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FRT presentation of classical Askey–Wilson algebras

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Abstract
Automorphisms of the infinite-dimensional Onsager algebra are introduced. Certain quotients of the Onsager algebra are formulated using a polynomial in these automorphisms. In the simplest case, the quotient coincides with the classical analog of the Askey–Wilson algebra. In the general case, generalizations of the classical Askey–Wilson algebra are obtained. The corresponding class of solutions of the non-standard classical Yang–Baxter algebra is constructed, from which a generating function of elements in the commutative subalgebra is derived. We provide also another presentation of the Onsager algebra and of the classical Askey–Wilson algebras.

Keywords Onsager algebra · Non-standard Yang–Baxter algebra · Askey–Wilson algebras · Integrable systems

Mathematics Subject Classification 81R50 · 81R10 · 81U15

1 Introduction

The Onsager algebra is an infinite-dimensional Lie algebra with three known presentations. Introduced by Onsager [22] in the investigation of the exact solution of the two-dimensional Ising model, the original presentation is given in terms of generators $\{A_n, G_m | n, m \in \mathbb{Z}\}$ and relations (see Definition 2.1). The second presentation is given in terms of two generators $\{A_0, A_1\}$ satisfying the so-called Dolan–Grady relations (2.4) [8,9]. Recently [3], a third presentation has been identified. It is given

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in terms of elements of the non-standard classical Yang–Baxter algebra (2.7) with r-matrix (2.5).

The Askey–Wilson algebra has been introduced in [36], providing an algebraic scheme for the Askey–Wilson polynomials. This algebra is connected with the double affine Hecke algebra of type $(C_1^\vee, C_1)$ [17–20,33], the theory of Leonard pairs [21,28, 29] and $U_q(sl_2)$ [15,16,35]. A well-known presentation of the Askey–Wilson algebra[1] is given in terms of three generators satisfying the relations displayed in Definition 3.1. Generalizations of the Askey–Wilson algebra are an active field of investigation. Various examples of generalizations have been considered in the literature, see, for instance, [11,14,23,24].

In this note, it is shown that the class of quotients of the Onsager algebra considered by Davies in [8,9] generates a classical analog ($q = 1$) of the Askey–Wilson algebra and generalizations of this algebra. For each quotient, classical analogs of the automorphisms recently introduced in [6] are used to derive explicit polynomial expressions for the generators. Based on the results of [3] extended to these quotients, for the classical Askey–Wilson algebra and each of its generalization, a presentation *à la Faddeev–Reshetikhin–Takhtajan* is given. Using this presentation, for each quotient a commutative subalgebra is identified. To complete the analysis, we also give a new presentation of the Onsager algebra that can be understood as the specialization $q = 1$ of the infinite-dimensional quantum algebra $\mathcal{A}_q$ introduced in [4,5]. In this alternative presentation, the quotients of the Onsager algebra corresponding to Davies' prescription are determined.

2 The Onsager algebra, quotients and FRT presentation

In this section, three different presentations of the Onsager algebra $\mathscr{O}$ are first reviewed, and three different automorphisms Φ, τ_0, τ_1 of the Onsager algebra are introduced. Using these, the elements in $\mathscr{O}$ are written as simple polynomial expressions of the fundamental generators A_0, A_1. Then, we consider certain quotients of the Onsager algebra introduced by Davies [8,9]. Each quotient is formulated using an operator written as a polynomial in the automorphisms. Given a quotient, the FRT presentation is constructed from which a generating function for mutually commuting quantities is obtained.

2.1 The Onsager algebra

The Onsager algebra has been introduced in the context of mathematical physics [22]. The first presentation of this algebra which originates in Onsager's work [22] is now recalled.

Definition 2.1 The Onsager algebra $\mathscr{O}$ is generated by $\{A_n, G_m | n, m \in \mathbb{Z}\}$ subject to the following relations:

$$[A_n, A_m] = 4\, G_{n-m}, \tag{2.1}$$

[1] For the *universal* Askey–Wilson algebra introduced in [32], a second presentation is known.

$$[G_n, A_m] = 2A_{n+m} - 2A_{m-n}, \tag{2.2}$$

$$[G_n, G_m] = 0. \tag{2.3}$$

Remark 1 $\{A_n, G_m\}$ for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$ form a basis of $\mathscr{O}$. Note that $G_{-n} = -G_n$ and $G_0 = 0$.

Note that a second presentation is given in terms of two generators A_0, A_1 subject to a pair of relations, the so-called Dolan–Grady relations [10]. They read:

$$[A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1], \qquad [A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0]. \tag{2.4}$$

These two presentations define isomorphic Lie algebras, see [8,9,25].

In a recent paper [3], a third presentation of the Onsager algebra was proposed using the framework of the non-standard classical Yang–Baxter algebra. It is called a *FRT presentation* in honor of the authors Faddeev–Reshetikhin–Takhtajan [12,13]. Let us introduce the r-matrix (u, v are formal variables, sometimes called "spectral parameters" in the literature on integrable systems)

$$r_{12}(u,v) = \frac{1}{(u-v)(uv-1)} \begin{pmatrix} u(1-v^2) & 0 & 0 & -2(u-v) \\ 0 & -u(1-v^2) & -2v(uv-1) & 0 \\ 0 & -2u(uv-1) & -u(1-v^2) & 0 \\ -2uv(u-v) & 0 & 0 & u(1-v^2) \end{pmatrix} \tag{2.5}$$

solution of the non-standard classical Yang–Baxter equation

$$\begin{aligned}[][\, r_{13}(u_1,u_3),\ r_{23}(u_2,u_3)\,] = {}& [\, r_{21}(u_2,u_1),\ r_{13}(u_1,u_3)\,] \\ &+ [\, r_{23}(u_2,u_3),\ r_{12}(u_1,u_2)\,],\end{aligned} \tag{2.6}$$

where we denote $r_{12}(u) = r(u) \otimes \mathbb{I}$, $r_{23}(u) = \mathbb{I} \otimes r(u)$ and so on.

Theorem 1 [3] *The non-standard classical Yang–Baxter algebra*

$$[\, B_1(u),\ B_2(v)\,] = [\, r_{21}(v,u),\ B_1(u)\,] + [\, B_2(v),\ r_{12}(u,v)\,] \tag{2.7}$$

for the r-matrix (2.5) *and*

$$B(u) = \begin{pmatrix} \mathcal{G}(u) & \mathcal{A}^-(u) \\ \mathcal{A}^+(u) & -\mathcal{G}(u) \end{pmatrix} \tag{2.8}$$

with

$$\mathcal{G}(u) = \sum_{n\geq 1} u^n G_n, \quad \mathcal{A}^-(u) = \sum_{n\geq 0} u^n A_{-n}, \quad \mathcal{A}^+(u) = \sum_{n\geq 1} u^n A_n\ , \tag{2.9}$$

provides an FRT presentation of the Onsager algebra.

This type of "twisted" classical r-matrix has been studied in [26] and their associated algebras in [27].

2.2 Automorphisms of the Onsager algebra

We are interested in three algebra automorphisms of $\mathcal{O}$. Let $\Phi : \mathcal{O} \to \mathcal{O}$ denote the algebra automorphism defined by $\Phi(A_0) = A_1$ and $\Phi(A_1) = A_0$. Observe that $\Phi^2 = \text{id}$. We now introduce two other automorphisms of $\mathcal{O}$.

