Letters in Mathematical Physics (2019) 109:2625–2648
https://doi.org/10.1007/s11005-019-01200-z

A note on the Schrödinger operator with a long-range potential

D. R. Yafaev[1,2]

Received: 9 October 2018 / Revised: 5 July 2019 / Accepted: 8 July 2019 / Published online: 15 July 2019

Abstract
Our goal is to develop spectral and scattering theories for the one-dimensional Schrödinger operator with a long-range potential $q(x)$, $x \geq 0$. Traditionally, this problem is studied with a help of the Green–Liouville approximation. This requires conditions on the first two derivatives $q'(x)$ and $q''(x)$. We suggest a new Ansatz that allows us to develop a consistent theory under the only assumption $q' \in L^1$.

Keywords Schrödinger equation · Dimension one · Modified Green–Liouville Ansatz · Limiting absorption principle · Eigenfunction expansion

Mathematics Subject Classification 34E20 · 34L10 · 34L15 · 47A40 · 81U05

1 Introduction

1.1 Short- and long-range potentials

The classical result of Weyl [16,17] (see also the book [14]) states that under very general circumstances a differential equation

$$-f''(x,z)+q(x)f(x,z)=zf(x,z), \quad x \geq 0, \quad \overline{q(x)}=q(x), \tag{1.1}$$

where $\operatorname{Im} z \neq 0$, has a solution $f(\cdot,z) \in L^2(\mathbb{R}_+)$. This fact, however, has no direct spectral consequences for the Schrödinger operator $H = -d^2/dx^2 + q(x)$ (with some boundary condition at the point $x = 0$) in the space $L^2(\mathbb{R}_+)$, except that its spectrum

Supported by project Russian Science Foundation 17-11-01126.

✉ D. R. Yafaev
yafaev@univ-rennes1.fr

1 CNRS, IRMAR-UMR 6625, Univ Rennes, 35000 Rennes, France

2 SPGU, Univ. Nab. 7/9, Saint Petersburg, Russia 199034

is simple. An advanced spectral analysis of the operator H and scattering theory for the pair $H_0 = -\mathrm{d}^2/\mathrm{d}x^2$, H require the continuity of the solutions $f(\cdot, z)$ as $\operatorname{Im} z \to 0$ which can be verified only under some specific assumptions on the potential $q(x)$.

We suppose that $q(x) \to 0$ as $x \to \infty$ and distinguish the short- and long-range cases. In the short-range case, it is assumed that $q \in L^1(\mathbb{R}_+)$. This allows one to construct a solution $\vartheta(x, z)$, known as the Jost solution, of Eq. (1.1) with asymptotics $\vartheta(x, z) \sim e^{-\sqrt{-z}x}$ as $x \to \infty$; it is supposed here that $\operatorname{Re} \sqrt{-z} > 0$ so that $\vartheta(\cdot, z) \in L^2(\mathbb{R}_+)$. It turns out that the function $\vartheta(x, z)$ is continuous in z as $\operatorname{Im} z \to 0$ (with a possible exception of the point $z = 0$). This result is crucial for the analysis of the operator H. It permits (see, e.g., Sects. 3.1 and 3.2 of the book [20]) to show that the structure of its positive spectrum is essentially the same as that of the "free" operator H_0. In particular, the positive spectrum of the operator H is absolutely continuous. This fact follows from the continuity in an appropriate topology of the resolvent $R(z) = (H - z)^{-1}$ as z approaches the positive spectrum of H. The last result is known as the limiting absorption principle. Note that the Jost solutions first appeared in the papers [6] by Jost and [7] by Levinson. They are widely discussed in the physics literature (see, e.g., §1 and §2 in Chapter 12 of the book [11]).

In the long-range case when $q \notin L^1(\mathbb{R}_+)$, the definition of the Jost solution has to be modified. It was shown by Matveev and Skriganov in [8] that, under some assumptions on the first two derivatives $q'(x)$ and $q''(x)$, Eq. (1.1) has a solution $\theta(x, z)$ described by the Green–Liouville Ansatz:

$$\theta(x, z) \sim (q(x) - z)^{-1/4} \exp\Big(- \int_0^x (q(y) - z)^{1/2} \mathrm{d}y \Big), \quad x \to \infty; \qquad (1.2)$$

it is supposed here that $\operatorname{Re} (q(y) - z)^{1/2} > 0$ so that again $\theta(\cdot, z) \in L^2(\mathbb{R}_+)$. We refer to the book [12], Chapter 6, for a careful presentation of the Green-Liouville method. Given the existence of the solutions $\theta(x, z)$ and their continuity in z up to the positive half-line, the spectral analysis of the operator H is performed in [8] essentially in the same way as in the short-range case. The best possible conditions on $q(x)$ required by this method are probably $q' \in L^2$ and $q'' \in L^1$ (see [19]).

1.2 Modified Green–Liouville Ansatz

Our goal is to develop spectral and stationary scattering theories for the Schrödinger operator H with a long-range potential $q(x)$ under the only assumption $q' \in L^1(\mathbb{R}_+)$. Note that, for functions $q(x)$ not satisfying the short-range assumption $q \in L^1(\mathbb{R}_+)$, some conditions on their derivatives are inavoidable. Indeed, the Wigner-von Neumann potential (see, e.g., Section XIII.13 of the book [13]) has asymptotics $q(x) \sim x^{-1} \sin x$ as $x \to \infty$ and the corresponding operator H has a positive eigenvalue. Thus, the positive spectrum of H is not absolutely continuous. More than that, Naboko constructed in [10] examples of Schrödinger operators with dense in $[0, \infty)$ point spectrum whose potentials decay only slightly slower than x^{-1}.

From analytic point of view, the present paper relies on a modification of the classical Green–Liouville Ansatz. Actually, removing the factor in front of the exponential, we replace the formula (1.2) by a simpler one

$$\theta(x,z) \sim \exp\Big(-\int_0^x (q(y)-z)^{1/2}\mathrm{d}y\Big), \quad x\to\infty. \tag{1.3}$$

The construction of solutions $\theta(x,z)$ of Eq. (1.1) with such asymptotic behavior under the only assumption $q' \in L^1(\mathbb{R}_+)$ is the main new point of the paper. Then spectral and stationary scattering theories for the operator H can be developed along essentially the same lines as in the short-range case.

In the problem we consider, the Ansatz (1.3) is more efficient (and is much simpler) than the Green–Liouville one. However, the classical Ansatz also has numerous advantages. For example, it was used in [19] to study low energy (as $z\to 0$) asymptotics of spectral and scattering data for potentials $q(x)$ decaying slower than x^{-2} as $x\to\infty$. Another important application of the Green–Liouville method is to differential equations (1.1) with coefficients $q(x)$ tending to $+\infty$ or to $-\infty$ as $x\to\infty$ (see [12], Chapter 6) as well as to some slowly oscillating $q(x)$.

The approach we use works equally well for more general, than Schrödinger, differential operators. In the paper, we consider the operator

$$H = -\frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x}+q(x), \quad p(x)>0, \quad \overline{q(x)}=q(x), \tag{1.4}$$

with some boundary condition at the point $x=0$ in the space $L^2(\mathbb{R}_+)$. We choose the condition $f(0)=0$.

1.3 Other methods

The limiting absorption principle under assumptions very close to $q'\in L^1$ was obtained long ago by the powerful Mourre method [9]; we refer to §6.9 of [20] where the conditions on q were stated explicitly. The Mourre method works for very general operators (for example, for the Schrödinger operator in all dimensions), but the only spectral information it gives is the absence of the singular continuous spectrum. For example, it says nothing about multiplicity of the spectrum and does not exclude positive eigenvalues of H.

There are also specific one-dimensional methods for a proof of the absolute continuity of the positive spectrum. Thus Weidmann in [15], see Theorem 14.25, proved this fact by the Gilbert–Pearson method [4] under somewhat more general assumptions on $q(x)$ than $q'\in L^1$.

One could probably also study asymptotics of eigenfunctions of H as $x\to\infty$ using Levinson's results (see §III.8 of the book [3]). However, the scattering theory approach we use here seems to be better adapted for a detailed spectral analysis of the operator H.

1.4 Structure of the paper

Section 2 is central. Here we construct modified Jost solutions, introduce a multiplicative change of variables and reduce differential equation (1.1) to a Volterra integral equation.

We study a regular solution $\varphi(x, z)$ of Eq. (1.1) in Sect. 3. In particular, we find its asymptotic behavior as $x \to \infty$. Of course, the answers are quite different for $z > 0$ and $z \notin [0, \infty)$. The result for $z \notin [0, \infty)$ seems not to be well known even in the short-range case.

