators and Gromov–Witten invariants of the Riemann sphere are under consideration.

Keywords Toda lattice hierarchy · Tau-function · Pair of wave functions · Matrix resolvent · Generating series

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Contents

1 Introduction . . . 556
1.1 Toda lattice hierarchy and tau-function . . . 556
1.2 Matrix resolvent . . . 559
1.3 From wave functions to correlation functions . . . 560
2 Matrix resolvent and tau-structure . . . 561
3 Pair of wave functions . . . 565
4 The k-point generating series . . . 568
5 Applications . . . 574
5.1 Application I: enumeration of ribbon graphs . . . 574
5.2 Application II: Gromov–Witten invariants of $\mathbb{P}^1$ in the stationary sector . . . 576
Appendix A: Pair of abstract pre-wave functions . . . 578
References . . . 582

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1 Introduction

The Toda lattice hierarchy, which contains the Toda lattice equation

$$\ddot{\sigma}(n) = e^{\sigma(n-1)-\sigma(n)} - e^{\sigma(n)-\sigma(n+1)}, \tag{1}$$

is an important *integrable hierarchy* of nonlinear differential–difference equations [18,19,22,27]. In this paper, following the idea of [13], we derive new formulae for generating series of k-point correlation functions for the Toda lattice hierarchy by using the matrix resolvent approach [10] and by introducing *a pair of wave functions*.

1.1 Toda lattice hierarchy and tau-function

Let

$$\mathcal{A} := \mathbb{Z}[v_0, w_0, v_{\pm 1}, w_{\pm 1}, v_{\pm 2}, w_{\pm 2}, \ldots] \tag{2}$$

be the polynomial ring. Define the shift operator $\Lambda : \mathcal{A} \to \mathcal{A}$ via

$$\Lambda(1) = 1, \quad \Lambda(v_i) = v_{i+1}, \quad \Lambda(w_i) = w_{i+1}, \quad \Lambda(fg) = \Lambda(f)\,\Lambda(g)$$

$\forall\, i \in \mathbb{Z}$ and $f, g \in \mathcal{A}$. Denote by Λ^{-1} the inverse of Λ satisfying $\Lambda^{-1}(v_i) = v_{i-1}$, $\Lambda^{-1}(w_i) = w_{i-1}$, and $\Lambda^{-1}(fg) = \Lambda^{-1}(f)\,\Lambda^{-1}(g)$. For a difference operator P on $\mathcal{A}$, we mean an operator of the form $P = \sum_{m\in\mathbb{Z}} P_m\,\Lambda^m$, where $P_m \in \mathcal{A}$. Denote $P_+ := \sum_{m\geq 0} P_m\,\Lambda^m$, $P_- := \sum_{m<0} P_m\,\Lambda^m$, $\mathrm{Coef}(P, m) := P_m$. A linear operator $D : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$, if

$$D(fg) = D(f)\,g + f\,D(g), \quad \forall\, f, g \in \mathcal{A}.$$

The derivation D is called *admissible* if it commutes with Λ. Clearly, every admissible derivation D is uniquely determined by the values $D(v_0)$ and $D(w_0)$. Let

$$L := \Lambda + v_0 + w_0\,\Lambda^{-1} \tag{3}$$

be a difference operator, and define a sequence of difference operators A_k, $k \geq 0$ by

$$A_k := \left(L^{k+1}\right)_+. \tag{4}$$

We associate with A_k a sequence of admissible derivations $D_k : \mathcal{A} \to \mathcal{A}$ defined via

$$D_k(v_0) := \mathrm{Coef}\big([A_k, L], 0\big), \quad D_k(w_0) := \mathrm{Coef}\big([A_k, L], -1\big), \qquad k \geq 0. \tag{5}$$

The first few $D_k(v_0)$ and $D_k(w_0)$ are $D_0(v_0) = w_1 - w_0$, $D_0(w_0) = w_0\,(v_0 - v_{-1})$; $D_1(v_0) = w_1(v_1 + v_0) - w_0(v_0 + v_{-1})$, $D_1(w_0) = w_0\big(w_1 - w_{-1} + v_0^2 - v_{-1}^2\big)$, etc.

Lemma 1 *The operators D_k, $k \geq 0$ pairwise commute.*

This lemma was known. We call D_k the Toda lattice derivations, and (5) the abstract Toda lattice hierarchy.

A *tau-structure* associated with the derivations $(D_k)_{k\geq 0}$ is a collection of polynomials $\big(\Omega_{p,q}, S_p\big)_{p,q\geq 0}$ in $\mathcal{A}$ satisfying

$$\Omega_{p,q} = \Omega_{q,p}, \quad D_r\big(\Omega_{p,q}\big) = D_q\big(\Omega_{p,r}\big), \tag{6}$$
$$(\Lambda - 1)\big(\Omega_{p,q}\big) = D_q\big(S_p\big), \tag{7}$$
$$w_0\big(1 - \Lambda^{-1}\big)\big(S_p\big) = D_p(w_0) \tag{8}$$

for all $p, q, r \geq 0$. It can be shown (e.g., [10]) that the tau-structure exists and is unique up to replacing $\Omega_{p,q}$, S_p by $\Omega_{p,q} + c_{p,q}$ and $S_p + a_p$ respectively, where $c_{p,q} = c_{q,p}$ and a_p are arbitrary constants. The tau-structure $\Omega_{p,q}$, S_p is called canonical if

$$\Omega_{p,q}\big|_{v_i=0,\, w_i=0,\, i\in\mathbb{Z}} = 0, \quad S_p\big|_{v_i=0,\, w_i=0,\, i\in\mathbb{Z}} = 0.$$

Let us take $\Omega_{p,q}$, S_p the canonical tau-structure. For $m \geq 3$, define

$$\Omega_{p_1,\dots,p_m} := D_{p_1}\cdots D_{p_{m-2}}\big(\Omega_{p_{m-1}p_m}\big) \in \mathcal{A}, \qquad p_1,\dots,p_m \geq 0. \tag{9}$$

By (6), we know that the $\Omega_{p_1,\dots,p_m}$, $m \geq 2$, are totally symmetric with respect to permutations of the indices $p_1,\dots,p_m$. The first few of these polynomials are

$$S_0 = v_0, \quad S_1 = w_1 + w_0 + v_0^2, \tag{10}$$
$$\Omega_{0,0} = w_0, \quad \Omega_{0,1} = \Omega_{1,0} = w_1(v_1 + v_0). \tag{11}$$

If we think of v_0, w_0 as two functions $v(n)$, $w(n)$ of n, respectively, and v_i, w_i as $v(n+i)$, $w(n+i)$, then the Toda lattice derivations D_k lead to a hierarchy of evolutionary differential–difference equations, called the Toda lattice hierarchy, given by

$$\frac{\partial v(n)}{\partial t_k} = D_k(v_0)(n), \qquad \frac{\partial w(n)}{\partial t_k} = D_k(w_0)(n), \tag{12}$$

where $k \geq 0$, and the $D_k(v_0)(n)$, $D_k(w_0)(n)$ are defined as $D_k(v_0)$, $D_k(w_0)$ with v_i, w_i replaced by $v(n+i)$, $w(n+i)$, respectively. Lemma 1 implies that the flows (12) all commute. So we can solve the whole Toda lattice hierarchy (12) together, which yields solutions of the form $(v = v(n,\mathbf{t}), w = w(n,\mathbf{t}))$. Here, $\mathbf{t} := (t_0, t_1, \dots)$ denotes the infinite time vector. Note that the $k = 0$ equations read

$$\dot v(n) = w(n+1) - w(n), \qquad \dot w(n) = w(n)\big(v(n) - v(n-1)\big), \tag{13}$$

which are equivalent to Eq. (1) via the transformation

$$w(n) = e^{\sigma(n-1)-\sigma(n)}, \qquad v(n) = -\dot\sigma(n).$$

Here, dot, “˙”, is identified with $\partial/\partial t_0$.

Let V be a ring of functions of n closed under shifting n by ± 1. For two given $f(n), g(n) \in V$, consider the initial value problem for (12) with the initial condition:

$$v(n, \mathbf{0}) = f(n), \quad w(n, \mathbf{0}) = g(n). \tag{14}$$

The solution $(v(n, \mathbf{t}), w(n, \mathbf{t})) \in V[[\mathbf{t}]]^2$ exists and is unique, which gives the following 1–1 correspondence:

$$\left\{\text{solution } (v, w) \text{ of (12) in } V[[\mathbf{t}]]^2\right\} \longleftrightarrow \left\{\text{initial data } (f, g)\right\}. \tag{15}$$

Example 1 $f(n) = 0$, $g(n) = n$. (For this case, one can take $V = \mathbb{Q}[n]$.) The corresponding unique solution governs the enumerations of ribbon graphs in all genera.

Example 2 $f(n) = (n + \frac{1}{2})\epsilon$, $g(n) = 1$. (For this case, one can take $V = \mathbb{Q}[\epsilon][n]$.) The corresponding unique solution governs the Gromov–Witten invariants of $\mathbb{P}^1$ in the stationary sector in all genera and all degrees.

Let $(v, w) \in V[[\mathbf{t}]]^2$ be an arbitrary solution to the Toda lattice hierarchy (12). Write $\Omega_{p,q}(n, \mathbf{t})$ and $S_p(n, \mathbf{t})$ as the images of $\Omega_{p,q}$ and S_p under the substitutions

$$v_i \mapsto v(n+i, \mathbf{t}), \quad w_i \mapsto w(n+i, \mathbf{t}), \qquad i \in \mathbb{Z}, \tag{16}$$

respectively. (Similar notations will be used for other elements of $\mathcal{A}$.) Equalities (6) then imply the existence of a function $\tau = \tau(n, \mathbf{t})$ such that for $p, q \geq 0$,

$$\Omega_{p,q}(n, \mathbf{t}) = \frac{\partial^2 \log \tau(n, \mathbf{t})}{\partial t_p \partial t_q}, \tag{17}$$

$$S_p(n, \mathbf{t}) = \frac{\partial}{\partial t_p} \log \frac{\tau(n+1, \mathbf{t})}{\tau(n, \mathbf{t})}, \tag{18}$$

$$w(n, \mathbf{t}) = \frac{\tau(n+1, \mathbf{t})\, \tau(n-1, \mathbf{t})}{\tau(n, \mathbf{t})^2}. \tag{19}$$

We call $\tau(n, \mathbf{t})$ the *Dubrovin–Zhang (DZ)-type tau-function* [10,15] of the solution (v, w), in short the tau-function of the solution. The symmetry in (9) is more obvious: the image $\Omega_{p_1,\ldots,p_m}(n, \mathbf{t})$ of $\Omega_{p_1,\ldots,p_m}$ under (16) satisfies

$$\Omega_{p_1,\ldots,p_m}(n, \mathbf{t}) = \frac{\partial^m \log \tau(n, \mathbf{t})}{\partial t_{p_1} \ldots \partial t_{p_m}}, \qquad m \geq 2,\ p_1, \ldots, p_m \geq 0. \tag{20}$$

Define $\Omega_p(n, \mathbf{t}) = \partial_{t_p} \log \tau(n, \mathbf{t})$, $p \geq 0$. These logarithmic derivatives of $\tau(n, \mathbf{t})$ are called *correlation functions* of the solution (v, w). The specializations $\Omega_{p_1,\ldots,p_m}(n, \mathbf{0})$ are called *m-point partial correlation functions* of (v, w).

Remark 1 The tau-function $\tau(n, \mathbf{t})$ of the solution (v, w) is unique up to multiplying it by the exponential of a linear function of $n, t_0, t_1, t_2, \ldots$.

1.2 Matrix resolvent

The matrix resolvent (MR) method for computing correlation functions for integrable hierarchies was introduced in [1–3], and was extended to the discrete case in [10] (in particular to the Toda lattice hierarchy). Denote

$$U(\lambda) := \begin{pmatrix} v_0 - \lambda & w_0 \\ -1 & 0 \end{pmatrix}.$$

The following lemma for the Toda lattice hierarchy was proven in [10].