Proposition 2.1 *There exist two involutive algebra automorphisms $\tau_0, \tau_1 : \mathcal{O} \to \mathcal{O}$ such that*

$$\tau_0(A_0) = A_0, \tag{2.10}$$

$$\tau_0(A_1) = -\frac{1}{8}\left(A_1A_0^2 - 2A_0A_1A_0 + A_0^2A_1\right) + A_1 = -\frac{1}{8}[A_0, [A_0, A_1]] + A_1, \tag{2.11}$$

$$\tau_1(A_1) = A_1, \tag{2.12}$$

$$\tau_1(A_0) = -\frac{1}{8}\left(A_0A_1^2 - 2A_1A_0A_1 + A_1^2A_0\right) + A_0 = -\frac{1}{8}[A_1, [A_1, A_0]] + A_0. \tag{2.13}$$

Proof Firstly, we show that τ_0 leaves invariant the first relation in (2.4). This follows immediately from the fact that

$$[A_0, \tau_0(A_1)] = [A_1, A_0]. \tag{2.14}$$

Secondly, we show that τ_0 leaves invariant the second relation in (2.4). Observe that:

$$[\tau_0(A_1), [\tau_0(A_1), A_0]] = -8\tau_0\tau_1(A_0) + 8A_0. \tag{2.15}$$

It follows:

$$[\tau_0(A_1), [\tau_0(A_1), [\tau_0(A_1), A_0]]] = 8\underbrace{[\tau_0\tau_1(A_0), \tau_0(A_1)]}_{=\tau_0([\tau_1(A_0), A_1])} + 8\underbrace{[\tau_0(A_1), A_0]}_{=[A_0, A_1]} = 16[A_0, A_1].$$

So, we conclude that τ_0 leaves invariant both relations in (2.4).

$$\tau_0(\tau_0(A_0)) = A_0, \tag{2.16}$$

$$\tau_0(\tau_0(A_1)) = -\frac{1}{8}[A_0, [A_0, \tau_0(A_1)]] + \tau_0(A_1) = A_1\ . \tag{2.17}$$

This proves that τ_0 is involutive and by consequence is a bijection.

The same holds for τ_1, using $\tau_1 = \Phi \circ \tau_0 \circ \Phi$. □

Remark 2 $(\tau_0\Phi)(\tau_1\Phi) = (\tau_1\Phi)(\tau_0\Phi) = \text{id}$.

Let us mention that the automorphisms Φ, τ_0, τ_1 can be viewed as the classical analogs $q = 1$ of the automorphisms considered in [6] (see also [34]). Using τ_0, τ_1 and Φ, the elements of the Onsager algebra admit simple expressions as polynomials of the two fundamental generators A_0, A_1.

Proposition 2.2 *In the Onsager algebra $\mathcal{O}$, one has:*

$$A_m = (\tau_1\Phi)^m(A_0) \quad \textit{and} \quad G_n = \frac{1}{4}[(\tau_1\Phi)^n(A_0), A_0]. \tag{2.18}$$

Proof By definition (2.1), one has $G_1 = [A_1, A_0]/4$. By Remark 2, one has $(\tau_0\Phi) = (\tau_1\Phi)^{-1}$. According to (2.10)–(2.13), it follows:

$$[G_1, A_0] = 2(A_1 - \tau_0(A_1)), \quad [G_1, A_1] = 2(\tau_1(A_0) - A_0)\ . \tag{2.19}$$

Comparing (2.19) with (2.2), we see that the identification (2.18) holds for $m = -1, 2$. Then, we note that $\tau_1(G_1) = -G_1$ by (2.4). Acting with $(\tau_1\Phi)^k$ on (2.19), one derives (2.2) for $n = 1$. The second relation in (2.18) follows from (2.1). □

Remark 3 $\Phi(A_{-n}) = A_{n+1}$ and $\Phi(G_n) = -G_n$.

In the FRT presentation displayed in Theorem 1, the action of the automorphisms is easily identified. The action of τ_0, τ_1 on the currents is such that:

$$\begin{aligned}
&(\tau_0\Phi)(\mathcal{A}^-(u)) = u^{-1}(\mathcal{A}^-(u) - A_0), \quad (\tau_0\Phi)(\mathcal{A}^+(u)) = u(\mathcal{A}^+(u) + A_0),\\
&(\tau_1\Phi)(\mathcal{A}^-(u)) = u(\mathcal{A}^-(u) + A_1), \qquad (\tau_1\Phi)(\mathcal{A}^+(u)) = u^{-1}(\mathcal{A}^+(u) - A_1),\\
&\quad (\tau_0\Phi)(\mathcal{G}(u)) = (\tau_1\Phi)(\mathcal{G}(u)) = \mathcal{G}(u).
\end{aligned} \tag{2.20}$$

2.3 Quotients of the Onsager algebra

In Davies' paper on the Onsager algebra and superintegrability [8,9], Davies considers certain quotients of the Onsager algebra. Below, we characterize the relations considered by Davies in terms of an operator which is a polynomial in two automorphisms $\overline{\tau}_0, \overline{\tau}_1$. As will be shown later, these quotients can be viewed as generalizations of the classical ($q = 1$) Askey–Wilson algebra.

Definition 2.2 Let $\{\alpha_n | n = 0, \ldots, N\}$ be nonzero scalars with N any nonzero positive integer. The algebra $\overline{\mathcal{O}}_N$ is defined as the quotient of the Onsager algebra $\mathcal{O}$ by the relations

$$\sum_{n=-N}^{N} \alpha_n A_{-n} = 0 \quad \text{and} \quad \sum_{n=-N}^{N} \alpha_n A_{n+1} = 0 \quad \text{with} \quad \alpha_{-n} = \alpha_n. \tag{2.21}$$

There exists an algebra homomorphism $\varphi_N : \mathcal{O} \to \overline{\mathcal{O}}_N$ that sends $A_0 \mapsto A_0$, $A_1 \mapsto A_1$. We now introduce three automorphisms $\overline{\tau}_0, \overline{\tau}_1$ and $\overline{\Phi}$ of $\overline{\mathcal{O}}_N$ such that

$\overline{\tau}_0\varphi_N = \varphi_N\tau_0$, $\overline{\tau}_1\varphi_N = \varphi_N\tau_1$ and $\overline{\Phi}(A_0) = A_1$. According to Proposition 2.2, introduce the operator:

$$S_N = \sum_{n=-N}^{N} \alpha_n (\overline{\tau}_1\overline{\Phi})^n. \tag{2.22}$$

The relations in (2.21) simply read $S_N(A_0) = 0$ and $S_N(A_1) = 0$, respectively. These results allow us to give an alternative presentation of the quotients $\overline{\mathscr{O}}_N$:

Proposition 2.3 *The quotient $\overline{\mathscr{O}}_N$ is generated by A_0 and A_1 subject to the Dolan Grady relations*

$$[A_0, [A_0, [A_0, A_1]]] = 16[A_0, A_1] \quad \textit{and} \\ [A_1, [A_1, [A_1, A_0]]] = 16[A_1, A_0], \tag{2.23}$$

and to the relations

$$S_N(A_0) = 0 \quad \textit{and} \quad S_N(A_1) = 0, \tag{2.24}$$

where S_N is defined by (2.22).

Furthermore, one has $[(\overline{\tau}_1\overline{\Phi})^p, S_N] = 0$ for any $p \in \mathbb{Z}$. Together with the second relation in (2.18), it follows:

Remark 4 The relations (2.21) imply:

$$\sum_{n=-N}^{N} \alpha_n A_{n+p} = 0, \quad \sum_{n=-N}^{N} \alpha_n G_{n+p} = 0 \quad \text{for any} \quad p \in \mathbb{Z}. \tag{2.25}$$

It follows that the algebra $\overline{\mathscr{O}}_N$ has only $3N$ linearly independent elements. We choose the set $\{A_n, G_m | n = -N+1, \ldots, N; m = 1, \ldots, N\}$.