Given the results of Sects. 2 and 3, we follow the standard approach to spectral analysis of the operator H in Sect. 4. First, we define H as a self-adjoint operator. Then we build its eigenfunctions of the continuous spectrum and establish an expansion theorem over these eigenfunctions. The limiting absorption principle is a by-product of these considerations.

Here is the list of miscellaneous results of Sect. 5: inclusion of an additional short-range term $q_{\rm sr} \in L^1$, a general boundary condition at the point $x = 0$, the problem on the whole line.

Occasionally, the dependence of various functions on x and z is omitted in notation; c and C are different constants whose precise values are of no importance. We also use notation: $(A)_+ = (|A| + A)/2$ for $A \in \mathbb{R}$.

2 Modified Jost solutions

2.1 Ansatz

Let us consider a more general than (1.1) differential equation

$$-(p(x)f'(x,z))' + q(x)f(x,z) = zf(x,z) \tag{2.1}$$

for $z \in \mathbb{C}\setminus\mathbb{R} =: \Pi$. We admit also that the spectral parameter z belongs to the closure $\operatorname{clos}\Pi$ of Π, that is, $z = \lambda \pm i0$ where $\lambda \in \mathbb{R}\setminus\{0\}$. With respect to the functions $p(x) > 0$ and $q(x) = \overline{q(x)}$, we accept the following

Assumption 2.1 (i) The function $p(x)$ is absolutely continuous on $\mathbb{R}_+$. The function $q(x)$ is absolutely continuous for $x \geq x_0$ where x_0 may be arbitrary large and $q \in L^1(0, x_0)$.

(ii) The derivatives

$$p' \in L^1(\mathbb{R}_+), \quad q' \in L^1(x_0, \infty)$$

(iii) The limits

$$\lim_{x\to\infty} p(x) =: p_0 > 0, \quad \lim_{x\to\infty} q(x) = 0. \tag{2.2}$$

It is sufficient to construct solutions of the differential equation (2.1) for large x. Then they can be standardly extended to all $x \geq 0$. A more general situation of this type is discussed in Sect. 5.1.

Our goal in this section is to distinguish a solution $\theta(x, z)$ of Eq. (2.1) by its behavior as $x \to \infty$. To define it, we first exhibit an explicit function $a(x, z)$ such that the (relative) remainder

$$r(x,z) := -a(x,z)^{-1}(p(x)a'(x,z))' + q(x) - z \tag{2.3}$$

in Eq. (2.1) is in L^1 (at infinity). Let us seek the Ansatz $a(x, z)$ in the form

$$a(x, z) = e^{-\Omega(x,z)} \tag{2.4}$$

and put $\omega(x, z) = \Omega'(x, z)$. Since

$$a(x, z)^{-1} a'(x, z) = -\omega(x, z), \tag{2.5}$$

we can rewrite (2.3) as

$$\begin{aligned} r(x, z) &= a(x, z)^{-1}(p(x)\omega(x, z)a(x, z))' + q(x) - z \\ &= (p(x)\omega(x, z))' - p(x)\omega(x, z)^2 + q(x) - z. \end{aligned} \tag{2.6}$$

Let us set

$$\omega(x, z) = \sqrt{\frac{q(x) - z}{p(x)}}, \tag{2.7}$$

where we suppose that $\operatorname{Re} \omega(x, z) > 0$ for all $z \in \Pi$. Clearly, the function $\omega(x, z)$ is analytic in $z \in \Pi$ and continuous up to the cut along $\mathbb{R}$. For $z \neq 0$, we choose $x_1 = x_1(z)$ such that $|q(x)| \leq |z|/2$ for all $x \geq x_1$. Then estimates

$$0 < c \leq |\omega(x, z)| \leq C < \infty \tag{2.8}$$

and

$$|\omega'(x, z)| \leq C(|p'(x)| + |q'(x)|) \tag{2.9}$$

are true for all $x \geq x_1$, and hence $\omega'(\cdot, z) \in L^1(x_1, \infty)$ under Assumption 2.1. Here and below all estimates are uniform in z from compact subsets of $\operatorname{clos} \Pi \setminus \{0\}$ (including the values of $z = \lambda \pm i0$ on the cut).

For the choice (2.7), the remainder (2.6) equals

$$r(x, z) = (p(x)\omega(x, z))'. \tag{2.10}$$

In view (2.8), (2.9), this yields the following result.

Lemma 2.2 *Define the functions $\Omega(x, z)$ and $a(x, z)$ by the formulas*

$$\Omega(x, z) = \int_0^x \sqrt{\frac{q(y) - z}{p(y)}} dy \tag{2.11}$$

and (2.4). Then the remainder (2.3) is given by formula (2.10) and $r \in L^1(x_1(z), \infty)$.

We emphasize that the classical Green–Liouville Ansatz differs from (2.4) by the additional factor $\omega(x, z)^{-1/2}$ in the right-hand side.

Since $\operatorname{Re}\omega(x,z)\geq 0$, it follows from (2.4), (2.11) that

$$\Big|\frac{a(y,z)}{a(x,z)}\Big|\leq 1,\quad y\geq x. \tag{2.12}$$

Moreover, $\operatorname{Re}\omega(x,z)\geq c(z)>0$ for $\operatorname{Im} z\neq 0$, so that we have a stronger estimate

$$\Big|\frac{a(y,z)}{a(x,z)}\Big|\leq e^{-c(z)(y-x)},\quad y\geq x,\quad c(z)>0,\quad \operatorname{Im} z\neq 0. \tag{2.13}$$

2.2 Multiplicative substitution

Instead of a solution $\theta(x,z)$ of Eq. (2.1), we introduce a function

$$u(x,z)=a(x,z)^{-1}\theta(x,z). \tag{2.14}$$

Lemma 2.3 *Let $r(x,z)$ be given by formula (2.10). Then Eq. (2.1) for $\theta(x,z)$ is equivalent to the equation*

$$(p(x)u'(x,z))'-2\omega(x,z)p(x)u'(x,z)=r(x,z)u(x,z) \tag{2.15}$$

for the function (2.14).

Proof A direct differentiation of the relation $\theta=au$ shows that

$$(p\theta')'=(pa'u)'+(pau')'=(pa')'u+(pu')'a-2\omega pau'$$

where we have taken (2.5) into account. It follows that

$$-(p\theta')'+(q-z)\theta=\big(-(pa')'+(q-z)a\big)u+\big(-(pu')'+2\omega pu'\big)a.$$

Since the first term on the right equals rau, we see that

$$-(p\theta')'+(q-z)\theta=\big(-(pu')'+2\omega pu'+ru\big)a.$$

Thus Eq. (2.1) for θ and (2.15) coincide. □

Next, we reduce differential equation (2.15) to an integral equation.

Lemma 2.4 *Let $z\in\Pi$, and let $u(x,z)$ be a solution of differential equation (2.15) such that*

$$\lim_{x\to\infty}u(x,z)=1. \tag{2.16}$$

Then

$$u(x,z)=1+\int_x^\infty G(x,y,z)r(y,z)u(y,z)\mathrm{d}y \tag{2.17}$$

where

$$G(x, y, z) = a(y, z)^2 \int_x^y p(s)^{-1} a(s, z)^{-2} \mathrm{d}s. \tag{2.18}$$

Proof Set

$$v(x, z) = p(x)u'(x, z) \quad \text{and} \quad \rho(x, z) = r(x, z)u(x, z). \tag{2.19}$$

Then (2.15) yields a differential equation

$$v'(x, z) - 2\omega(x, z)v(x, z) = \rho(x, z)$$

of first order for the function $v(x, z)$. In view of (2.5), its solution is given by the equality

$$v(x, z) = -a(x, z)^{-2}\Big(\int_x^\infty a(y, z)^2 \rho(y, z)\mathrm{d}y + c\Big)$$

where we have to choose $c = 0$ because $a(x, z)^{-2}$ exponentially grows as $x \to \infty$. Therefore the function $u(x, z)$ satisfying (2.16) and (2.19) can be recovered by the formula

$$u(x, z) = 1 - \int_x^\infty p(s)^{-1} v(s, z)\mathrm{d}s = 1 + \int_x^\infty p(s)^{-1} a(s, z)^{-2}\Big(\int_s^\infty a(y, z)^2 \rho(y, z)\mathrm{d}y\Big)\mathrm{d}s.$$

By virtue of (2.13), Fubini's theorem allows us to interchange the order of integrations here. This yields Eq. (2.17). □

2.3 Integral equation

The following assertion plays the crucial role in the analysis of Eq. (2.17) as z approaches the half-axis $(0, \infty)$ (the continuous spectrum of H).