Lemma 2 [10] *There exists a unique series* $R(\lambda) \in \mathrm{Mat}\left(2, \mathcal{A}\left[\left[\lambda^{-1}\right]\right]\right)$ *satisfying*

$$\Lambda\big(R(\lambda)\big)\, U(\lambda) - U(\lambda)\, R(\lambda) = 0, \tag{21}$$

$$\mathrm{Tr}\, R(\lambda) = 1, \quad \det R(\lambda) = 0, \tag{22}$$

$$R(\lambda) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}\left(2, \mathcal{A}\left[\left[\lambda^{-1}\right]\right]\lambda^{-1}\right). \tag{23}$$

The unique series $R(\lambda)$ in Lemma 2 is called the *basic matrix resolvent*. The first few terms of $R(\lambda)$ are given by

$$R(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -w_0 \\ 1 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} w_0 & -v_0 w_0 \\ v_{-1} & -w_0 \end{pmatrix} \frac{1}{\lambda^2} + \begin{pmatrix} w_0(v_0 + v_{-1}) & -w_0(w_0 + w_1 + v_0^2) \\ w_0 + w_{-1} + v_{-1}^2 & -w_0(v_0 + v_{-1}) \end{pmatrix} \frac{1}{\lambda^3} + \cdots. \tag{24}$$

Proposition 1 [10] *For any* $k \geq 2$*, the following formula holds true:*

$$\sum_{i_1,\ldots,i_k \geq 0} \frac{\Omega_{i_1,\ldots,i_k}}{\lambda_1^{i_1+2} \ldots \lambda_k^{i_k+2}} = -\sum_{\pi \in \mathcal{S}_k / C_k} \frac{\mathrm{tr} \prod_{j=1}^k R\big(\lambda_{\pi(j)}\big)}{\prod_{j=1}^k \big(\lambda_{\pi(j)} - \lambda_{\pi(j+1)}\big)} - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}, \tag{25}$$

where $\mathcal{S}_k$ *denotes the symmetry group and* C_k *the cyclic group, and* $\pi(k+1) := \pi(1)$.

The meaning of (25) is the following: For any fixed permutation $(j_1, \ldots, j_k)$ of $(1, \ldots, k)$, expanding the right-hand side with respect to $|\lambda_{j_1}| > \cdots > |\lambda_{j_k}| >> 0$ gives identical formal power series with the left-hand side. This is because, after the summation over the $\mathcal{S}_k / C_k$ and subtracting $\frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}$, the poles in the diagonal cancel (cf. Proposition 2 of [12] for a straightforward proof of this point). We note that, as formal power series, the coefficients of the both sides of (25) are in $\mathcal{A}$. We give in Sect. 2 a new proof of (25), where we keep all derivations with coefficients in $\mathcal{A}$.

1.3 From wave functions to correlation functions

In [13], we introduced the notion of a tuple of wave functions (in many cases *a pair*) to the study of tau-function without using the Sato theory. Let us generalize it to the Toda lattice hierarchy. Our definition of a pair will be based on the standard construction of wave functions for the Toda lattice hierarchy [5,6,27]. For given $(f(n), g(n))$ a pair of arbitrary elements in V, let L be the linear difference operator $L = \Lambda + f(n) + g(n)\Lambda^{-1}$. Denote

$$s(n) := -\big(1-\Lambda^{-1}\big)^{-1}\big(\log g(n)\big). \tag{26}$$

The function $s(n)$ is in a certain extension $\widehat{V}$ of V and is uniquely determined by $\log g(n)$ up to a constant. Below we fix a choice of $s(n)$. An element $\psi_A(\lambda, n) = \big(1 + \mathrm{O}\big(\lambda^{-1}\big)\big)\,\lambda^n$ in the module $\widetilde{V}\left[\left[\lambda^{-1}\right]\right]\lambda^n$ is called a (formal) wave function of type A associated with $f(n), g(n)$, if $L\big(\psi_A(\lambda, n)\big) = \lambda\psi_A(\lambda, n)$. Here, $\widetilde{V}$ is a ring of functions of n satisfying

$$V \subset (\Lambda - 1)\big(\widetilde{V}\big) \subset \widetilde{V}.$$

An element $\psi_B(\lambda, n) = \big(1 + \mathrm{O}\big(\lambda^{-1}\big)\big)\,e^{-s(n)}\lambda^{-n}$ in the module $\widetilde{V}\left[\left[\lambda^{-1}\right]\right]e^{-s(n)}\lambda^{-n}$ is called a (formal) wave function of type B, if $L\big(\psi_B(\lambda, n)\big) = \lambda\psi_B(\lambda, n)$. Let $\psi_A \in \widetilde{V}\left[\left[\lambda^{-1}\right]\right]\lambda^n$ and $\psi_B \in \widetilde{V}\left[\left[\lambda^{-1}\right]\right]e^{-s(n)}\lambda^{-n}$ be two wave functions of type A and of type B associated with $(f(n), g(n))$, respectively. Define

$$d(\lambda, n) := \psi_A(\lambda, n)\,\psi_B(\lambda, n-1) - \psi_B(\lambda, n)\,\psi_A(\lambda, n-1). \tag{27}$$

We call ψ_A, ψ_B form *a pair* if the following normalization condition holds:

$$e^{s(n-1)}d(\lambda, n) = \lambda. \tag{28}$$

The existence of a pair of wave functions is proved in Sect. 3.

Denote by $(v(n, \mathbf{t}), w(n, \mathbf{t}))$ the unique solution in $V[[\mathbf{t}]]^2$ to the Toda lattice hierarchy with $(f(n), g(n))$ as its initial value, by $\psi_A(\lambda, n)$ and $\psi_B(\lambda, n)$ a pair of wave functions associated with $(f(n), g(n))$ and by $\tau(n, \mathbf{t})$ the DZ-type tau-function of $(v(n, \mathbf{t}), w(n, \mathbf{t}))$. Introduce

$$D(\lambda, \mu, n) := \frac{\psi_A(\lambda, n)\,\psi_B(\mu, n-1) - \psi_A(\lambda, n-1)\,\psi_B(\mu, n)}{\lambda - \mu}. \tag{29}$$

Theorem 1 *Fix $k \geq 2$ being an integer. The generating series of k-point partial correlation functions has the following expression:*

$$\sum_{i_1,\dots,i_k\geq 0} \frac{\partial^k \log\tau}{\partial t_{i_1}\dots\partial t_{i_k}}(n,\mathbf{0})\,\frac{1}{\lambda_1^{i_1+2}\dots\lambda_k^{i_k+2}} = (-1)^{k-1}\frac{e^{ks(n-1)}}{\prod_{j=1}^k\lambda_j}\sum_{\pi\in\mathcal{S}_k/C_k}\prod_{j=1}^k D(\lambda_{\pi(j)},\lambda_{\pi(j+1)},n) - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}. \tag{30}$$

Theorem 1 gives an algorithm with the initial value $(f(n), g(n))$ as the only input for computing the k_{th}-order logarithmic derivatives of the tau-function $\tau(n,\mathbf{t})$ evaluated at $\mathbf{t}=\mathbf{0}$ for $k\geq 2$. Indeed, by solving the spectral problem $L(\psi)=\lambda\psi$ with $L=\Lambda+f(n)+g(n)\Lambda^{-1}$ and with the normalization condition (28), one constructs a pair of wave functions; the coefficients in the $\mathbf{t}$-expansion of $\log\tau(n,\mathbf{t})$ are then obtained through algebraic manipulations by using (85). (Recall that in the inverse scattering method (cf., e.g., [18,19]), an additional integral equation needs to be solved.) Two applications of Theorem 1 are given in Sect. 5. For a certain class of bispectral solutions (cf. [20]), it would be possible to give a *canonical* way of constructing a pair of wave functions, which was briefly mentioned in [13] for the KdV hierarchy; we plan to do this for KdV and for Toda lattice in a future publication.

Organization of the paper In Sect. 2, we review the MR method of studying tau-structure for the Toda lattice hierarchy. In Sect. 3, we prove the existence of a pair of wave functions. In Sect. 4, we prove Theorem 1 and several other theorems. Applications to the computations of GUE correlators and Gromov–Witten invariants of $\mathbb{P}^1$ are given in Sect. 5. In Appendix A, we give an extension of $\mathcal{A}$, define a pair of abstract pre-wave functions, and prove an abstract version for Theorem 1.

2 Matrix resolvent and tau-structure

We continue in this section with more details in reviewing the MR method [10] to the Toda lattice hierarchy. Denote by $\mathcal{L}$ the matrix Lax operator for the Toda lattice:

$$\mathcal{L} := \begin{pmatrix}\Lambda & 0\\ 0 & \Lambda\end{pmatrix} + \begin{pmatrix} v_0-\lambda & w_0\\ -1 & 0\end{pmatrix} = \Lambda + U(\lambda).$$

Let $R(\lambda)$ be the basic matrix resolvent (of $\mathcal{L}$). Write

$$R(\lambda) = \begin{pmatrix}1+\alpha(\lambda) & \beta(\lambda)\\ \gamma(\lambda) & -\alpha(\lambda)\end{pmatrix}, \tag{31}$$

$$\alpha(\lambda) = \sum_{i\geq 0}\frac{a_i}{\lambda^{i+1}},\quad \beta(\lambda) = \sum_{i\geq 0}\frac{b_i}{\lambda^{i+1}},\quad \gamma(\lambda) = \sum_{i\geq 0}\frac{c_i}{\lambda^{i+1}}, \tag{32}$$

where $a_i, b_i, c_i\in\mathcal{A}$. From the defining Eqs. (21)–(23), we see that the series α,β,γ satisfy the equations

$$\beta(\lambda) \;=\; -w_0\,\Lambda\big(\gamma(\lambda)\big), \tag{33}$$

$$\gamma(\lambda) \;=\; \frac{1 \,+\, \alpha(\lambda) \,+\, \Lambda^{-1}\big(\alpha(\lambda)\big)}{\lambda - v_{-1}}, \tag{34}$$

$$\Big(\alpha(\lambda) - \Lambda\big(\alpha(\lambda)\big)\Big)(\lambda - v_0) \,-\, w_0\,\frac{1 + \alpha(\lambda) + \Lambda^{-1}\big(\alpha(\lambda)\big)}{\lambda - v_{-1}} + w_1\,\frac{1 + \Lambda\big(\alpha(\lambda)\big) + \Lambda^2\big(\alpha(\lambda)\big)}{\lambda - v_1} \;=\; 0, \tag{35}$$

$$\alpha(\lambda) + \alpha(\lambda)^2 \,+\, \beta(\lambda)\gamma(\lambda) \;=\; 0. \tag{36}$$

These equalities give rise to the following recursion relation for a_i, b_i, c_i:

$$b_j \;=\; -w_0\,\Lambda(c_j), \qquad c_{j+1} \;=\; v_{-1}\,c_j \,+\, \big(1 \,+\, \Lambda^{-1}\big)\,(a_j), \tag{37}$$

$$\big(1 - \Lambda\big)\,(a_{j+1}) \,+\, v_0\,\big(\Lambda - 1\big)\,(a_j) \,+\, w_1\,\Lambda^2(c_j) \,-\, w_0\,c_j \;=\; 0, \tag{38}$$

$$a_\ell = \sum_{i+j=\ell-1} \Big(w_0\,c_i\,\Lambda(c_j) \,-\, a_i\,a_j\Big) \tag{39}$$

along with

$$a_0 \;=\; 0, \qquad c_0 \;=\; 1. \tag{40}$$

Equations (37)–(40) are called the matrix resolvent recursion relation.