Note that above relations can be derived using the commutation relations (2.1)–(2.3) [8,9].

Remark 5 It is possible to introduce a slightly more general quotient $\overline{\mathscr{O}}_N(c_1, c_2)$. The algebra $\overline{\mathscr{O}}_N(c_1, c_2)$ is defined as the quotient of the Onsager algebra $\mathscr{O}$ by the relations

$$\sum_{n=-N}^{N} \alpha_n A_{-n} = c_1 \quad \text{and} \quad \sum_{n=-N}^{N} \alpha_n A_{n+1} = c_2 \quad \text{with} \quad \alpha_{-n} = \alpha_n, \tag{2.26}$$

where c_1, c_2 are central elements. In this quotient, the following relations hold

$$\sum_{n=-N}^{N} \alpha_n A_{n+2p} = c_1, \quad \sum_{n=-N}^{N} \alpha_n A_{n+2p+1} = c_2, \quad \sum_{n=-N}^{N} \alpha_n G_{n+p} = 0 \quad \text{for any} \quad p \in \mathbb{Z}. \tag{2.27}$$

We recover $\overline{\mathscr{O}}_N$ by putting $c_1 = c_2 = 0$.

In the algebra $\overline{\mathscr{O}}_N$, all higher elements can be written in terms of the elements $\{A_n, G_m | n = -N+1, \ldots, N; m = 1, \ldots, N\}$. Without loss of generality, choose $\alpha_N \equiv 1$. By induction using (2.25), one finds:

$$A_{-N-p} = (-1)^{p+N} \sum_{j=-N+1}^{N} \mathbb{U}_{p,j}^{(N)}(\alpha_0, \ldots, \alpha_{N-1}) A_j \quad \text{for any} \quad p \geq 0, \tag{2.28}$$

where $\mathbb{U}_{p,j}^{(N)}(\alpha_0, \ldots, \alpha_{N-1})$ is a $N-$variable polynomial that is determined recursively through the relation:

$$\mathbb{U}_{p+1,j}^{(N)}(\alpha_0, \ldots, \alpha_{N-1}) = \sum_{k=0}^{p} (-1)^k \alpha_{k-N+1} \mathbb{U}_{p-k,j}^{(N)}(\{\alpha_l\})$$
$$+ \begin{cases} (-1)^{N+p} \alpha_{j+p+1} & \text{for } -N+1 \leq j \leq N-p-1 \\ 0 & \text{for } N-p \leq j \leq N \end{cases},$$

with the convention $\alpha_{-N+1+k} \equiv 0$ if $k \geq 2N$ and initial conditions:

$$\mathbb{U}_{0,j}^{(N)}(\alpha_0, \ldots, \alpha_{N-1}) = (-1)^{N+1} \alpha_j.$$

Similarly, one gets:

$$A_{N+p+1} = (-1)^{p+N} \sum_{j=-N+1}^{N} \mathbb{U}_{p,j}^{(N)}(\alpha_0, \ldots, \alpha_{N-1}) A_{1-j},$$
$$G_{N+p+1} = (-1)^{p+N+1} \sum_{j=-N+1}^{N} \mathbb{U}_{p,j}^{(N)}(\alpha_0, \ldots, \alpha_{N-1}) G_{j-1} \quad \text{for any} \quad p \geq 0,$$

where (2.1) has been used to derive the second relation. For $N = 1$, one finds that $\mathbb{U}_{n-j,j}^{(1)}(\alpha_0) = U_n(\alpha_0)$ is the Chebyshev polynomial of second kind.

2.4 FRT presentation of the quotients $\overline{\mathscr{O}}_N$

For the class of quotients $\overline{\mathscr{O}}_N$ of the Onsager algebra, the corresponding solutions of the non-standard Yang–Baxter algebra (2.7) are now constructed.

Proposition 2.4 *The non-standard classical Yang–Baxter algebra* (2.7) *for the r-matrix* (2.5) *and*

$$B^{(N)}(u) = \frac{1}{p^{(N)}(u)}\begin{pmatrix} \mathcal{G}^{(N)}(u) & \mathcal{A}^{-(N)}(u) \\ \mathcal{A}^{+(N)}(u) & -\mathcal{G}^{(N)}(u) \end{pmatrix} \quad \text{with} \quad p^{(N)}(u) = \sum_{p=-N}^{N} \alpha_p u^{-p} \tag{2.29}$$

where, by setting $f_p^{(N)}(u) = \sum_{q=p}^{N} \alpha_q u^{p-q}$,

$$\mathcal{A}^{+(N)}(u) = \sum_{p=1}^{N} \big(f_p^{(N)}(u) A_p - u f_p^{(N)}(u^{-1}) A_{-p+1} \big), \tag{2.30}$$

$$\mathcal{A}^{-(N)}(u) = \sum_{p=1}^{N} \big(u^{-1} f_p^{(N)}(u) A_{-p+1} - f_p^{(N)}(u^{-1}) A_p \big), \tag{2.31}$$

$$\mathcal{G}^{(N)}(u) = \sum_{p=1}^{N} \big(f_p^{(N)}(u) + f_p^{(N)}(u^{-1}) \big) G_p - \sum_{p=1}^{N} \alpha_p G_p, \tag{2.32}$$

provides an FRT presentation of the algebra $\overline{\mathcal{O}}_N$.

Proof The goal consists in expressing all the elements $\{A_n, G_m | n, m \in \mathbb{Z}\}$ present in the FRT presentation of the Onsager algebra (see Theorem 1) in terms of the $3N$ linearly independent elements of $\overline{\mathcal{O}}_N$ $\{A_n, G_m | n = -N+1, \dots, N; m = 1, \dots, N\}$. For instance, let us consider the current $\mathcal{A}^+(u)$ in (2.8). Imposing the first relation of (2.25), it follows:

$$\begin{aligned}
\mathcal{A}^+(u) &= \sum_{p=1}^{N} u^p A_p + \sum_{p=N+1}^{\infty} u^p A_p \\
&= \sum_{p=1}^{N} u^p A_p - \frac{1}{\alpha_N} \sum_{p=1}^{\infty} u^{p+N} \sum_{q=-N}^{N-1} \alpha_q A_{p+q} \\
&= \sum_{p=1}^{N} u^p A_p - \frac{1}{\alpha_N} \sum_{q=-N}^{-1} \alpha_q u^{N-q} \\
&\quad \underbrace{\sum_{p=1}^{\infty} u^{p+q} A_{p+q}}_{=\mathcal{A}^+(u) + \sum_{p=q+1}^{0} u^p A_p} - \frac{\alpha_0}{\alpha_N} u^N \underbrace{\sum_{p=1}^{\infty} u^p A_p}_{=\mathcal{A}^+(u)} - \frac{1}{\alpha_N} \sum_{q=1}^{N-1} \alpha_q u^{N-q} \underbrace{\sum_{p=1}^{\infty} u^{p+q} A_{p+q}}_{=\mathcal{A}^+(u) - \sum_{p=1}^{q} u^p A_p} .
\end{aligned}$$

By factorizing $\mathcal{A}^+(u)$ in the last equation and after simplifications, one gets:

$$\mathcal{A}^+(u)\underbrace{\sum_{q=-N}^{N}\alpha_q u^{N-q}}_{\equiv u^N p^{(N)}(u)} = \sum_{q=1}^{N}\alpha_q u^{N-q}\sum_{p=1}^{q}u^p A_p - \sum_{q=-N}^{-1}\alpha_q u^{N-q}\sum_{p=q+1}^{0}u^p A_p. \tag{2.33}$$

It follows:

$$\mathcal{A}^+(u) = \frac{1}{p^{(N)}(u)}\sum_{q=1}^{N}\sum_{p=1}^{q}\big(\alpha_q u^{p-q}A_p - \alpha_{-q}u^{q-p+1}A_{-p+1}\big),$$

which leads to the formula (2.30). Applying the same procedure to $\mathcal{A}^-(u)$ and $\mathcal{G}(u)$, we obtain the other formulae. □

Using the FRT presentation, a commutative subalgebra of $\overline{\mathcal{O}}_N$ can be easily identified. Note that the result below is a straightforward restriction of [3, Proposition 2.5] to the quotients of the Onsager algebra.