Lemma 2.5 *For* $z \in \operatorname{clos}\Pi\setminus\{0\}$ *and* $y \geq x \geq x_1(z)$*, kernel* (2.18) *is uniformly bounded:*

$$|G(x, y, z)| \leq C < \infty. \tag{2.20}$$

Proof Set $\tau = (p\omega)^{-1}$. According to (2.8) and (2.9), we have

$$\tau \in L^\infty \quad \text{and} \quad \tau' \in L^1 \tag{2.21}$$

Integrating by parts, we see that

$$2\int_x^y p(s)^{-1} a(s)^{-2}\mathrm{d}s = \int_x^y \tau(s)\mathrm{d}a(s)^{-2} = \tau(y)a(y)^{-2} - \tau(x)a(x)^{-2} - \int_x^y \tau'(s)a(s)^{-2}\mathrm{d}s.$$

Multiplying this equality by $a(y)^2$ and using estimate (2.12) and relations (2.21), we get bound (2.20). □

Lemmas 2.2 and 2.5 allow us to solve the Volterra equation (2.17) by iterations. Let us state the corresponding result.

Lemma 2.6 *For $z \in \operatorname{clos}\Pi\setminus\{0\}$, equation (2.17) has a (unique) bounded solution $u(x,z)$. For every $x \geq 0$, this function is analytic in $z \in \Pi$ and is continuous up to the cut along $\mathbb{R}$ with possible exception of the point $z = 0$. The function $u(x,z)$ obeys an estimate*

$$|u(x,z) - 1| \leq C\varepsilon(x) \tag{2.22}$$

where

$$\varepsilon(x) = \int_x^\infty \big(|p'(y)| + |q'(y)|\big)\mathrm{d}y$$

and the constant C does not depend on z in compact subsets of the set $\operatorname{clos}\Pi\setminus\{0\}$.

The next assertion is converse to Lemma 2.4.

Lemma 2.7 *For $z \in \operatorname{clos}\Pi\setminus\{0\}$, a solution $u(x,z)$ of integral equation (2.17) satisfies also differential equation (2.15).*

Proof According to (2.18), we have

$$G(x,x) = 0 \quad \text{and} \quad G'_x(x,y) = -p(x)^{-1}a(x)^{-2}a(y)^2.$$

Therefore it follows from (2.17) that

$$u'(x) = -p(x)^{-1}a(x)^{-2}\int_x^\infty a(y)^2 r(y)u(y)\mathrm{d}y \tag{2.23}$$

and hence, by (2.5),

$$(p(x)u'(x))' = -2a(x)^{-2}\omega(x)\int_x^\infty a(y)^2 r(y)u(y)\mathrm{d}y + r(x)u(x). \tag{2.24}$$

Substituting expressions (2.23) and (2.24) into the left-hand side of (2.15), we see that it equals $r(x)u(x)$. □

Now we are in a position to give a precise

Definition 2.8 For $z \in \operatorname{clos}\Pi\setminus\{0\}$, define the function $a(x,z)$ by formulas (2.4) and (2.11). Denote by $u(x,z)$ the function constructed in Lemma 2.6. The (modified) Jost solution of equation (2.1) is defined by the formula

$$\theta(x,z) = a(x,z)u(x,z) \tag{2.25}$$

for $x \geq x_1(z)$, and then $\theta(x,z)$ is extended to all $x \geq 0$ as a solution of Eq. (2.1).

It follows from (2.22) that

$$\theta(x,z)=a(x,z)\big(1+O(\varepsilon(x))\big),\quad x\to\infty. \tag{2.26}$$

Note also that

$$\theta(x,\bar z)=\overline{\theta(x,z)}$$

and, in particular,

$$\theta(x,\lambda-i0)=\overline{\theta(x,\lambda+i0)},\quad \lambda\in\mathbb{R}\setminus\{0\}. \tag{2.27}$$

Let us summarize the results obtained.

Theorem 2.9 *Let Assumption 2.1 be satisfied, and let $z\in\operatorname{clos}\Pi\setminus\{0\}$. Denote by $u(x,z)$ the function constructed in Lemma 2.6. Then the function $\theta(x,z)$ defined by equality (2.14) satisfies Eq. (2.1), and it has asymptotics (2.26). For every $x\geq 0$, the function $\theta(x,z)$ is analytic in $z\in\Pi$ and is continuous up to the cut along $\mathbb{R}$ with possible exception of the point $z=0$. Asymptotics (2.26) is uniform in z from compact subsets of the set* $\operatorname{clos}\Pi\setminus\{0\}$.

Corollary 2.10 *For $\lambda>0$, set*

$$\Phi(x,\lambda)=\int_0^x\sqrt{\Big(\frac{\lambda-q(y)}{p(y)}\Big)_+}\,dy,\quad K(\lambda)=\exp\Big(-\int_0^\infty\sqrt{\Big(\frac{q(y)-\lambda}{p(y)}\Big)_+}\,dy\Big). \tag{2.28}$$

Then

$$\theta(x,\lambda\pm i0)=K(\lambda)\exp\Big(\pm i\Phi(x,\lambda)\Big)(1+O(\varepsilon(x)))\quad \textit{as}\quad x\to\infty.$$

Let us make several additional observations.

Remark 2.11 (i) Unlike the Jost solution in the short-range case, the function $\theta(x,z)$ is not analytic in the whole half-plane $\operatorname{Re}z<0$ because, in general, $\theta(x,\lambda-i0)\neq\theta(x,\lambda+i0)$ even for $\lambda<0$. This circumstance is, however, inessential. In particular, it follows from (2.27) that $\theta(x,\lambda-i0)=0$ if and only if $\theta(x,\lambda+i0)=0$. This subject is further discussed in Sect. 4.2.

(ii) For $z\in\Pi$, relation (2.26) distinguishes a unique solution of equation (2.1). Indeed, the differential equations (2.1) and the integral equation (2.17) are equivalent, and Lemma 2.6 ensures that the solution of (2.17) satisfying (2.22) is unique.

(iii) According to (2.23), $u'(x)=O(\varepsilon(x))$ as $x\to\infty$, and hence the derivative $\theta'=a(-\omega u+u')$ has asymptotics

$$\theta'(x,z)=-\sqrt{-z/p_0}\,a(x,z)\big(1+O(\varepsilon(x))\big),\quad x\to\infty. \tag{2.29}$$

(iv) It follows from definition (2.7) that

$$\omega(x,z)=\sqrt{-z/p_0}+o(1) \quad \text{whence} \quad \Omega(x,z)=x\sqrt{-z/p_0}+o(x)$$

as $x\to\infty$. In particular,

$$\theta(\cdot,z)\in L^2(\mathbb{R}_+) \quad \text{for} \quad z\in\Pi.$$

2.4 Non-uniqueness of Jost solutions

For long-range perturbations, there is no canonical choice of the Jost solution $\theta(x,z)$: one can replace $\theta(x,z)$ by

$$\widetilde{\theta}(x,z)=\theta(x,z)b(z), \quad b(\bar{z})=\overline{b(z)}, \tag{2.30}$$

where $b(z)$ is some function analytic in Π and continuous up to the cut along $\mathbb{R}$. Thus the function $\Omega(x,z)$ can be replaced in (2.4) by a function $\widetilde{\Omega}(x,z)$ provided the difference $\Omega(x,z)-\widetilde{\Omega}(x,z)$ has a finite limit as $x\to\infty$.

This observation allows one to simplify expression (2.11) if additional information on decay of $p_1(x)=p(x)-p_0$ and of $q(x)$ is available. Note, first, that in the short-range case when $p_1\in L^1$ and $q\in L^1$ one can choose $\Omega_0(x,z)=x\sqrt{-z/p_0}$ instead of $\Omega(x,z)$. Indeed, in this case we have

$$\lim_{x\to\infty}\big(\Omega(x,z)-\Omega_0(x,z)\big)=\int_0^\infty \frac{p_0q(x)+zp_1(x)}{p_0\sqrt{(q(x)-z)p(x)}+p(x)\sqrt{-zp_0}}\mathrm{d}x=:\beta(z).$$

Recall that in the short-range case the standard (non-modified) Jost solution $\vartheta(x,z)$ is distinguished by the asymptotics $\vartheta(x,z)\sim e^{-x\sqrt{-z/p_0}}$ as $x\to\infty$. Therefore we have $\theta(x,z)=e^{-\beta(z)}\vartheta(x,z)$.