It was proven [10] that the abstract Toda lattice hierarchy (5) can be equivalently written as

$$D_j\,(v_0) \;=\; \big(\Lambda - 1\big)\,(a_{j+1}),$$
$$D_j\,(w_0) \;=\; w_0\,\big(\Lambda - 1\big)\,(c_{j+1}),$$

where $j \geq 0$. Define an operator $\nabla(\lambda)$ by

$$\nabla(\lambda) \;:=\; \sum_{j\geq 0} \frac{D_j}{\lambda^{j+2}}. \tag{41}$$

We have

$$\nabla(\lambda)\,(v_0) \;=\; \big(\Lambda - 1\big)\,\big(\alpha(\lambda)\big), \tag{42}$$

$$\nabla(\lambda)\,(w_0) \;=\; w_0\,\big(\Lambda - 1\big)\,\big(\gamma(\lambda) - 1\big). \tag{43}$$

Lemma 3 *There exists a unique element* $W(\lambda, \mu)$ *in* $\mathcal{A} \otimes \mathrm{sl}_2(\mathbb{C})\,\big[\big[\lambda^{-1}, \mu^{-1}\big]\big]\,\lambda^{-1}\mu^{-1}$ *of the form*

$$W(\lambda, \mu) \;=\; \begin{pmatrix} X(\lambda, \mu) & Y(\lambda, \mu) \\ Z(\lambda, \mu) & -X(\lambda, \mu) \end{pmatrix}$$

satisfying the following linear inhomogeneous equations for the entries of W:

$$\Lambda\big(W(\lambda,\mu)\big)\,U(\lambda) - U(\lambda)\,W(\lambda,\mu) + \Lambda\big(R(\lambda)\big)\,\nabla(\mu)\big(U(\lambda)\big) - \nabla(\mu)\big(U(\lambda)\big)\,R(\lambda) = 0, \tag{44}$$

$$X(\lambda,\mu) + 2\alpha(\lambda)\,X(\lambda,\mu) + \gamma(\lambda)\,Y(\lambda,\mu) + \beta(\lambda)\,Z(\lambda,\mu) = 0. \tag{45}$$

Proof The existence part of this lemma follows from Lemma 2. Indeed, if we define

$$W(\lambda,\mu) := \nabla(\mu)\big(R(\lambda)\big),$$

then $W(\lambda,\mu)$ satisfies (44)–(45). To see the uniqueness part, we first note that the (1,2)-entry and the (2,1)-entry of the matrix equation (44) imply that Y and Z can be uniquely expressed in terms of X. Indeed, we have

$$Z(\lambda,\mu) = \tfrac{(1+\Lambda^{-1})(X(\lambda,\mu))}{\lambda-v_{-1}} + \gamma(\lambda)\tfrac{\Lambda^{-1}\circ\nabla(\mu)(v_0)}{\lambda-v_{-1}}, \tag{46}$$

$$Y(\lambda,\mu) = -\nabla(\mu)(w_0)\,\tfrac{1+\alpha(\lambda)+\Lambda(\alpha(\lambda))}{\lambda-v_0} - w_0\,\tfrac{(1+\Lambda)(X(\lambda,\mu))}{\lambda-v_0} + \beta(\lambda)\tfrac{\nabla(\mu)(v_0)}{\lambda-v_0}. \tag{47}$$

Substituting these two expressions in (45), we obtain the following linear inhomogeneous difference equation for X:

$$\Big(1+2\alpha(\lambda)+\tfrac{\beta(\lambda)}{\lambda-v_{-1}}-\tfrac{w_0\gamma(\lambda)}{\lambda-v_0}\Big)X(\lambda,\mu) - \tfrac{w_0\gamma(\lambda)}{\lambda-v_0}\Lambda\big(X(\lambda,\mu)\big) + \tfrac{\beta(\lambda)}{\lambda-v_{-1}}\Lambda^{-1}\big(X(\lambda,\mu)\big) = \Big(1+\alpha(\lambda)+\Lambda\big(\alpha(\lambda)\big)\Big)\gamma(\lambda)\tfrac{\nabla(\mu)(w_0)}{\lambda-v_0} - \beta(\lambda)\gamma(\lambda)\big(1+\Lambda^{-1}\big)\Big(\tfrac{\nabla(\mu)(v_0)}{\lambda-v_0}\Big). \tag{48}$$

Suppose this equation has two solutions X_1, X_2 in $\mathcal{A}\big[\big[\lambda^{-1},\mu^{-1}\big]\big]\,\lambda^{-1}\mu^{-1}$. Let $X_0 = X_1 - X_2$, then $X_0 \in \mathcal{A}\big[\big[\lambda^{-1},\mu^{-1}\big]\big]\,\lambda^{-1}\mu^{-1}$, and it satisfies the following equation:

$$\Big(1+2\alpha(\lambda)+\tfrac{\beta(\lambda)}{\lambda-v_{-1}}-\tfrac{w_0\gamma(\lambda)}{\lambda-v_0}\Big)X_0(\lambda,\mu) - \tfrac{w_0\gamma(\lambda)}{\lambda-v_0}\Lambda\big(X_0(\lambda,\mu)\big) + \tfrac{\beta(\lambda)}{\lambda-v_{-1}}\,\Lambda^{-1}\big(X_0(\lambda,\mu)\big) = 0. \tag{49}$$

It follows that X_0 vanishes. Indeed, write $X_0 = \sum_{j\ge 0} X_{0,j}(\mu)\lambda^{-(j+1)}$. Observe that

$$\tfrac{1}{\lambda-v_m} = \tfrac{1}{\lambda} + \tfrac{v_m}{\lambda^2} + \cdots \in \mathcal{A}\Big[\Big[\lambda^{-1}\Big]\Big]\lambda^{-1}, \qquad m=-1,0,$$

and recall that $\alpha(\lambda),\beta(\lambda),\gamma(\lambda) \in \mathcal{A}\big[\big[\lambda^{-1}\big]\big]\,\lambda^{-1}$. Then, by comparing the coefficients of powers of λ^{-1} consecutively, we find that $X_{0,0}(\mu)=0$, $X_{0,1}(\mu)=0$, $X_{0,2}(\mu)=0$, ... So $X_0 = 0$. Hence, $X_1 = X_2$. The lemma is proved. □

Based on this lemma, we now give a new proof for the following proposition.

Proposition 2 [10] *The following equation holds true:*

$$\nabla(\mu)\, R(\lambda) \;=\; \frac{1}{\mu-\lambda}\big[R(\mu),\, R(\lambda)\big] + \big[Q(\mu),\, R(\lambda)\big], \tag{50}$$

where

$$Q(\mu) \;:=\; -\frac{\mathrm{id}}{\mu} \;+\; \begin{pmatrix} 0 & 0 \\ 0 & \gamma(\mu) \end{pmatrix}.$$

Proof Define W^* as the right-hand side of (50), i.e.,

$$W^* \;:=\; \frac{1}{\mu-\lambda}\big[R(\mu),\, R(\lambda)\big] + \big[Q(\mu),\, R(\lambda)\big].$$

More precisely, the entries of W^* have the expressions:

$$X^* = \frac{w_0}{\mu-\lambda}\Big(\frac{(\alpha(\lambda)+\Lambda(\alpha(\lambda))+1)(\Lambda^{-1}(\alpha(\mu))+\alpha(\mu)+1)}{(\lambda-v_0)(\mu-v_{-1})} - \frac{(\Lambda^{-1}(\alpha(\lambda))+\alpha(\lambda)+1)(\alpha(\mu)+\Lambda(\alpha(\mu))+1)}{(\lambda-v_{-1})(\mu-v_0)}\Big), \tag{51}$$

$$Y^* = \frac{w_0}{\lambda-\mu}\Big(\frac{(\alpha(\lambda)+\Lambda(\alpha(\lambda))+1)(\Lambda^{-1}(\alpha(\mu))(\lambda-\mu)+\alpha(\mu)(\lambda+\mu-2v_{-1})+\lambda-v_{-1})}{(\lambda-v_0)(\mu-v_{-1})} + \frac{(2\alpha(\lambda)+1)(\alpha(\mu)+\Lambda(\alpha(\mu))+1)}{v_0-\mu}\Big), \tag{52}$$

$$Z^* = \frac{1}{\lambda-\mu}\Big(\frac{(\Lambda^{-1}(\alpha(\lambda))+\alpha(\lambda)+1)(\Lambda^{-1}(\alpha(\mu))-\alpha(\mu))}{v_{-1}-\lambda} + \frac{(\Lambda^{-1}(\alpha(\lambda))-\alpha(\lambda))(\Lambda^{-1}(\alpha(\mu))+\alpha(\mu)+1)}{\mu-v_{-1}}\Big). \tag{53}$$

We can then verify that $W^* \in \mathcal{A}\otimes \mathrm{sl}_2(\mathbb{C})\big[\big[\lambda^{-1},\mu^{-1}\big]\big]\,\lambda^{-1}\mu^{-1}$, as well as that $W := W^*$ satisfies Eqs. (44), (45). The latter is done by a lengthy but straightforward calculation. The proposition is proved due to Lemma 3. □

If we define $\widetilde{\Omega}_{i,j}, \widetilde{S}_i$ by

$$\sum_{i,j\ge 0} \frac{\widetilde{\Omega}_{i,j}}{\lambda^{i+2}\mu^{j+2}} \;=\; \frac{\mathrm{Tr}\,\big(R(\lambda)R(\mu)\big)}{(\lambda-\mu)^2} \;-\; \frac{1}{(\lambda_1-\lambda_2)^2}, \tag{54}$$

$$\Lambda\big(\gamma(\lambda)\big) \;=\; \lambda^{-1} \;+\; \sum_{i\ge 0} \widetilde{S}_i\,\lambda^{-i-2}, \tag{55}$$

then according to [10], $\widetilde{\Omega}_{i,j}, \widetilde{S}_i$ gives the canonical tau-structure for the Toda lattice, i.e.,

$$\widetilde{\Omega}_{i,j} \;=\; \Omega_{i,j}, \quad \widetilde{S}_i \;=\; S_i.$$

These equalities together with Proposition 2 lead to Proposition 1; see [10] for the detailed proof of Proposition 1.

Before ending this section, we will make two remarks. The first remark is that all the entries of $R(\lambda)$ can be expressed by the canonical tau-structure. Indeed, we have

$$\alpha(\lambda) = \sum_{p\geq 0} \Omega_{p,0}\, \lambda^{-p-2}, \quad \beta(\lambda) = -w_0\, \Lambda\big(\gamma(\lambda)\big), \tag{56}$$

$$\Lambda\big(\gamma(\lambda)\big) = \lambda^{-1} + \sum_{p\geq 0} S_p\, \lambda^{-p-2}. \tag{57}$$

The proof was in [10]. The second remark is that existence of a tau-structure in general implies Lemma 1, and note that the proof in [10] of the fact that $\widetilde{\Omega}_{i,j}, \widetilde{S}_i$ is a tau-structure does not use the commutativity of the abstract Toda lattice hierarchy, so as a by-product of the matrix resolvent method we get a new proof of Lemma 1 together with a simple construction of the Toda lattice hierarchy. Similar idea was in [3].

3 Pair of wave functions

As in the Introduction, we start with the linear operator $L(n) = \Lambda + f(n) + g(n)\, \Lambda^{-1}$, where $f(n)$ and $g(n)$ are two given arbitrary elements in V. We show in this section the existence of pairs of wave functions associated with $(f(n), g(n))$. Let us write

$$\psi_A(\lambda, n) = e^{(\Lambda-1)^{-1} y(\lambda,n)} \lambda^n, \quad y(\lambda, n) := \sum_{i\geq 1} \frac{y_i(n)}{\lambda^i}, \tag{58}$$

$$\psi_B(\lambda, n) = e^{(\Lambda-1)^{-1} z(\lambda,n)} e^{-s(n)}\, \lambda^{-n}, \quad z(\lambda, n) := \sum_{i\geq 1} \frac{z_i(n)}{\lambda^i}. \tag{59}$$

Then, the spectral problems $L(n)\big(\psi(\lambda, n)\big) = \lambda \psi(\lambda, n)$ for $\psi = \psi_A$ and for $\psi = \psi_B$ recast into the following equations:

$$\lambda\, e^{y(\lambda,n)} + f(n) - \lambda + g(n)\, \lambda^{-1} e^{-y(\lambda,n-1)} = 0, \tag{60}$$

$$\lambda\, e^{-z(\lambda,n-1)} + f(n) - \lambda + g(n+1)\, \lambda^{-1} e^{z(\lambda,n)} = 0, \tag{61}$$

yielding recursions of the form (as equivalent conditions to (60)–(61))

$$y_{k+1}(n) = -\sum_{\substack{m_1,\dots,m_k\geq 0 \\ \sum_{i=1}^k i m_i = k+1}} \frac{\prod_{i=1}^k y_i(n)^{m_i}}{\prod_{i=1}^k m_i!} - f(n)\delta_{k,0}$$
$$- g(n) \sum_{\substack{m_1,\dots,m_{k-1}\geq 0 \\ \sum_{i=1}^{k-1} i m_i = k-1}} \frac{\prod_{i=1}^{k-1} (-1)^{m_i} y_i(n-1)^{m_i}}{\prod_{i=1}^{k-1} m_i!}, \tag{62}$$

$$z_{k+1}(n) = \sum_{\substack{m_1,\dots,m_k\geq 0\\ \sum_{i=1}^k i m_i = k+1}} \frac{\prod_{i=1}^k (-1)^{m_i} z_i(n)^{m_i}}{\prod_{i=1}^k m_i!} + f(n+1)\delta_{k,0}$$
$$+ g(n+2) \sum_{\substack{m_1,\dots,m_{k-1}\geq 0\\ \sum_{i=1}^{k-1} i m_i = k-1}} \frac{\prod_{i=1}^{k-1} z_i(n+1)^{m_i}}{\prod_{i=1}^{k-1} m_i!}, \tag{63}$$

where $k \geq 0$. From these recursions, it easily follows that $y_k, z_k \in V$, $k \geq 0$. This proves the existence of wave functions of type A and of type B meeting the definitions in Sect. 1.3. Clearly, ψ_A and ψ_B are unique up to multiplying by arbitrary series $G(\lambda)$ and $E(\lambda)$ of λ^{-1} with constant coefficient of the form $G(\lambda) \in 1 + \mathbb{C}\left[\left[\lambda^{-1}\right]\right]\lambda^{-1}$ and $E(\lambda) \in 1 + \mathbb{C}\left[\left[\lambda^{-1}\right]\right]\lambda^{-1}$. Since $\psi_A(\lambda, n) = \left(1 + \mathrm{O}(\lambda^{-1})\right)\lambda^n$ and since $\psi_B(\lambda, n) = \left(1 + \mathrm{O}\left(\lambda^{-1}\right)\right) e^{-s(n)}\lambda^{-n}$, we find that the $d(\lambda, n)$ defined in (27) must have the form

$$d(\lambda, n) = \lambda\, e^{-s(n-1)}\, e^{\sum_{k\geq 1} d_k(n)\, \lambda^{-k}}.$$

Then, by using the definitions of wave functions and of $s(n)$, one easily derives that

$$e^{s(n)}\, d(\lambda, n+1) = e^{s(n-1)}\, d(\lambda, n). \tag{64}$$

It follows that all $d_k(n)$, $k \geq 1$, are constants. Therefore, for any fixed choice of ψ_A, we can suitably choose the factor $E(\lambda)$ for ψ_B such that ψ_A, ψ_B form a pair. This proves the existence of pair of wave functions associated with $f(n)$, $g(n)$.

