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Feynman integral relations from parametric annihilators

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Abstract
We study shift relations between Feynman integrals via the Mellin transform through parametric annihilation operators. These contain the momentum space integration by parts relations, which are well known in the physics literature. Applying a result of Loeser and Sabbah, we conclude that the number of master integrals is computed by the Euler characteristic of the Lee–Pomeransky polynomial. We illustrate techniques to compute this Euler characteristic in various examples and compare it with numbers of master integrals obtained in previous works.

Keywords Feynman integrals · Master integrals · Integration by parts · IBP · D-module · Euler characteristic

1 Introduction

At higher orders in perturbative quantum field theory, the computation of observables via Feynman diagrams involves a rapidly growing number of Feynman integrals. Fortunately, the number of integrals which need to be computed explicitly can be reduced drastically by use of linear relations

$$\sum_i c_i \mathcal{I}_i = 0 \qquad (*)$$

between different Feynman integrals $\mathcal{I}_i$, with coefficients c_i that are rational functions of the space-time dimension d and the kinematic invariants characterizing the physical process (masses and momenta of elementary particles).

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The most commonly used method to derive such identities is the integration by parts (IBP) method introduced in [22,92]. In this approach, the relations (∗) are obtained as integrals of total derivatives in the momentum space representation of Feynman integrals. Combining these relations, any integral of interest can be expressed as a linear combination of some finite, preferred set of *master integrals*.[1] Laporta's algorithm [50] provides a popular approach to obtain such reductions, and various implementations of it are available [2,62,75,76,80,86,98]. However, the increase of complexity of todays computations has recently motivated considerable theoretical effort to improve our understanding of the IBP approach and the efficiency of automated reductions [35,40, 52,55,57,70,78,97,99]. This includes a method by Baikov [5,7,8,82], which is based on a parametric representation of Feynman integrals.

It is interesting to ask for the number of master integrals which remain after such reductions. This number provides an estimate for the complexity of the computation and informs the problem of constructing a basis of master integrals by an Ansatz. In the recent literature, algorithms to count master integrals were proposed and implemented in the computer programs Mint [58] and AZURITE [32].

We propose an unambiguous definition for the number of master integrals as the dimension of an appropriate vector space. This definition and our entire discussion are independent of the method of the reduction. The main result of this article shows that this number is a well-understood topological invariant: the Euler characteristic of the complement of a hypersurface $\{\mathcal{G} = 0\}$ associated to the Feynman graph. Therefore, many powerful tools are available for its computation.

To arrive at our result, we follow Lee and Pomeransky [58] and view Feynman integrals as a Mellin transform of $\mathcal{G}^{-d/2}$, where $\mathcal{G}$ is a certain polynomial in the Schwinger parameters. Each of these parameters corresponds to a denominator (inverse propagators or irreducible scalar products) of the momentum space integrand. The classical IBP relations relate Feynman integrals which differ from each other by integer shifts of the exponents of these denominators. As Lee [56] and Baikov [7] pointed out, such shift relations correspond to annihilation operators of the integrands of parametric representations.[2] In our set-up, these are differential operators P satisfying

$$P\mathcal{G}^{-d/2} = 0.$$

We recall that such parametric annihilators provide all shift relations between Feynman integrals, in particular the ones known from the classical IBP method in momentum space. The obvious question, whether the latter suffice to obtain all shift relations, seems to remain open. As a positive indication in this direction, we show that the momentum space relations contain the inverse dimension shift.

Ideals of parametric annihilators are examples of *D*-modules. Loeser and Sabbah studied the *algebraic Mellin transform* [60] of holonomic *D*-modules and proved a dimension formula in [59,61], which, applied to our case, identifies the number of master integrals as an Euler characteristic. The key property here is holonomicity,

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[1] For different applications, various criteria for choosing the master integrals have been suggested. These include uniform transcendentality [39], finiteness [96] or finiteness of coefficients [21].

[2] Tkachov's idea [93] to insert Bernstein–Sato operators in the integrand with two Symanzik polynomials was used for numerical computations of one- and two-loop integrals [9,27,28,68,69].

which was studied in the context of Feynman integrals already in [46] and of course is crucial in the proof [77] that there are only finitely many master integrals.

It is furthermore worthwhile to notice that algorithms [65,67] have been developed to compute generators for the ideal of all annihilators of $\mathcal{G}^{-d/2}$, see also [72, section 5.3]. Today, efficient implementations of these algorithms via Gröbner bases are available in specialized computer algebra systems such as SINGULAR [4,25]. We hope that these improvements may stimulate further progress in the application of D-module theory to Feynman integrals [30,78,79,90].

We begin our article with a review of the momentum space and parametric representations of scalar Feynman integrals and recall how the Mellin transform translates shift relations to differential operators that annihilate the integrand. In Sect. 2.4 we illustrate how the classical IBP identities obtained in momentum space supply special examples of such annihilators. The relations between integrals in different dimensions are addressed in Sect. 2.5, where we relate them to the Bernstein–Sato operators and show that these can be obtained from momentum space IBPs. Our main result is presented in Sect. 3, where we apply the theory of Loeser and Sabbah to count the master integrals in terms of the Euler characteristic. Practical applications of this formula are presented in Sect. 4, which includes a comparison to other approaches and results in the literature. Finally, we discuss some open questions and future directions.

In "Appendix A", we give an example to illustrate our definitions in momentum space, present proofs of the parametric representations and demonstrate algebraically that momentum space IBPs are parametric annihilators. The theory of Loeser and Sabbah is reviewed in "Appendix B", which includes complete, simplified proofs of those theorems that we invoke in Sect. 3. Finally, "Appendix C" discusses the parametric annihilators of a two-loop example in detail.

2 Annihilators and integral relations

In this section, we elaborate a method to obtain relations between Feynman integrals from differential operators with respect to the Feynman parameters and show that these relations include the well-known IBP relations from momentum space.

2.1 Feynman integrals and Schwinger parameters

At first we fix conventions and notation for Feynman integrals in momentum space and recall their representations using Schwinger parameters. While the former is the setting for most traditional approaches to study IBP identities, it is the latter (in particular in its form with a single polynomial) which provides the direct link to the theory of D-modules that our subsequent discussion will be based on.

We consider integrals (also called *integral families* [98]) that are defined by an dL-fold integral (L is the *loop number*), over so-called *loop momenta* $\ell_1, \ldots, \ell_L$ in d-dimensional Minkowski space, of a product of powers of *denominators* $\mathsf{D} = (\mathsf{D}_1, \ldots, \mathsf{D}_N)$:

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{j=1}^{L} \int \frac{\mathrm{d}^d \ell_j}{\mathrm{i}\pi^{d/2}} \right) \prod_{a=1}^{N} \mathsf{D}_a^{-\nu_a}. \tag{2.1}$$

The denominators are (at most) quadratic forms in the L loop momenta and some number E of linearly independent *external momenta* $p_1, \ldots, p_E$. In most applications, the denominators are inverse Feynman propagators associated with the momentum flow through a Feynman graph that arises from imposing momentum conservation at each vertex, see Example 3. However, we will only restrict ourselves to graphs from Sect. 3.2 onwards, and keep our discussion completely general until then.

An integral (2.1) is a function of the *indices* $\nu = (\nu_1, \ldots, \nu_N)$ (denominator exponents), the dimension d of spacetime and kinematical invariants (masses and scalar products of external momenta). However, we suppress the dependence on kinematics in the notation and treat kinematical invariants as complex numbers throughout. The dimension and indices are understood as free variables; that is, we consider Feynman integrals as meromorphic functions of $(d, \nu) \in \mathbb{C}^{1+N}$ in the sense of Speer [83].

Definition 1 To each denominator D_a of a list $\mathsf{D} = (\mathsf{D}_1, \ldots, \mathsf{D}_N)$, we associate a variable x_a ($1 \leq a \leq N$) called *Schwinger parameter*. The linear combination

$$\sum_{a=1}^{N} x_a \mathsf{D}_a = - \sum_{i,j=1}^{L} \Lambda_{ij} (\ell_i \cdot \ell_j) + \sum_{i=1}^{L} 2(Q_i \cdot \ell_i) + J \tag{2.2}$$

decomposes into quadratic, linear and constant terms in the loop momenta.[3] This defines a symmetric $L \times L$ matrix Λ, a vector Q of L linear combinations of external momenta and a scalar J. We define furthermore the polynomials

$$\mathcal{U} := \det \Lambda, \quad \mathcal{F} := \mathcal{U} \left(Q^{\mathsf{T}} \Lambda^{-1} Q + J \right) \quad \text{and} \quad \mathcal{G} := \mathcal{U} + \mathcal{F}. \tag{2.3}$$

Schwinger parameters yield useful representations of the Feynman integrals (2.1):

Proposition 2 *Let us denote the* superficial degree of convergence *by*

$$\omega := |\nu| - L \frac{d}{2} \quad where \quad |\nu| := \sum_{i=1}^{N} \nu_i. \tag{2.4}$$
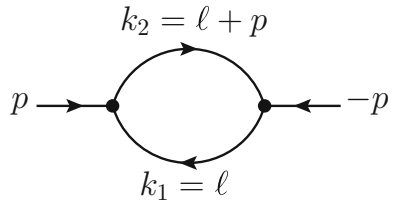
Then, the Feynman integral (2.1) *can be written as*

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i-1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}}, \tag{2.5}$$

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \Gamma(\omega) \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i-1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \frac{\delta \left(1 - \sum_{j=1}^{N} x_j \right)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^\omega} \quad and \tag{2.6}$$

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[3] The scalar products $\ell_i \cdot \ell_j = \ell_i^1 \ell_j^1 - \sum_{\mu=2}^{d} \ell_i^\mu \ell_j^\mu$ are understood with respect to the Minkowski metric.

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^{N} \int_0^{\infty} \frac{x_i^{\nu_i - 1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \mathcal{G}^{-d/2}. \tag{2.7}$$

Formulas (2.5) and (2.6) are known since the sixties, and we refer to [64,81] for detailed discussions and for the original references. The trivial consequence (2.7) was popularized only much more recently by Lee and Pomeransky [58], and it is this representation that we will use in the following. In Appendix A.1, we include proofs for these equations and provide further technical details.

Example 3 Consider the graph in Fig. 1 with massless Feynman propagators, $\mathsf{D}_1 = -\ell^2$ and $\mathsf{D}_2 = -(\ell + p)^2$. We find $\Lambda = x_1 + x_2$, $Q = -x_2 p$ and $J = -x_2 p^2$ according to (2.2). Hence, the graph polynomials (2.3) become $\mathcal{U} = x_1 + x_2$ and $\mathcal{F} = (-p^2)x_1 x_2$ such that

$$\mathcal{I}(\nu_1, \nu_2) = \frac{\Gamma(\frac{d}{2})}{\Gamma(d - \nu_1 - \nu_2)} \int_0^{\infty} \frac{x_1^{\nu_1 - 1} \mathrm{d}x_1}{\Gamma(\nu_1)} \int_0^{\infty} \frac{x_2^{\nu_2 - 1} \mathrm{d}x_2}{\Gamma(\nu_2)}$$
$$\times \left(x_1 + x_2 - p^2 x_1 x_2 \right)^{-d/2}. \tag{2.8}$$

Remark 4 (meromorphicity) Depending on the values of d and ν, integrals (2.1) and (2.5)–(2.7) can be divergent. However, there exists a non-empty, open domain in $\mathbb{C}^{1+N} \ni (d, \nu)$ where all of them are convergent and agree with each other.[4] From there, analytic continuation defines a unique, meromorphic extension of every Feynman integral to the whole parameter space $\mathbb{C}^{1+N}$. The poles are simple and located on affine hyperplanes defined by linear equations with integer coefficients. For these foundations of analytic regularization, we refer to [83,84].

For a number of reasons, in particular the preservation of fundamental symmetries, the most widely used scheme in quantum field theory is *dimensional regularization* [23,87]. It consists of specializing the $\nu \in \mathbb{Z}^N$ to integers and only keeps the dimension d as a regulator. In this case, the poles are not necessarily simple anymore.

The uniqueness of the analytic continuation of a Feynman integral (as a function of d and ν) is very important; in particular, it means that an identity between Feynman integrals is already proven once it has been established locally in the non-empty domain of convergence of the involved integral representations. In other words, in any calculation with analytically regularized Feynman integrals, we may simply assume,

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[4] The only exception are cases of zero-scale subintegrals, like massless tadpoles. Such integrals are zero in dimensional regularization and therefore irrelevant for our considerations.

without loss of generality, that the parameters are such that the integrals converge. The resulting relation then necessarily remains true everywhere by analytic continuation.

Example 5 The one-loop propagator in Example 3 can be computed in terms of Γ-functions:

$$\mathcal{I}(\nu_1, \nu_2) = (-p^2)^{d/2-\nu_1-\nu_2} \frac{\Gamma(d/2 - \nu_1)\Gamma(d/2 - \nu_2)\Gamma(\nu_1 + \nu_2 - d/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(d - \nu_1 - \nu_2)}. \quad (2.9)$$

Integral (2.8) converges only in a certain domain of (ν, d), but there it evaluates to (2.9), which has a unique meromorphic continuation. Its poles lie on the infinite family of hyperplanes defined by $\{d/2 - \nu_1 = k\}$, $\{d/2 - \nu_2 = k\}$ and $\{\nu_1 + \nu_2 - d/2 = k\}$, indexed by $k \in \mathbb{Z}_{\leq 0}$.

2.2 Integral relations and the Mellin transform

In this section, we summarize how relations between ν-shifted Feynman integrals can be identified with differential operators that annihilate $\mathcal{G}^{-d/2}$. This method was suggested in [56] (and in [7] for the Baikov representation).

The parametric representation (2.7) can be interpreted as a multi-dimensional Mellin transform. For our purposes, we slightly deviate from the standard definition, as for example given in [18] and include the factors $\Gamma(\nu_i)$ that occur in (2.7).