Proposition 2.5 *Let κ, κ^*, μ be generic scalars. A generating function of mutually commuting elements in $\overline{\mathcal{O}}_N$ is given by:*

$$b^{(N)}(u) = \frac{1}{p^{(N)}(u)}\sum_{p=0}^{N-1}\big(f_p^{(N)}(u) - f_p^{(N)}(u^{-1})\big)I_p, \tag{2.34}$$

where

$$\begin{aligned} I_p &= \kappa(A_p + A_{-p}) + \kappa^*(A_{p+1} + A_{-p+1}) + \mu(G_{p+1} - G_{p-1}),\\ I_0 &= \kappa A_0 + \kappa^* A_1 + \mu G_1. \end{aligned} \tag{2.35}$$

Proof Introduce the 2×2 matrix:

$$M(x) = \begin{pmatrix} \mu/x & \kappa + \kappa^*/x \\ \kappa + \kappa^* x & \mu x \end{pmatrix} \tag{2.36}$$

which is a solution of

$$[tr_1(\overline{r}_{12}(u,v)M_1(u)),\ M_2(v)] = 0. \tag{2.37}$$

Then, by using the result [3, Proposition 2.5], one shows that $b^{(N)}(u) = tr M(u)B^{(N)}(u)$ satisfies $[b^{(N)}(u), b^{(N)}(v)] = 0$. Inserting (2.30)–(2.32) in $b^{(N)}(u) = tr\big(M(u)B^{(N)}(u)\big)$, one derives (2.34). □

3 $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$ and generalized classical Askey–Wilson algebras

The defining relations of the algebra $\overline{\mathcal{O}}_N$ are easily extracted from the defining relations of the non-standard classical Yang–Baxter algebra (2.7). For instance, we consider the cases $N = 1, 2$ below. For $N = 1$, we prove that $\overline{\mathcal{O}}_1$ is isomorphic to the Askey–Wilson algebra introduced by [36] specialized at $q = 1$.

3.1 The classical Askey–Wilson algebra $aw(3)$

We treat here in detail the case of the quotient $\overline{\mathcal{O}}_1$. To simplify the notations, we choose $\alpha_0 = \alpha$ and $\alpha_{\pm 1} = 1$. Equation (2.29) becomes

$$B^{(1)}(u) = \frac{1}{p^{(1)}(u)} \begin{pmatrix} G_1 & u^{-1}A_0 - A_1 \\ -uA_0 + A_1 & -G_1 \end{pmatrix} \tag{3.1}$$

where $p^{(1)}(u) = u + \alpha + u^{-1}$. Then, the non-standard Yang–Baxter algebra (2.7) provides the following defining relations of $\overline{\mathcal{O}}_1$

$$\begin{aligned} &[G_1,\ A_0] = 2\alpha A_0 + 4A_1, \qquad [A_1,\ G_1] = 2\alpha A_1 + 4A_0, \\ &[A_1,\ A_0] = 4G_1. \end{aligned} \tag{3.2}$$

Remark 6 The r-matrix (2.5) allows us to construct a representation of $\overline{\mathcal{O}}_1$. Indeed, the map $\pi(B_1^{(1)}(u)) = r_{13}(u, w)$ satisfies the non-standard Yang–Baxter algebra (2.7) and the expansion w.r.t. u is same. By comparing the expansions, one gets the following representation, for $\alpha = -w - w^{-1}$,

$$\pi(G_1) = (w^{-1} - w) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(A_0) = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \pi(A_1) = 2 \begin{pmatrix} 0 & w^{-1} \\ w & 0 \end{pmatrix}. \tag{3.3}$$

By Proposition 2.3, there is another presentation of the algebra $\overline{\mathcal{O}}_1$. Indeed, $\overline{\mathcal{O}}_1$ is generated by A_0 and A_1 subject to

$$[A_0, [A_0, A_1]] - 8\alpha A_0 - 16A_1 = 0, \qquad [A_1, [A_1, A_0]] - 8\alpha A_1 - 16A_0 = 0. \tag{3.4}$$

Let us remark that the Dolan–Grady relations (2.23) are not necessary in this case since they are implied by (3.4).

In [36], Zhedanov introduced the Askey–Wilson algebra with three generators K_0, K_1, K_2 and deformation parameter q. More recently, a central extension of the original Askey–Wilson algebra [36] called the universal Askey–Wilson algebra has been introduced [32]. In that paper, besides the original presentation of [36], a second presentation of the universal Askey–Wilson algebra is given. Below, we show that the quotient of the Onsager algebra $\overline{\mathcal{O}}_1$ is isomorphic to the classical ($q = 1$) analog of the Askey–Wilson algebra, denoted $aw(3)$. The first presentation of the original Askey–Wilson algebra is now recalled.

Definition 3.1 [36] The Askey–Wilson algebra has three generators K_0, K_1, K_2 that satisfy the commutation relations[2]:

$$\begin{aligned} \left[K_0, K_1\right]_q &= K_2, \quad \left[K_2, K_0\right]_q = BK_0 + C_1K_1 + D_1, \\ \left[K_1, K_2\right]_q &= BK_1 + C_0K_0 + D_0, \end{aligned} \tag{3.5}$$

where B, C_0, C_1, D_0, D_1 are the structure constants of the algebra.

Remark 7 In terms of the generators K_0, K_1, the defining relations of the Askey–Wilson algebra read:

$$\begin{aligned} \left[K_0, \left[K_0, K_1\right]_q\right]_{q^{-1}} + BK_0 + C_1K_1 + D_1 = 0, \\ \left[K_1, \left[K_1, K_0\right]_q\right]_{q^{-1}} + BK_1 + C_0K_0 + D_0 = 0. \end{aligned}$$

Definition 3.2 The classical Askey–Wilson algebra, denoted $aw(3)$, is the Askey–Wilson algebra specialized to $q = 1$. We keep the same notations for the classical Askey–Wilson algebra than for the usual Askey–Wilson algebra.

Proposition 3.1 *The algebra* $\overline{\mathscr{O}}_1$ *and the algebra* $aw(3)$ *are isomorphic. The isomorphism* $aw(3) \to \overline{\mathscr{O}}_1$ *is given by:*

$$K_0 \mapsto a_0A_0 + b_0, \quad K_1 \mapsto a_1A_1 + b_1, \quad K_2 \mapsto -\frac{a_0a_1}{4}G_1, \quad q \mapsto 1$$

with the identification of the structure constants:

$$B = -8\alpha/a_0a_1, \quad C_0 = -16/a_0^2, \quad C_1 = -16/a_1^2, \quad D_0 = -\frac{8\alpha b_0 + 16b_1}{a_0^2a_1},$$
$$D_1 = -\frac{8\alpha b_1 + 16b_0}{a_1^2a_0}.$$

Proof By direct computation. □

A corollary of this proposition is that Proposition 2.4 provides an FRT presentation of $aw(3)$. Note that for a specialization of the structure constants $B = D_0 = D_1$ in (3.5), one recovers the q-deformation of the Cartesian presentation of the sl_2 Lie algebra [37]. From that point of view, the representation (3.3) is natural.