We proceed with the time dependence. Let $(v(n, \mathbf{t}), w(n, \mathbf{t}))$ be the unique solution in $V[[\mathbf{t}]]^2$ to the Toda lattice hierarchy satisfying the initial condition $v(n, \mathbf{0}) = f(n)$, $w(n, \mathbf{0}) = g(n)$. Let $L(n, \mathbf{t}) := \Lambda + v(n, \mathbf{t}) + w(n, \mathbf{t})\,\Lambda^{-1}$. Define $\sigma(n, \mathbf{t})$ as the unique up to a constant function satisfying the following equations:

$$w(n, \mathbf{t}) = e^{\sigma(n-1, \mathbf{t}) - \sigma(n, \mathbf{t})}, \tag{65}$$
$$\frac{\partial \sigma(n, \mathbf{t})}{\partial t_p} = -S_p(n, \mathbf{t}), \quad p \geq 0. \tag{66}$$

An element $\psi_A(n, \mathbf{t}, \lambda) = \left(1 + \mathrm{O}\left(\lambda^{-1}\right)\right)\lambda^n\, e^{\sum_{k\geq 0} t_k \lambda^{k+1}}$ in $\widetilde{V}\left[\left[\mathbf{t}, \lambda^{-1}\right]\right]\lambda^n e^{\sum_{k\geq 0} t_k \lambda^{k+1}}$ is called a wave function of type A associated with $(v(n, \mathbf{t}), w(n, \mathbf{t}))$ if

$$L(n, \mathbf{t})\left(\psi_A(\lambda, n, \mathbf{t})\right) = \lambda\, \psi_A(\lambda, n, \mathbf{t}), \quad \frac{\partial \psi_A}{\partial t_k} = \left(L^{k+1}\right)_+\left(\psi_A\right). \tag{67}$$

An element $\psi_B(n, \mathbf{t}, \lambda) = \left(1 + \mathrm{O}\left(\lambda^{-1}\right)\right)\lambda^{-n} e^{-\sum_{k\geq 0} t_k \lambda^{k+1}}$ in $\widetilde{V}\left[\left[\mathbf{t}, \lambda^{-1}\right]\right] e^{-\sigma(n,\mathbf{t})}\lambda^{-n}$ $e^{-\sum_{k\geq 0} t_k \lambda^{k+1}}$ is called a wave function of type B associated with $(v(n, \mathbf{t}), w(n, \mathbf{t}))$ if

$$L(n, \mathbf{t})\left(\psi_B(\lambda, n, \mathbf{t})\right) = \lambda\, \psi_B(\lambda, n, \mathbf{t}), \quad \frac{\partial \psi_B}{\partial t_k} = -\left(L^{k+1}\right)_-\left(\psi_B\right). \tag{68}$$

The existence of wave functions ψ_A and ψ_B of type A and of type B associated with $(v(n,\mathbf{t}), w(n,\mathbf{t}))$ is a standard result in the theory of integrable systems (cf. [5,6,13,27]); therefore, we omit its details. Denote

$$d(\lambda,n,\mathbf{t}) := \psi_A(\lambda,n,\mathbf{t})\,\psi_B(\lambda,n-1,\mathbf{t}) - \psi_B(\lambda,n,\mathbf{t})\,\psi_A(\lambda,n-1,\mathbf{t}), \tag{69}$$

and introduce

$$m(\mu,\lambda,n,\mathbf{t}) := \frac{R(\mu,n,\mathbf{t})}{\mu-\lambda} + Q(\mu,n,\mathbf{t}), \tag{70}$$

where $Q(\mu,n,\mathbf{t}) := -\frac{\mathrm{id}}{\mu} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma(\mu,n,\mathbf{t}) \end{pmatrix}$. We know from, e.g., [10] that the wave function $\psi_A(\lambda,n,\mathbf{t})$ satisfies

$$\nabla(\mu)\begin{pmatrix} \psi_A(\lambda,n,\mathbf{t}) \\ \psi_A(\lambda,n-1,\mathbf{t}) \end{pmatrix} = m(\mu,\lambda,n,\mathbf{t})\begin{pmatrix} \psi_A(\lambda,n,\mathbf{t}) \\ \psi_A(\lambda,n-1,\mathbf{t}) \end{pmatrix}. \tag{71}$$

Similarly, the wave function $\psi_B(\lambda,n,\mathbf{t})$ satisfies

$$\nabla(\mu)\begin{pmatrix} \psi_B(\lambda,n,\mathbf{t}) \\ \psi_B(\lambda,n-1,\mathbf{t}) \end{pmatrix} = \left(m(\mu,\lambda,n,\mathbf{t}) - \frac{\lambda}{\mu(\mu-\lambda)}I\right)\begin{pmatrix} \psi_B(\lambda,n,\mathbf{t}) \\ \psi_B(\lambda,n-1,\mathbf{t}) \end{pmatrix}. \tag{72}$$

Here, I denotes the 2×2 identity matrix.

Lemma 4 *The following formula holds true:*

$$\nabla(\mu)\big(d(\lambda,n,\mathbf{t})\big) = \left(-\frac{1}{\mu} + \gamma(\mu,n,\mathbf{t})\right)d(\lambda,n,\mathbf{t}). \tag{73}$$

Proof Recalling definition (69) for d and using (71)–(72), we find

$$\nabla(\mu)\big(d(\lambda,n,\mathbf{t})\big) = \left(\mathrm{tr}\big(m(\mu,\lambda,n,\mathbf{t})\big) - \frac{\lambda}{\mu(\mu-\lambda)}\right)d(\lambda,n,\mathbf{t}). \tag{74}$$

The lemma is then proved via a straightforward computation. □

Definition 1 We say ψ_A, ψ_B form *a pair* if $e^{\sigma(n-1,\mathbf{t})}d(\lambda,n,\mathbf{t}) = \lambda$.

The next lemma shows the existence of a pair.

Lemma 5 *There exist a pair of wave functions ψ_A, ψ_B associated with $(v(n,\mathbf{t}), w(n,\mathbf{t}))$. Moreover, the freedom of the pair is characterized by a factor $G(\lambda)$ via*

$$\psi_A(\lambda,n,\mathbf{t}) \mapsto G(\lambda)\,\psi_A(\lambda,n,\mathbf{t}), \quad \psi_B(\lambda,n,\mathbf{t}) \mapsto \frac{1}{G(\lambda)}\,\psi_B(\lambda,n,\mathbf{t}), \tag{75}$$

$$G(\lambda) = \sum_{j\geq 0} G_j \lambda^{-j}, \quad G_0 = 1 \tag{76}$$

with G_j, $j \geq 1$ being arbitrary constants.

Proof Firstly, the freedom of a wave function ψ_A associated with (v, w) is characterized by the multiplication by a factor $G(\lambda)$ of the form (76). Fix an arbitrary choice of ψ_A. For ψ_B being a wave function of type B associated with (v, w), from (69) and the definitions of wave functions, we know $e^{\sigma(n-1,\mathbf{t})}d(\lambda, n, \mathbf{t})$ must have the form

$$e^{\sigma(n-1,\mathbf{t})}d(\lambda, n, \mathbf{t}) = \lambda\, e^{\sum_{k\geq 1} d_k(n,\mathbf{t})\,\lambda^{-k}} \tag{77}$$

for some $d_k(n, \mathbf{t})$, $k \geq 1$. By using (67), (68), (69), we find

$$d(\lambda, n+1, \mathbf{t}) = w(n, \mathbf{t})\, d(\lambda, n, \mathbf{t}) = e^{\sigma(n-1,\mathbf{t})-\sigma(n,\mathbf{t})}\, d(\lambda, n, \mathbf{t}),$$

i.e.,

$$e^{\sigma(n,\mathbf{t})}d(\lambda, n+1, \mathbf{t}) = e^{\sigma(n-1,\mathbf{t})}\, d(\lambda, n, \mathbf{t}), \tag{78}$$

Using Lemma 4 and (66), we have

$$\begin{aligned}
&\nabla(\mu)\Big(e^{\sigma(n-1,\mathbf{t})}d(\lambda, n, \mathbf{t})\Big)\\
&= e^{\sigma(n-1,\mathbf{t})}\nabla(\mu)\big(\sigma(n-1, \mathbf{t})\big)d(\lambda, n, \mathbf{t}) + e^{\sigma(n-1,\mathbf{t})}\nabla(\mu)\big(d(\lambda, n, \mathbf{t})\big)\\
&= -e^{\sigma(n-1,\mathbf{t})}\sum_{p\geq 0}\frac{S_p(n-1, \mathbf{t})}{\mu^{p+2}}d(\lambda, n, \mathbf{t})\\
&\quad + e^{\sigma(n-1,\mathbf{t})}d(\lambda, n, \mathbf{t})\left(-\frac{1}{\mu} + \gamma(\mu, n, \mathbf{t})\right) = 0.
\end{aligned}$$

So we have

$$\frac{\partial(e^{\sigma(n-1,\mathbf{t})}d(\lambda, n, \mathbf{t}))}{\partial t_p} = 0, \quad \forall\, p \geq 0. \tag{79}$$

We deduce from (77), (78), (79) that $d_k(n, \mathbf{t})$, $k \geq 1$, are all constants. Therefore, there exists a unique choice of ψ_B such that ψ_A, ψ_B form a pair. The lemma is proved. □

4 The *k*-point generating series

Let $(v, w) = (v(n, \mathbf{t}), w(n, \mathbf{t})) \in V[[\mathbf{t}]]^2$ be the unique solution to the Toda lattice hierarchy with the initial value $(v(n, \mathbf{0}), w(n, \mathbf{0})) = (f(n), g(n))$, and (ψ_A, ψ_B) a pair of wave functions associated with (v, w). Define

$$\Psi_{\text{pair}}(\lambda, n, \mathbf{t}) = \begin{pmatrix} \psi_A(\lambda, n, \mathbf{t}) & \psi_B(\lambda, n, \mathbf{t}) \\ \psi_A(\lambda, n-1, \mathbf{t}) & \psi_B(\lambda, n-1, \mathbf{t}) \end{pmatrix}. \tag{80}$$

Proposition 3 *The following identity holds true:*

$$R(\lambda, n, \mathbf{t}) \equiv \Psi_{\text{pair}}(\lambda, n, \mathbf{t}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_{\text{pair}}^{-1}(\lambda, n, \mathbf{t}). \tag{81}$$

Proof Define

$$M = M(\lambda, n, \mathbf{t}) := \Psi_{\text{pair}}(\lambda, n, \mathbf{t}) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_{\text{pair}}^{-1}(\lambda, n, \mathbf{t}).$$

It is easy to verify that M satisfies

$$[\mathcal{L}, M]\left(\Psi_{\text{pair}}\right) = 0, \qquad \det M = 0.$$

The entries of M in terms of the pair of wave functions read

$$M = \frac{1}{d(\lambda, n, \mathbf{t})} \begin{pmatrix} \psi_A(\lambda, n, \mathbf{t})\, \psi_B(\lambda, n-1, \mathbf{t}) & -\psi_A(\lambda, n, \mathbf{t})\, \psi_B(\lambda, n, \mathbf{t}) \\ \psi_A(\lambda, n-1, \mathbf{t})\, \psi_B(\lambda, n-1, \mathbf{t}) & -\psi_A(\lambda, n-1, \mathbf{t})\, \psi_B(\lambda, n, \mathbf{t}) \end{pmatrix}, \tag{82}$$

where we recall that $d(\lambda, n, \mathbf{t}) = \psi_A(\lambda, n, \mathbf{t})\, \psi_B(\lambda, n-1, \mathbf{t}) - \psi_B(\lambda, n, \mathbf{t})\, \psi_A(\lambda, n-1, \mathbf{t})$, which coincides with the determinant of $\Psi(\lambda, n, \mathbf{t})$. It follows from $\psi_A(\lambda, n, \mathbf{t}) = \left(1+\mathrm{O}\left(\lambda^{-1}\right)\right) \lambda^n\, e^{\sum_{k\geq 0} t_k \lambda^{k+1}}$ and $\psi_B(\lambda, n, \mathbf{t}) = \left(1+\mathrm{O}\left(\lambda^{-1}\right)\right) e^{-\sigma(n,\mathbf{t})} \lambda^{-n}\, e^{-\sum_{k\geq 0} t_k \lambda^{k+1}}$ that

$$M(\lambda) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{Mat}\left(2, \widetilde{V}\left[\left[\mathbf{t}, \lambda^{-1}\right]\right] \lambda^{-1}\right). \tag{83}$$