Definition 6 Let $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{C}^N$. The twisted (multi-dimensional) Mellin transform of a function $f : \mathbb{R}_+^N \longrightarrow \mathbb{C}$ is defined as

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\right) f(x_1, \ldots, x_N), \quad (2.10)$$

whenever this integral exists. As a special case, we define

$$\widetilde{\mathcal{I}}(\nu) := \mathcal{M}\left\{\mathcal{G}^{-d/2}\right\}(\nu) \quad \text{such that} \quad \mathcal{I}(\nu) = \frac{\Gamma(d/2)}{\Gamma(d/2 - \omega)}\widetilde{\mathcal{I}}(\nu). \quad (2.11)$$

Recall that, as mentioned in Remark 4, we do not have to worry about the actual domain of convergence of (2.10) in the algebraic derivations below. The key features of the Mellin transform for us are the following elementary relations; see [18] for their form without the Γ's in (2.10).

Lemma 7 Let $\alpha, \beta \in \mathbb{C}$, $\nu \in \mathbb{C}^N$, $1 \leq i \leq N$ and $f, g : \mathbb{R}_+^N \longrightarrow \mathbb{C}$. Writing e_i for the i-th unit vector, the (twisted) Mellin transform has the following properties:

1. Linearity: $\mathcal{M}\{\alpha f + \beta g\}(\nu) = \alpha \mathcal{M}\{f\}(\nu) + \beta \mathcal{M}\{g\}(\nu)$,
2. Multiplication: $\mathcal{M}\{x_i f\}(\nu) = \nu_i \mathcal{M}\{f\}(\nu + e_i)$ and
3. Differentiation: $\mathcal{M}\{\partial_i f\}(\nu) = -\mathcal{M}\{f\}(\nu - e_i)$.

Proof The linearity is immediate from (2.10), and the functional equation $\Gamma(\nu_i + 1) = \nu_i \Gamma(\nu_i)$ provides the multiplication rule $\left[x_i^{\nu_i-1}/\Gamma(\nu_i)\right] x_i = \nu_i x_i^{\nu_i}/\Gamma(\nu_i + 1)$. The differentiation rule is a consequence of integration by parts,

$$\int_0^\infty \frac{x_i^{\nu_i-1}\mathrm{d}x_i}{\Gamma(\nu_i)}\partial_i f = \left[\frac{x_i^{\nu_i-1}}{\Gamma(\nu_i)}f\right]_{x_i=0}^\infty - \int_0^\infty \frac{x_i^{\nu_i-2}\mathrm{d}x_i}{\Gamma(\nu_i-1)}f,$$

because the boundary terms vanish inside the convergence domain of both integrals. This just says that if $\lim_{x_i\to 0}(x_i^{\nu_i-\epsilon-1}f)$ is finite for some $\epsilon > 0$, then $\lim_{x_i\to 0}(x_i^{\nu_i-1}f) = 0$ vanishes and the analogous argument applies to the upper bound $x_i \to \infty$. □

2.3 Operator algebras and annihilators

The Mellin transform relates differential operators acting on $\mathcal{G}^{-d/2}$ with operators that shift the indices ν of the Feynman integrals $\mathcal{I}(\nu)$. In this section, we formalize this connection algebraically in the language of D-modules. For the most part, we only need basic notions which we will introduce below, and point out [24] as a particularly accessible introduction to the subject.

Definition 8 The Weyl algebra A^N in N variables $x_1, \ldots, x_N$ is the non-commutative algebra of polynomial differential operators

$$A^N := \mathbb{C}\langle x_1, \ldots, x_N, \partial_1, \ldots, \partial_N \mid \partial_i x_j = x_j \partial_i + \delta_{ij} \text{ for all } 1 \leq i, j \leq N\rangle, \tag{2.12}$$

such that the commutators are $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{i,j}$ (Kronecker delta).

Note that with the multi-index notations $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ and $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_N^{\beta_N}$, every operator $P \in A^N$ can be written uniquely in the form

$$P = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha \partial^\beta \quad \text{with} \quad \alpha, \beta \in \mathbb{N}_0^N,$$

by commuting all derivatives to the right (only finitely many of the coefficients $c_{\alpha\beta} \in \mathbb{C}$ are nonzero). Extending the coefficients $c_{\alpha\beta}$ from $\mathbb{C}$ to polynomials $\mathbb{C}[s]$ in a further, commuting variable s, we obtain the algebra $A^N[s] := A^N \otimes_\mathbb{C} \mathbb{C}[s]$. Later, we will also consider the case $A_k^N := A^N \otimes_\mathbb{C} k$ of coefficients that are rational functions $k := \mathbb{C}(s)$. The integrands of the Mellin transform (2.10) naturally form a $A^N[s]$-module $A^N[s]f^s$, which we will introduce now.

Definition 9 Given a polynomial $f \in \mathbb{C}[x]$, the $A^N[s]$-module $\mathbb{C}[s, x, 1/f] \cdot f^s$ consists of elements of the form $(p/f^k) \cdot f^s$ (where $p \in \mathbb{C}[s, x], k \in \mathbb{N}_0$) with the $A^N[s]$-action

$$q\left(\frac{p}{f^k} \cdot f^s\right) := \frac{qp}{f^k} \cdot f^s, \quad \partial_i\left(\frac{p}{f^k} \cdot f^s\right) := \frac{f(\partial_i p) + (s-k)p(\partial_i f)}{f^{k+1}} \cdot f^s \tag{2.13}$$

for any polynomial $q \in \mathbb{C}[s, x]$. This is just the natural action by multiplication and differentiation, $\partial_i \mapsto \partial/(\partial x_i)$. We abbreviate $f^{s+k} := f^k \cdot f^s$ for $k \in \mathbb{Z}$. The cyclic

submodule generated by f^s is denoted as $A^N[s]f^s$. If we extend the coefficients to the rational functions in s, we write $A_k^N f^s = A^N[s]f^s \otimes_{\mathbb{C}[s]} \mathbb{C}(s)$.

Definition 10 The s-parametric annihilators of a polynomial f in $x_1, \ldots, x_N$ are the elements of the (left) ideal of operators in $A^N[s]$ whose action on f^s is zero:

$$\mathrm{Ann}_{A^N[s]}(f^s) := \left\{ P \in A^N[s] \colon Pf^s = 0 \right\}.$$

Example 11 Given $f \in \mathbb{C}[x_1, \ldots, x_N]$, we always have the trivial annihilators

$$f \partial_i - s(\partial_i f) \in \mathrm{Ann}\left(f^s\right) \quad \text{for} \quad 1 \le i \le N. \tag{2.14}$$

Note that an annihilator ideal is a module over $A^N[s]$, that is, whenever $Pf^s = 0$, also $(QP)f^s = Q(Pf^s) = 0$ for any operator $Q \in A^N[s]$. These ideals are studied in D-module theory [24,72] and in principle annihilators can be computed algorithmically with computer algebra systems such as SINGULAR [3,4,25]. For the study of the Feynman integrals (2.7), we set $s = -d/2$ and $f = \mathcal{G} = \mathcal{U} + \mathcal{F}$ is the polynomial from (2.3). Via the Mellin transform, the elements of $A^N[s]\mathcal{G}^s$ are the integrands of shifts of the Feynman integral $\mathcal{M}\{\mathcal{G}^s\}$. Due to Lemma 7, every annihilator $P \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ corresponds to an identity of Feynman integrals with shifted indices ν.

Definition 12 The algebra S^N of shift operators in N variables is defined by

$$S^N := \mathbb{C}\left\langle \hat{\mathbf{1}}^+, \ldots, \hat{\mathbf{N}}^+, \mathbf{1}^-, \ldots, \mathbf{N}^- \mid [\hat{\mathbf{i}}^+, \hat{\mathbf{j}}^+] = [\mathbf{i}^-, \mathbf{j}^-] = 0 \text{ and } [\hat{\mathbf{i}}^+, \mathbf{j}^-] = \delta_{i,j} \right\rangle. \tag{2.15}$$

This algebra is clearly isomorphic to the Weyl algebra A^N, since under the identifications $\hat{\mathbf{i}}^+ \leftrightarrow \partial_i$ and $\mathbf{j}^- \leftrightarrow x_j$ the commutation relations are identical. In fact, a different isomorphism is given by $\hat{\mathbf{i}}^+ \leftrightarrow x_i$ and $\mathbf{j}^- \leftrightarrow -\partial_j$, and it is this identification that corresponds to the Mellin transform (see Lemma 7). We therefore denote it by[5]

$$\mathcal{M}\{\cdot\} : A^N \xrightarrow{\cong} S^N, \quad P \mapsto \mathcal{M}\{P\} := P\Big|_{x_i \mapsto \hat{\mathbf{i}}^+, \partial_i \mapsto -\mathbf{i}^- \text{ for all } 1 \le i \le N}. \tag{2.16}$$

The conceptual difference between S^N and A^N is that we think of A^N as acting on functions $f(x)$ by differentiation, whereas S^N acts on functions $F(\nu)$ of a different set $\nu = (\nu_1, \ldots, \nu_N)$ of variables (the indices of Feynman integrals) by shifts of the argument and, in case of $\hat{\mathbf{i}}^+$, a multiplication with ν_i:

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[5] The use of the symbol $\mathcal{M}\{\cdot\}$ for both the analytic Mellin transform (2.10) and isomorphism (2.16) should not lead to any confusion. It is suggestive of, and justified by, Corollary 13.

$$(\mathbf{i}^- F)(\nu) := F(\nu - \mathbf{e}_i) \quad \text{and} \quad (\hat{\mathbf{i}}^+ F)(\nu) := \nu_i F(\nu + \mathbf{e}_i). \qquad (2.17)$$

These operators are used very frequently in the literature on IBP relations, as for example in [35,52,78,81]. An important role is played by the operators

$$\mathbf{n}_i := \hat{\mathbf{i}}^+ \mathbf{i}^-, \quad \text{which act by simple multiplication:} \quad (\mathbf{n}_i F)(\nu) = \nu_i F(\nu). \quad (2.18)$$

Their commutation relations are

$$[\hat{\mathbf{i}}^+, \mathbf{n}_j] = \hat{\mathbf{i}}^+ \delta_{i,j} \quad \text{and} \quad [\mathbf{i}^-, \mathbf{n}_j] = -\mathbf{i}^- \delta_{i,j}. \qquad (2.19)$$

In terms of the S^N action (2.17), we can rephrase the essence of Lemma 7 as

Corollary 13 *The (twisted) Mellin transform* (2.10) *is compatible with the actions of the algebras* A^N *and* S^N *under their identification* (2.16). *In other words,*

$$\mathcal{M}\{Pf^s\} = \mathcal{M}\{P\}[\mathcal{M}\{f^s\}] \quad \textit{for all operators} \ \ P \in A^N[s]. \qquad (2.20)$$

Corollary 14 *Every annihilator* $P \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ *of* $\mathcal{G}^s = \mathcal{G}^{-d/2}$ *yields a shift relation* $\mathcal{M}\{P\} \in S^N[s] := S^N \otimes \mathbb{C}[s]$ *of the Feynman integral* $\widetilde{\mathcal{I}}$ *from* (2.11): $\mathcal{M}\{P\}\widetilde{\mathcal{I}} = 0.$

Example 15 For the bubble graph in Fig. 1 with $\mathcal{G} = x_1 + x_2 - p^2 x_1 x_2$ from Example 3,

$$(-p^2)x_1(s - x_1 \partial_1) + (s - x_1 \partial_1 - x_2 \partial_2) \in \mathrm{Ann}(\mathcal{G}^s)$$

is easily checked to annihilate $\mathcal{G}^s$. We therefore get the shift relation

$$(s + \mathbf{n}_1 + \mathbf{n}_2)\widetilde{\mathcal{I}} = p^2 \hat{\mathbf{1}}^+ (s + \mathbf{n}_1)\widetilde{\mathcal{I}} = p^2 (s + \mathbf{n}_1 + 1)\hat{\mathbf{1}}^+ \widetilde{\mathcal{I}}.$$

According to (2.18), this relation can also be written as

$$(-p^2)\nu_1 \widetilde{\mathcal{I}}(\nu_1 + 1, \nu_2) = -\frac{s + \nu_1 + \nu_2}{s + \nu_1 + 1}\widetilde{\mathcal{I}}(\nu_1, \nu_2). \qquad (2.21)$$

We prefer to work with the modified Feynman integral $\widetilde{\mathcal{I}}$ from (2.11), because it is directly related to the Mellin transform. However, it is straightforward to translate relations between $\widetilde{\mathcal{I}}$ into relations for the actual Feynman integral $\mathcal{I}$. Namely, if $P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$, we substitute (2.11) to see

$$0 = \mathcal{M}\{P\}\widetilde{\mathcal{I}} = \sum_{\alpha,\beta} c_{\alpha,\beta} \left(\prod_{i,j=1}^N (\hat{\mathbf{i}}^+)^{\alpha_i} (-\mathbf{j}^-)^{\beta_j} \right) \frac{\Gamma(-s - \omega)}{\Gamma(-s)} \mathcal{I}$$

and then recall from (2.4) that $\omega = \sum_i \nu_i + Ls$ to conclude

$$0 = \frac{\Gamma(-s)\mathcal{M}\{P\}\widetilde{\mathcal{I}}}{\Gamma(-s-\omega)} = \sum_{\alpha,\beta} c_{\alpha,\beta} \frac{\Gamma(-s-\omega-|\alpha|+|\beta|)}{\Gamma(-s-\omega)} \left(\prod_{i,j=1}^{N} (\hat{\mathbf{i}}^+)^{\alpha_i}(-\mathbf{j}^-)^{\beta_j} \right) \mathcal{I}.$$

(2.22)

Notice that the fraction $\Gamma(-s-\omega-|\alpha|+|\beta|)/\Gamma(-s-\omega)$ is a rational function in ω (and thus in ν and s), due to the functional equation for Γ, since $|\alpha|$ and $|\beta|$ are integers.

Example 16 Substituting $\widetilde{\mathcal{I}}(\nu_1, \nu_2) = \mathcal{I}(\nu_1, \nu_2)\Gamma(-2s-\nu_1-\nu_2)/\Gamma(-s)$ and $\widetilde{\mathcal{I}}(\nu_1 + 1, \nu_2) = \mathcal{I}(\nu_1 + 1, \nu_2)\Gamma(-2s-\nu_1-1-\nu_2)/\Gamma(-s)$ from (2.11) into (2.21) results in

$$(-p^2)\nu_1 \mathcal{I}(\nu_1 + 1, \nu_2) = \frac{(s+\nu_1+\nu_2)(2s+\nu_1+\nu_2+1)}{s+\nu_1+1} \mathcal{I}(\nu_1, \nu_2),$$

which also follows from expression (2.9) of $\mathcal{I}(\nu)$ in terms of Γ-functions.