The universal Askey–Wilson algebra has been introduced in [32]. For this algebra, a second presentation is known [32, Theorem 2.2]. It is given in terms of the quotient of the q-deformed analog of the Dolan–Grady relations (2.4) by a relation of quartic order in the two fundamental generators. These relations correspond to the presentation given by relations (3.4). Let us mention also that, from the second relation of (2.25) with $N = p = 1$, one gets $\alpha G_1 + G_2 = 0$. In terms of A_0, A_1, this relation reads:

$$8\alpha[A_1, A_0] + 2(A_1A_0A_1A_0 - A_0A_1A_0A_1) - A_1^2A_0^2 + A_0^2A_1^2 = 0. \tag{3.6}$$

[2] We denote the q-commutator $[X, Y]_q = qXY - q^{-1}YX$.

Note that (3.6) is not necessary: it follows from the commutator of the first (resp. second) relation in (3.4) with A_0 (resp. A_1). We would like to point out that the relations (3.4) coincide with (2.2), (2.3) of [32] for the specialization $q = 1$ and central elements evaluated to scalar values. Also, the Dolan–Grady relations (2.4) together with (3.6) coincide with the specialization $q = 1$ (and a suitable identification of the central element γ in terms of α) of the relation given in [32, Theorem 2.2].

3.2 The generalized classical Askey–Wilson algebra *aw*(6)

For $N = 2$, choose $\alpha_0 = \alpha'$, $\alpha_{\pm 1} = \alpha$ and $\alpha_{\pm 2} = 1$, Eq. (2.29) reads

$$B^{(2)}(u) = \frac{1}{p^{(2)}(u)} \begin{pmatrix} G_2 + (u + \alpha + u^{-1})G_1 & u^{-1}A_{-1} + u^{-1}(\alpha + u^{-1})A_0 - (u + \alpha)A_1 - A_2 \\ -uA_{-1} - u(u + \alpha)A_0 + u(\alpha + u^{-1})A_1 + A_2 & -G_2 - (u + \alpha + u^{-1})G_1 \end{pmatrix} \tag{3.7}$$

where $p^{(2)}(u) = u^2 + \alpha u + \alpha' + \alpha u^{-1} + u^{-2}$. One gets the following defining relations for $\overline{\mathscr{O}}_2$ from (2.7)

$$[A_0, A_{-1}] = [A_2, A_1] = [A_1, A_0] = 4G_1,$$

$$[A_1, A_{-1}] = [A_2, A_0] = 4G_2, \tag{3.8}$$

$$[A_2, A_{-1}] = 4(1 - \alpha')G_1 - 4\alpha G_2, \tag{3.9}$$

$$[G_1, A_0] = 2A_1 - 2A_{-1}, \quad [G_1, A_1] = 2A_2 - 2A_0, \tag{3.10}$$

$$[G_1, A_{-1}] = 2\alpha A_{-1} + 2(1 + \alpha')A_0 + 2\alpha A_1 + 2A_2, \tag{3.11}$$

$$[G_1, A_2] = -2A_{-1} - 2\alpha A_0 - 2(1 + \alpha')A_1 - 2\alpha A_2, \tag{3.12}$$

$$[G_2, A_0] = 2\alpha A_{-1} + 2\alpha' A_0 + 2\alpha A_1 + 4A_2, \tag{3.13}$$

$$[G_2, A_1] = -4A_{-1} - 2\alpha A_0 - 2\alpha' A_1 - 2\alpha A_2, \tag{3.14}$$

$$[G_2, A_{-1}] = 2(\alpha' - \alpha^2)A_{-1} + 2\alpha(1 - \alpha')A_0 + 2(2 - \alpha^2)A_1 - 2\alpha A_2, \tag{3.15}$$

$$[G_2, A_2] = 2\alpha A_{-1} + 2(\alpha^2 - 2)A_0 + 2\alpha(\alpha' - 1)A_1 + 2(\alpha^2 - \alpha')A_2, \tag{3.16}$$

$$[G_1, G_2] = 0. \tag{3.17}$$

Remark 8 As previously, a representation of $\overline{\mathscr{O}}_2$ is obtained from the r-matrix as follows:

$$\pi(B_1^{(2)}(u)) = r_{13}(u, w_1) + r_{14}(u, w_2) \tag{3.18}$$

with $\alpha = -w_1 - w_1^{-1} - w_2 - w_2^{-1}$ and $\alpha' = w_1 w_2 + w_1 w_2^{-1} + 2 + w_1^{-1} w_2 + w_1^{-1} w_2^{-1}$. By expanding w.r.t. the formal variable u, one gets a 4×4 representation for A_{-1}, A_0, A_1, A_2, G_1 and G_2.

By analogy with the classical Askey–Wilson algebra $aw(3)$ with defining relations (3.2), we call the algebra generated by the 6 elements A_{-1}, A_0, A_1, A_2, G_1, G_2 subject to the relations (3.8)–(3.17) the generalized classical Askey–Wilson $aw(6)$. By construction, this algebra is isomorphic to $\overline{\mathcal{O}}_2$.

By using Proposition 2.3 for $N = 2$, we get another presentation of the algebra $\overline{\mathcal{O}}_2 \cong aw(6)$: it is generated by A_0 and A_1 subject to the Dolan–Grady relation (2.23) with the additional following relations

$$[A_0, [A_1, [A_0, [A_1, A_0]]]] - 16[A_1, [A_1, A_0]] - 8\alpha[A_0, [A_0, A_1]] \\ +64(\alpha' + 2)A_0 + 128\alpha A_1 = 0, \tag{3.19}$$

$$[A_1, [A_0, [A_1, [A_0, A_1]]]] - 16[A_0, [A_0, A_1]] - 8\alpha[A_1, [A_1, A_0]] \\ +64(\alpha' + 2)A_1 + 128\alpha A_0 = 0. \tag{3.20}$$

By analogy with both previous examples, we define the generalization of the classical Askey–Wilson algebra, denoted $aw(3N)$, as the algebra $\overline{\mathcal{O}}_N$ generated by $3N$ generators $\{A_{-N+1}, \ldots, A_N\}$ and $\{G_1, \ldots, G_N\}$ and subject to the relations projecting the FRT relation (2.7). The number of defining relations $3N(3N - 1)/2$ and we do not write them explicitly. Using the FRT presentation, these relations can be easily extracted. We can alternatively define $aw(3N)$ with the help of Proposition 2.3, as the algebra generated by A_0, A_1 and subject to the Dolan–Grady relations (2.23) and relations (2.24). Finally, let us recall that a generating function of elements of its commutative subalgebra is given in Proposition 2.5.

4 Another presentation of the Onsager algebra and its quotients

In this section, a Lie algebra denoted $\mathcal{A}$ is introduced. It is shown to be isomorphic with the Onsager algebra. The corresponding FRT presentation is given, and polynomial expressions for the elements in $\mathcal{A}$ are obtained in terms of the two fundamental generators using the automorphisms introduced in Sect. 2. Then, we introduce the algebra $\overline{\mathcal{A}}_N$ as a quotient of $\mathcal{A}$ by the classical analog of the relations derived in [4,5]. The FRT presentation of $\overline{\mathcal{A}}_N$ is given.