If $p_1\in L^2$ and $q\in L^2$ (but $p_1\notin L^1$, $q\notin L^1$), we set

$$\Omega_1(x,z)=\sqrt{-z/p_0}\Big(x-(2z)^{-1}\int_0^x q(y)dy-(2p_0)^{-1}\int_0^x p_1(y)\mathrm{d}y\Big). \tag{2.31}$$

Then $\Omega(x,z)-\Omega_1(x,z)$ has a finite limit as $x\to\infty$ so that function (2.11) can be replaced by (2.31). Similarly, in the case $p_1\in L^3$ and $q\in L^3$ (but $p_1\notin L^2$, $q\notin L^2$), we set

$$\Omega_2(x,z)=\Omega_1(x,z)+8^{-1}\sqrt{-z/p_0}\Big(z^{-2}\int_0^x q(y)^2\mathrm{d}y+3p_0^{-2}\int_0^x p_1(y)^2\mathrm{d}y\Big) \tag{2.32}$$

where $\Omega_1(x,z)$ is given by (2.31). Then $\Omega(x,z)-\Omega_2(x,z)$ has a finite limit as $x\to\infty$ so that function (2.11) can be replaced by (2.32). This procedure can be continued to treat a general case where $p_1\in L^n$ and $q\in L^n$ for some integer n.

3 Regular solution

3.1 Uniform estimates

In addition to $\theta(x,z)$, we introduce a regular solution $\varphi(x,z)$ of Eq. (2.1) distinguished by conditions at the point $x=0$:

$$\varphi(0,z)=0, \quad \varphi'(0,z)=1. \tag{3.1}$$

For every $x\geq 0$, the function $\varphi(x,z)$ is analytic in $z\in\mathbb{C}$.

Let us obtain estimates on $\varphi(x,z)$ for large x. These estimates will be uniform in z from compact subsets of $\mathbb{C}\setminus\{0\}$. Define again the functions $\Omega(x,z)$, $a(x,z)$ by formulas (2.11), (2.4), but instead of (2.14) we now set

$$\mathsf{u}(x,z)=a(x,z)\varphi(x,z). \tag{3.2}$$

Similarly to Lemma 2.3, it is straightforward to check that Eq. (2.1) for $\varphi(x,z)$ is equivalent to the equation

$$(p(x)\mathsf{u}'(x,z))'+2\omega(x,z)p(x)\mathsf{u}'(x,z)=-r(x,z)\mathsf{u}(x,z)$$

for the function (3.2) with the remainder $r(x,z)$ given by formula (2.10). The initial conditions (3.1) for the functions $\varphi(x,z)$ and $\mathsf{u}(x,z)$ coincide. Suppose that $|z|\geq c>0$ and choose x_0 such that $|q(x)|\leq c/2$ for $x\geq x_0$. Similarly to Lemma 2.4, it is easy to derive a Volterra integral equation for the function $\mathsf{u}(x,z)$:

$$\mathsf{u}(x,z)=\mathsf{u}^{(0)}(x,z)-\int_{x_0}^{x}K(x,y,z)r(y,z)\mathsf{u}(y,z)\mathrm{d}y \tag{3.3}$$

where

$$\mathsf{u}^{(0)}(x,z)=p(x_0)\int_{x_0}^{x}p(y)^{-1}a(y,z)^2\mathrm{d}y \quad \text{and}$$
$$K(x,y,z)=a(y,z)^{-2}\int_{y}^{x}p(s)^{-1}a(s,z)^2\mathrm{d}s.$$

If $x\geq y\geq x_0(z)$, we can integrate here by parts which shows that the functions $\mathsf{u}_0(x,z)$ and $K(x,y,z)$ are uniformly bounded. Therefore solving Eq. (3.3) by iterations, we see that its solution $\mathsf{u}(x,z)$ is also bounded. Coming back to relation (3.2), we can state the following result.

Theorem 3.1 *Let Assumption 2.1 be satisfied. Then for z in compact subsets of $\mathbb{C}\setminus\{0\}$, the regular solution $\varphi(x,z)$ of Eq. (2.1) obeys an estimate*

$$|\varphi(x,z)|\leq C\exp\left(\operatorname{Re}\int_0^x\sqrt{\frac{q(y)-z}{p(y)}}\,\mathrm{d}y\right), \quad \forall x\in\mathbb{R}_+.$$

3.2 Asymptotics at infinity

For $z \in \operatorname{clos} \Pi \setminus \{0\}$, let us consider the Wronskian

$$w(z) = \{\varphi(\cdot, z), \theta(\cdot, z)\} := p(x)(\varphi'(x, z)\theta(x, z) - \varphi(x, z)\theta'(x, z)) \tag{3.4}$$

of the regular $\varphi(x, z)$ and Jost $\theta(x, z)$ solutions of Eq. (2.1). Since the right-hand side of (3.4) does not depend on $x \geq 0$, we can set $x = 0$ whence

$$w(z) = p(0)\theta(0, z). \tag{3.5}$$

Let us now consider the Jost solutions $\theta(x, \lambda \pm i0)$ on the cut along $(0, \infty)$. Using Corollary 2.10 and calculating the Wronskian of $\theta(x, \lambda + i0)$ and $\theta(x, \lambda - i0)$ for $x \to \infty$, we find that

$$w_0(\lambda) := \{\theta(\cdot, \lambda + i0), \theta(\cdot, \lambda - i0)\} = 2i\sqrt{p_0\lambda}K(\lambda)^2.$$

In particular, the Jost solutions are linearly independent. It is easy to see that

$$\varphi(x, \lambda) = \frac{\theta(x, \lambda + i0)w(\lambda - i0) - \theta(x, \lambda - i0)w(\lambda + i0)}{\{\theta(\cdot, \lambda + i0), \theta(\cdot, \lambda - i0)\}}. \tag{3.6}$$

Indeed, by (3.5), the right-hand side of (3.6) equals 0 for $x = 0$ and its derivative in x equals 1 for $x = 0$. Thus the right-hand side of (3.6) satisfies equation (2.1) where $z = \lambda$ and conditions (3.1). It follows from (3.6) that

$$w(\lambda \pm i0) \neq 0, \quad \lambda > 0,$$

since otherwise we would have $\varphi(x, \lambda) = 0$ for all $x \geq 0$.

Now we set

$$\kappa(\lambda) = |w(\lambda \pm i0)| \quad \text{and} \quad w(\lambda + i0) = \kappa(\lambda)e^{i\eta(\lambda)}. \tag{3.7}$$

By analogy with the short-range case, we use the terms “limit amplitude” for $\kappa(\lambda)$ and “limit phase” (or scattering phase, or phase shift) for $\eta(\lambda)$. According to Theorem 2.9 the amplitude $\kappa(\lambda)$ is a continuous function of $\lambda > 0$. The phase $\eta(\lambda)$ is defined by equations (3.7) up to a term $2\pi n$ where n is an integer. Since $\kappa(\lambda) > 0$, the function $\eta(\lambda)$ can also be chosen continuous in $\lambda > 0$.

The next result is a direct consequence of Corollary 2.10 and representation (3.6).

Theorem 3.2 *Let Assumption 2.1 be satisfied. For each $\lambda > 0$, the regular solution $\varphi(x, \lambda)$ of Eq. (2.1) has asymptotics*

$$\varphi(x, \lambda) = \frac{\kappa(\lambda)}{\sqrt{p_0\lambda}\, K(\lambda)} \sin\big(\Phi(x, \lambda) - \eta(\lambda)\big) + O(\varepsilon(x)) \tag{3.8}$$

as $x \to \infty$. In particular, the operator H has no positive eigenvalues.

Remark 3.3 In the short-range case, the Jost solution $\vartheta(x, \lambda \pm i0)$ is distinguished by the asymptotics $\vartheta(x, \lambda \pm i0) \sim e^{\pm ix\sqrt{\lambda/p_0}}$ as $x \to \infty$. Then the limit amplitude $\kappa(\lambda)$ and the limit phase $\eta(\lambda)$ are defined by relations (3.7) where $w(\lambda) = \{\varphi(\cdot, \lambda), \vartheta(\cdot, \lambda + i0)\}$.

Remark 3.4 Expressions (3.6) and hence (3.8) are of course invariant with respect to the change (2.30) of the Jost solution. This means that the amplitude factor and the phase $\Phi(x, \lambda) - \eta(\lambda)$ in (3.8) do not depend on the regularization of the Jost solution. However, we emphasize that, separately, the terms $\Phi(x, \lambda)$ and $\eta(\lambda)$ do depend on it. Hence, in the long-range case, the definition of the scattering phase $\eta(\lambda)$ is not intrinsic.

3.3 Exponentially growing solutions

Generically, for $z \notin [0, \infty)$, the regular solution $\varphi(x, z)$ grows exponentially as $x \to \infty$. Here we find its asymptotics. The method below was used in the short-range case in §4.1 of the book [20] but seems not to be commonly known.