Let us recapitulate these observations: By the Mellin transform (2.16), every annihilator $P \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ gives rise to a linear relation $\mathcal{M}\{P\}\widetilde{\mathcal{I}} = 0$ of the rescaled Feynman integral $\widetilde{\mathcal{I}}$ via a shift operator $\mathcal{M}\{P\} \in S^N[s]$. Also we noted that according to $\widetilde{\mathcal{I}} = \mathcal{I}\,\Gamma(-s-\omega)/\Gamma(-s)$ from (2.11), such a relation is equivalent to a relation of the original Feynman integral $\mathcal{I}$ as in (2.22).

On the other hand, if we are given a shift relation $R\widetilde{\mathcal{I}} = 0$ where $R \in S^N[s]$, then $R = \mathcal{M}\{P\}$ corresponds to the differential operator $P = \mathcal{M}^{-1}\{R\} \in A^N[s]$ under the Mellin transform (2.16). From the vanishing $\mathcal{M}\{P\mathcal{G}^s\} = 0$ we can conclude that this operator must be an annihilator, $P \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$. This follows from

Theorem 17 (Inverse Mellin transform, e.g. [18, Theorem 3.5]) *Suppose that the (twisted) Mellin transform $f^\star(\nu) := \mathcal{M}\{f\}(\nu)$ of $f(x)$ from (2.10) converges in a domain of the form $a_i \leq \mathrm{Re}(\nu_i) \leq b_i$ for all $1 \leq i \leq N$, where $a, b \in \mathbb{R}^N$. Then, its inverse is given by*

$$f(x) = \mathcal{M}^{-1}\{f^\star\}(x) = \left(\prod_{i=1}^{N} \int_{\sigma_i + i\mathbb{R}} \mathrm{d}\nu_i \, \frac{\Gamma(\nu_i)}{x_i^{\nu_i} \cdot 2\pi i} \right) f^\star(\nu), \quad \text{where} \quad x \in \mathbb{R}_+^N. \text{ (2.23)}$$

This multiple integral along lines parallel to the imaginary axis converges for $a_i \leq \sigma_i \leq b_i$ $(1 \leq i \leq N)$ and does not depend on the concrete choice of σ_i.

So not only do we get relations for Feynman integrals from parametric annihilators of $\mathcal{G}^s$, but in fact *every* relation of the form $R\widetilde{\mathcal{I}} = 0$ for a polynomial shift operator $R \in S^N[s]$ does arise in this way.

Corollary 18 *Let $\mathrm{Ann}_{S^N[s]}(\widetilde{\mathcal{I}}) \subseteq S^N[s]$ denote the $S^N[s]$-module of polynomial shift operators that annihilate a (rescaled) Feynman integral (2.11). Then the Mellin*

transform (2.16) *restricts to a bijection between these relations and parametric anni-hilators:*

$$\mathcal{M}\{\cdot\} : \ \mathrm{Ann}_{A^N[s]}\left(\mathcal{G}^s\right) \xrightarrow{\cong} \mathrm{Ann}_{S^N[s]}\left(\widetilde{\mathcal{I}}\right).$$ (2.24)

Instead of focussing on the annihilators themselves, we can also look at the $A^N[s]$-module $A^N[s] \cdot \mathcal{G}^s \cong A^N[s]/\mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ of the integrands and the $S^N[s]$-module $S^N[s] \cdot \widetilde{\mathcal{I}} \cong S^N[s]/\mathrm{Ann}_{S^N[s]}(\widetilde{\mathcal{I}})$ of all (shifted) Feynman integrals. The Mellin transform gives an isomorphism

$$\mathcal{M}\{\cdot\} : \ A^N[s] \cdot \mathcal{G}^s \xrightarrow{\cong} S^N[s] \cdot \widetilde{\mathcal{I}}.$$ (2.25)

2.4 On the correspondence to momentum space

In this section, we first recall the integration by parts (IBP) relations for Feynman integrals that are derived in momentum space, following [35]. We then note that these provide a special set of parametric annihilators and discuss some open questions in regard of this comparison of IBPs in parametric and momentum space.

Since the denominators D_a from (2.1) are quadratic forms in the $M = L + vE$ momenta $q = (q_1, \ldots, q_M) := (\ell_1, \ldots, \ell_L, p_1, \ldots, p_E)$, we can write them in the form

$$\mathsf{D}_a = \sum_{\{i,j\} \in \Theta} \mathcal{A}_a^{\{i,j\}} s_{\{i,j\}} + \lambda_a$$ (2.26)

such that the coefficients $\mathcal{A}_a^{\{i,j\}}$ and λ_a are independent of loop momenta and the pairs

$$\Theta := \{\{i,j\} : 1 \leq i \leq L \text{ and } 1 \leq j \leq M\}$$ (2.27)

label the $|\Theta| = \frac{L(L+1)}{2} + LE$ loop-momentum-dependent scalar products

$$s_{\{i,j\}} := q_i q_j = q_j q_i.$$ (2.28)

In order to express the IBP relations coming from momentum space in terms of integrals (2.1), we need to assume, for this and the following section, that we consider $N = |\Theta|$ denominators such that the $N \times N$ square matrix $\mathcal{A}$ defined by (2.26) is invertible.[6] We think of $\mathcal{A}_a^{\{i,j\}}$ as the element of $\mathcal{A}$ in row a and column $\{i, j\}$ and write $\mathcal{A}_{\{i,j\}}^a$ for the entry in row $\{i, j\}$ and column a of $\mathcal{A}^{-1}$, such that the inverse of (2.26) can be written as

$$s_{\{i,j\}} = \sum_{a=1}^{N} \mathcal{A}_{\{i,j\}}^a \left(\mathsf{D}_a - \lambda_a\right) \quad \text{for all} \quad \{i,j\} \in \Theta.$$ (2.29)

---

[6] For integrals associated to Feynman graphs, the number of edges is often less than $|\Theta|$. In this case, one augments the list of inverse propagators by an appropriate choice of additional quadratic forms in the loop momenta, called *irreducible scalar products* (ISPs), to achieve $N = |\Theta|$. See Example 62.

We are interested in relations of the Feynman integral $\mathcal{I}$ from (2.1), that is,

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{j=1}^{L} \int \frac{\mathrm{d}^d \ell_j}{i\pi^{d/2}} \right) f \quad \text{with the integrand} \quad f = \prod_{i=1}^{N} \mathsf{D}_i^{-\nu_i}. \quad (2.30)$$

Definition 19 The *momentum space IBP relations* of $\mathcal{I}(\nu_1, \ldots, \nu_N)$ are those relations between scalar Feynman integrals that are obtained from Stokes' theorem

$$\left(\prod_{n=1}^{L} \int \mathrm{d}^d \ell_n \right) \mathbf{o}_j^i f = 0, \quad (2.31)$$

where the operators $\mathbf{o}_j^i$ are defined in terms of the momenta[7] as

$$\mathbf{o}_j^i := \frac{\partial}{\partial q_i} \cdot q_j = \sum_{\mu=1}^{d} \frac{\partial}{\partial q_i^\mu} q_j^\mu \quad \text{for} \quad i \in \{1, \ldots, L\}, j \in \{1, \ldots, M\}. \quad (2.32)$$

The following, explicit form of these relations as difference equations is essentially due to Baikov [6,7]; see also Grozin [35]. For completeness, we include the proof in appendix A.1.

Proposition 20 *Given a set of $N = |\Theta|$ denominators D such that the matrix $\mathcal{A}$ defined by (2.26) is invertible, every momentum space IBP relation can be written explicitly as*

$$O_j^i \mathcal{I}(\nu_1, \ldots, \nu_N) = 0 \quad (2.33)$$

where O_j^i denotes shift operators, indexed by $1 \leq i \leq L$ and $1 \leq j \leq M$, that are given by

$$O_j^i := \begin{cases} d\delta_{ij} - \sum_{a,b=1}^{N} C_{aj}^{bi} \hat{a}^+ \left(b^- - \lambda_b \right) & \text{for } j \leq L \text{ and} \\ - \sum_{a,b=1}^{N} C_{aj}^{bi} \hat{a}^+ \left(b^- - \lambda_b \right) - \sum_{a=1}^{N} \sum_{m=L+1}^{M} A_a^{\{i,m\}} q_j q_m \hat{a}^+ & \text{for } j > L. \end{cases}$$

$$(2.34)$$

The coefficients C_{aj}^{bi} are defined as

$$C_{aj}^{bi} := \begin{cases} \sum_{m=1}^{M} A_a^{\{i,m\}} A_{\{m,j\}}^b \left(1 + \delta_{mi}\right) & \text{if } j \leq L \text{ and} \\ \sum_{m=1}^{L} A_a^{\{i,m\}} A_{\{m,j\}}^b \left(1 + \delta_{mi}\right) & \text{if } j > L. \end{cases} \quad (2.35)$$

---

[7] Recall that $q_1, \ldots, q_L$ denote the loop momenta, whereas $q_{L+1}, \ldots, q_M$ are the external momenta.

Corollary 21 *To every difference equation $\mathbf{O}^i_j \mathcal{I} = 0$ from momentum space IBP, there corresponds a parametric annihilator $\widetilde{\mathbf{O}}^i_j \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ of the form*

$$\widetilde{\mathbf{O}}^i_j = d\delta_{ij} + \sum_{a,b=1}^{N} C^{bi}_{aj} x_a (\partial_b + \lambda_b H) \quad for \ \ i, j \leq L \quad and \tag{2.36}$$

$$\widetilde{\mathbf{O}}^i_j = \sum_{a,b=1}^{N} C^{bi}_{aj} x_a (\partial_b + \lambda_b H) - \sum_{a=1}^{N} \sum_{m=L+1}^{M} \mathcal{A}^{\{i,m\}}_a q_j q_m x_a H \quad for \ \ i \leq L < j, \tag{2.37}$$

where $H := \frac{(L+1)d}{2} + \sum_{c=1}^{N} x_c \partial_c$.

Proof First recall rescaling (2.11) between the Feynman integral $\mathcal{I}$ and the Mellin transform $\widetilde{\mathcal{I}}$ of $\mathcal{G}^s$. As we saw in (2.22), this means that

$$\frac{\Gamma(-s-\omega)}{\Gamma(-s)} \hat{\mathbf{a}}^+ \mathcal{I} = \Gamma(-s-\omega) \hat{\mathbf{a}}^+ \frac{\widetilde{\mathcal{I}}}{\Gamma(-s-\omega)} = (-s-\omega-1)\,\hat{\mathbf{a}}^+ \widetilde{\mathcal{I}} = \hat{\mathbf{a}}^+ (-s-\omega)\widetilde{\mathcal{I}},$$

and so if we substitute (2.11) into $\mathbf{O}^i_j \mathcal{I} = 0$ for the operators from (2.34), then apart from the substitution $\hat{\mathbf{a}}^+ \mathbf{b}^- \mapsto x_a(-\partial_b)$ which does not change ω, the remaining terms with shifts $\hat{\mathbf{a}}^+$ do increment ω by one and thus acquire an additional factor of

$$-s-\omega = (L+1)(-s) - \sum_i \mathbf{n}_i \mapsto (L+1)(-s) + \sum_{c=1}^{N} x_c \partial_c = H.$$

This proves that $\mathcal{M}\left\{\widetilde{\mathbf{O}}^i_j\right\} \widetilde{\mathcal{I}} = 0$ for the operators in (2.36) and (2.37) and Theorem 17 concludes the proof. □

Note that the proof of the identity $\widetilde{\mathbf{O}}^i_j \mathcal{G}^s = 0$ given in Corollary 21 rests on the inverse Mellin transform. An alternative, direct algebraic proof is given in Appendix A.2.[8]

Definition 22 By Mom we denote the left $A^N[s]$-module generated by the annihilators $\widetilde{\mathbf{O}}^i_j$ from Corollary 21, corresponding to the momentum space IBP identities:

$$\mathrm{Mom} := \sum_{i,j} A^N[s] \cdot \widetilde{\mathbf{O}}^i_j \subseteq \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s). \tag{2.38}$$

Since the $\widetilde{\mathbf{O}}^i_j$ are first-order differential operators, we have the inclusions

$$\mathrm{Mom} \subseteq \mathrm{Ann}^1_{A^N[s]}(\mathcal{G}^s) \subseteq \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s), \tag{2.39}$$

---

[8] For an algebraic proof of the analogous statement in the Baikov representation, see [35, section 9].

where Ann^1 denotes the $A^N[s]$-module generated by all first-order annihilators. Note that for a generic polynomial $\mathcal{G}$, one would not expect that all of its annihilators can be obtained from linear ones ($\mathrm{Ann}^1 \subsetneq \mathrm{Ann}$). It is therefore interesting that we observe the equality $\mathrm{Ann}^1 = \mathrm{Ann}$ in all cases of Feynman integrals that we checked.

Question 23 *For a Feynman integral with a complete set of irreducible scalar products, is the second inclusion in* (2.39) *an equality? In other words, are the s-parametric annihilators of Lee–Pomeransky polynomials $\mathcal{G}$ linearly generated?*

Regarding the first inclusion in (2.39), we do know that it is strict (see the example in Appendix C.4). However, it seems that Mom does provide all identities once we enlarge the coefficients to rational functions in the dimension $s = -d/2$ and the indices ν_e. Let us write $\theta = (\theta_1, \ldots, \theta_N)$ and $\theta_e := x_e \partial_e$ such that $\nu_e = \mathcal{M}\{-\theta_e\}$.