The proposition then follows from the uniqueness theorem proved in Sect. 2. □

Define

$$D(\lambda, \mu, n, \mathbf{t}) := \frac{\psi_A(\lambda, n, \mathbf{t})\, \psi_B(\mu, n-1, \mathbf{t}) - \psi_A(\lambda, n-1, \mathbf{t})\, \psi_B(\mu, n, \mathbf{t})}{\lambda - \mu}. \tag{84}$$

Theorem 2 *Fix $k \geq 2$ being an integer. The generating series of k-point correlation functions of the solution $(v(n, \mathbf{t}), w(n, \mathbf{t}))$ has the following expression:*

$$\sum_{i_1,\dots,i_k \geq 0} \frac{\Omega_{i_1,\dots,i_k}(n, \mathbf{t})}{\lambda_1^{i_1+2} \cdots \lambda_k^{i_k+2}} = (-1)^{k-1} \frac{e^{k\sigma(n-1,\mathbf{t})}}{\prod_{j=1}^k \lambda_j} \sum_{\pi \in \mathcal{S}_k / C_k} \prod_{j=1}^k D(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n, \mathbf{t}) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \tag{85}$$

Proof It follows from (81) that

$$R(\lambda, n, \mathbf{t}) \;=\; \frac{r_1(\lambda, n, \mathbf{t})^T r_2(\lambda, n, \mathbf{t})}{d(\lambda, n, \mathbf{t})}, \tag{86}$$

where $r_1(\lambda, n, \mathbf{t}) \;:=\; (\psi_A(\lambda, n, \mathbf{t}), \psi_A(\lambda, n-1, \mathbf{t}))$, $r_2(\lambda, n, \mathbf{t}) \;:=\; (\psi_B(\lambda, n-1, \mathbf{t}), -\psi_B(\lambda, n, \mathbf{t}))$. Substituting this expression into the identity

$$\sum_{i_1, i_2 \geq 0} \frac{\Omega_{i_1,i_2}(n, \mathbf{t})}{\lambda_1^{i_1+2} \lambda_2^{i_2+2}} \;=\; \frac{\mathrm{Tr}\left(R_1(\lambda_1, n, \mathbf{t})\, R_2(\lambda_2, n, \mathbf{t})\right)}{(\lambda_1 - \lambda_2)^2} - \frac{1}{(\lambda_1 - \lambda_2)^2}, \tag{87}$$

we obtain

$$\begin{aligned} \sum_{i_1, i_2 \geq 0} \frac{\Omega_{i_1,i_2}(n, \mathbf{t})}{\lambda_1^{i_1+2} \lambda_2^{i_2+2}} \;&=\; \frac{\mathrm{Tr}\left(r_1(\lambda_1, n, \mathbf{t})^T r_2(\lambda_1, n, \mathbf{t})\, r_1(\lambda_2, n, \mathbf{t})^T r_2(\lambda_2, n, \mathbf{t})\right)}{d(\lambda_1, n, \mathbf{t}) d(\lambda_2, n, \mathbf{t})(\lambda_1 - \lambda_2)^2} \\ &\quad - \frac{1}{(\lambda_1 - \lambda_2)^2} \\ &= \frac{\left(r_2(\lambda_2, n, \mathbf{t})\, r_1(\lambda_1, n, \mathbf{t})^T\right)\left(r_2(\lambda_1, n, \mathbf{t})\, r_1(\lambda_2, n, \mathbf{t})^T\right)}{d(\lambda_1, n, \mathbf{t}) d(\lambda_2, n, \mathbf{t})(\lambda_1 - \lambda_2)^2} \\ &\quad - \frac{1}{(\lambda_1 - \lambda_2)^2} \\ &= -\frac{D(\lambda_1, \lambda_2, n, \mathbf{t})\, D(\lambda_2, \lambda_1, n, \mathbf{t})}{\lambda_1 \lambda_2\, e^{-2\sigma(n-1,\mathbf{t})}} - \frac{1}{(\lambda_1 - \lambda_2)^2}, \end{aligned} \tag{88}$$

where we used definition (84) and

$$\begin{aligned} &\frac{\psi_A(\lambda, n, \mathbf{t})\, \psi_B(\mu, n-1, \mathbf{t}) \;-\; \psi_A(\lambda, n-1, \mathbf{t})\, \psi_B(\mu, n, \mathbf{t})}{\lambda - \mu} \\ &\quad = \frac{r_2(\mu, n, \mathbf{t})\, r_1(\lambda, n, \mathbf{t})^T}{\lambda - \mu}. \end{aligned}$$

This proves the $k = 2$ case of (85). For $k \geq 3$, the proof is similar. Indeed,

$$\begin{aligned} &\sum_{i_1, \ldots, i_k \geq 0} \frac{\Omega_{i_1,\ldots,i_k}(n, \mathbf{t})}{\lambda_1^{i_1+1} \ldots \lambda_k^{i_k+1}} \\ &\quad = -\sum_{\pi \in \mathcal{S}_k/C_k} \frac{\mathrm{Tr}\left(\prod_{j=1}^k r_1\left(\lambda_{\pi(j)}, n, \mathbf{t}\right)^T r_2\left(\lambda_{\pi(j)}, n, \mathbf{t}\right)\right)}{e^{-k\,\sigma(n-1,\mathbf{t})} \prod_{j=1}^k \left(\lambda_{\pi(j)} - \lambda_{\pi(j+1)}\right)} \\ &\quad = -\sum_{\pi \in \mathcal{S}_k/C_k} \frac{r_2\left(\lambda_{\pi(k)}, n, \mathbf{t}\right) r_1\left(\lambda_{\pi(1)}, n, \mathbf{t}\right)^T \ldots r_2\left(\lambda_{\pi(k-1)}, n, \mathbf{t}\right) r_1\left(\lambda_{\pi(k)}, n, \mathbf{t}\right)^T}{e^{-k\,\sigma(n-1,\mathbf{t})} \prod_{j=1}^k \left(\lambda_{\pi(j)} - \lambda_{\pi(j+1)}\right)} \\ &\quad = -\frac{(-1)^k}{e^{-k\,\sigma(n-1,\mathbf{t})}} \sum_{\pi \in \mathcal{S}_k/C_k} \prod_{j=1}^k D\left(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n, \mathbf{t}\right). \end{aligned} \tag{89}$$

This proves the $k \geq 3$ case of (85). The theorem is proved. □

Remark 2 In (85) or (30), the freedom (75) affects the $D(\lambda, \mu)$ through multiplying it by a factor of the form $\frac{G(\lambda)}{G(\mu)}$, but the product $\prod_{j=1}^{k} D(\lambda_{\pi(j)}, \lambda_{\pi(j+1)})$ remains unchanged.

In Appendix A, the abstract form of (85) is obtained, where a pair of abstract pre-wave functions are introduced.

Proof of Theorem 1 Taking $\mathbf{t} = \mathbf{0}$ on the both sides of (85) gives (30). □

Write

$$\psi_A(\lambda, n, \mathbf{t}) \ = \ \phi_A(\lambda, n, \mathbf{t})\, \lambda^n, \qquad \psi_B(\lambda, n, \mathbf{t}) \ = \ \phi_B(\lambda, n, \mathbf{t})\, e^{-\sigma(n,\mathbf{t})}\, \lambda^{-n}. \tag{90}$$

Theorem 1 can then be alternatively written in terms of ϕ_A, ϕ_B by the following corollary.

Corollary 1 *The following formula holds true for $k \geq 2$:*

$$\sum_{i_1,\dots,i_k \geq 0} \frac{\Omega_{i_1,\dots,i_k}(n, \mathbf{t})}{\lambda_1^{i_1+2} \dots \lambda_k^{i_k+2}} = (-1)^{k-1} \sum_{\pi \in \mathcal{S}_k / C_k} \prod_{j=1}^{k} B(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n, \mathbf{t}) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}, \tag{91}$$

where $B(\lambda, \mu, n, \mathbf{t})$ is defined by

$$B(\lambda, \mu, n, \mathbf{t}) := \frac{\phi_A(\lambda, n, \mathbf{t})\, \phi_B(\mu, n-1, \mathbf{t}) \ - \ w(n, \mathbf{t})\, \phi_A(\lambda, n-1, \mathbf{t})\, \phi_B(\mu, n, \mathbf{t})}{\lambda - \mu}. \tag{92}$$

In particular, let $\phi_A(\lambda, n) := e^{(\Lambda-1)^{-1}(y(\lambda,n))}$, $\phi_B(\lambda, n) := e^{(\Lambda-1)^{-1}(z(\lambda,n))} e^{-s(n)}$ (cf. (60)–(61)), and let $B(\lambda, \mu, n) := \frac{\phi_A(\lambda,n)\, \phi_B(\mu,n-1) - g(n)\, \phi_A(\lambda,n-1)\, \phi_B(\mu,n)}{\lambda-\mu}$, then we have

$$\sum_{i_1,\dots,i_k \geq 0} \frac{\Omega_{i_1,\dots,i_k}(n, \mathbf{0})}{\lambda_1^{i_1+2} \dots \lambda_k^{i_k+2}} = (-1)^{k-1} \sum_{\pi \in \mathcal{S}_k / C_k} \prod_{j=1}^{k} B(\lambda_{\pi(j)}, \lambda_{\pi(j+1)}, n) - \frac{\delta_{k,2}}{(\lambda_1 - \lambda_2)^2}. \tag{93}$$

For some particular examples related to matrix models, it turns out that the suitable chosen D coincides, possibly up to simple factors, with certain kernel of the matrix model. However, the D is not unique. We now introduce a formal series $K(\lambda, \mu)$ such that the generating series of multi-point correlation functions still has an explicit expression, but this time K is *local* and is therefore unique for the given solution. The series K is defined by

$$K(\lambda, \mu) := \frac{(1 + \alpha(\lambda))(1 + \alpha(\mu)) - w_0\, \gamma(\lambda)\, \Lambda\big(\gamma(\mu)\big)}{\lambda - \mu}, \tag{94}$$

where $1+\alpha(\lambda)$ is the (1,1) entry of the basic matrix resolvent $R(\lambda)$, and $\gamma(\lambda)$ is the (2,1) entry. The next theorem expresses the left-hand side of (85) in terms of K.