Let $\theta(x, z)$ be the Jost solution of Eq. (2.1). For $z = \lambda < 0$, we can pick either $\theta(x, \lambda + i0)$ or $\theta(x, \lambda - i0)$. We choose $\varrho = \varrho(z)$ in such a way that $\theta(x, z) \neq 0$ for all $x \geq \varrho(z)$. If $\operatorname{Im} z \neq 0$, we can set $\varrho(z) = 0$ because the equality $\theta(x_0, z) = 0$ would imply that the self-adjoint operator (1.4) in the space $L^2(x_0, \infty)$ with the boundary condition $f(x_0) = 0$ has complex eigenvalue z. Let us introduce an exponentially growing solution $\xi(x, z)$ of Eq. (2.1).

Theorem 3.5 *Let Assumption 2.1 be satisfied, and let $z \in \mathbb{C}\setminus[0, \infty)$. Then the function*

$$\xi(x, z) = \theta(x, z) \int_{\varrho(z)}^{x} \theta(y, z)^{-2} p(y)^{-1} \mathrm{d}y, \quad x \geq \varrho(z), \tag{3.9}$$

satisfies Eq. (2.1) and

$$\xi(x, z) = \Big(2\sqrt{-p_0 z}\, a(x, z)\Big)^{-1} (1 + o(1)), \quad \xi'(x, z) = \Big(2 p_0 a(x, z)\Big)^{-1} (1 + o(1)) \tag{3.10}$$

as $x \to \infty$.

Proof Differentiating (3.9), we find that

$$\begin{aligned} &-(p(x)\xi'(x))' + (q(x) - z)\xi(x) \\ &= \Big(- (p(x)\theta'(x))' + (q(x) - z)\theta(x)\Big) \int_{\varrho}^{x} \theta(y)^{-2} p(y)^{-1} \mathrm{d}y \end{aligned}$$

which implies Eq. (2.1) for $\xi(x)$.

Integrating by parts, we see that

$$2\int_{\varrho}^{x} \theta(y)^{-2} p(y)^{-1} \mathrm{d}y = t(x)a(x)^{-2} - t(\varrho)a(\varrho)^{-2} - \int_{\varrho}^{x} t'(y)a(y)^{-2} \mathrm{d}y \tag{3.11}$$

where $t = (p\omega u^2)^{-1}$. Let us multiply this equality by $a(x)\theta(x)$ and consider the limit $x \to \infty$. It follows from Lemma 2.6 and Theorem 2.9 that the first term on the right

$$t(x)\theta(x)a(x)^{-1} = (-p_0 z)^{-1/2} + O(\varepsilon(x)). \tag{3.12}$$

The second term tends to 0 exponentially. Finally, using estimate (2.13), we find that

$$\Big|a(x)\theta(x)\int_{\varrho}^{x} t'(y)a(y)^{-2}\mathrm{d}y\Big| \le C\int_0^x e^{-c(z)(x-y)}|t'(y)|\mathrm{d}y. \tag{3.13}$$

Since according to Lemma 2.6 (see also equality (2.23)) $t'(x) \to 0$ as $x \to \infty$, the same is true for expression (3.13). Therefore relations (3.11) and (3.12) imply asymptotic formulas (3.10). ◻

Using asymptotics (2.26), (2.29) and (3.10), we can calculate the Wronskian of the solutions $\theta(x,z)$ and $\xi(x,z)$:

$$\{\theta(\cdot,z),\xi(\cdot,z)\} = -1.$$

It follows that

$$\varphi(x,z) = \{\varphi(\cdot,z),\theta(\cdot,z)\}\xi(x,z) - \{\varphi(\cdot,z),\xi(\cdot,z)\}\theta(x,z) \tag{3.14}$$

where $\{\varphi(\cdot,z),\xi(\cdot,z)\} = p(0)\xi(0,z)$. In view of Theorems 2.9 and 3.5, relation (3.14) yields the asymptotic behavior of the regular solution.

Theorem 3.6 *Let Assumption 2.1 be satisfied, and let $z \in \mathbb{C}\setminus[0,\infty)$. Then*

$$\varphi(x,z) = \frac{w(z)}{2\sqrt{-p_0 z}}\exp\Big(\int_0^x\sqrt{\frac{q(y)-z}{p(y)}}\,\mathrm{d}y\Big)(1+o(1)), \quad x\to\infty, \tag{3.15}$$

if $w(z) = \{\varphi(\cdot,z),\theta(\cdot,z)\} \neq 0$ and

$$\varphi(x,z) = -\{\varphi(\cdot,z),\xi(\cdot,z)\}\exp\Big(-\int_0^x\sqrt{\frac{q(y)-z}{p(y)}}\,\mathrm{d}y\Big)(1+o(1)), \quad x\to\infty,$$

if $w(z) = 0$.

Thus $\varphi(x,z)$ exponentially grows if z is not an eigenvalue of H, and it exponentially decays in the opposite case.

Remark 3.7 Estimates of the remainders in (3.10) and (3.15) are not uniform in z as it approaches the half-axis $(0,\infty)$. As a consequence, we cannot put $z = \lambda \in \mathbb{R}_+$ in (3.15). Such a relation would contradict (3.8). On the contrary, the estimate on $\varphi(x,z)$ of Theorem 3.1 is uniform in z.

4 Spectral results

4.1 Differential operator

First, we define differential operator (1.4) as a self-adjoint operator. We choose the boundary condition $f(0) = 0$; see Sect. 5.2 for other conditions.

The simplest possibility is to define H via the quadratic form

$$h[f, f] = \int_0^\infty \big(p(x)|f'(x)|^2 + q(x)|f(x)|^2\big)\mathrm{d}x. \tag{4.1}$$

As is well known, this form is closed on the Sobolev space $\mathsf{H}_0^1(\mathbb{R}_+) =: \mathcal{D}[h]$ of functions satisfying the condition $f(0) = 0$ provided

$$0 < p_0 \le p(x) \le p_1 < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}_+} \int_x^{x+1} |q(y)|\mathrm{d}y < \infty. \tag{4.2}$$

Therefore H may be defined as a self-adjoint operator with domain $\mathcal{D}(H) \subset \mathcal{D}[h]$ corresponding to this form (see Chapter 10 of the book [1]). This operator satisfies the relation

$$(Hf, g) = h[f, g] \tag{4.3}$$

for all $f \in \mathcal{D}(H)$ and all $g \in \mathcal{D}[h]$.

It turns out that its domain $\mathcal{D}(H)$ can be described explicitly.

Lemma 4.1 *In addition to (4.2), assume that the function $p(x)$ is absolutely continuous on $\mathbb{R}_+$. Then $f \in \mathcal{D}(H)$ if and only if $f \in \mathsf{H}_0^1(\mathbb{R}_+)$, the function $p(x)f'(x)$ is absolutely continuous and*

$$(Hf)(x) := -(p(x)f'(x))' + q(x)f(x) \in L^2(\mathbb{R}_+). \tag{4.4}$$

Proof Suppose that $f(x)$ satisfies these conditions. Using that the function $p(x)f'(x)$ is absolutely continuous and integrating by parts, we see that

$$\begin{aligned}&\int_0^\infty \Big(-(p(x)f'(x))' + q(x)f(x)\Big)\overline{g(x)}\mathrm{d}x\\ &\quad= \int_0^\infty \big(p(x)f'(x)\overline{g'(x)} + q(x)f(x)\overline{g(x)}\big)\mathrm{d}x\end{aligned} \tag{4.5}$$

at least for all $g \in C_0^\infty(\mathbb{R}_+)$. An arbitrary $g \in \mathsf{H}_0^1(\mathbb{R}_+)$ can be approximated by functions $g_n \in C_0^\infty(\mathbb{R}_+)$ in the norm of $\mathsf{H}^1(\mathbb{R}_+)$. Passing to the limit $n \to \infty$ in the equality (4.5) for g_n and using (4.4), we obtain relation (4.3) whence $f \in \mathcal{D}(H)$.