Question 24 *Given any annihilator $P \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$, does there exist a polynomial $q \in \mathbb{C}[s, \theta]$ such that $q P \in \mathrm{Mom}$? In other words, does*

$$\mathbb{C}(s, \theta) \otimes_{\mathbb{C}[s,\theta]} \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s) = \mathbb{C}(s, \theta) \otimes_{\mathbb{C}[s,\theta]} \mathrm{Mom} \quad hold? \qquad (2.40)$$

To test this conjecture, we should take all known shift relations for Feynman integrals, and check if they can be realized as elements of Mom (after localizing at $\mathbb{C}(s, \theta)$). In the remainder of this section, we will address such a relation, namely the one originating from the well-known dimension shifts.

2.5 Dimension shifts

The representation $\mathcal{I} = \mathcal{M}\{e^{-\mathcal{F}/\mathcal{U}} \cdot \mathcal{U}^s\}$ from (2.5) shows, through Corollary 13, that

$$\mathcal{I}(d) = \mathcal{M}\{\mathcal{U}\}\,\mathcal{I}(d+2) \qquad (2.41)$$

where $\mathcal{M}\{\mathcal{U}\} = \mathcal{U}(\hat{\mathbf{1}}^+, \ldots, \hat{\mathbf{N}}^+)$ is obtained from the polynomial $\mathcal{U}(x_1, \ldots, x_N)$ by substituting $x_i \mapsto \hat{\mathbf{i}}^+$. This *raising* dimension shift was pointed out by Tarasov [89],[9] and had been observed before in special cases [26]. For $\widetilde{\mathcal{I}}(d) = \mathcal{I}(d) \cdot \Gamma(d/2 - \omega)/\Gamma(d/2)$, the relation takes the form

$$\widetilde{\mathcal{I}}(d) = \frac{s}{s + \omega}\mathcal{U}(\hat{\mathbf{1}}^+, \ldots, \hat{\mathbf{N}}^+)\widetilde{\mathcal{I}}(d+2). \qquad (2.42)$$

At the same time, the representation $\widetilde{\mathcal{I}} = \mathcal{M}\{\mathcal{G}^s\}$ implies also that

$$\widetilde{\mathcal{I}}(d) = \mathcal{G}(\hat{\mathbf{1}}^+, \ldots, \hat{\mathbf{N}}^+)\widetilde{\mathcal{I}}(d+2) = \mathcal{M}\{\mathcal{G}\}\,\widetilde{\mathcal{I}}(d+2). \qquad (2.43)$$

---

[9] Tarasov considered the special case where all inverse propagators are of the form $\mathsf{D}_e = k_e^2 - m_e^2$, and hence, $\mathcal{U}$ is just the graph (first Symanzik) polynomial from (3.22).

Remark 25 The equality of (2.42) and (2.43) implies, via the Mellin transform, that

$$H(s)\mathcal{G} + s\mathcal{U} = -\left(s - \sum_a x_a \partial_a + Ls\right)\mathcal{G} + s\mathcal{U} \in \mathrm{Ann}(\mathcal{G}^{s-1}).$$

Indeed, $H(s)\mathcal{G}^s = s(L\mathcal{U} + (L+1)\mathcal{F})\mathcal{G}^{s-1} - s(L+1)\mathcal{G}^s = -s\mathcal{U}\mathcal{G}^{s-1}$ follows from the homogeneity of $\mathcal{U}$ and $\mathcal{F}$, see (A.5).[10]

A *lowering* dimension shift, expressing $\mathcal{I}(d+2)$ in terms of $\mathcal{I}(d)$, corresponds to a Bernstein–Sato operator of $\mathcal{G}$ under the Mellin transform $\widetilde{\mathcal{I}}(d) = \mathcal{M}\left\{\mathcal{G}^{-d/2}\right\}$[11]:

Definition 26 A Bernstein–Sato operator $P(s) \in A^N[s]$ for a non-constant polynomial f is a polynomial differential operator such that there exists a polynomial $b(s) \in \mathbb{C}[s]$ with

$$P(s)f^{s+1} = b(s)f^s. \tag{2.44}$$

Such operators always exist, and the *Bernstein–Sato polynomial* is the unique monic polynomial $b(s)$ of smallest degree for which (2.44) has a nonzero solution [10,73]. Given a solution of (2.44) for $f = \mathcal{G}$, we get a lowering dimension shift relation:

$$\widetilde{\mathcal{I}}(d+2) = \frac{1}{b(s-1)}\mathcal{M}\{P(s-1)\}\widetilde{\mathcal{I}}(d). \tag{2.45}$$

Corollary 27 *If we allow the coefficients to be rational functions $k = \mathbb{C}(s)$, every integral in $d + 2n$ dimensions can be written as an integral in d dimensions. In other words, the multiplication with f is invertible on $A_k^N f^s$. Put still differently, f^s generates the full module*

$$A_k^N \cdot f^s = k[x, 1/f] \cdot f^s. \tag{2.46}$$

Proof By (2.44), $f^{s-n} = \frac{P(s-n)}{b(s-n)} \cdots \frac{P(s-1)}{b(s-1)} f^s \in A_k^N \cdot f^s$ for all $n \in \mathbb{N}$.   □

In general, computing a Bernstein operator is not at all trivial. But in the case of a complete set of irreducible scalar products ($N = |\Theta|$ and $\mathcal{A}$ is invertible), an explicit formula for the lowering dimension shift follows from Baikov's representation [7] of Feynman integrals. We use form (2.48) given by Lee in [53,54]:

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[10] In fact, $H(s)\mathcal{G} + s\mathcal{U} = \sum_e x_e (\mathcal{G}\partial_e - (s-1)(\partial_e\mathcal{G}))$ follows from the trivial annihilators (2.14) of $\mathcal{G}^{s-1}$.

[11] Tkachov proposed in [93] to use a generalization of (2.44) to several polynomials (the individual Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$, instead of $\mathcal{G} = \mathcal{U} + \mathcal{F}$), which in the physics literature is referred to as Bernstein–Tkachov theorem. However, this result is in fact due to Sabbah [71] (see also [36]).

Recall that $(q_1, \ldots, q_M) = (\ell_1, \ldots, \ell_L, p_1, \ldots, p_E)$ denotes the combined loop-and external momenta $(M = L + E)$. We introduce the Gram determinants

$$\mathsf{Gr}_n(s) := \det \begin{pmatrix} q_n \cdot q_n & \cdots & q_n \cdot q_M \\ \vdots & \ddots & \vdots \\ q_M \cdot q_n & \cdots & q_M \cdot q_M \end{pmatrix} = \det \left(s_{\{i,j\}} \right)_{n \leq i, j \leq M} \qquad (2.47)$$

and remark that $\mathsf{Gr} := \mathsf{Gr}_{L+1}(s) = \det(p_i \cdot p_j)_{1 \leq i, j \leq E}$ depends only on the external momenta. Furthermore, note that $\mathsf{Gr}_1(s)$ is a polynomial in the scalar products $s_{\{i,j\}} = q_i \cdot q_j$. By (2.29), we can think of it also as a polynomial in the denominators D.

Definition 28 The *Baikov polynomial* $\mathcal{P}(y) \in \mathbb{C}[y_1, \ldots, y_N]$ is the polynomial defined by $\mathcal{P}(\mathsf{D}_1, \ldots, \mathsf{D}_N) = \mathsf{Gr}_1(s)$.

Theorem 29 (Baikov representation [53]) *The Feynman integral (2.1) can be written as*

$$\mathcal{I}(d) = \frac{c \cdot \pi^{-LE/2 - L(L-1)/4}}{\Gamma\left(\frac{d-E-L+1}{2}\right) \cdots \Gamma\left(\frac{d-E}{2}\right)} \cdot \frac{(-1)^{Ld/2}}{\mathsf{Gr}^{(d-E-1)/2}} \left(\prod_{e=1}^{N} \int \frac{\mathrm{d}\, y_e}{y_e^{\nu_e}} \right) \cdot \{\mathcal{P}(y)\}^{(d-N-1)/2}$$

$$(2.48)$$

where $c \in \mathbb{Q}$ is a rational constant and the Baikov polynomial $\mathcal{P}(y)$ has degree at most $M = L + E$. The contour of integration in (2.48) is such that $\mathcal{P}$ vanishes on its boundary.

We include the proof in "Appendix A.3". For us, the interesting feature of this alternative formula is that the dimension appears with a positive sign in the exponent of the integrand. We can therefore directly read off

Corollary 30 (lowering dimension shift [54]) *A Feynman integral in $d + 2$ dimensions can be expressed as an integral in d dimensions by*[12]

$$\mathcal{I}(d + 2) = \frac{(-1)^L}{\left(\frac{d-L-E+1}{2}\right) \cdots \left(\frac{d-E}{2}\right)} \frac{\mathcal{P}(\mathbf{1}^-, \ldots, \mathbf{N}^-)}{\mathsf{Gr}} \mathcal{I}(d). \qquad (2.49)$$

Proof According to (2.48), $\mathcal{I}(d + 2)$ is obtained by multiplying the integrand of $\mathcal{I}(d)$ with $(-1)^L \mathcal{P}(y)/\mathsf{Gr}$ and adjusting the Γ-factors in the prefactor as $\Gamma\left(\frac{d+2-E}{2}\right) = \frac{d-E}{2} \Gamma\left(\frac{d-E}{2}\right)$ and so on. Multiplying the integrand of the Baikov representation with y_e is equivalent to decrementing ν_e; hence, the multiplication of the integrand by $\mathcal{P}(y)$ can be written as the action of $\mathcal{P}(\mathbf{1}^-, \ldots, \mathbf{N}^-)$ on the integral. $\qquad \square$

Equation (2.49) can be thought of as a Bernstein equation for $\mathcal{I}(d)$, or, equivalently, as a special type of integral relation: Combining (2.49) with the raising dimension shift (2.49), we find that

---

[12] The first fraction can also be written as $2^L / (d - L - E + 1)_L$ in terms of the Pochhammer symbol (raising factorial) $a_L = a(a + 1) \cdots (a + L - 1)$.

$$\left\{ \frac{\mathcal{P}(\mathbf{1}^-, \dots, \mathbf{N}^-)}{\mathrm{Gr}} \mathcal{U}(\hat{\mathbf{1}}^+, \dots, \hat{\mathbf{N}}^+) - \frac{E - d + 2}{2} \cdots \frac{E - d + (L + 1)}{2} \right\} \mathcal{I}(d) = 0.$$

(2.50)

We ask if this annihilator is contained in Mom; in other words, whether the lowering dimension shift relation (2.49) is a consequence of the momentum space IBP identities. This is what we will establish in the following Proposition 31. First let us write the Baikov polynomial $\mathcal{P}(y)$ explicitly: The block decomposition $(s_{i,j})_{1 \leq i, j \leq M} = \begin{pmatrix} V & B \\ B^\mathsf{T} & G \end{pmatrix}$ with $V = (\ell_i \cdot \ell_j)_{i,j \leq L}$, $B = (\ell_i \cdot p_j)_{i \leq L, j \leq E}$ and $G = (p_i \cdot p_j)_{i,j \leq E}$ shows that

$$\frac{\mathcal{P}(y)}{\mathrm{Gr}} = \det Q(y) \quad \text{where} \quad Q(\mathsf{D}) := V - B G^{-1} B^\mathsf{T}$$

is an $L \times L$ matrix whose entries are quadratic in the denominators. By (2.29),

$$Q_{i,j} = \mathcal{A}^a_{\{i,j\}}(\mathsf{D}_a - \lambda_a) - \mathcal{A}^a_{\{i,r\}}(\mathsf{D}_a - \lambda_a) G^{-1}_{r,s} \mathcal{A}^b_{\{j,s\}}(\mathsf{D}_b - \lambda_b) \qquad (2.51)$$

where we suppress the explicit summation signs over $a = 1, \dots, N$ in the first summand and over $a, b = 1, \dots, N$ and $r, s = L + 1, \dots, M$ in the second summand.

Proposition 31 *Annihilator* (2.50)*, corresponding to the lowering dimension shift* (2.49)*, is contained in the ideal of shift operators generated by the momentum space IBP's from Proposition* 20*:*

$$\det Q(\mathbf{1}^-, \dots, \mathbf{N}^-) \cdot \mathcal{U}(\hat{\mathbf{1}}^+, \dots, \hat{\mathbf{N}}^+) - \prod_{j=2}^{L+1} \frac{(E + j - d)}{2} \in \sum_{i,j} S^N[d] \cdot \mathbf{O}^i_j.$$

(2.52)

Proof Let us abbreviate $\tilde{\Lambda} := \Lambda(\hat{\mathbf{1}}^+, \dots, \hat{\mathbf{N}}^+)$ from (A.2) such that $\mathcal{U} = \det \Lambda$ and similarly $\tilde{Q} := Q(\mathbf{1}^-, \dots, \mathbf{N}^-)$ for matrix (2.51). Using (2.51) and (2.15), we compute

$$\begin{aligned}
\left[\tilde{Q}_{i,j}, \hat{\mathbf{c}}^+ \right] &= \mathcal{A}^a_{\{i,j\}} \left[\mathbf{a}^-, \hat{\mathbf{c}}^+ \right] - G^{-1}_{r,s} \mathcal{A}^a_{\{i,r\}} \mathcal{A}^b_{\{j,s\}} \left((\mathbf{a}^- - \lambda_a) \left[\mathbf{b}^- - \lambda_b, \hat{\mathbf{c}}^+ \right] \right. \\
&\quad \left. + \left[\mathbf{a}^- - \lambda_a, \hat{\mathbf{c}}^+ \right] (\mathbf{b}^- - \lambda_b) \right) \\
&= - \mathcal{A}^c_{\{i,j\}} + G^{-1}_{r,s} \left(\mathcal{A}^a_{\{i,r\}} \mathcal{A}^c_{\{j,s\}} (\mathbf{a}^- - \lambda_a) + \mathcal{A}^c_{\{i,r\}} \mathcal{A}^b_{\{j,s\}} (\mathbf{b}^- - \lambda_b) \right).
\end{aligned}$$