4.1 Another presentation of the Onsager algebra

In [4,5] (see also [7]), an infinite-dimensional quantum algebra denoted $\mathcal{A}_q$ has been introduced. Recently, it has been conjectured that a certain quotient of $\mathcal{A}_q$ is isomorphic to the q-Onsager algebra[3] [2]. We now introduce the classical analog of $\mathcal{A}_q$ ($q = 1$).

Definition 4.1 $\mathcal{A}$ is a Lie algebra with generators $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{k+1} | k \in \mathbb{Z}_{\geq 0}\}$ satisfying the following relations, for $k, l \geq 0$:

$$\big[\mathcal{W}_{-l}, \mathcal{W}_{k+1}\big] = \tilde{\mathcal{G}}_{k+l+1}, \tag{4.1}$$

$$\big[\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_{-l}\big] = 16\mathcal{W}_{-k-l-1} - 16\mathcal{W}_{k+l+1}, \tag{4.2}$$

[3] The q-Onsager algebra is defined in terms of generators and q-analogs of the Dolan–Grady relations (2.4), see [1,31]. Note that the same relations showed up earlier in the context of polynomial association schemes [30].

$$\big[\mathcal{W}_{l+1}, \tilde{\mathcal{G}}_{k+1}\big] = 16\mathcal{W}_{l+k+2} - 16\mathcal{W}_{-k-l}, \tag{4.3}$$

$$\big[\mathcal{W}_{-k}, \mathcal{W}_{-l}\big] = 0, \quad \big[\mathcal{W}_{k+1}, \mathcal{W}_{l+1}\big] = 0, \quad \big[\tilde{\mathcal{G}}_{k+1}, \tilde{\mathcal{G}}_{l+1}\big] = 0. \tag{4.4}$$

Remark 9 The generators $\mathcal{W}_0, \mathcal{W}_1$ satisfy the Dolan–Grady relations (2.4).

Indeed, inserting the relations (4.1) into (4.2), (4.3) for $k = l = 0$, from the first two equalities in (4.4) for $k = 1, l = 0$ one gets:

$$\begin{aligned}
&[\mathcal{W}_0, [\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]]] = 16[\mathcal{W}_0, \mathcal{W}_1],\\
&[\mathcal{W}_1, [\mathcal{W}_1, [\mathcal{W}_1, \mathcal{W}_0]]] = 16[\mathcal{W}_1, \mathcal{W}_0].
\end{aligned} \tag{4.5}$$

Proposition 4.1 *The non-standard classical Yang–Baxter algebra* (2.7) *for the r-matrix* (2.5) *and*

$$B(u) = \frac{1}{2}\begin{pmatrix} -\frac{1}{4}\,\tilde{\mathcal{G}}(u) & u^{-1}\mathcal{W}_+(u) - \mathcal{W}_-(u) \\ -u\mathcal{W}_+(u) + \mathcal{W}_-(u) & \frac{1}{4}\,\tilde{\mathcal{G}}(u) \end{pmatrix} \tag{4.6}$$

with, by setting $U = (u + u^{-1})/2$,

$$\begin{aligned}
\mathcal{W}_+(u) &= \sum_{k=0}^{\infty} \mathcal{W}_{-k} U^{-k-1}, \quad \mathcal{W}_-(u) = \sum_{k=0}^{\infty} \mathcal{W}_{k+1} U^{-k-1},\\
\tilde{\mathcal{G}}(u) &= \sum_{k=0}^{\infty} \tilde{\mathcal{G}}_{k+1} U^{-k-1}\ ,
\end{aligned} \tag{4.7}$$

provides an FRT presentation of the algebra $\mathcal{A}$.

Proof Insert (4.6) into (2.7) with (2.5). Define the formal variables $U = (u + u^{-1})/2$ and $V = (v + v^{-1})/2$. One obtains equivalently:

$$\begin{aligned}
&(U - V)\big[\mathcal{W}_+(u), \mathcal{W}_-(v)\big] = \tilde{\mathcal{G}}(v) - \tilde{\mathcal{G}}(u),\\
&(U - V)\big[\tilde{\mathcal{G}}(u), \mathcal{W}_\pm(v)\big] \pm 16\big(U\mathcal{W}_\pm(u) - V\mathcal{W}_\pm(v) - \mathcal{W}_\mp(u) + \mathcal{W}_\mp(v)\big) = 0,\\
&\big[\mathcal{W}_\pm(u), \mathcal{W}_\pm(v)\big] = 0, \quad \big[\tilde{\mathcal{G}}(u), \tilde{\mathcal{G}}(v)\big] = 0.
\end{aligned}$$

Expanding the currents as (4.7), the above equations are equivalent to (4.1)–(4.4). □

Theorem 2 *The Onsager algebra* $\mathscr{O}$ *(see Definition* 2.1*) and the algebra* $\mathcal{A}$ *(see Definition* 4.1*) are isomorphic.*

Proof By Theorem 1 and Proposition 4.1, the Onsager algebra $\mathscr{O}$ and the algebra $\mathcal{A}$ have the same FRT presentation (2.7) with the same r-matrix (2.5). Then, the isomorphism between $\mathscr{O}$ and $\mathcal{A}$ follows from the fact that the power series of the entries in (2.8), (4.6) have same expansions w.r.t. the formal variable. □

The explicit relation between the generators $\{A_k, G_l | k \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 0}\}$ of the Onsager algebra $\mathscr{O}$ and the generators $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{l+1} | k, l \in \mathbb{Z}_{\geq 0}\}$ of the algebra $\mathcal{A}$ is obtained as follows. By comparison between (2.8) and (4.6), we get:

$$\mathcal{A}^+(u) \equiv \frac{1}{2}\big(-u\mathcal{W}_+(u) + \mathcal{W}_-(u)\big), \quad \mathcal{A}^-(u) \equiv \frac{1}{2}\big(u^{-1}\mathcal{W}_+(u) - \mathcal{W}_-(u)\big),$$
$$\mathcal{G}(u) \equiv -\frac{1}{8}\tilde{\mathcal{G}}(u) \tag{4.8}$$

with (2.9) and (4.7). Then, one can prove that one has the following expansion around $u = 0$, for $k \geq 0$:

$$U^{-k-1} = 2\sum_{p=0}^{\infty} c_p^{2p+k} u^{2p+k+1} \quad \text{with} \quad c_p^k = (-1)^p 2^{k-2p} \frac{(k-p)!}{(p)!(k-2p)!}.$$

By direct comparison of the l.h.s and r.h.s in (4.8), it follows, for $k \geq 0$,

$$A_{k+1} = \sum_{p=0}^{\left[\frac{k}{2}\right]} c_p^k \mathcal{W}_{k-2p+1} - \sum_{p=0}^{\left[\frac{k-1}{2}\right]} c_p^{k-1} \mathcal{W}_{-k+2p+1}, \tag{4.9}$$

$$A_{-k} = \sum_{p=0}^{\left[\frac{k}{2}\right]} c_p^k \mathcal{W}_{2p-k} - \sum_{p=0}^{\left[\frac{k-1}{2}\right]} c_p^{k-1} \mathcal{W}_{k-2p}, \tag{4.10}$$

$$G_{k+1} = -\frac{1}{4}\sum_{p=0}^{\left[\frac{k}{2}\right]} c_p^k \tilde{\mathcal{G}}_{k-2p+1}. \tag{4.11}$$

Conversely, one has:

$$\mathcal{W}_{-k} = \frac{1}{2^k}\sum_{p=0}^{k} \frac{k!}{p!(k-p)!} A_{k-2p}, \quad \mathcal{W}_{k+1} = \frac{1}{2^k}\sum_{p=0}^{k} \frac{k!}{p!(k-p)!} A_{k+1-2p}, \tag{4.12}$$

$$\tilde{\mathcal{G}}_{k+1} = \frac{1}{2^{k-2}}\sum_{p=0}^{k} \frac{k!}{p!(k-p)!} G_{2p-k-1}. \tag{4.13}$$

Here, $[n]$ is the integer part of n (with the convention $[-1/2] = -1$). For small values of k, explicit relations between the first few elements are reported in "Appendix A."