Conversely, suppose that $f \in \mathcal{D}(H)$. It follows from (4.3) that

$$|h[f, g]| \le C\|g\|, \quad \forall g \in \mathcal{D}[h],$$

and hence there exists $t \in L^2(\mathbb{R}_+)$ such that

$$h[f,g] = \int_0^\infty t(x)\overline{g(x)}\mathrm{d}x.$$

Comparing this expression with (4.1), we see that

$$\int_0^\infty p(x) f'(x)\overline{g'(x)}\mathrm{d}x = \int_0^\infty \big(t(x) - q(x) f(x)\big)\overline{g(x)}\mathrm{d}x. \tag{4.6}$$

If $g \in C_0^\infty(\mathbb{R}_+)$, then (4.6) is the definition of the distributional derivative of the function pf'. This derivative equals $qf - t$ where $qf \in L^1_{\rm loc}(\mathbb{R}_+)$ because $f \in C(\mathbb{R}_+) \subset \mathsf{H}^1(\mathbb{R}_+)$. Therefore the function pf' is absolutely continuous and $-(pf')' + qf = t \in L^2(\mathbb{R}_+)$. □

4.2 Resolvent

Theorem 2.9 allows us to perform spectral analysis of the operator H in a sufficiently standard way. We start with a construction of its resolvent. Let $R(z)$, $\operatorname{Im} z \neq 0$, be an integral operator defined by the formula

$$(R(z)g)(x) = \frac{\varphi(x,z)}{w(z)} \int_x^\infty \theta(y,z) g(y)\mathrm{d}y + \frac{\theta(x,z)}{w(z)} \int_0^x \varphi(y,z) g(y)\mathrm{d}y. \tag{4.7}$$

Using that the functions φ and θ satisfy the Eq. (2.1), one easily verifies that, for example, for $g \in C_0^\infty(\mathbb{R}_+)$, the function $f(x) = (R(z)g)(x)$ belongs to the domain $\mathcal{D}(H)$ of the operator H and $(H - z) f = g$ whence $R(z)g = (H - z)^{-1} g$. It follows that $R(z) = (H - z)^{-1}$ is the resolvent of the operator H. In particular, the operator $R(z)$ is bounded. Relation (4.7) means that the integral kernel of $R(z)$ equals

$$R(x,y;z) = w(z)^{-1}\varphi(x,z)\theta(y,z) \quad \text{for} \quad x \leq y \quad \text{and} \quad R(y,x;z) = R(x,y;z). \tag{4.8}$$

Recall that, for $\lambda < 0$, the values $w(\lambda \pm i0)$ and, more generally, $\theta(x, \lambda \pm i0)$ are different, but satisfy (2.27). In particular, $w(\lambda + i0) = 0$ if and only if $w(\lambda - i0) = 0$. Nevertheless, the function $\theta(x,z)/w(z)$ is analytic in $\mathbb{C} \setminus [0,\infty)$ because

$$\frac{\theta(x,\lambda + i0)}{w(\lambda + i0)} = \frac{\theta(x,\lambda - i0)}{w(\lambda - i0)}, \quad \lambda < 0. \tag{4.9}$$

Indeed, consider an auxiliary function

$$\Delta(x,\lambda) = \theta(x,\lambda + i0) w(\lambda - i0) - \theta(x,\lambda - i0) w(\lambda + i0).$$

It satisfies Eq. (2.1), $\Delta(0,\lambda) = 0$ and

$$p(0)\Delta'(0,\lambda) = \{\theta(\lambda + i0), \theta(\lambda - i0)\}.$$

Using asymptotics (2.26) and (2.29), we find that $\Delta'(0,\lambda)=0$ if $\lambda<0$ and hence $\Delta(x,\lambda)=0$ for all $x\geq 0$. This proves equality (4.9).

Formula (4.8) (as well as (3.5)) implies that $w(z)=0$ if and only if z is an eigenvalues of the operator H. In particular, zeros of the function $w(z)$ are negative (it is also not excluded that $w(0)=0$). Recall also that $w(\lambda\pm i0)\neq 0$ for $\lambda>0$.

Let us summarize these results.

Theorem 4.2 *Let Assumption 2.1 be satisfied. Then*

(i) *The resolvent $R(z)=(H-z)^{-1}$ of the operator H is an integral operator with kernel (4.8). For all $x,y\geq 0$, it is an analytic function of $z\in\mathbb{C}\setminus[0,\infty)$ with simple poles at eigenvalues of the operator H. A point $z\in\mathbb{C}\setminus[0,\infty)$ is an eigenvalue of H if and only if $w(z)=0$.*

(ii) *For all $x,y\geq 0$, the integral kernel $R(x,y;z)$ is a continuous function of z up to the cut along $[0,\infty)$ with possible exception of the point $z=0$.*

(iii) *The integral kernel of $R(z)$ satisfies an estimate*

$$|R(x,y;z)|\leq C|w(z)|^{-1}\exp\Big(-2\,\mathrm{Re}\int_x^y\sqrt{\frac{q(s)-z}{p(s)}}\,\mathrm{d}s\Big),\quad x\leq y,$$

where C does not depend on $x,y\geq 0$ and on z in compact subsets of the closed set $\operatorname{clos}\big(\mathbb{C}\setminus[0,\infty)\big)$ as long as z is separated from the point 0. In particular, $R(x,y;z)$ is a bounded function of z away from eigenvalues of the operator H and the point 0.

This statement (ii) is known as the limiting absorption principle. It implies

Corollary 4.3 *The spectrum of the operator H on the half-axis $(0,\infty)$ is absolutely continuous.*

Let us now consider the spectral projector $E(\lambda)$ of the operator H. It is also an integral operator with kernel $E(x,y;\lambda)$ related to the resolvent kernel of H by the Cauchy-Stieltjes formula

$$2\pi i dE(x,y;\lambda)/d\lambda=R(x,y;\lambda+i0)-R(x,y;\lambda-i0). \tag{4.10}$$

The following assertion is a direct consequence of Theorem 4.2, part (ii).

Corollary 4.4 *For all $x,y\geq 0$, the integral kernel $E(x,y;\lambda)$ is continuously differentiable in $\lambda>0$.*

4.3 Eigenfunction expansion

Putting together formulas (4.8) and (4.10) for $z = \lambda \pm i0$, $\lambda > 0$, and taking into account relation (3.6), we see that

$$
\begin{aligned}
\mathrm{d}E(x, y; \lambda)/\mathrm{d}\lambda &= (2\pi i)^{-1}\varphi(x, \lambda)\Big(\frac{\theta(y, \lambda + i0)}{w(\lambda + i0)} - \frac{\theta(y, \lambda - i0)}{w(\lambda - i0)}\Big) \\
&= \sqrt{p_0\lambda}\, K(\lambda)^2 \frac{\varphi(x, \lambda)\varphi(y, \lambda)}{\pi|w(\lambda \pm i0)|^2}, \quad \lambda > 0. \qquad (4.11)
\end{aligned}
$$

Of course this representation extends to all $x, y \geq 0$.

Relation (4.11) allows us to diagonalize the operator H in the same way as in the short-range case (see, for example, §4.2 of the book [20]). Keeping in mind scattering theory framework, we introduce two sets of eigenfunctions $\psi_\pm$ and two diagonalizations $\Psi_\pm$ of the absolutely continuous part $HE(\mathbb{R}_+)$ of H. Let us set

$$
\psi_\pm(x, \lambda) = \frac{\sqrt[4]{p_0\lambda}\, K(\lambda)}{\sqrt{\pi}\, w(\lambda \mp i0)} \varphi(x, \lambda) \qquad (4.12)
$$

and

$$
(\Psi_\pm f)(\lambda) = \int_0^\infty \overline{\psi_\pm(x, \lambda)} f(x)\mathrm{d}x, \quad f \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+). \qquad (4.13)
$$

It follows from (4.11) that

$$
\|E(\Lambda)f\|^2 = \int_\Lambda |(\Psi_\pm f)(\lambda)|^2 d\lambda \qquad (4.14)
$$

for every interval Λ such that clos $\Lambda \subset \mathbb{R}_+$ and hence

$$
\Psi_\pm^* \Psi_\pm = E(\mathbb{R}_+). \qquad (4.15)
$$

In particular, $\Psi_\pm$ extends to a bounded operator on $L^2(\mathbb{R}_+)$.

Let A be the operator of multiplication by λ in the space $L^2(\mathbb{R}_+)$. Since the functions $\psi_\pm(x, \lambda)$ satisfy Eq. (2.1), the intertwining property

$$
\Psi_\pm H = \mathsf{A}\Psi_\pm \qquad (4.16)
$$

holds.

Relation (4.15) is naturally interpreted as the completeness of the eigenfunctions $\psi_\pm(x, \lambda)$ of the operator H. Their orthogonality means that the adjoint operator $\Psi_\pm^*$ is isometric:

$$
\Psi_\pm \Psi_\pm^* = I. \qquad (4.17)
$$

This relation can be checked exactly as in the short-range case.

Let us summarize these results.

Theorem 4.5 *Let Assumption 2.1 be satisfied. Then the operators* $\Psi_\pm : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ *defined by formulas* (4.12), (4.13) *are bounded and relations* (4.15)–(4.17) *hold true.*

Corollary 4.6 *The positive spectrum of the operator* H *covers* $\mathbb{R}_+$. *It is absolutely continuous and simple.*

4.4 Wave operators

Let us now consider the differential operator

$$H_0 = -p_0 \mathrm{d}^2/\mathrm{d}x^2$$

with the same boundary condition $f(0) = 0$ in the space $L^2(\mathbb{R}_+)$. In this case, the two operators $\Psi_\pm$ defined by (4.12), (4.13) reduce to the single operator (the Fourier sine transform)

$$(\Psi_0 f)(\lambda) = \frac{1}{\sqrt{\pi}\sqrt[4]{p_0\lambda}} \int_0^\infty \sin\left(\sqrt{\lambda/p_0}\, x\right) f(x) dx.$$

This operator possesses of course all properties enumerated in Theorem 4.5.