Contracting with the matrix $\mathcal{A}^{\{k,l\}}_c$ (for $k, l \leq L$) by summing over c, we conclude that

$$\begin{aligned}
\left[\tilde{Q}_{i,j}, \mathcal{A}^{\{k,l\}}_c \hat{\mathbf{c}}^+ \right] &= - \delta_{\{i,j\},\{k,l\}} + G^{-1}_{r,s} \left(\mathcal{A}^a_{\{i,r\}} \delta_{\{j,s\},\{k,l\}} (\mathbf{a}^- \right. \\
&\quad \left. - \lambda_a) + \delta_{\{i,r\},\{k,l\}} \mathcal{A}^b_{\{j,s\}} (\mathbf{b}^- - \lambda_b) \right).
\end{aligned}$$

Note that the indices r and s take values $> L$, whereas i and j are $\leq L$. Hence, $\delta_{\{j,s\},\{k,l\}} = \delta_{\{i,r\},\{k,l\}} = 0$. So, recalling (A.2), we finally arrive at

$$\left[\tilde{Q}_{i,j}, \tilde{\Lambda}_{k,l} \right] = \frac{1 + \delta_{k,l}}{2} \delta_{\{i,j\},\{k,l\}} = \frac{\delta_{i,k}\delta_{j,l} + \delta_{i,l}\delta_{j,k}}{2}. \tag{2.53}$$

Recall (2.3), that $\mathcal{U}(x) = \det \Lambda(x)$, such that

$$\det Q(\mathbf{1}^-, \ldots, \mathbf{N}^-) \cdot \mathcal{U}(\hat{\mathbf{1}}^+, \ldots, \hat{\mathbf{N}}^+) = \det \tilde{Q} \cdot \det \tilde{\Lambda}.$$

We can now invoke an identity of Turnbull [94], see also [29] for a combinatorial and [20] for an algebraic proof, which relates this product of determinants to a determinant of the product $\tilde{Q} \cdot \tilde{\Lambda}$. This is non-trivial, because the elements of these two matrices do not commute, according to (2.53). Turnbull's identity, as stated in [20, Proposition 1.4], applies precisely to this kind of very mild non-commutativity (2.53) and states that

$$\det \tilde{Q} \cdot \det \tilde{\Lambda} = \text{col-det} \left(\tilde{Q} \cdot \tilde{\Lambda} + Q_{col} \right), \quad \text{where} \quad (Q_{col})_{i,j} := -\frac{L-i}{2} \delta_{i,j} \tag{2.54}$$

is a simple diagonal matrix and col-det denotes the column-ordered determinant

$$\text{col-det } A := \sum_{\sigma \in S_N} \text{sgn}(\sigma) A_{\sigma(1),1} \ldots A_{\sigma(N),N}. \tag{2.55}$$

So let us now compute the entries of the product of $\tilde{Q}$ from (2.51) with $\tilde{\Lambda}$. Firstly,

$$\mathcal{A}^b_{\{j,s\}}(\mathbf{b}^- - \lambda_b)\tilde{\Lambda}_{j,k} = -\frac{1}{2}\mathcal{A}^b_{\{j,s\}}(1 + \delta_{j,k})\mathcal{A}^{\{j,k\}}_c(\mathbf{b}^- - \lambda_b)\hat{\mathbf{c}}^+$$

$$= -\frac{1}{2}C^{bk}_{cs}\left\{ \hat{\mathbf{c}}^+(\mathbf{b}^- - \lambda_b) - \delta_{b,c} \right\}$$

according to (2.35). Note that $C^{bk}_{bs} = \sum_{b,j}(1 + \delta_{j,k})\mathcal{A}^{\{j,k\}}_b \mathcal{A}^b_{\{j,s\}} = \sum_j (1 + \delta_{j,k})\delta_{\{j,k\},\{j,s\}} = 0$ due to $s > L \geq k$. So we can rewrite, due to (2.34),

$$\mathcal{A}^b_{\{j,s\}}(\mathbf{b}^- - \lambda_b)\tilde{\Lambda}_{j,k} = \frac{1}{2}\mathbf{O}^k_s + \frac{1}{2}\sum_{m>L} \mathcal{A}^{\{k,m\}}_b \hat{\mathbf{b}}^+ (q_s \cdot q_m).$$

Note that $q_s \cdot q_m = G_{s,m}$ such that contraction of the second summand with $G^{-1}_{r,s}$ produces $\delta_{r,m}$. So the sum over m collapses, and up to the term with $\mathbf{O}^k_s$, $\tilde{Q}_{i,j}\tilde{\Lambda}_{j,k}$ is

$$-\frac{1 + \delta_{j,k}}{2}\mathcal{A}^a_{\{i,j\}}(\mathbf{a}^- - \lambda_a)\mathcal{A}^{\{j,k\}}_c\hat{\mathbf{c}}^+ - \frac{1}{2}\mathcal{A}^a_{\{i,r\}}(\mathbf{a}^- - \lambda_a)\mathcal{A}^{\{k,r\}}_b\hat{\mathbf{b}}^+$$

$$= -\frac{1}{2}C^{ak}_{bi}(\mathbf{a}^- - \lambda_a)\hat{\mathbf{b}}^+$$

$$= \frac{1}{2} \left\{ \mathbf{O}_i^k - d\delta_i^k + C_{ai}^{ak} \right\} = \frac{1}{2} \left\{ \mathbf{O}_i^k - d\delta_i^k + (M+1)\delta_i^k \right\},$$

where the term $C_{ai}^{ak} = \sum_{a,j}(1 + \delta_{j,k})\mathcal{A}_a^{\{j,k\}}\mathcal{A}_{\{j,i\}}^a = \sum_j (1 + \delta_{j,k})\delta_{\{j,k\},\{j,i\}} = \delta_i^k(M+1)$ comes from commuting $\mathbf{a}^-$ with $\hat{\mathbf{b}}^+$. Putting our results together, we arrive at

$$\tilde{Q}_{i,j}\tilde{\Lambda}_{j,k} = \frac{1}{2} \left\{ (M+1-d)\delta_i^k + \mathbf{O}_i^k - \mathcal{A}_{\{i,r\}}^a(\mathbf{a}^- - \lambda_a)G_{r,s}^{-1}\mathbf{O}_s^k \right\}. \quad (2.56)$$

So if we ignore all terms that lie in the (left) ideal generated by the momentum space (shift) operators $\mathbf{O}_j^i$, the column determinant (2.55) of the matrix $\tilde{Q} \cdot \tilde{\Lambda} + Q_{col}$ from (2.54) can be replaced by an ordinary determinant $\det B$ of the diagonal matrix

$$B_{i,j} = \frac{\delta_{i,j}}{2}(M+1-d-L+i) = \frac{\delta_{i,j}}{2}(E+1-d+i),$$

such that indeed we conclude with the result that

$$\det \tilde{Q} \cdot \det \tilde{\Lambda} \equiv \prod_{i=1}^{L} \frac{E+1-d+i}{2} \quad \mathrm{mod} \sum_{i,j} S^N[d] \cdot \mathbf{O}_j^i.$$

$\square$

The Mellin transform $\mathcal{I}(d) = \mathcal{M}\left\{\mathcal{U}^s e^{-\mathcal{F}/\mathcal{U}}\right\}$ identifies (2.50) with the annihilator

$$\left\{ \det Q(-\partial) \cdot \mathcal{U} - \tilde{b}(s) \right\} \cdot \mathcal{U}^s e^{-\mathcal{F}/\mathcal{U}} = 0, \quad \mathrm{where} \quad \tilde{b}(s) := \prod_{j=2}^{L+1} \left(s + \frac{E+j}{2} \right). \quad (2.57)$$

To phrase this in terms of $\widetilde{\mathcal{I}}(d) = \mathcal{M}\{\mathcal{G}^s\} = \mathcal{I}(d) \cdot \Gamma(-s-\omega)/\Gamma(-s)$, we can use $\mathcal{U}\mathcal{G}^s = -H(s+1)\mathcal{G}^{s+1}/(s+1)$ from Remark 25 to conclude that

$$\Gamma(H(s)) \cdot \det Q(-\partial) \cdot \frac{1}{\Gamma(H(s)-L-1)} \cdot \mathcal{G}^{s+1} = -(s+1)\tilde{b}(s)\mathcal{G}^s, \quad (2.58)$$

where $H(s) = \mathcal{M}^{-1}\{-s-\omega\} = -s(L+1) + \sum_{i=1}^{N} \theta_i$. Recall from (2.22) that the left-hand side of (2.58) can be written, in terms of the homogeneous components Q_r (with degree r) of $\det Q(-\partial) = \sum_r Q_r$, as

$$\sum_r \frac{\Gamma(H(s))}{\Gamma(H(s)-L-1+r)} Q_r = \sum_{r \leq L+1} \left[\prod_{i=1}^{L+1-r} (H(s)-i) \right] Q_r$$
$$+ \sum_{r > L+1} \left[\prod_{i=0}^{r-L-1} \frac{1}{H(s)+i} \right] Q_r.$$

If $r \leq L+1$, this is a polynomial differential operator, and we thus obtained an explicit Bernstein–Sato operator as in Definition 26.

Corollary 32 *If the degree of the Baikov polynomial $\mathcal{P}(y)$ is not more than $L+1$, then the Bernstein–Sato polynomial $b(s)$ of the Lee–Pomeransky polynomial $\mathcal{G}$ is a divisor of $(s+1)\tilde{b}(s)$. In particular, all roots of $b(s)/(s+1)$ are simple and at half-integers.*

Note that $\deg \mathcal{P}(y) \leq \min\{2L, M\} = L + \min\{L, E\}$ by Definition 28 and Eq. (2.51), so in particular, the corollary applies to all propagator graphs ($E = 1$) and to all graphs with one loop ($L = 1$).

3 Euler characteristic as number of master integrals

Here, we will show, using the theory of Loeser and Sabbah [59], that the number of master integrals equals the Euler characteristic of the complement of the hypersurface defined by $\mathcal{G} = 0$ inside the torus $\mathbb{G}_m^N$. (We write $\mathbb{G}_m = \mathbb{A} \setminus \{0\}$ for the multiplicative group and $\mathbb{A}$ for the affine line.) For a full understanding of this section, some knowledge of basic D-module theory is indispensable, but we tried to include sufficient detail for the main ideas to become clear to non-experts as well. In particular, we will give self-contained proofs that only use D-module theory at the level of [24].

Definition 33 By $V_\mathcal{G}$ we denote the vector space of all Feynman integrals associated to $\mathcal{G}$, over the field $\mathbb{C}(s, \nu) := \mathbb{C}(s, \nu_1, \ldots, \nu_N)$ of rational functions (in the dimension and indices). More precisely, with $\widetilde{\mathcal{I}}_\mathcal{G} := \mathcal{M}\{\mathcal{G}^s\}$,

$$V_\mathcal{G} := \sum_{n \in \mathbb{Z}^N} \mathbb{C}(s, \nu) \cdot \widetilde{\mathcal{I}}_\mathcal{G}(\nu + n) = \mathbb{C}(s, \nu) \otimes_{\mathbb{C}[s,\nu]} \left(S^N[s] \cdot \widetilde{\mathcal{I}}_\mathcal{G} \right). \qquad (3.1)$$

The *number of master integrals* is the dimension of this vector space:

$$\mathfrak{C}(\mathcal{G}) := \dim_{\mathbb{C}(s,\nu)} V_\mathcal{G}. \qquad (3.2)$$

Note that this is the same as the dimension of the space $\sum_n \mathbb{C}(s, \nu) \mathcal{I}_\mathcal{G}(\nu + n)$ of Feynman integrals (2.1), because the ratios $\mathcal{I}_\mathcal{G}(\nu + n)/\widetilde{\mathcal{I}}_\mathcal{G}(\nu + n) = \Gamma(-s)/\Gamma(-s - \omega - |n|)$ with $|n| = n_1 + \cdots + n_N$ are all related by a rational function in $\mathbb{C}(s, \nu)$, see (2.22).

Remark 34 The phrase "master integrals" is used with different meanings in the physics literature. The main sources for discrepancies are:

1. Almost always the integrals are considered only for integer indices $\nu \in \mathbb{Z}^N$, instead of as functions of arbitrary indices. In this setting, integrals with at least one $\nu_e = 0$ can be identified with quotient graphs ("subtopologies") and are often discarded from the counting of master integrals.
2. We only discuss relations of integrals that are expressible as linear shift operators acting on a single integral. This set-up cannot account for relations of integrals of different graphs (with some fixed values of the indices), as for example discussed

in [48]. It also excludes symmetry relations, which are represented by permutations of the indices ν_e.

3. Some authors do not count integrals if they can be expressed in terms of Γ-functions or products of simpler integrals, for example [41,48].

Taking care of these subtleties, we will demonstrate in Sect. 4 that our definition gives results that do match the counting of master integrals obtained by other methods.

A fundamental result for methods of integration by parts reduction is that the number of master integrals is finite. This was proven in [77] for the case of integer indices $\nu \in \mathbb{Z}^N$, using the momentum space representation. Below we will show that this result holds much more generally, for unconstrained ν, and that it becomes a very natural statement once it is viewed through the parametric representation. Notably, it remains true for Mellin transforms $\mathcal{M}\{\mathcal{G}^s\}$ of *arbitrary* polynomials $\mathcal{G}$—the fact that $\mathcal{G}$ comes from a (Feynman) graph is completely irrelevant for this section.

Recall that, by the Mellin transform, we can rephrase statements about integrals in terms of the parametric integrands. In line with (2.25) and (3.1), we can rewrite (3.2) as

$$\mathfrak{C}(\mathcal{G}) = \dim_{\mathbb{C}(s,\theta)}\left(\mathbb{C}(s,\theta) \otimes_{\mathbb{C}[s,\theta]} A^N[s] \cdot \mathcal{G}^s\right),$$

where $\mathbb{C}[s,\theta] = \mathbb{C}[s,\theta_1,\ldots,\theta_N]$ denotes the polynomials in the dimension $s = -d/2$ and the operators $\theta_e := x_e \partial_e = \mathcal{M}^{-1}\{-\nu_e\}$, and $F := \mathbb{C}(s,\theta)$ stands for their fraction field (the rational functions in these variables). Since F contains $k := \mathbb{C}(s)$, we can equivalently work over this base field throughout and write

$$\mathfrak{C}(\mathcal{G}) = \dim_F(F \otimes_R \mathcal{M}) \tag{3.3}$$

in terms of $R := k[\theta]$ and the module $\mathcal{M} = A_k^N \cdot \mathcal{G}^s$ over the Weyl algebra $A_k^N := A^N \otimes_{\mathbb{C}} k = A^N[s] \otimes_{\mathbb{C}[s]} k$ over the field $k = \mathbb{C}(s)$. Crucially, $A_k^N \cdot \mathcal{G}^s$ is a *holonomic* A_k^N-module, which is a fundamental result due to Bernstein [10].