According to Theorem 2, (4.12), (4.13) and (4.4), the following three lemmas are easily shown.

Lemma 4.1 *The following subsets form a basis for the same subspace of* $\mathscr{O}$*:*

$$(i)\ A_0, \quad A_1 + A_{-1}, \quad A_2 + A_{-2}, \quad A_3 + A_{-3}, \ldots$$

$$(ii)\ \mathcal{W}_0, \quad \mathcal{W}_{-1}, \quad \mathcal{W}_{-2}, \quad \mathcal{W}_{-3}, \ldots$$

Lemma 4.2 *The following subsets form a basis for the same subspace of $\mathcal{O}$:*

$$(i)\ A_1, \quad A_2 + A_0, \quad A_3 + A_{-1}, \quad A_4 + A_{-2}, \ldots$$
$$(ii)\ \mathcal{W}_1, \quad \mathcal{W}_2, \quad \mathcal{W}_3, \quad \mathcal{W}_4, \ldots$$

Lemma 4.3 *The following subsets form a basis for the same subspace of $\mathcal{O}$:*

$$(i)\ G_1, \quad G_2, \quad G_3, \quad G_4, \ldots$$
$$(ii)\ \tilde{\mathcal{G}}_1, \quad \tilde{\mathcal{G}}_2, \quad \tilde{\mathcal{G}}_3, \quad \tilde{\mathcal{G}}_4, \ldots$$

Remark 10 From [25], we know that the generators of the Onsager algebra $\mathcal{O}$ can be represented by the matrices

$$A_k = 2\begin{pmatrix} 0 & t^k \\ t^{-k} & 0 \end{pmatrix}, \qquad G_k = (t^k - t^{-k})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.14}$$

where t is an indeterminate. Then, by using (4.12) and (4.13), we deduce that the generators of $\mathcal{A}$ can be represented by the matrices

$$\mathcal{W}_{-k} = \left(\frac{t+t^{-1}}{2}\right)^k A_0, \quad \mathcal{W}_{k+1} = \left(\frac{t+t^{-1}}{2}\right)^k A_1, \quad \tilde{\mathcal{G}}_{k+1} = -4\left(\frac{t+t^{-1}}{2}\right)^k G_1. \tag{4.15}$$

4.2 Automorphisms of the algebra $\mathcal{A}$

In view of the isomorphism between $\mathcal{O}$ and $\mathcal{A}$, the action of the automorphisms τ_0, τ_1, Φ introduced in Proposition 2.1 is now described in the alternative presentation $\mathcal{A}$. Inverting the correspondence (4.8), one has:

$$\begin{aligned} \mathcal{W}_+(u) &\equiv \frac{2}{(u^{-1}-u)}\big(\mathcal{A}^+(u) + \mathcal{A}^-(u)\big), \\ \mathcal{W}_-(u) &\equiv \frac{2}{(u^{-1}-u)}\big(u^{-1}\mathcal{A}^+(u) + u\mathcal{A}^-(u)\big), \\ \tilde{\mathcal{G}}(u) &\equiv -8\mathcal{G}(u). \end{aligned} \tag{4.16}$$

Using (2.20), it yields to:

$$\begin{aligned} &\tau_0(\mathcal{W}_+(u)) = \mathcal{W}_+(u), \quad \tau_0(\mathcal{W}_-(u)) = 2U\mathcal{W}_+(u) - \mathcal{W}_-(u) - 2\mathcal{W}_0, \\ &\tau_1(\mathcal{W}_-(u)) = \mathcal{W}_-(u), \quad \tau_1(\mathcal{W}_+(u)) = 2U\mathcal{W}_-(u) - \mathcal{W}_+(u) - 2\mathcal{W}_1, \\ &\tau_0(\tilde{\mathcal{G}}(u)) = \tau_1(\tilde{\mathcal{G}}(u)) = -\tilde{\mathcal{G}}(u). \end{aligned}$$

Using (4.7), it follows:

Proposition 4.2 *The action of the automorphisms* τ_0, τ_1 *on the elements of* $\mathcal{A}$ *is such that:*

$$\tau_0(\mathcal{W}_{-k}) = \mathcal{W}_{-k}, \quad \tau_0(\mathcal{W}_{k+1}) = 2\mathcal{W}_{-k-1} - \mathcal{W}_{k+1}, \tag{4.17}$$

$$\tau_1(\mathcal{W}_{k+1}) = \mathcal{W}_{k+1}, \quad \tau_1(\mathcal{W}_{-k}) = 2\mathcal{W}_{k+2} - \mathcal{W}_{-k}, \tag{4.18}$$

$$\tau_0(\tilde{\mathcal{G}}_{k+1}) = \tau_1(\tilde{\mathcal{G}}_{k+1}) = -\tilde{\mathcal{G}}_{k+1}. \tag{4.19}$$

From (4.2), (4.3) note that

$$\mathcal{W}_{-k-1} = \frac{1}{16}[\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_0] + \mathcal{W}_{k+1}, \quad \mathcal{W}_{k+2} = \frac{1}{16}[\mathcal{W}_1, \tilde{\mathcal{G}}_{k+1}] + \mathcal{W}_{-k}.$$

Inserting $\tilde{\mathcal{G}}_{k+1} = [\mathcal{W}_0, \mathcal{W}_{k+1}]$ in the first equation above, from (4.17), (4.19) one recovers the classical ($q = 1$) analogs of the formulae given in Proposition 7.4 of [34]. Similarly, $\tilde{\mathcal{G}}_{k+1} = [\mathcal{W}_{-k}, \mathcal{W}_1]$ can be inserted into the second equation above in order to rewrite (4.18).

Combining above relations, one gets:

$$(\tau_0 + \tau_1)(\mathcal{W}_+(u)) = 2U\mathcal{W}_-(u) - 2\mathcal{W}_1, \quad (\tau_0 + \tau_1)(\mathcal{W}_-(u)) = 2U\mathcal{W}_+(u) - 2\mathcal{W}_0.$$

From the expansions (4.7), it follows (note that $\mathcal{W}_1 = \tau_1\Phi(\mathcal{W}_0)$):

Proposition 4.3 *In the algebra* $\mathcal{A}$*, one has:*

$$\mathcal{W}_{-k} = \left(\frac{\tau_0\Phi + \tau_1\Phi}{2}\right)^k (\mathcal{W}_0), \quad \mathcal{W}_{k+1} = \left(\frac{\tau_0\Phi + \tau_1\Phi}{2}\right)^k (\mathcal{W}_1) \quad \textit{and}$$
$$\tilde{\mathcal{G}}_{k+1} = \Big[\mathcal{W}_0, \left(\frac{\tau_0\Phi + \tau_1\Phi}{2}\right)^k (\mathcal{W}_1)\Big].$$

Remark 11 $\Phi(\mathcal{W}_{-k}) = \mathcal{W}_{k+1}$, $\Phi(\tilde{\mathcal{G}}_{k+1}) = -\tilde{\mathcal{G}}_{k+1}$.