Stationary wave operators $U_\pm$ for the pair H_0, H are defined by the relation

$$U_\pm = \Psi_\pm^* \Psi_0.$$

The following result is a direct consequence of Theorem 4.5.

Theorem 4.7 *Under Assumption 2.1, the wave operators* $U_\pm$ *are isometric,* $U_\pm^* U_\pm = I$, *complete, i.e.,* $U_\pm U_\pm^* = E(\mathbb{R}_+)$, *and enjoy the intertwining property* $HU_\pm = U_\pm H_0$.

It follows from relations (4.12), (4.13) that, for all $f \in L^2(\mathbb{R}_+)$,

$$(\Psi_+ f)(\lambda) = S(\lambda)(\Psi_- f)(\lambda) \tag{4.18}$$

where the coefficient

$$S(\lambda) = \frac{w(\lambda - i0)}{w(\lambda + i0)}$$

is known as the stationary scattering matrix.

Time-dependent wave operators $W_\pm$ are defined as strong limits

$$W_\pm = \text{s-lim}_{t\to\pm\infty} e^{iHt} \mathsf{U}_0(t) \tag{4.19}$$

where $\mathsf{U}_0(t)$ is a suitable (see, for example, [2] or [5,18]) unitary regularization of $e^{-iH_0 t}$. If the limits (4.19) exist, then the operators $W_\pm$ are automatically isometric and enjoy the intertwining property. Moreover, the operators $W_\pm$ are complete because the spectrum of the operator H is simple according to the classical Weyl result.

Consideration of the operators $W_\pm$ is out of the scope of the present article. We note, however, that various proofs of the existence of limits in (4.19) require somewhat more stringent conditions on p and q compared to Assumption 2.1. Under such conditions, the equality $W_\pm = U_\pm$ also holds.

5 Miscellaneous

5.1 Short-range perturbations

Let us now consider a more general case where $q(x)$ is replaced by a function

$$q(x) + q_{\rm sr}(x)$$

with a short-range term $q_{\rm sr} \in L^1(\mathbb{R}_+)$. We suppose that p and q satisfy Assumption 2.1 and define the functions Ω and θ, etc., by formulas of Sect. 2 neglecting $q_{\rm sr}$. Then for the remainder (2.3), we have the expression

$$r(x, z) = (p(x)\omega(x, z))' + q_{\rm sr}(x) \tag{5.1}$$

instead of (2.10); as before, $r(\cdot, z) \in L^1(\mathbb{R}_+)$. Therefore the integral equation (2.17) with kernel (2.18) and remainder (5.1) has a solution $u(x, z)$ satisfying estimate (2.22). Then the Jost solution of the differential equation (2.1) with q replaced by $q + q_{\rm sr}$ is again defined by equality (2.25). Thus all the results obtained in the previous sections for the particular case $q_{\rm sr} = 0$ remain true.

5.2 General boundary conditions

Here, we briefly discuss the differential operator (1.4) in $L^2(\mathbb{R}_+)$ with a boundary condition

$$f'(0) = hf(0), \quad h = \bar{h}. \tag{5.2}$$

As before, $\theta(x, z)$ is the Jost solution of Eq. (2.1) constructed in Theorem 2.9. Instead of (3.1), the regular solution $\varphi(x, z)$ of this equation will now be distinguished by the conditions

$$\varphi(0, z) = 1, \quad \varphi'(0, z) = h. \tag{5.3}$$

This is again an analytic function of $z \in \mathbb{C}$. Formula (3.6) remains true where according to (5.3) the Wronskian

$$w(z) = \{\varphi(\cdot, z), \theta(\cdot, z)\} = p(0)(h\theta(0, z) - \theta'(0, z)). \tag{5.4}$$

The resolvent kernel is still given by the relation (4.8) which yields representation (4.11) with the functions $\varphi(x, \lambda)$ and $w(z)$ defined by (5.3) and (5.4). As before, the functions $\psi_\pm(x, \lambda)$ are given by formula (4.12) and the operators $\Psi_\pm$ – by formula

(4.13). Then Theorems 4.5 and 4.7 remain true for the operator H corresponding to the boundary condition (5.2).

5.3 Problem on the whole line

Consider now the operator (1.4) in the space $L^2(\mathbb{R})$. We here follow closely the scheme described for short-range potentials, for example, in §5.1 of [20].

Suppose that the conditions on $p(x)$ and $q(x)$ are imposed for all $x \in \mathbb{R}$; in particular, the limits in (2.2) are taken for $|x| \to \infty$. In addition to the Jost solution $\theta(x,z) =: \theta_1(x,z)$ of Eq. (2.1) built in Theorem 2.9, we distinguish a solution $\theta_2(x,z)$ by its asymptotics as $x \to -\infty$. For the function $\Omega(x,z)$ defined by (2.11), we set $a_2(x,z) = e^{\Omega(x,z)}$. Then the construction of Theorem 2.9 leads to the solution $\theta_2(x,z)$ of Eq. (2.1) with asymptotics $\theta_2(x,z) \sim a_2(x,z)$ as $x \to -\infty$ (cf. (2.26)). We also introduce the Wronskian of the solutions θ_1 and θ_2:

$$w(z) = \{\theta_2(\cdot,z), \theta_1(\cdot,z)\}.$$

The construction of the resolvent $R(z) = (H-z)^{-1}$, $z \in \Pi$, is similar to Sect. 4.2. Since $\theta_1 \in L^2$ as $x \to \infty$ and $\theta_2 \in L^2$ as $x \to -\infty$, the resolvent kernel equals (cf. (4.8))

$$R(x,y;z) = w(z)^{-1}\theta_2(x,z)\theta_1(y,z) \quad \text{for} \quad x \le y \quad \text{and} \quad R(y,x;z) = R(x,y;z). \tag{5.5}$$

Suppose now that $z = \lambda \pm i0$ where $\lambda > 0$. Calculating the Wronskians for $x \to \infty$ (if $j=1$) or for $x \to -\infty$ (if $j=2$), we find that

$$\{\theta_j(\lambda+i0), \theta_j(\lambda-i0)\} = (-1)^{j-1} 2i\sqrt{p_0\lambda}, \quad j = 1,2.$$

Thus, these solutions are linearly independent for all $\lambda > 0$, and we have

$$\begin{aligned}\theta_1(x,\lambda+i0) = (2i)^{-1}(p_0\lambda)^{-1/2}K_1(\lambda)^{-2}\big(&\overline{\mathbf{w}(\lambda)}\theta_2(x,\lambda+i0)\\ &- w(\lambda+i0)\theta_2(x,\lambda-i0)\big)\end{aligned} \tag{5.6}$$

$$\begin{aligned}\theta_2(x,\lambda+i0) = (2i)^{-1}(p_0\lambda)^{-1/2}K_2(\lambda)^{-2}\big(&\mathbf{w}(\lambda)\theta_1(x,\lambda+i0)\\ &- w(\lambda+i0)\theta_1(x,\lambda-i0)\big)\end{aligned} \tag{5.7}$$

where

$$\mathbf{w}(\lambda) = \{\theta_2(\cdot,\lambda+i0), \theta_1(\cdot,\lambda-i0)\},$$

$K_1(\lambda) = K(\lambda)$ is defined by (2.28) and

$$K_2(\lambda) = \exp\left(-\int_{-\infty}^{0}\sqrt{\left(\frac{q(y)-\lambda}{p(y)}\right)_+}\,\mathrm{d}y\right).$$

Note the identity

$$|w(\lambda \pm i0)|^2 = 4p_0\lambda K_1(\lambda)^2 K_2(\lambda)^2 + |\mathbf{w}(\lambda)|^2. \tag{5.8}$$

For its proof, we substitute expression (5.6) for $\theta_1(x, \lambda + i0)$ and $\overline{\theta_1(x, \lambda + i0)} = \theta_1(x, \lambda - i0)$ into the right-hand side of (5.7) and observe that the coefficient in front of $\theta_2(x, \lambda + i0)$ should be equal to 1.

We need an analogue of representation (4.11).