Holonomic modules are, in a precise sense, the most constrained and behave in many ways like finite-dimensional vector spaces. For example, sub- and quotient modules, direct and inverse images of holonomic modules are again holonomic [44,45], and holonomic modules in zero variables are precisely the finite-dimensional vector spaces. The holonomicity of the parametric integrand was already exploited in [46] to show that Feynman integrals fulfil a holonomic system of differential equations, and it is also a key ingredient in the proof in [77].

The number defined in (3.3) has been studied by Loeser and Sabbah [59] in a slightly different setting, namely for holonomic modules over the algebra

$$\mathcal{D}_k^N := k[x_1^{\pm 1},\ldots,x_N^{\pm 1}]\langle \partial_1,\ldots,\partial_N\rangle = k[x^{\pm 1}] \otimes_{k[x]} A_k^N = A_k^N[x^{-1}] \tag{3.4}$$

of linear differential operators on the torus $\mathbb{G}_m^N$. Note that $\mathcal{D}_k^N$ is just the localization of A_k^N at the coordinate hyperplanes $x_i = 0$; that is, the coefficients of the derivations are extended from polynomials $\mathcal{O}(\mathbb{A}_k^N) = k[x]$ to Laurent polynomials $\mathcal{O}(\mathbb{G}_m^N) = k[x^{\pm 1}]$

in the coordinates x_i. Equivalently, we can also view $\mathcal{D}_k^N = \iota^* A_k^N$ as the pull-back under the (open) inclusion

$$\iota \colon \mathbb{G}_m^N \hookrightarrow \mathbb{A}_k^N.$$

The pull-back along ι turns every A_k^N-module $\mathcal{M}$ into a $\mathcal{D}_k^N$-module $\iota^*\mathcal{M}$, namely the localization $\iota^*\mathcal{M} = k[x^{\pm 1}] \otimes_{k[x]} \mathcal{M} = \mathcal{M}[x^{-1}]$. Importantly, if $\mathcal{M}$ is holonomic, so is its pull-back $\iota^*\mathcal{M}$. The starting point for this section is

Theorem 35 (Loeser and Sabbah [59,61]) *Let $\mathcal{M}$ denote a holonomic A_k^N-module. Then, $F \otimes_R \mathcal{M}$ is a finite-dimensional vector space over F. Moreover, its dimension is given by the Euler characteristic $\dim_F(F \otimes_R \mathcal{M}) = \chi(\iota^*\mathcal{M})$.*

In "Appendix B", we provide a self-contained proof of this crucial theorem, simpler and more explicit than in [61]. For now, let us content ourselves with reducing it to the known situation on the torus.

Proof We can invoke $\dim_F(F \otimes_R \iota^*\mathcal{M}) = \chi(\iota^*\mathcal{M}) < \infty$ from [61, Théorème 2]. To conclude, we just need to note that $F \otimes_R \mathcal{M}$ and $F \otimes_R \mathcal{M}[x^{-1}] = F \otimes_R \iota^*\mathcal{M}$ are isomorphic vector spaces (over F). This is clear since each coordinate x_i is invertible after localizing at F: Due to $\partial_i x_i = 1 + x_i \partial_i$, we find that $(1 + x_i \partial_i)^{-1} \otimes \partial_i \in F \otimes_R A_k^N$ is an inverse to $1 \otimes x_i$.   □

This result not only implies the mere finiteness of the number of master integrals, but in addition gives a formula for this number—it is the Euler characteristic, given by

$$\chi(\mathcal{M}') := \chi(\mathrm{DR}(\mathcal{M}')) = \sum_i (-1)^i \dim_k H^i(\mathrm{DR}(\mathcal{M}')), \qquad (3.5)$$

of the algebraic de Rham complex of $\mathcal{M}' := \iota^*\mathcal{M}$. This is the complex

$$\mathrm{DR}(\mathcal{M}') := \left(\Omega_{\mathbb{G}_m^N}^\bullet \otimes_{\mathcal{O}(\mathbb{G}_m^N)} \mathcal{M}'[N], \mathrm{d} \right) \qquad (3.6)$$

of $\mathcal{M}'$-valued differential forms on the torus $\mathbb{G}_m^N$, with the connection $\mathrm{d}(\omega \otimes m) = \mathrm{d}\omega \otimes m + \sum_{i=1}^N (\mathrm{d}x_i \wedge \omega) \otimes \partial_i m$. Note that the r-forms ω are shifted to sit in degree $r - N$ of the complex, which is thus supported in degrees between $-N$ and 0; hence (3.5) is a finite sum over $-N \leq i \leq 0$. The extremal cohomology groups are easily identified as

$$H^{-N}(\mathrm{DR}(\mathcal{M}')) = \bigcap_{i=1}^N \ker \partial_i \quad \text{and} \quad H^0(\mathrm{DR}(\mathcal{M}')) \cong \mathcal{M}' \Big/ \sum_{i=1}^N \partial_i \mathcal{M}' = \pi_*\mathcal{M}',$$
$$(3.7)$$

with the latter also known as push-forward of $\mathcal{M}'$ under the projection $\pi \colon \mathbb{G}_m^N \longrightarrow \mathbb{A}_k^0$ to the point. Since holonomicity is preserved under direct images, we conclude

that $\dim_k H^0\left(\mathrm{DR}(\mathcal{M}')\right)$ is finite.[13] In fact, the same is true for the other de Rham cohomology groups, which shows that (3.5) is indeed well defined.[14]

Since we are interested in Feynman integrals, we consider the special case where the A_k^N-module $\mathcal{M}$ is simply $\mathcal{M} = A_k^N \cdot \mathcal{G}^s$ from Definition 9. Its elements can be written uniquely in the form $h \cdot \mathcal{G}^s$, where $h \in k[x, \mathcal{G}^{-1}]$, such that $A_k^N \cdot \mathcal{G}^s \cong k[x, \mathcal{G}^{-1}]$ by (2.46) are isomorphic as $k[x]$-modules: $x_i(h\mathcal{G}^s) = (x_i h)\mathcal{G}^s$. The action (2.13) of the derivatives, however, is twisted by a term proportional to s: $\partial_i(h\mathcal{G}^s) = \mathcal{G}^s(\partial_i h + sh(\partial_i \mathcal{G})/\mathcal{G})$. Despite this twist, we find that the Euler characteristic stays the same:

Proposition 36 *Let $\mathcal{G} \in \mathbb{C}[x_1, \ldots, x_N]$ be a polynomial and set $k = \mathbb{C}(s)$. Then, the Euler characteristics of the algebraic de Rham complexes of the holonomic A_k^N-module $\iota^* A_k^N \mathcal{G}^s$ and the holonomic $A_{\mathbb{C}}^N$-module $\mathbb{C}[x^{\pm 1}, \mathcal{G}^{-1}] = \mathcal{O}(\mathbb{G}_m^N \backslash \mathbb{V}(\mathcal{G}))$ coincide:*

$$\chi(\iota^* A_k^N \mathcal{G}^s) = \chi\left(\mathbb{C}[x^{\pm 1}, \mathcal{G}^{-1}]\right). \tag{3.8}$$

In particular, we can dispose of the parameter s completely and compute with the algebraic de Rham complex of $\mathbb{C}[x^{\pm 1}, \mathcal{G}^{-1}]$, which is the ring of regular functions of the complement of the hypersurface $\mathbb{V}(\mathcal{G}) = \{x : \mathcal{G}(x) = 0\}$ in the torus $\mathbb{G}_m^N$. Combining Theorem 35 with Proposition 36, we thus obtain our main result:

Corollary 37 *The number of master integrals of an integral family with N denominators is*

$$\mathfrak{C}(\mathcal{G}) = \chi\left(\mathbb{C}[x^{\pm 1}, \mathcal{G}^{-1}]\right), \tag{3.9}$$

the Euler characteristic of the algebraic de Rham complex of the complement of the hypersurface $x_1 \ldots x_N \cdot \mathcal{G} = 0$ inside the affine plane $\mathbb{A}^N$. Via Grothendieck's comparison isomorphism, this is the same as the topological Euler characteristic, up to a sign:[15]

$$\mathfrak{C}(\mathcal{G}) = (-1)^N \chi\left(\mathbb{C}^N \backslash \{x_1 \ldots x_N \cdot \mathcal{G} = 0\}\right) = (-1)^N \chi\left(\mathbb{G}_m^N \backslash \{\mathcal{G} = 0\}\right). \tag{3.10}$$

Remark 38 We stress that this geometric interpretation of the number of master integrals is valid for dimensionally regulated Feynman integrals, that is, we consider them as meromorphic functions in d (and ν). This is reflected in our treatment of $s = -d/2$ as a symbolic parameter.

If, instead, one specializes to a fixed dimension like $d = 2$ ($s = -1$) or $d = 4$ ($s = -2$), then (2.46) is no longer true in general.[16] It can then happen that $A^N \cdot \mathcal{G}^s \subsetneq$

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[13] Recall that a holonomic module over the point $\mathbb{A}_k^0$ is the same as a finite-dimensional k-vector space.

[14] The de Rham complex $\mathrm{DR}(A_k^N)$ is a resolution of $k[x]$ by free A_k^N-modules, such that $H^\bullet(\mathrm{DR}(\mathcal{M}'))$ are the (left) derived functors of $\pi_* \mathcal{M}' = H^0(\mathrm{DR}(\mathcal{M}'))$. In the language of derived categories, saying that $\pi_* \mathcal{M}'$ is holonomic actually means precisely that $\mathrm{DR}(\mathcal{M}')$ is a complex with cohomology groups that are holonomic modules over the point—that is, finite-dimensional vector spaces over k.

[15] This sign arises from the shift by N in the Definition (3.6) of the de Rham complex DR.

[16] It fails precisely if, for some $r \in \mathbb{N}$, $s - r$ is a zero of the Bernstein–Sato polynomial of $\mathcal{G}$.

$\mathbb{C}[x, \mathcal{G}^{-1}]$ is a proper submodule (note $k = \mathbb{C}(s) = \mathbb{C}$). While Theorem 35 still applies and relates the number of master integrals in a fixed dimension to the Euler characteristic of $\mathcal{D}^N \cdot \mathcal{G}^s$, this is not always equal to the topological Euler characteristic (3.10). This is expected, since the number of master integrals is known to be different in fixed dimensions [88].[17]

Proof of Proposition 36 Given an $A_{\mathbb{C}}^N$-module $\mathcal{M}$ and a polynomial $f \in \mathbb{C}[x]$, set $\mathcal{M}' := \mathcal{M}[f^{-1}]$ and consider the $A_{\mathbb{C}}^N$-module $\mathcal{M}' f^s$ formed by products of f^s with elements $m \in \mathcal{M}'[s] := \mathcal{M}' \otimes_{\mathbb{C}} \mathbb{C}[s]$.[18] As vector spaces, $\mathcal{M}' f^s \cong \mathcal{M}'[s]$ via $m f^s \mapsto m$, but the $A_{\mathbb{C}}^N$ action on $\mathcal{M}' f^s$ has twisted derivatives to take into account the factor f^s:

$$x_i \cdot m f^s := x_i m f^s \quad \text{and} \quad \partial_i \cdot m f^s := \left\{ (\partial_i m) + sm \frac{\partial_i f}{f} \right\} f^s. \quad (3.11)$$

Following Malgrange [63], we introduce the action of a further variable t by setting

$$t \cdot m(s) f^s := m(s+1) f^{s+1} \quad \text{and} \quad \partial_t \cdot m(s) f^s := -sm(s-1) f^{s-1}, \quad (3.12)$$

where we use the intuitive abbreviation $f^{s+r} := f^r \cdot f^s$ for $r \in \mathbb{Z}$. One easily verifies $[\partial_t, t] = 1$ and $[\partial_i, \partial_t] = [\partial_i, t] = [x_i, \partial_t] = [x_i, t] = 0$, such that $\mathcal{M}' f^s$ becomes an $A_{\mathbb{C}}^{N+1}$-module in the $N+1$ variables $(x_1, \ldots, x_N, t)$. Note that the operator t acts invertibly on $\mathcal{M}'$, such that the identity $\partial_t t = -s$ gives $\partial_t = -st^{-1}$ and thus

$$\frac{\mathcal{M}' f^s}{\partial_t \mathcal{M}' f^s} = \frac{\mathcal{M}' f^s}{st^{-1} \mathcal{M}' f^s} = \frac{\mathcal{M}' f^s}{s \mathcal{M}' f^s} \cong \mathcal{M}'$$

is an isomorphism of $A_{\mathbb{C}}^N$-modules.[19] Since ∂_t is injective on $\mathcal{M}' f^s$ (it raises the degree in s), the de Rham complex $\mathrm{DR}(\mathcal{M}' f^s)$ is quasi-isomorphic to $\mathrm{DR}(\mathcal{M}' f^s / \partial_t \mathcal{M}' f^s) = \mathrm{DR}(\mathcal{M}')$, see Corollary 70. So we can conclude the equality

$$\chi(\mathcal{M}' f^s) = \chi(\mathcal{M}'), \quad (3.13)$$

once we assume that $\mathcal{M}$ is holonomic to ensure that these Euler characteristics are well defined. Indeed, the holonomicity of $\mathcal{M}'$ and $\mathcal{M}' f^s$ holds because

- $\mathcal{M}' = j^* \mathcal{M}$ is the pull-back of $\mathcal{M}$ under the inclusion $j : \mathbb{A}_{\mathbb{C}}^N \setminus \{f = 0\} \hookrightarrow \mathbb{A}_{\mathbb{C}}^N$,

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[17] In these references, the indices are restricted to integers. However, our calculations in Sect. 4 show agreement of this way of counting with our set-up where the indices are treated as free parameters.