Note that the polynomial expressions for the elements $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{k+1}\}$ computed here using the action of the automorphisms can be viewed as the classical ($q = 1$) analogs of the expressions computed in [2], where the elements of the algebra $\mathcal{A}_q$ are derived as polynomials of the fundamental generators $\mathcal{W}_0, \mathcal{W}_1$ satisfying the q-deformed version of (4.5).

4.3 Quotients of the Lie algebra $\mathcal{A}$ and of the Onsager algebra

By analogy with the analysis of the previous section, we now introduce certain quotients of the algebra $\mathcal{A}$. These quotients can be viewed as the classical analogs of the quotients of algebra $\mathcal{A}_q$ considered in [4,5, Eq. 11].

Definition 4.2 Let $\{\beta_n|n = 0, \ldots, N\}$ be nonzero scalars with N any nonzero positive integer. The algebra $\overline{\mathcal{A}}_N$ is defined as the quotient of the algebra $\mathcal{A}$ by the relations

$$\sum_{k=0}^{N} \beta_k \mathcal{W}_{-k} = 0 \quad \text{and} \quad \sum_{k=0}^{N} \beta_k \mathcal{W}_{k+1} = 0. \tag{4.20}$$

According to Proposition 4.3, introduce the operator:

$$S'_N = \sum_{n=0}^{N} \beta_n (\overline{\tau}_0 \overline{\Phi} + \overline{\tau}_1 \overline{\Phi})^n. \tag{4.21}$$

Then, Eq. (4.20) simply reads $S'_N(\mathcal{W}_0) = 0$ and $S'_N(\mathcal{W}_1) = 0$, respectively. Furthermore, one has $[(\overline{\tau}_0\overline{\Phi} + \overline{\tau}_1\overline{\Phi})^p, S'_N] = 0$ for any $p \in \mathbb{Z}$. It follows:

Remark 12 The relations (4.20) imply:

$$\sum_{k=0}^{N} \beta_k \mathcal{W}_{-k-p} = 0, \quad \sum_{k=0}^{N} \beta_k \mathcal{W}_{k+1+p} = 0, \quad \sum_{k=0}^{N} \beta_k \tilde{\mathcal{G}}_{k+1+p} = 0 \qquad \text{for any} \quad p \in \mathbb{Z}_{\geq 0}. \tag{4.22}$$

The algebra $\overline{\mathcal{A}}_N$ has $3N$ generators $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \tilde{\mathcal{G}}_{k+1}|k = 0, 1, \ldots, N-1\}$.

Note that above relations (4.22) can be derived using the commutation relations (4.1)–(4.3).

Theorem 3 *The algebra $\overline{\mathcal{A}}_N$ is isomorphic to the quotient of the Onsager algebra $\overline{\mathscr{O}}_N$ with the identification*

$$\beta_{2k} = \frac{2^{2k}}{(2k)!} \sum_{p=k}^{\left[\frac{N}{2}\right]} 2p(-1)^{p-k} \frac{(k+p-1)!}{(p-k)!} \alpha_{2p}, \tag{4.23}$$

$$\beta_{2k+1} = \frac{2^{2k+1}}{(2k+1)!} \sum_{p=k+1}^{\left[\frac{N+1}{2}\right]} (2p-1)(-1)^{p-k-1} \frac{(k+p-1)!}{(p-k-1)!} \alpha_{2p-1}. \tag{4.24}$$

Proof By Theorem 2, $\mathscr{O}$ and $\mathcal{A}$ are isomorphic, and the isomorphism is given by (4.9)–(4.11). To show that $\overline{\mathcal{A}}_N$ and $\overline{\mathscr{O}}_N$ are isomorphic, it is necessary and sufficient to show that (2.21) and (4.20) are equivalent if relations (4.23)–(4.24) hold. By inserting (4.9) and (4.10) in (2.21), one gets equivalently (4.20) by using (4.23)–(4.24). □

The corresponding class of solutions of the non-standard Yang–Baxter algebra (2.7) is now considered.

Proposition 4.4 *Let* $\{\beta_p | p = 0, \ldots, N-1\}$ *be nonzero scalars with* $N \in \mathbb{N}_{\geq 1}$. *Then, the non-standard classical Yang–Baxter algebra* (2.7) *for the r-matrix* (2.5) *and*

$$B^{(N)}(u) = \frac{1}{2\tilde{p}^{(N)}(U)} \begin{pmatrix} -\frac{1}{4}\,\tilde{\mathcal{G}}^{(N)}(u) & u^{-1}\mathcal{W}_+^{(N)}(u) - \mathcal{W}_-^{(N)}(u) \\ -u\mathcal{W}_+^{(N)}(u) + \mathcal{W}_-^{(N)}(u) & \frac{1}{4}\,\tilde{\mathcal{G}}^{(N)}(u) \end{pmatrix}$$

$$\tilde{p}^{(N)}(U)) = \sum_{p=0}^{N} \beta_p U^p, \tag{4.25}$$

where

$$\mathcal{W}_+^{(N)}(u)) = \sum_{k=0}^{N-1} \tilde{f}_k^{(N)}(U)\mathcal{W}_{-k}, \quad \mathcal{W}_-^{(N)}(u) = \sum_{k=0}^{N-1} \tilde{f}_k^{(N)}(U)\mathcal{W}_{k+1},$$

$$\tilde{\mathcal{G}}^{(N)}(u) = \sum_{k=0}^{N-1} \tilde{f}_k^{(N)}(U)\tilde{\mathcal{G}}_{k+1} \tag{4.26}$$

and

$$\tilde{f}_k^{(N)}(U) = \sum_{p=k+1}^{N} \beta_p U^{p-k-1}, \tag{4.27}$$

provides an FRT presentation of the algebra $\overline{\mathcal{A}}_N$.

Proof The proof is similar to the one of Proposition 2.4 by replacing the relations (2.8) and (2.21) by (4.6) and (4.20). □

Remark 13 Note that (4.25) can be interpreted as the classical analog of the Sklyanin's operators constructed in [4,5] satisfying the reflection algebra.

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Appendix A

From (4.9) to (4.11), for $k = 0, 1, 2$ one has:

$$A_0 = \mathcal{W}_0, \quad A_1 = \mathcal{W}_1, \quad G_1 = -\frac{1}{4}\tilde{\mathcal{G}}_1,$$

$$A_{-1} = 2\mathcal{W}_{-1} - \mathcal{W}_1, \quad A_2 = 2\mathcal{W}_2 - \mathcal{W}_0, \quad G_2 = -\frac{1}{2}\tilde{\mathcal{G}}_2,$$

$$A_{-2} = 4\mathcal{W}_{-2} - \mathcal{W}_0 - 2\mathcal{W}_2, \quad A_3 = 4\mathcal{W}_3 - \mathcal{W}_1 - 2\mathcal{W}_{-1}, \quad G_3 = -\tilde{\mathcal{G}}_3 + \frac{1}{4}\tilde{\mathcal{G}}_1.$$

Conversely, from (4.12)–(4.13) for $k = 1, 2$ one has:

$$\mathcal{W}_{-1} = \frac{A_1 + A_{-1}}{2}, \quad \mathcal{W}_2 = \frac{A_0 + A_2}{2}, \quad \tilde{\mathcal{G}}_2 = -2G_2,$$
$$\mathcal{W}_{-2} = \frac{A_2 + 2A_0 + A_{-2}}{4}, \quad \mathcal{W}_2 = \frac{A_3 + 2A_1 + A_{-1}}{4}, \quad \tilde{\mathcal{G}}_3 = -G_3 - 2G_1 .$$

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