Lemma 5.1 *For all $x, y \in \mathbb{R}$ and $\lambda > 0$, we have the representation*

$$\begin{aligned}\mathrm{d}E(x, y; \lambda)/\mathrm{d}\lambda \;&= \frac{\sqrt{p_0\lambda}}{\pi|w(\lambda \pm i0)|^2}\Big(K_1(\lambda)^2\theta_1(x, \lambda + i0)\theta_1(y, \lambda - i0) \\ &\quad + K_2(\lambda)^2\theta_2(x, \lambda + i0)\theta_2(y, \lambda - i0)\Big).\end{aligned} \tag{5.9}$$

Proof It follows from the Cauchy–Stieltjes formula (4.10) and the representation (5.5) that

$$\begin{aligned}\mathrm{d}E(x, y; \lambda)/\mathrm{d}\lambda = \frac{1}{2\pi i|w(\lambda \pm i0)|^2}\Big(&w(\lambda - i0)\theta_2(x, \lambda + i0)\theta_1(y, \lambda + i0) \\ &-w(\lambda + i0)\theta_2(x, \lambda - i0)\theta_1(y, \lambda - i0)\Big), \quad x \le y.\end{aligned} \tag{5.10}$$

Let us show that the right-hand sides of (5.9) and (5.10) coincide. Replacing $\theta_1(x, \lambda + i0)$ in (5.9) by its expression (5.6), we see that it suffices to check the identity

$$\begin{aligned}&\big(\mathbf{w}(\lambda)\theta_2(x, \lambda + i0) - w(\lambda + i0)\theta_2(x, \lambda - i0)\big)\theta_1(y, \lambda - i0) \\ &\quad + 2i\sqrt{p_0\lambda}\, K_2(\lambda)^2\theta_2(x, \lambda + i0)\theta_2(y, \lambda - i0) \\ &= w(\lambda - i0)\theta_2(x, \lambda + i0)\theta_1(y, \lambda + i0) - w(\lambda + i0)\theta_2(x, \lambda - i0)\theta_1(y, \lambda - i0).\end{aligned} \tag{5.11}$$

The coefficients in front of $\theta_2(x, \lambda+i0)$ in the left and right sides of (5.11) coincide by virtue of (5.7). The terms containing $\theta_2(x, \lambda - i0)$ in the left and right sides of (5.11) are the same. Of course representation (5.9) extends to $x \ge y$ since its right-hand side becomes complex conjugated if x and y are interchanged. □

Remark 5.2 The left-hand side of (5.9) is real and symmetric in (x, y). Therefore in addition to (5.9), we have the representation

$$\begin{aligned}\mathrm{d}E(x, y; \lambda)/\mathrm{d}\lambda \;&= \frac{\sqrt{p_0\lambda}}{\pi|w(\lambda \pm i0)|^2}\Big(K_1(\lambda)^2\theta_1(x, \lambda - i0)\theta_1(y, \lambda + i0) \\ &\quad + K_2(\lambda)^2\theta_2(x, \lambda - i0)\theta_2(y, \lambda + i0)\Big).\end{aligned}$$

Instead of (4.12), we now define eigenfunctions of the continuous spectrum of the operator H by the relation

$$\psi_j(x,\lambda) = \frac{\sqrt[4]{p_0\lambda}\, K_j(\lambda)}{i\sqrt{\pi}\, w(\lambda + i0)} \theta_j(x, \lambda + i0), \quad j = 1, 2. \tag{5.12}$$

Then (cf. (4.13)) we introduce the mappings $\Psi_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+; \mathbb{C}^2)$ by formulas

$$\begin{aligned}(\Psi_+ f)(\lambda) &= \Big(\int_{-\infty}^{\infty} \psi_2(x,\lambda) f(x)\mathrm{d}x, \int_{-\infty}^{\infty} \psi_1(x,\lambda) f(x)\mathrm{d}x\Big)^\top \\ (\Psi_- f)(\lambda) &= \Big(\int_{-\infty}^{\infty} \overline{\psi_1(x,\lambda)} f(x)\mathrm{d}x, \int_{-\infty}^{\infty} \overline{\psi_2(x,\lambda)} f(x)\mathrm{d}x\Big)^\top .\end{aligned} \tag{5.13}$$

It follows from (5.9) that relation (4.14) holds. Therefore, similarly to the proof of Theorem 4.5, one obtains the following result. Note that the multiplication operator A acts now in the space $L^2(\mathbb{R}_+; \mathbb{C}^2)$.

Theorem 5.3 *Let Assumption 2.1 be satisfied for all $x \in \mathbb{R}$. Then the operators $\Psi_\pm$ are bounded and satisfy relations (4.15)–(4.17).*

Corollary 5.4 *The positive spectrum of the operator H covers $\mathbb{R}_+$. It is absolutely continuous and has multiplicity two.*

In terms of functions (5.12), relations (5.6) and (5.7) can equivalently be rewritten as

$$\begin{pmatrix} \psi_2(x,\lambda) \\ \psi_1(x,\lambda) \end{pmatrix} = S(\lambda) \begin{pmatrix} \overline{\psi_1(x,\lambda)} \\ \overline{\psi_2(x,\lambda)} \end{pmatrix} \tag{5.14}$$

where

$$S(\lambda) = w(\lambda + i0)^{-1} \begin{pmatrix} i\gamma(\lambda) & \mathbf{w}(\lambda) \\ \overline{\mathbf{w}(\lambda)} & i\gamma(\lambda) \end{pmatrix} \quad \text{and} \quad \gamma(\lambda) = 2\sqrt{p_0\lambda} K_1(\lambda) K_2(\lambda). \tag{5.15}$$

According to (5.8), the 2×2 matrix $S(\lambda)$ is unitary. It is known as the scattering matrix for the problem on the whole line. It follows from (5.14) that, for all $f \in L^2(\mathbb{R})$, the identity (4.18) holds with the operators $\Psi_\pm$ defined by (5.13) and the matrix (5.15).

Finally, we note that formulas (5.6), (5.7) yield asymptotics of the eigenfunctions (5.12) as $x \to \pm\infty$.

References

1. Birman, M. Sh. Solomyak, M.Z.: Spectral Theory of Selfadjoint Operators in Hilbert Space. Reidel, Doldrecht (1987)
2. Buslaev, V.S., Matveev, V.B.: Wave operators for the Schrödinger equation with a slowly decreasing potential. Theor. Math. Phys. **2**, 266–274 (1970)
3. Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. McGraw-Hill, New York (1955)

4. Gilbert, D., Pearson, D.B.: On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators. J. Math. Anal. Appl. **128**(2), 30–56 (1987)
5. Hörmander, L.: The existence of wave operators in scattering theory. Math. Z. **146**, 69–91 (1976)
6. Jost, R.: Über die falschen Nullstellen der Eigenwerte des S-matrix. Helv. Phys. Acta **20**, 250–266 (1947)
7. Levinson, N.: On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase. Danske Videnskab. Selskab, Mat.-Fyz. Medd. **25**(9), 29 (1949)
8. Matveev, V.B., Skriganov, M.M.: Scattering problem for radial Schrödinger equation with a slowly decreasing potential. Theor. Math. Phys. **10**(2), 156–164 (1972)
9. Mourre, E.: Absence of singular spectrum for certain self-adjoint operators. Commun. Math. Phys. **78**, 391–400 (1981)
10. Naboko, S.N.: On the dense point spectrum of Schrödinger and Dirac operators. Theor. Math. Phys. **68**(1), 18–28 (1986)
11. Newton, R.: Scattering Theory of Waves and Particles. Springer, Berlin (1982)
12. Olver, F.W.J.: Asymptotics and Special Functions. Academic Press, New York (1974)
13. Reed, M., Simon, B.: Methods of Modern Mathematical Physics IV. Academic Press, New York (1978)
14. Titchmarsh, E.C.: Eigenfunction Expansions Associated with Second-Order Differential Equations, vol. 1. Oxford, (1946)
15. Weidmann, J.: Lineare Operatoren in Hilberträumen, Teil II: Anwendungen. Teubner Verlag, Stuttgart (2003)
16. Weyl, H.: Über gewöhnliche lineare Differentialgleichungen mit singulären Stellen und ihre Eigenfunktionen, Nachr. Ges. Wiss. Göttingen, 37-64 (1909)
17. Weyl, H.: Über gewöhnliche Differentialgleichungen mit Singularitatten und die zugehörigen Entwicklungen willkürlicher Funktionen. Math. Ann. **68**, 220–269 (1910)
18. Yafaev, D.R.: Wave operators for the Schrödinger equation. Theor. Math. Phys. **45**(2), 992–998 (1981)
19. Yafaev, D.R.: The low energy scattering for slowly decreasing potentials. Commun. Math. Phys. **85**, 177–196 (1982)
20. Yafaev, D.R.: Mathematical Scattering Theory: Analytic Theory. American Mathematical Society, Providence, Rhode Island (2010)