Theorem 3 *For any $k \geq 2$, the following formula holds true:*

$$\sum_{i_1,\ldots,i_k\geq 0} \frac{\Omega_{i_1,\ldots,i_k}}{\lambda_1^{i_1+2}\ldots\lambda_k^{i_k+2}} = (-1)^{k-1} \frac{\sum_{\pi\in\mathcal{S}_k/C_k}\prod_{j=1}^{k} K\left(\lambda_{\pi(j)},\lambda_{\pi(j+1)}\right)}{\prod_{i=1}^{k}\left(1+\alpha(\lambda_i)\right)} - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}. \tag{95}$$

Proof The identity (81) gives

$$\begin{aligned}
\psi_B(\lambda,n-1,\mathbf{t}) &= \frac{(1+\alpha(\lambda,n,\mathbf{t}))\,d(\lambda,n,\mathbf{t})}{\psi_A(\lambda,n,\mathbf{t})},\\
\psi_B(\lambda,n,\mathbf{t}) &= -\frac{\beta(\lambda,n,\mathbf{t})\,d(\lambda,n,\mathbf{t})}{\psi_A(\lambda,n,\mathbf{t})} = w_n\,\frac{\gamma(\lambda,n+1,\mathbf{t})\,d(\lambda,n,\mathbf{t})}{\psi_A(\lambda,n,\mathbf{t})},\\
\psi_A(\lambda,n-1,\mathbf{t}) &= \psi_A(\lambda,n,\mathbf{t})\,\frac{\gamma(\lambda,n,\mathbf{t})}{1+\alpha(\lambda,n,\mathbf{t})}.
\end{aligned}$$

Substituting these expressions into (84), we obtain

$$D(\lambda,\mu,n,\mathbf{t}) = d(\mu,n,\mathbf{t})\frac{\psi_A(\lambda,n,\mathbf{t})}{\psi_A(\mu,n,\mathbf{t})}e(\lambda,\mu,n,\mathbf{t}), \tag{96}$$

where

$$e(\lambda,\mu,n,\mathbf{t}):=\frac{(1+\alpha(\lambda,n,\mathbf{t}))(1+\alpha(\mu,n,\mathbf{t}))-w_n(\mathbf{t})\,\gamma(\lambda,n,\mathbf{t})\,\gamma(\mu,n+1,\mathbf{t})}{(\lambda-\mu)\,(1+\alpha(\lambda,n,\mathbf{t}))}. \tag{97}$$

Combining with the definition of $K(\lambda,\mu,n,\mathbf{t})$ and Theorem 1, we find

$$\begin{aligned}
&\sum_{i_1,\ldots,i_k\geq 0}\frac{\Omega_{i_1,\ldots,i_k}(n,\mathbf{t})}{\lambda_1^{i_1+2}\ldots\lambda_k^{i_k+2}}\\
&= (-1)^{k-1}\sum_{\pi\in\mathcal{S}_k/C_k}\prod_{j=1}^{k} K\left(\lambda_{\pi(j)},\lambda_{\pi(j+1)},n,\mathbf{t}\right) - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}.
\end{aligned} \tag{98}$$

The theorem is proved. □

It seems to be an interesting question to study the geometric and algebraic meaning of the kernel K (as well as D). Below we give without proof some of their properties.

Proposition 4 *The functions K and D are related to*

$$K(\lambda,\mu,n,\mathbf{t}) = \frac{e^{\sigma(n-1,\mathbf{t})}}{\mu}\left(1+\alpha(\lambda,n,\mathbf{t})\right)\frac{\psi_A(\mu,n,\mathbf{t})}{\psi_A(\lambda,n,\mathbf{t})}\,D(\lambda,\mu,n,\mathbf{t})$$

$$= \frac{e^{2\sigma(n-1,\mathbf{t})}}{\lambda\,\mu}\,\psi_A(\mu,n,\mathbf{t})\,\psi_B(\lambda,n-1,\mathbf{t})\,D(\lambda,\mu,n,\mathbf{t})$$
$$= \frac{e^{\sigma(n-1,\mathbf{t})}}{\lambda}\,\big(1+\alpha(\mu,n,\mathbf{t})\big)\,\frac{\psi_B(\lambda,n-1,\mathbf{t})}{\psi_B(\mu,n-1,\mathbf{t})}\,D(\lambda,\mu,n,\mathbf{t}).$$

We observe that the following three formal series

$$K(\lambda,\mu)-\frac{1+\alpha(\lambda)}{\lambda-\mu},\quad K(\lambda,\mu)-\frac{1+\alpha(\mu)}{\lambda-\mu},\quad K(\lambda,\mu)-\frac{2+\alpha(\lambda)+\alpha(\mu)}{2\,(\lambda-\mu)}$$

all belong to $\mathcal{A}\big[\big[\lambda^{-1},\mu^{-1}\big]\big]$. Therefore, the following three formal series

$$K(\lambda,\mu,n,\mathbf{t})-\frac{1+\alpha(\lambda,n,\mathbf{t})}{\lambda-\mu},\quad K(\lambda,\mu,n,\mathbf{t})-\frac{1+\alpha(\mu,n,\mathbf{t})}{\lambda-\mu},$$
$$K(\lambda,\mu,n,\mathbf{t})-\frac{2+\alpha(\lambda,n,\mathbf{t})+\alpha(\mu,n,\mathbf{t})}{2(\lambda-\mu)}$$

all belong to $V[[\mathbf{t}]]\big[\big[\lambda^{-1},\mu^{-1}\big]\big]$. It follows from this observation and Proposition 4 that

$$\frac{e^{s(n-1)}}{\mu}\,D(\lambda,\mu,n,\mathbf{0})\left(\frac{\mu}{\lambda}\right)^{n}-\frac{1}{\lambda-\mu}\ \in\ \widetilde{V}\left[\left[\lambda^{-1},\mu^{-1}\right]\right]. \tag{99}$$

Remark 3 We could loosen both the conditions for wave functions and the pair condition. Let us say ψ_A and ψ_B are pre-wave functions of type A and of type B, respectively, if they satisfy the first equations of (67) and (68). Define $d_{\rm pre}(\lambda,n,\mathbf{t})$ and $D_{\rm pre}(\lambda,\mu,n,\mathbf{t})$ by (129) and (140). Then, the following formula holds true:

$$\sum_{i_1,\ldots,i_k\geq 0}\frac{\Omega_{i_1,\ldots,i_k}(n,\mathbf{t})}{\lambda_1^{i_1+2}\cdots\lambda_k^{i_k+2}}$$
$$=\frac{(-1)^{k-1}}{\prod_{j=1}^{k}d_{\rm pre}(\lambda_j,n,\mathbf{t})}\sum_{\pi\in\mathcal{S}_k/C_k}\prod_{j=1}^{k}D_{\rm pre}(\lambda_{\pi(j)},\lambda_{\pi(j+1)},n,\mathbf{t})-\frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}. \tag{100}$$

Now, ψ_A and ψ_B are determined by $(v(n,\mathbf{t}),w(n,\mathbf{t}))$ up to

$$\psi_A(\lambda,n,\mathbf{t})\ \mapsto\ G(\lambda,\mathbf{t})\,\psi_A(\lambda,n,\mathbf{t}),\quad \psi_B(\lambda,n,\mathbf{t})\ \mapsto\ E(\lambda,\mathbf{t})\,\psi_B(\lambda,n,\mathbf{t}),$$

where $G(\lambda,\mathbf{t})=1+\sum_{k\geq1}G_k(\mathbf{t})\lambda^{-k}$, $E(\lambda,\mathbf{t})=1+\sum_{k\geq1}E_k(\mathbf{t})\lambda^{-k}$ with $G_k(\mathbf{t}),E_k(\mathbf{t})\in\mathbb{C}[[\mathbf{t}]]$, $k\geq1$. This freedom affects $D_{\rm pre}(\lambda,\mu,n,\mathbf{t})$ and $d_{\rm pre}(\lambda,n,\mathbf{t})$ into

$$D_{\rm pre}(\lambda,\mu,n,\mathbf{t})\mapsto G(\lambda,\mathbf{t})\,E(\mu,\mathbf{t})\,D_{\rm pre}(\lambda,\mu,\mathbf{t}),$$
$$d_{\rm pre}(\lambda,n,\mathbf{t})\mapsto G(\lambda,\mathbf{t})\,E(\lambda,\mathbf{t})\,d_{\rm pre}(\lambda,\mathbf{t}).$$

Therefore, it gives rise to each summand of (100) the factor

$$\frac{\prod_{j=1}^{k} G(\lambda_{\pi(j)}, \mathbf{t}) E(\lambda_{\pi(j+1)}, \mathbf{t})}{\prod_{j=1}^{k} G(\lambda_j, \mathbf{t}) E(\lambda_j, \mathbf{t})},$$

which is equal to 1. Hence, the right-hand side of (100) remains unchanged.

5 Applications

Partition functions in some matrix models and enumerative models are particular tau-functions for the Toda lattice hierarchy. Theorem 1 can then be used for computing their logarithmic derivatives. In this section, we do two explicit computations.

5.1 Application I: enumeration of ribbon graphs

The initial data of the GUE solution to the Toda lattice hierarchy are given by $f(n) = 0$ and $g(n) = n$; see, for example, [10] for the proof. For this case, we can take $V = \mathbb{Q}[n]$ and $\widetilde{V} = V$. Substituting the initial data in (26), we find

$$\begin{aligned} s(n) \;&=\; -\big(1 - \Lambda^{-1}\big)^{-1} \log g(n) \\ &=\; -\big(1 - \Lambda^{-1}\big)^{-1} \log n \;=\; -\log \Gamma(n+1) \;+\; C, \end{aligned} \tag{101}$$

where C is a constant. Below, we fix this constant as 0.

Proposition 5 *The ψ_A, ψ_B defined by*

$$\psi_A(\lambda, n) = \sum_{j\ge 0} (-1)^j \frac{(n-2j+1)_{2j}}{2^j\, j!\, \lambda^{2j}} \lambda^n, \tag{102}$$

$$\psi_B(\lambda, n) = \Gamma(n+1) \sum_{j\ge 0} \frac{(n+1)_{2j}}{2^j\, j!\, \lambda^{2j}} \lambda^{-n} \tag{103}$$

form a particular pair of wave functions associated with $(f(n) = 0, g(n) = n)$. Here and below $(a)_i$ denotes the increasing Pochhammer symbol defined by $(a)_i = a(a+1)\ldots(a+i-1)$.

Proof It is straightforward to verify that both ψ_A and ψ_B satisfy the equation

$$\psi(\lambda, n+1) \;+\; n\, \psi(\lambda, n-1) \;=\; \lambda\, \psi(\lambda, n). \tag{104}$$

Moreover, from definitions (102)–(103), we see that

$$\psi_A \in \widetilde{V}\big(\!\big(\lambda^{-1}\big)\!\big)\, \lambda^n, \quad \psi_B \in \widetilde{V}\big(\!\big(\lambda^{-1}\big)\!\big) e^{-s(n)} \lambda^{-n}.$$

We are left to show that

$$\Gamma(n)^{-1}\Big(\psi_A(\lambda,n)\,\psi_B(\lambda,n-1)-\psi_B(\lambda,n)\,\psi_A(\lambda,n-1)\Big) \;=\; \lambda. \tag{105}$$

Clearly, the meaning of this identity is the following: Both sides of (105) are Laurent series of λ^{-1} with coefficients in $\widetilde{V} = V = Q[n]$, and the equality means all the coefficients should be equal. More precisely, the identity (105) can be equivalently written as the following sequence of identities:

$$\begin{aligned}\frac{n}{j+1}\sum_{j_1=0}^{j+1}\frac{(-1)^{j_1}}{2}\binom{j+1}{j_1}\binom{n+2j_1-1}{2j+1}\\ +\sum_{j_1=0}^{j}(-1)^{j_1}\binom{j}{j_1}\binom{n+2j_1}{2j+1} \;=\; 0, \quad j\geq 0.\end{aligned} \tag{106}$$

From (64), we know that the left-hand side of (106) as a polynomial of n is a constant for any $j \geq 0$. Note that the value of the left-hand side of (106) at $n = 0$ is obviously 0 for any $j \geq 0$. The proposition is proved. □

It follows from the above proposition an explicit expression for the $D(\lambda,\mu,n,\mathbf{0})$ (cf. Eq. (84)) associated with the pair (102)–(103):

$$\frac{e^{s(n-1)}}{\mu}\,D(\lambda,\mu,n,\mathbf{0})\left(\frac{\mu}{\lambda}\right)^{n} \;=\; \frac{1}{\lambda-\mu} \;+\; A(\lambda,\mu,n), \tag{107}$$

with $A(\lambda,\mu,n)$ given by

$$\begin{aligned}A(\lambda,\mu,n) = \sum_{k\geq 1}\frac{(2k-1)!!}{(2k)!}\sum_{p=0}^{2k-1}(-1)^{p+[(p+1)/2]}\binom{k-1}{[p/2]}\\ \cdot\prod_{j=-p}^{2k-1-p}(n+j)\,\lambda^{-p-1}\mu^{-(2k-p)}.\end{aligned} \tag{108}$$

This explicit expression (108) first appeared in [31]. Denote

$$\widehat{A}(\lambda,\mu,n) \;=\; \frac{1}{\lambda-\mu} \;+\; A(\lambda,\mu,n). \tag{109}$$

As a corollary of Proposition 5, Theorem 1, and the above (107), we have achieved a new proof of the following theorem of Jian Zhou.