[18] It would be more consistent with Definition 9 to write $\mathcal{M}'[s] f^s$ instead of $\mathcal{M}' f^s$, but we prefer the shorter form to avoid clutter and to stress that we view it as an $A_{\mathbb{C}}^N$-module, not a $A_{\mathbb{C}[s]}^N$-module.

[19] This is also clear from the fact that $\mathcal{M}' f^s / (\partial_t \mathcal{M}' f^s) = \pi_*(\mathcal{M}' f^s)$ is the push-forward of $\mathcal{M}' f^s$ under the projection $\pi : \mathbb{A}^{N+1} \longrightarrow \mathbb{A}^N$ forgetting the last coordinate. Namely, since $\mathcal{M}' f^s = F_* \mathcal{M}'$ as we discuss below, $\pi_*(F_* \mathcal{M}') = (\pi \circ F)_* \mathcal{M}' = \mathrm{id}_* \mathcal{M}' = \mathcal{M}'$.

- $\mathcal{M}' f^s = F_* \mathcal{M}'$ is the push-forward of $\mathcal{M}'$ under the closed embedding $F \colon \mathbb{A}_{\mathbb{C}}^N \hookrightarrow \mathbb{A}_{\mathbb{C}}^{N+1}$ which sends x to $(x, f(x))$.[20]

Alternatively, the filtration $\Gamma_j \mathcal{M}' f^s := f^{-j} \sum_{i \le j} s^i \Gamma_{2j(\deg f)-i} \mathcal{M}$ induced by any good filtration $\Gamma_\bullet$ on $\mathcal{M}$ directly shows the holonomicity of $\mathcal{M}' f^s$, since its dimension grows like j^{N+1} for large j. We now invoke the theory of Loeser–Sabbah to deduce that

$$\chi(\mathcal{M}' f^s) = \dim_{\mathbb{C}(\theta, t\partial_t)} \mathcal{M}' f^s (\theta, t\partial_t) = \dim_{k(\theta)} \mathcal{N}(\theta) = \chi(\mathcal{N})$$

where $\mathcal{N} := \mathcal{M}' f^s (t\partial_t)$ denotes the algebraic Mellin transform (B.1) of $\mathcal{M}' f^s$ with respect to the coordinate t. But note that, according to (3.12), localizing at $t\partial_t = -s-1$ just extends the coefficients to $k = \mathbb{C}(s)$. So $\mathcal{N} = \mathcal{M}' \otimes_{\mathbb{C}} \mathbb{C}(s) f^s = \mathcal{M} \otimes_{\mathbb{C}} k f^s$ is just the holonomic A_k^N-module on the left-hand side of (3.8), because, over k, f is invertible by Corollary 27. We have proven $\chi(\mathcal{M} \otimes_{\mathbb{C}} k f^s) = \chi(\mathcal{M}[f^{-1}])$, and the special case of $\mathcal{M} = \mathbb{C}[x^{\pm 1}]$ with $f = \mathcal{G}$ proves the claim. □

Remark 39 More abstractly, Proposition 36 can also be seen as an application of the theory of characteristic cycles [33]: It is known that the Euler characteristic only depends on the characteristic cycle of a A_k^N-module, which follows from the Dubson–Kashiwara formula [51, equation (6.6.4)]. Therefore, it is sufficient to show that the A_k^N-modules $k[x^{\pm 1}, \mathcal{G}^{-1}]$ and $k[x^{\pm 1}]\mathcal{G}^s$ have the same characteristic cycles, via [33, Theorem 3.2]. This follows from the fact that these modules are identical up to the twist by the isomorphism $\partial_i \mapsto \partial_i + s(\partial_i \mathcal{G})/\mathcal{G}$ of $\mathcal{D}_k^N[\mathcal{G}^{-1}]$.

3.1 No master integrals

Corollary 37 shows in particular that there are no master integrals, $\mathfrak{C}(\mathcal{G}) = 0$, precisely when the Euler characteristic $\chi(\mathbb{G}_m^N \setminus \mathbb{V}(f)) := \chi(\mathbb{C}[x^{\pm 1}, f^{-1}]) = \chi((\mathbb{C}^*)^N \setminus \mathbb{V}(f))$ vanishes for $f = \mathcal{G}$. For example, this happens if f is homogeneous in a generalized sense: Suppose we can find $\lambda_0, \ldots, \lambda_N \in \mathbb{Z}$, not all zero, such that

$$f\left(x_1 t^{\lambda_1}, \ldots, x_N t^{\lambda_N}\right) = t^{\lambda_0} f(x_1, \ldots, x_N) \quad \text{in} \quad \mathbb{C}[x, t^{\pm 1}]; \tag{3.14}$$

which is equivalent (apply ∂_t and set $t = 1$) to the existence of a linear annihilator,

$$P_s^\lambda \cdot f^s = 0, \quad \text{of the form} \quad P_s^\lambda := \sum_{i=1}^N \lambda_i \theta_i - s\lambda_0 \in \mathbb{Z}[s, \theta] \setminus \{0\}. \tag{3.15}$$

---

[20] It follows from (3.12) that $\mathcal{M}' f^s = \bigoplus_{n \ge 0} \partial_t^n \mathcal{M}' \cong F_* \mathcal{M}$ as $\mathbb{C}[x]$-modules, since $\partial_t^n \mathcal{M}' = \partial_t^n t^n \mathcal{M}' \equiv s^n \mathcal{M}' \mod s^{<n} \mathcal{M}'$. Furthermore, the derivatives act on $F_* \mathcal{M}$ by $\partial_i \bullet m(s) = (\partial_i - (\partial_i f)\partial_t)m(s) = (\partial_i + s(\partial_i f)t^{-1})m(s) = (\partial_i + s(\partial_i f)/f)m(s-1)$ in accordance with (3.11).

Lemma 40 *Given* $f \in \mathbb{C}[x_1, \ldots, x_N]$, *the algebraic Mellin transform* $\mathfrak{M} :=$ $\mathscr{M} \otimes_{\mathbb{C}(s)[\theta]} \mathbb{C}(s, \theta)$ *of* $\mathscr{M} := \mathcal{D}_k^N \cdot f^s$ *is zero if, and only if,* f^s *is annihilated by a polynomial in the Euler operators* θ:

$$\text{Ann}_{A_k^N}(f^s) \cap \mathbb{C}[s, \theta] \neq \{0\}. \tag{3.16}$$

Proof Clearly, $\mathfrak{M} = \{0\}$ requires f^s to be mapped to zero in the localization $\mathfrak{M}$ of $\mathscr{M}$ at $\mathbb{C}[s, \theta] \setminus \{0\}$, and therefore the existence of a nonzero polynomial $P(\theta, s) \in \mathbb{C}[s, \theta]$ with $P(\theta, s) \cdot f^s = 0$. Conversely, given such an operator, its shifts $P(\theta - \alpha, s + r)$ by $(r, \alpha) \in \mathbb{Z}^{1+N}$ annihilate the elements $x^\alpha \cdot f^{s+r}$, which are therefore all mapped to zero in $\mathfrak{M}$. By linearity, this proves $\mathfrak{M} = \{0\}$, because every element of $\mathfrak{M}$ can be written as $gf^{s+r} \otimes h$ for some $r \in \mathbb{Z}$, $h \in \mathbb{C}[s, \theta] \setminus \{0\}$ and a Laurent polynomial $g \in k[x^{\pm 1}]$. $\qquad\square$

In particular, the presence of a linear annihilator (3.15) implies $\mathfrak{M} = \{0\}$ and hence $\chi(\mathbb{G}_m^N \setminus \mathbb{V}(f)) = 0$ via Corollary 37. Note that we could equally phrase this in terms of the hypersurface $\mathbb{V}(f) \subset \mathbb{G}_m^N$ itself as $\chi(\mathbb{V}(f)) = 0$, because the Euler characteristics are related through $\chi(\mathbb{V}(f)) = -\chi(\mathbb{G}_m^N \setminus \mathbb{V}(f))$ (see Sect. 3.3).

When $f = \mathcal{G}$ comes from Feynman graph G (as in the next section), it is not difficult to see that homogeneity (3.14) occurs precisely when G has a *tadpole*.[21] If this is the case, the integrals from Proposition 2 do not converge for any values of s and ν. In fact, $\mathfrak{M} = \{0\}$ dictates that the only value one can assign to $\mathcal{M}\{f^s\}(\nu)$ which is consistent with integration by parts relations is zero.

The purpose of this section is to show that the simple homogeneity condition (3.14) is not only sufficient for a vanishing Mellin transform, but it is also necessary:

Proposition 41 *Let* $f \in \mathbb{C}[x_1, \ldots, x_N]$ *denote a polynomial. Then, the hypersurface* $\{f = 0\}$ *inside the torus* $\mathbb{G}_m^N$ *has vanishing Euler characteristic precisely when there are* $\lambda_0, \ldots, \lambda_N \in \mathbb{Z}$, *not all zero, such that (3.14) holds.*

Geometrically, homogeneity (3.14) can be interpreted as follows: Dividing by the greatest common divisor, we may assume that $\lambda_0, \ldots, \lambda_N$ are relatively prime. Thus, we may extend $(\lambda_1, \ldots, \lambda_N)$ to a basis of the lattice $\mathbb{Z}^N$ and hence construct a matrix $A \in \mathsf{GL}_N(\mathbb{Z})$ with first row $A_{1i} = \lambda_i$. In the associated coordinates y, defined by

$$x_i = \prod_{j=1}^N y_j^{A_{ji}} \quad \text{and} \quad y_i = \prod_{j=1}^N x_j^{A_{ji}^{-1}} \quad \text{where} \quad A_{ji}^{-1} := \left(A^{-1}\right)_{ji},$$

the polynomial f takes the form $f(x) = y_1^{\lambda_0} g(\bar{y})$ for some Laurent polynomial $g \in \mathbb{C}[\bar{y}^{\pm 1}]$ in the remaining variables $\bar{y} = (y_2, \ldots, y_N)$. In particular, the hypersurface $\{f = 0\} = \{g = 0\}$ can be defined by an equation independent of the coordinate y_1.

---

[21] A tadpole here means a proper subgraph $\gamma \subsetneq G$ which shares only a single vertex with the rest of G and does not depend on masses or external momenta. In this case, $\mathcal{G}_G = \mathcal{U}_\gamma \mathcal{F}_{G/\gamma}$ factorizes such that the variables x_i with $i \in \gamma$ only appear in the homogeneous polynomial $\mathcal{U}_\gamma$ of degree $\lambda_0 := L_\gamma$. Thus, we obtain (3.14) by setting $\lambda_e = 1$ if $e \in \gamma$ and $\lambda_e = 0$ otherwise.

Corollary 42 *Let* $f \in \mathbb{C}[x_1, \ldots, x_N]$ *denote a polynomial. Then,* $\mathbb{V}(f) \subset \mathbb{G}_m^N$ *has Euler characteristic zero if and only if it is isomorphic to a product of* $\mathbb{G}_m$ *times a hypersurface* $\{g = 0\} \subset \mathbb{G}_m^{N-1}$.

To prove Proposition 41, we will look at the *Newton polytope* NP (f) of f, which is defined as the convex hull of the exponents of monomials that appear in f:

$$\text{NP}\left(\sum_{\alpha \in \mathbb{Z}^N} c_\alpha x^\alpha\right) := \text{conv}\left\{\alpha \in \mathbb{Z}^N : c_\alpha \neq 0\right\} \subset \mathbb{R}^N. \tag{3.17}$$

Since every monomial x^α is an eigenvector of operators (3.15), $P_s^\lambda(\theta) \cdot x^\alpha = P_s^\lambda(\alpha) x^\alpha$, it is annihilated by P_1^λ exactly when α belongs to $F_\lambda := \{P_1^\lambda(\alpha) = 0\}$, the hyperplane $F_\lambda = \{\alpha : \alpha_1 \lambda_1 + \cdots + \alpha_N \lambda_N = \lambda_0\}$. In particular, $0 = P_s^\lambda \cdot f^s = s f^{s-1} P_1^\lambda \cdot f$ is equivalent to the Newton polytope NP $(f) \subset F_\lambda$ being contained in that hyperplane. We can therefore reformulate the equivalent conditions (3.14) and (3.15) as

$$\dim \text{NP}(f) < N. \tag{3.18}$$

Such polytopes have zero N-dimensional volume, and we call them *degenerate*.

Proof of Proposition 41 We proceed by induction over the dimension N, and we will assume f to be non-constant. (The proposition holds trivially for any constant $f \in \mathbb{C}$.) In the case $N = 1$, the variety $\mathbb{V}(f) \subset \mathbb{C}^*$ is a finite set, and hence, its Euler characteristic coincides with its cardinality. Therefore, $\chi(\mathbb{V}(f)) = 0$ if and only if f has no zero inside the torus. This is only possible if f is proportional to a monomial x_1^r; in particular, f must be homogeneous and we are done.

Now consider $N > 1$ and assume that $\chi(\mathbb{G}_m^N \setminus \mathbb{V}(f)) = \chi(\mathbb{V}(f)) = 0$. Recall that (3.14) is equivalent to degeneracy (3.18) of NP (f), so we only need to rule out the non-degenerate case. We achieve this by exploiting the hypothesis $\dim \text{NP}(f) = N$ to construct a linear annihilator P_s^λ of f^s, which implies (3.18) in contradiction to the non-degeneracy of NP (f).