Theorem 4 [31] *Fix $k \geq 2$ being an integer. The generating series of k-point connected GUE correlators has the following expression:*

$$\sum_{i_1,\ldots,i_k\geq 1} \frac{\left\langle \operatorname{tr} M^{i_1}\ldots\operatorname{tr} M^{i_k}\right\rangle_{\rm c}}{\lambda_1^{i_1+1}\ldots\lambda_k^{i_k+1}} = (-1)^{k-1}\sum_{\pi\in\mathcal{S}_k/C_k}\prod_{j=1}^{k}\widehat{A}\left(\lambda_{\pi(j)},\lambda_{\pi(j+1)},n\right) - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}, \tag{110}$$

where $\widehat{A}$ is defined by (108)–(109)*. Here, we recall that for any fixed $i_1,\ldots,i_k$, the connected GUE correlator $\langle \operatorname{tr} M^{i_1}\ldots\operatorname{tr} M^{i_k}\rangle_{\rm c}$ is a polynomial of n (cf.* [4,10,17,21]*).*

5.2 Application II: Gromov–Witten invariants of $\mathbb{P}^1$ in the stationary sector

The initial data for the Gromov–Witten solution to the Toda lattice hierarchy were, for example, derived in [10–12]. It has the following explicit expression:

$$f(n) = n\epsilon + \frac{\epsilon}{2}, \quad g(n) = 1. \tag{111}$$

We have

$$s(n) = -\left(1-\Lambda^{-1}\right)^{-1}\log 1 = C,$$

where C is an arbitrary constant. Below, we take $C=0$.

Proposition 6 *The ψ_1, ψ_2 defined by*

$$\psi_A(\lambda,n) = \epsilon^{\frac{\lambda}{\epsilon}-\frac12}\,\Gamma\left(\frac{\lambda}{\epsilon}+\frac12\right) J_{\frac{\lambda}{\epsilon}-n-\frac12}\left(\frac{2}{\epsilon}\right), \tag{112}$$

$$\psi_B(\lambda,n) = (-1)^{n+1}\epsilon^{-\frac{\lambda}{\epsilon}-\frac12}\,\lambda\,\Gamma\left(-\frac{\lambda}{\epsilon}+\frac12\right) J_{-\frac{\lambda}{\epsilon}+n+\frac12}\left(\frac{2}{\epsilon}\right) \tag{113}$$

form a particular pair of wave functions associated with $f(n)=n\epsilon+\frac{\epsilon}{2}$, $g(n)=1$. Here, $J_\nu(y)$ denotes the Bessel function, and the right-hand sides of (112)–(113) *are understood as the large λ asymptotics of the corresponding analytic functions.*

Proof Firstly, using the properties of Bessel functions, we can verify that $\psi_A(\lambda,n)$ and $\psi_B(\lambda,n)$ defined from the above asymptotics satisfy

$$\psi_A(\lambda,n+1) + \left(n\epsilon+\frac{\epsilon}{2}\right)\psi_A(\lambda,n) + \psi_A(\lambda,n-1) = \lambda\,\psi_A(\lambda,n),$$

$$\psi_B(\lambda,n+1) + \left(n\epsilon+\frac{\epsilon}{2}\right)\psi_B(\lambda,n) + \psi_B(\lambda,n-1) = \lambda\,\psi_B(\lambda,n).$$

Secondly, as λ goes to ∞, the following asymptotics hold true:

$$\epsilon^{\frac{\lambda}{\epsilon}-\frac{1}{2}}\,\Gamma\Big(\frac{\lambda}{\epsilon}+\frac{1}{2}\Big)\,J_{\frac{\lambda}{\epsilon}-n-\frac{1}{2}}\Big(\frac{2}{\epsilon}\Big)\;\sim\;\lambda^n\Big(1\,+\,\mathrm{O}\big(\lambda^{-1}\big)\Big),$$
$$(-1)^{n+1}\epsilon^{-\frac{\lambda}{\epsilon}-\frac{1}{2}}\,\lambda\,\Gamma\Big(-\frac{\lambda}{\epsilon}+\frac{1}{2}\Big)\,J_{-\frac{\lambda}{\epsilon}+n+\frac{1}{2}}\Big(\frac{2}{\epsilon}\Big)\;\sim\;\lambda^{-n}\Big(1\,+\,\mathrm{O}\big(\lambda^{-1}\big)\Big).$$

Thirdly, ψ_A and ψ_B also satisfy

$$\psi_A(\lambda,n)\,\psi_B(\lambda,n-1)\;-\;\psi_B(\lambda,n)\,\psi_A(\lambda,n-1)\;=\;\lambda.$$

We have verified all the defining properties for a pair of wave functions associated with $f(n)=n\epsilon+\frac{\epsilon}{2}$, $g(n)=1$. The proposition is proved. □

Note that

$$\psi_A(\lambda,n-1)\;=\;\epsilon^{\frac{\lambda}{\epsilon}-\frac{1}{2}}\,\Gamma\Big(\frac{\lambda}{\epsilon}+\frac{1}{2}\Big)\,J_{\frac{\lambda}{\epsilon}-n+\frac{1}{2}}\Big(\frac{2}{\epsilon}\Big),\tag{114}$$
$$\psi_B(\lambda,n-1)\;=\;(-1)^n\epsilon^{-\frac{\lambda}{\epsilon}-\frac{1}{2}}\,\lambda\,\Gamma\Big(-\frac{\lambda}{\epsilon}+\frac{1}{2}\Big)\,J_{-\frac{\lambda}{\epsilon}+n-\frac{1}{2}}\Big(\frac{2}{\epsilon}\Big),\tag{115}$$

and denote

$$J_\nu(y)\;=:\;\frac{(y/2)^\nu}{\Gamma(\nu+1)}\,j_{\nu+\frac{1}{2}}(y^2/4).$$

It follows from (84), (112)–(115) that the $D(\lambda,\mu,0,\mathbf{0})$ associated with the pair (112)–(113) has the following explicit expression:

$$\frac{1}{\mu}D(\lambda,\mu,0,\mathbf{0})\;=\;-\frac{1}{\epsilon}\,\frac{j_{-\frac{\mu}{\epsilon}}\big(\frac{1}{\epsilon^2}\big)\,j_{\frac{\lambda}{\epsilon}}\big(\frac{1}{\epsilon^2}\big)+\frac{\epsilon^{-2}}{(\frac{1}{2}-\frac{\mu}{\epsilon})(\frac{1}{2}+\frac{\lambda}{\epsilon})}\,j_{1-\frac{\mu}{\epsilon}}\big(\frac{1}{\epsilon^2}\big)\,j_{1+\frac{\lambda}{\epsilon}}\big(\frac{1}{\epsilon^2}\big)}{\mu/\epsilon-\lambda/\epsilon}.$$

Then, according to [12], the function $\frac{1}{\mu}D(\lambda,\mu,0,\mathbf{0})$ has the following expressions:

$$\frac{1}{\mu}D(\lambda,\mu,0,\mathbf{0})$$
$$=\;-\frac{1}{\epsilon}\sum_{k=0}^{\infty}\frac{(a-b-2k+1)_{k-1}}{k!\,(-a+\frac{1}{2})_k\,(b+\frac{1}{2})_k}\epsilon^{-2k}\tag{116}$$
$$=\;\frac{-1}{\epsilon(a-b)}\;{}_2F_3\Big(\frac{b-a}{2},\,\frac{b-a+1}{2};\;\frac{1}{2}-a\,,\;\frac{1}{2}+b,b-a+1;\;-4\epsilon^{-2}\Big)\tag{117}$$
$$\sim\;\frac{-1}{\epsilon(a-b)}\;-\;\sum_{p,q\geq 0}\frac{(-1)^{q+1}}{a^{p+1}b^{q+1}}\sum_{k\geq 1}\frac{\epsilon^{-2k-1}}{k!}$$
$$\sum_{1\leq i,j\leq k}(-1)^{i+j}\,\frac{(i+j-2k)_{k-1}\big(i-\frac{1}{2}\big)^p\big(j-\frac{1}{2}\big)^q}{(i-1)!(j-1)!(k-i)!(k-j)!}\;\;=:\;\widehat{A}(\lambda,\mu),\tag{118}$$

where $a := \frac{\mu}{\epsilon}$, $b := \frac{\lambda}{\epsilon}$, the $(a-b+1)_{-1}$ of (116) is defined as $1/(a-b)$, and $\sim$ in (118) is taken as $a, b \to \infty$ away from the half integers. The explicit expression (118) first appeared in [12]. So we have completed a new proof of the following theorem.

Theorem 5 [12] *The generating series of k-point ($k \geq 2$) Gromov–Witten invariants of $\mathbb{P}^1$ in the stationary sector has the following explicit expression:*

$$\epsilon^k \sum_{i_1,\ldots,i_k \geq 0} \frac{(i_1+1)!\ldots(i_k+1)!}{\lambda_1^{i_1+2}\ldots\lambda_k^{i_k+2}} \langle \tau_{i_1}(\omega)\ldots\tau_{i_k}(\omega)\rangle(\epsilon) = (-1)^{k-1} \sum_{\pi\in\mathcal{S}_k/C_k} \prod_{i=1}^{k} \widehat{A}\left(\lambda_{\pi(i)}, \lambda_{\pi(i+1)}\right) - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}, \tag{119}$$

where $\widehat{A}(\lambda,\mu)$ is explicitly defined in (118), and

$$\langle \tau_{i_1}(\omega)\ldots\tau_{i_k}(\omega)\rangle(\epsilon) := \sum_{g\geq 0} \epsilon^{2g-2} \sum_{d\geq 0} \int_{\left[\overline{\mathcal{M}}_{g,k}(\mathbb{P}^1,d)\right]^{\mathrm{virt}}} \mathrm{ev}_1^*(\omega)\ldots\mathrm{ev}_k^*(\omega)\,\psi_1^{i_1}\ldots\psi_k^{i_k}. \tag{120}$$

(See, for example, [12] for the notation about the integral in the right-hand side of (120).)

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Appendix A: Pair of abstract pre-wave functions

Here, we construct a ring that is suitable for defining abstract pre-wave functions. Recall that $\mathcal{A}$ is the ring of polynomials of $v_k, w_k, k \in \mathbb{Z}$. Instead of the $\mathbb{Z}$-coefficients, we will use in this appendix the $\mathbb{Q}$-coefficients, i.e., $\mathcal{A} = \mathbb{Q}\big[\{v_k, w_k \,|\, k \in \mathbb{Z}\}\big]$, is now under consideration. For each monic monomial $\alpha \in \mathcal{A}\backslash\mathbb{Q}$, we associate a symbol m_α. Denote by $\mathcal{B}$ the polynomial ring

$$\mathcal{B} := \mathbb{Q}\big[\{m_\alpha \,|\, \alpha \text{ is a monic monomial in } \mathcal{A}\backslash\mathbb{Q}\}\big]. \tag{121}$$

Define the action of Λ^k on $\mathcal{B}$ with $k \in \mathbb{Z}$ by

$$\Lambda^k\big(m_{\alpha_1}\ldots m_{\alpha_l}\big) = m_{\Lambda^k(\alpha_1)}\ldots m_{\Lambda^k(\alpha_l)} \tag{122}$$

for $\alpha_1,\ldots,\alpha_l$ being monic monomials in $\mathcal{A}\backslash\mathbb{Q}$, as well as by linearly extending it to other elements of $\mathcal{B}$. For a monic monomial $\alpha = v_{i_1}\ldots v_{i_r} w_{j_1}\ldots w_{j_s} \in \mathcal{A}\backslash\mathbb{Q}$ with $i_1 \leq \cdots \leq i_r$, $j_1 \leq \cdots \leq j_s$ and $r+s \geq 1$, let $k_\alpha := -i_1$ (if $r \geq 1$), $k_\alpha := -j_1$

(if $r = 0$); the monomial $\Lambda^{k_\alpha}(\alpha) \in \mathcal{A}$ is then called the (unique) reduced monomial (associated to α). Denote by $\mathcal{C}$ the polynomial ring generated by m_β, v_k, w_k with $\mathbb{Q}$-coefficients, where β are reduced monic monomials, and $k \in \mathbb{Z}$. Let us also define an action of Λ^k on $\mathcal{C}$, $k \in \mathbb{Z}$. To this end, we introduce some notations: For β a reduced monic monomial of $\mathcal{A}$, denote

$$n_{\Lambda^k(\beta)} := \begin{cases} m_\beta + \sum_{i=0}^{k-1} \Lambda^i(\beta), & k \geq 0, \\ m_\beta - \sum_{i=k}^{-1} \Lambda^i(\beta), & k \leq -1. \end{cases} \tag{123}$$

Then, for a monomial $\alpha \cdot m_{\beta_1} \dots m_{\beta_s}$ of $\mathcal{C}$ with α being a monomial in $\mathcal{A}$, define

$$\Lambda^k(\alpha \cdot m_{\beta_1} \dots m_{\beta_s}) = \Lambda^k(\alpha) \cdot \prod_{j=1}^{s} n_{\Lambda^k(\beta_j)}, \quad k \in \mathbb{Z}. \tag{124}$$

Define the action of Λ^k on other elements in $\mathcal{C}$ by requiring it as a linear operator. Denote by $p : \mathcal{B} \to \mathcal{C}$ the linear map which maps $m_{\alpha_1} \dots m_{\alpha_l} \in \mathcal{B}$ to $n_{\alpha_1} \dots n_{\alpha_l} \in \mathcal{C}$, for α_i, $i = 1, \dots, l$ being monic monomials in $\mathcal{A}\backslash\mathbb{Q}$. Denote by $\mathcal{B}^0$ the image of p. Clearly, $\mathcal{A} \subset \mathcal{B}^0$. Indeed, the element $(\Lambda - 1)\left(\sum_{i=1}^{l} \lambda_i\, m_{\alpha_i}\right) \in \mathcal{B}$ becomes $\sum_{i=1}^{l} \lambda_i \alpha_i \in \mathcal{A}$ under the map p. Here, $\alpha_1, \dots, \alpha_l$ are distinct monic monomials in $\mathcal{A}\backslash\mathbb{Q}$. Finally, we define an operator $\mathbb{S} : \mathcal{A}\backslash\mathbb{Q} \to \mathcal{B}^0$ by

$$\mathbb{S}\left(\lambda_1\alpha_1 + \cdots + \lambda_l\alpha_l\right) = \lambda_1 n_{\alpha_1} + \cdots + \lambda_l n_{\alpha_l} \tag{125}$$

for $\alpha_1, \dots, \alpha_l$ being distinct monic monomials and $\lambda_1, \dots, \lambda_l \in \mathbb{Q}$.

Motivated by (62) and (63), define two families of elements $y_i, z_i \in \mathcal{A}$, $i \geq 1$ by

$$\begin{aligned} y_{k+1} &= -\sum_{\substack{m_1,\dots,m_k \geq 0 \\ \sum_{i=1}^k i m_i = k+1}} \frac{\prod_{i=1}^k y_i^{m_i}}{\prod_{i=1}^k m_i!} - v_0 \delta_{k,0} \\ &\quad - w_0 \sum_{m_1,\dots,m_{k-1} \geq 0\, \sum_{i=1}^{k-1} i m_i = k-1} \frac{\prod_{i=1}^{k-1} (-1)^{m_i} \left(\Lambda^{-1}(y_i)\right)^{m_i}}{\prod_{i=1}^{k-1} m_i!}, \\ z_{k+1} &= \sum_{\substack{m_1,\dots,m_k \geq 0 \\ \sum_{i=1}^k i m_i = k+1}} \frac{\prod_{i=1}^k (-1)^{m_i} z_i^{m_i}}{\prod_{i=1}^k m_i!} + v_1 \delta_{k,0} \\ &\quad + w_2 \sum_{\substack{m_1,\dots,m_{k-1} \geq 0 \\ \sum_{i=1}^{k-1} i m_i = k-1}} \frac{\prod_{i=1}^{k-1} \left(\Lambda(z_i)\right)^{m_i}}{\prod_{i=1}^{k-1} m_i!}. \end{aligned}$$

Equivalently, the generating series $y(\lambda) := \sum_{i \geq 1} y_i/\lambda^i$, $z(\lambda) := \sum_{i \geq 1} z_i/\lambda^i$ satisfy

$$\lambda\, e^{y(\lambda)} \;+\; v_0 - \lambda \;+\; w_0\, \lambda^{-1} \Lambda^{-1}\big(e^{-y(\lambda)}\big) \;=\; 0,$$
$$\lambda\, \Lambda^{-1}\big(e^{-z(\lambda)}\big) \;+\; v_0 - \lambda \;+\; w_1\, \lambda^{-1} e^{z(\lambda)} \;=\; 0.$$

Define

$$\psi_A \;:=\; e^{\mathbb{S}(y(\lambda))} \otimes \lambda^n \otimes 1, \quad \psi_B \;:=\; e^{\mathbb{S}(z(\lambda))} \otimes \lambda^{-n} \otimes e^{-\sigma}, \tag{126}$$

where $e^{-\sigma}$ is a formal element satisfying $e^{(1-\Lambda^{-1})(-\sigma)} = w_0$, and λ^n, λ^{-n} are formal elements satisfying $\Lambda^k(1 \otimes \lambda^n) = \lambda^k \otimes \lambda^n$, $\Lambda^k(1 \otimes \lambda^{-n}) = \lambda^{-k} \otimes \lambda^{-n}$, $k \in \mathbb{Z}$. We have

$$L\big(\psi_A\big) \;=\; \lambda\, \psi_A, \quad L\big(\psi_B\big) \;=\; \lambda\, \psi_B,$$
$$\psi_A(\lambda) \;=\; \big(1 + \mathrm{O}\big(\lambda^{-1}\big)\big) \otimes \lambda^n \;\in\; \mathcal{C}\Big[\Big[\lambda^{-1}\Big]\Big] \otimes \lambda^n, \tag{127}$$
$$\psi_B(\lambda) \;=\; \big(1 + \mathrm{O}\big(\lambda^{-1}\big)\big) \otimes \lambda^{-n} \otimes e^{-\sigma} \;\in\; \mathcal{C}\Big[\Big[\lambda^{-1}\Big]\Big] \otimes \lambda^{-n} \otimes e^{-\sigma}, \tag{128}$$

where $L = \Lambda + v_0 + w_0\, \Lambda^{-1}$. We call ψ_A and ψ_B the abstract pre-wave functions of type A and of type B, respectively, associated with v_0, w_0.

Denote

$$d_{\mathrm{pre}}(\lambda) \;:=\; \psi_A(\lambda)\, \Lambda^{-1}\big(\psi_B(\lambda)\big) \;-\; \psi_B(\lambda)\, \Lambda^{-1}\big(\psi_A(\lambda)\big) \tag{129}$$

and

$$\Psi(\lambda) \;:=\; \begin{pmatrix} \psi_A(\lambda) & \psi_B(\lambda) \\ \Lambda^{-1}\big(\psi_A(\lambda)\big) & \Lambda^{-1}\big(\psi_B(\lambda)\big) \end{pmatrix}. \tag{130}$$

Then, we have the following identity:

$$R(\lambda) \;=\; \Psi(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi^{-1}(\lambda) \;=:\; M(\lambda). \tag{131}$$

The proof is similar to that of Proposition 3. (The main fact used in the proof is that from the definition, the coefficients of entries of $R(\lambda)$ are uniquely determined in an algebraic way.) We omit its details here. However, let us explain the equality (131) by an equivalent form. From definition, we have

$$M(\lambda) \;=\; \frac{1}{d_{\mathrm{pre}}(\lambda)} \begin{pmatrix} \psi_A(\lambda)\, \Lambda^{-1}\big(\psi_B(\lambda)\big) & -\psi_A(\lambda)\, \psi_B(\lambda) \\ \Lambda^{-1}\big(\psi_A(\lambda)\big)\, \Lambda^{-1}\big(\psi_B(\lambda)\big) & -\Lambda^{-1}\big(\psi_A(\lambda)\big)\, \psi_B(\lambda) \end{pmatrix}.$$

Then, from a straightforward calculation by using the definitions, we find

$$M_{11}(\lambda) \;=\; \frac{1}{1 \;-\; \frac{w_0}{\lambda^2}\, e^{\Lambda^{-1}(z(\lambda) - y(\lambda))}}, \tag{132}$$

$$M_{12}(\lambda) = \frac{1}{\lambda^{-1}\, e^{-\Lambda^{-1}(y(\lambda))} - \frac{\lambda}{w_0}\, e^{-\Lambda^{-1}(z(\lambda))}}, \tag{133}$$

$$M_{21}(\lambda) = \frac{1}{\lambda\, e^{\Lambda^{-1}(y(\lambda))} - \frac{w_0}{\lambda}\, e^{\Lambda^{-1}(z(\lambda))}}, \tag{134}$$

$$M_{22}(\lambda) = \frac{1}{1 - \frac{\lambda^2}{w_0}\, e^{\Lambda^{-1}(y(\lambda)-z(\lambda))}}. \tag{135}$$

Hence, the equality (131) means new expressions for the entries of the basic matrix resolvent $R(\lambda)$ explicitly in terms of $y(\lambda)$, $z(\lambda)$. Substituting the following expansions

$$y(\lambda) = -\frac{v_0}{\lambda} - \frac{\frac{1}{2}v_0^2 + w_0}{\lambda^2} + \cdots, \qquad z(\lambda) = \frac{v_1}{\lambda} + \frac{\frac{1}{2}v_1^2 + w_2}{\lambda^2} + \cdots \tag{136}$$

into (132)–(135), we find that the new expressions agree with (24). Combining with (56), (57), we obtain

$$\frac{1}{\frac{\lambda^2}{w_0}\, e^{\Lambda^{-1}(y(\lambda)-z(\lambda))} - 1} = \sum_{p\geq 0} \Omega_{p,0}\, \lambda^{-p-2} =: A, \tag{137}$$

$$\frac{1}{\lambda\, e^{\Lambda^{-1}(y(\lambda))} - \frac{w_0}{\lambda}\, e^{\Lambda^{-1}(z(\lambda))}} = \lambda^{-1} + \sum_{p\geq 0} \Lambda^{-1}\big(S_p\big)\, \lambda^{-p-2} =: B. \tag{138}$$

We therefore arrive at the following formulae:

$$e^{\Lambda^{-1}(y(\lambda))} = \frac{1}{\lambda}\,\frac{1+A}{B}, \qquad e^{\Lambda^{-1}(z(\lambda))} = \frac{\lambda}{w_0}\,\frac{A}{B}. \tag{139}$$

Let us proceed to the generating series of multi-point correlation functions. Define

$$D_{\rm pre}(\lambda,\mu) := \frac{\psi_A(\lambda)\, \Lambda^{-1}\big(\psi_B(\mu)\big) - \Lambda^{-1}\big(\psi_A(\lambda)\big)\, \psi_B(\mu)}{\lambda-\mu}. \tag{140}$$

Using (131), Proposition 1, and a similar argument to the proof of Theorem 2, we obtain

$$\sum_{i_1,\dots,i_k\geq 0} \frac{\Omega_{i_1,\dots,i_k}}{\lambda_1^{i_1+2}\cdots\lambda_k^{i_k+2}} = \frac{(-1)^{k-1}}{\prod_{j=1}^k d_{\rm pre}(\lambda_j)} \sum_{\pi\in\mathcal{S}_k/C_k} \prod_{j=1}^{k} D_{\rm pre}(\lambda_{\pi(j)},\lambda_{\pi(j+1)}) - \frac{\delta_{k,2}}{(\lambda_1-\lambda_2)^2}. \tag{141}$$

For the reader's convenience, we give the first few terms of the abstract pre-wave functions $\psi_A(\lambda)$ and $\psi_B(\lambda)$ as follows:

$$\psi_A = \Bigg(1 - \frac{m_{v_0}}{\lambda} + \frac{m_{v_0}^2 - m_{v_0^2} - 2m_{w_0}}{2\lambda^2} - \frac{1}{6\lambda^3}\Big(m_{v_0}^3 + 2m_{v_0^3} - 3m_{v_0}m_{v_0^2} + 6m_{v_0w_0} + 6m_{v_0w_1} - 6m_{v_0}m_{w_0} - 6v_{-1}w_0\Big) + \mathrm{O}\Big(\frac{1}{\lambda^4}\Big)\Bigg)\lambda^n, \tag{142}$$

$$\psi_B = \Bigg(1 + \frac{m_{v_0} + v_0}{\lambda} + \frac{m_{v_0}^2 + m_{v_0^2} + 2v_0m_{v_0} + 2m_{w_0} + 2v_0^2 + 2w_0 + 2w_1}{2\lambda^2} + \frac{1}{6\lambda^3}\Big(m_{v_0}^3 + 6m_{v_0}m_{w_0} + 3m_{v_0}m_{v_0^2} + 2m_{v_0^3} + 6m_{v_0w_1} + 6m_{v_0w_0} + 3v_0m_{v_0}^2 + 6v_0^2m_{v_0} + 6w_0m_{v_0} + 6w_1m_{v_0} + 3v_0m_{v_0^2} + 6v_0m_{w_0} + 6v_0^3 + 12v_0w_0 + 12v_0w_1 + 6v_1w_1\Big) + \mathrm{O}\Big(\frac{1}{\lambda^4}\Big)\Bigg)\lambda^{-n}e^{-\sigma}. \tag{143}$$

It turns out that the above abstract pre-wave functions form *a pair*. Namely, $d_{\mathrm{pre}}(\lambda) = \lambda\, e^{\Lambda^{-1}(-\sigma)}$. Interestingly, for given arbitrary initial value $(f(n), g(n))$, based on this statement, one obtains a constructive method for a pair of wave functions associated with $(f(n), g(n))$ (cf. (28) in Sect. 1.3 for the definition of a pair). This is important considering Theorem 1. We hope to confirm the statement on the pair property of the abstract pre-wave functions in another publication.

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