To start, we use Lemma 40 to find a polynomial $0 \neq P(\theta, s) \in \mathbb{C}[s, \theta]$ such that $P(\theta, s) \cdot f^s = 0$, and we choose one with minimal total degree in s and θ. Then pick an $(N-1)$ dimensional face $\sigma = \text{NP}(f) \cap F_\lambda$, which we can write as the intersection of NP (f) with a hyperplane F_λ for some integers $\lambda_0, \ldots, \lambda_N$ such that NP $(f) \subseteq \{\alpha : P_1^\lambda(\alpha) \leq 0\}$. Under rescaling (3.14), all monomials of $f = \sum_\alpha c_\alpha x^\alpha$ with $\alpha \in \sigma \subset F_\lambda$ acquire a factor of t^{λ_0}, while the remaining monomials with $\alpha \in \text{NP}(f) \setminus \sigma$ come with a smaller exponent $\sum_{i=1}^N \alpha_i \lambda_i < \lambda_0$ of t:

$$f(x_1 t^{\lambda_1}, \ldots, x_N t^{\lambda_N}) = t^{\lambda_0} f_\sigma(x)\left(1 + \mathcal{O}\left(t^{-1}\right)\right), \quad \text{where} \quad f_\sigma(x) := \sum_{\alpha \in \mathbb{Z}^N \cap \sigma} c_\alpha x^\alpha \tag{3.19}$$

and $\mathcal{O}\left(t^{-1}\right)$ denotes a rational function in $t^{-1} \mathbb{C}(x)[t^{-1}]$. Note that $P(\theta, s) \cdot f^s(\{x_i t^{\lambda_i}\}) = 0$ is still zero, because the rescaling of x commutes with the Euler

operators $\theta_i \cdot h(x_i t^{\lambda_i}) = (\theta_i \cdot h(x_i))|_{x_i \mapsto x_i t^{\lambda_i}}$. Therefore, applying $P(\theta, s)$ to the s-th power of the right-hand side of (3.19) and dividing by $t^{s\lambda_0}$ yields

$$0 = P(\theta, s) \cdot f_\sigma^s(x) \left(1 + \mathcal{O}\left(t^{-1}\right)\right)^s = P(\theta, s) \cdot f_\sigma^s + \mathcal{O}\left(t^{-1}\right) f_\sigma^s,$$

where $\mathcal{O}\left(t^{-1}\right)$ on the right-hand side denotes a formal series in $t^{-1}\mathbb{C}(x, s)[[t^{-1}]]$. In particular, the coefficient of t^0 must vanish, and we conclude that $P(\theta, s) \cdot f_\sigma^s = 0$. Label the variables such that $\lambda_N \neq 0$, then we can divide $P(\theta, s)$ by the linear form $P_s^\lambda(\theta, s)$ from (3.15), as a polynomial in θ_N, to obtain a decomposition

$$P(\theta, s) = P(\theta', 0, s) + P_s^\lambda(\theta, s) \cdot Q(\theta, s)$$

for some polynomial $Q(\theta, s) \in \mathbb{C}[\theta, s]$, such that the first summand depends only on $\theta' := (\theta_1, \ldots, \theta_{N-1})$ and s. Since $\mathrm{NP}(f_\sigma) = \sigma \subset F_\lambda$ is contained in the hyperplane $F_\lambda = \{\alpha : P_1^\lambda(\alpha) = 0\}$, we see $P_s^\lambda \cdot f_\sigma^s = f^{s-1} P_1^\lambda \cdot f_\sigma = 0$ and thus $Q(\theta, s)$ drops out in

$$0 = P(\theta, s) \cdot f_\sigma^s = P(\theta', 0, s) \cdot f_\sigma^s = P(\theta', 0, s) \cdot g^s,$$

where $g := f|_{x_N=1} \in \mathbb{C}[x_1, \ldots, x_{N-1}]$ is a polynomial in less than N variables. If $P(\theta', 0, s)$ were nonzero, Lemma 40 would show $\chi(\mathbb{G}_m^{N-1} \backslash \mathbb{V}(g)) = 0$, such that we could apply our induction hypothesis to g and conclude that g is homogeneous in our generalized sense. We saw that this is equivalent to the degeneracy of $\mathrm{NP}(g)$, which contradicts that $\mathrm{NP}(g) \cong \mathrm{NP}(f_\sigma) = \sigma$ is of dimension $N - 1$.[22]

Therefore, $P(\theta', 0, s)$ must be zero and we conclude that $P(\theta, s) = P_s^\lambda \cdot Q$ has a linear factor $P_s^\lambda(\theta, s)$.[23] Now set $m := Q(\theta, s) \cdot f^s$, which is nonzero, because $P(\theta, s)$ was chosen as an annihilator of f^s of minimal degree. We may write this element in the form $m = a \cdot f^{s+r}$ for some $r \in \mathbb{Z}$ and a Laurent polynomial $a \in \mathbb{C}(s)[x^\pm]$. After multiplying with a polynomial in $\mathbb{C}[s]$, we may even assume $0 \neq a \in \mathbb{C}[s, x^{\pm 1}]$ with $P_s^\lambda \cdot af^{s+r} = 0$. Applying the Leibniz rule and dividing by af^{s+r}, we find

$$0 = \frac{P_0^\lambda \cdot a}{a} - s\lambda_0 + (s + r)\frac{P_0^\lambda \cdot f}{f}.$$

Since the degree of $P_0^\lambda \cdot a$ in s is at most the degree (in s) of a itself, this first summand on the right has a finite limit as $s \to 0$. We therefore must have a cancellation of the terms linear in s, $P_0^\lambda \cdot f = \lambda_0 f$, which yields the sought-after $P_s^\lambda \cdot f^s = 0$. $\qquad\square$

Remark 43 In summary, we showed that the following six conditions on a polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ are equivalent: (1) homogeneity (3.14) of f, (2) existence (3.15) of an annihilator of f^s linear in θ, (3) existence (3.16) of an annihilator of f^s polynomial

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[22] Observe that $\mathrm{NP}(g)$ is the orthogonal projection of $\mathrm{NP}(f_\sigma)$ onto the coordinate hyperplane $\{\alpha_N = 0\}$. This projection restricts to an isomorphism between $\{\alpha_N = 0\}$ and F_λ (because $\lambda_N \neq 0$), and therefore $\dim \mathrm{NP}(g) = \dim \mathrm{NP}(f_\sigma)$.

[23] The existence of a linear annihilator could also be deduced from [31, Théorème 9.2].

in θ, (4) degeneracy (3.18) of the Newton polytope NP (f), (5) vanishing of the Euler characteristic $\chi(\mathbb{G}_m^N \backslash \mathbb{V}(f)) = \chi(\mathbb{V}(f)) = 0$ and (6) divisibility of $\mathbb{V}(f)$ by $\mathbb{G}_m$ as stated in Corollary 42.

To conclude, let us interpret our observation in the light of the well-known result, due to Kouchnirenko [49, Théorème IV] and Khovanskii [47, Theorem 2 in section 3], that relates the Euler characteristic to the volume of the Newton polytope:

Theorem 44 *For almost all polynomials $f \in \mathbb{C}[x_1, \ldots, x_N]$ with a fixed Newton polytope, we have $\chi(\mathbb{G}_m^N \backslash \mathbb{V}(f)) = (-1)^N \cdot N! \cdot \text{Vol NP}(f)$.*

For polynomials f whose nonzero coefficients are sufficiently generic, Proposition 41 follows from Theorem 44. Our proof shows that when $\text{Vol NP}(f) = 0$, the theorem applies without any constraints on the nonzero coefficients of f. In fact, this statement extends to the case when $N! \, \text{Vol NP}(f) = 1$, because it is known that $(-1)^N \cdot \chi(\mathbb{G}_m^N \backslash \mathbb{V}(f))$ is always bounded from above by $N! \, \text{Vol NP}(f)$, for all f, leaving only the possibilities $\{0, 1\}$ for the signed Euler characteristic $(-1)^N \cdot \chi(\mathbb{G}_m^N \backslash \mathbb{V}(f))$.

3.2 Graph polynomials

Our discussion so far applies to all integrals of type (2.1)—the defining data are thus the set $\mathsf{D} = (\mathsf{D}_1, \ldots, \mathsf{D}_N)$ of denominators, which is sometimes also called an *integral family* [98]. The denominators can be arbitrary quadratic forms in the loop momenta; decomposition (2.2) then defines the associated polynomials $\mathcal{U}$, $\mathcal{F}$ and $\mathcal{G}$ through (2.3). In particular, the denominators do *not* have to be related to the momentum flow through a (Feynman) graph in any way.

However, we will from now on consider the most common case in applications: integrals associated to a Feynman graph with Feynman propagators.

Definition 45 Given a connected Feynman graph G with N internal edges, $E + 1$ external legs and L loops, imposing momentum conservation at each vertex determines the momenta k_e flowing through each edge e in terms of the E external and L loop momenta.[24] The *Symanzik polynomials* $\mathcal{U}_G$ and $\mathcal{F}_G$ of the graph G are the polynomials $\mathcal{U}$ and $\mathcal{F}$ from (2.3) for the set $\mathsf{D} = (\mathsf{D}_1, \ldots, \mathsf{D}_N)$ of inverse Feynman propagators,[25]

$$\frac{1}{\mathsf{D}_e} = \frac{1}{-k_e^2 + m_e^2 - \mathrm{i}\epsilon} \quad (1 \le e \le N), \tag{3.20}$$

where m_e is the mass associated to the particle propagating along edge e. The number $\mathfrak{C}(G)$ of master integrals of the Feynman graph G is defined in terms of (3.2) as

$$\mathfrak{C}(G) := \mathfrak{C}(\mathcal{G}_G) \quad \text{where} \quad \mathcal{G}_G := \mathcal{U}_G + \mathcal{F}_G. \tag{3.21}$$

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[24] Momentum conservation implies that the sum $p_1 + \cdots + p_{E+1} = 0$ of the incoming momenta on all external legs vanishes; hence only E of them are independent.

[25] The infinitesimal imaginary part $\mathrm{i}\epsilon$ is irrelevant for our purpose of counting integrals and will be henceforth ignored.

This class of integrals (using only the propagators in the graph) is sometimes referred to as *scalar integrals* and might appear to be insufficient for applications, since in general one needs to augment the inverse propagators by additional denominators, called *irreducible scalar products* (ISPs), in order to be able to express arbitrary numerators of the momentum space integrand in terms of integrals (2.1); see Example 62. Therefore, one might expect that, in order to count all these integrals, one ought to replace $\mathcal{G}_G$ in (3.21) by the polynomial associated to the full set of denominators, including the ISPs.

However, it is well known since [89] that all such integrals with ISPs are in fact linear combinations of scalar integrals in higher dimensions $d + 2k$, for some $k \in \mathbb{N}$. Furthermore, those can be written as scalar integrals in the original dimension d by Corollary 27. Therefore, (3.21) is the correct definition to count the number of master integrals of (any integral family determined by) a Feynman graph.

We can therefore invoke the following, well-known combinatorial formulas for the Symanzik polynomials [14,81], which go back at least to [64].

Proposition 46 *The Symanzik polynomials of a graph G can be written as*

$$\mathcal{U}_G = \sum_T \prod_{e \notin T} x_e \quad \text{and} \quad \mathcal{F}_G = \mathcal{U}_G \sum_{e=1}^N x_e m_e^2 - \sum_F p_F^2 \prod_{e \notin T} x_e, \qquad (3.22)$$

where T runs over the spanning trees of G and F enumerates the spanning two-forests of G. (p_F denotes the sum of all external momenta flowing into one of the components of F.)

In Sect. 4.2, we will use these formulas to count the sunrise integrals.

Example 47 The graph polynomials of the bubble graph (Fig. 1) are, in general kinematics,

$$\mathcal{U} = x_1 + x_2 \quad \text{and} \quad \mathcal{F} = (x_1 + x_2)(x_1 m_1^2 + x_2 m_2^2) - p^2 x_1 x_2. \qquad (3.23)$$

It is important to keep in mind that, even with a fixed graph, the number of master integrals will vary depending on the kinematical configuration—e.g. whether a propagator is massive or massless, or whether an external momentum is non-exceptional or sits on a specific value (like zero or various *thresholds*). We will always explicitly state any assumptions on the kinematics, and hence stick with the simple notation (3.21).

3.3 The Grothendieck ring of varieties

Since we are from now on only interested in the Euler characteristic, we can simplify calculations by abstracting from the concrete variety $\mathbb{V}(\mathcal{G}) := \{\mathcal{G} = 0\}$ to its class $[\mathcal{G}]$ in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$. This ring is the free Abelian group generated by isomorphism classes $[X]$ of varieties over $\mathbb{C}$, modulo the inclusion–exclusion relation $[X] = [X \backslash Z] + [Z]$ for closed subvarieties $Z \subset X$. It is a unital ring for the product $[X] \cdot [Y] = [X \times Y]$ with unit $1 = [\mathbb{A}^0]$ given by the class of the point. Crucially,

the Euler characteristic factors through the Grothendieck ring, since it is compatible with these relations: $\chi(X) = \chi(X \backslash Z) + \chi(Z)$ and $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. The class $\mathbb{L} = [\mathbb{A}^1]$ of the affine line is called *Lefschetz motive* and fulfils $\chi(\mathbb{L}) = 1$. For several polynomials $P_1, \ldots, P_n$, we write $\mathbb{V}(P_1, \ldots, P_n) := \{P_1 = \cdots = P_n = 0\}$.

If a variety is described by polynomials that are linear in one of the variables, we can eliminate this variable to reduce the ambient dimension.[26] Let us state such a relation explicitly, since our setting is slightly different than usual: For us, the natural ambient space is the torus $\mathbb{G}_m^N$ and not the affine plane $\mathbb{A}^N$.

Lemma 48 *Let $A, B \in \mathbb{C}[x_1, \ldots, x_{N-1}]$ and consider the linear polynomial $A + x_N B$. Then*

$$[\mathbb{G}_m^N \backslash \mathbb{V}(A + x_N B)] = \mathbb{L} \cdot \left[\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A, B) \right]$$
$$- [\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A)] - [\mathbb{G}_m^{N-1} \backslash \mathbb{V}(B)] \qquad (3.24)$$

holds in the Grothendieck ring. In particular, the Euler characteristic is

$$\chi\left(\mathbb{G}_m^N \backslash \mathbb{V}(A + x_N B)\right) = -\chi\left(\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A \cdot B)\right). \qquad (3.25)$$

Proof Consider the hypersurface $\mathbb{V}(\mathcal{G}) \subset \mathbb{G}_m^N$, defined by $\mathcal{G} := A + x_N B$, under the projection $\pi \colon \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^{N-1}$ that forgets the last coordinate x_N.

As long as $AB \neq 0$, the unique solution of $\mathcal{G} = 0$ in the fibre is $x_N = -A/B$. If $A = 0$, the solution $x_N = 0$ is not in $\mathbb{G}_m$ and the fibre is empty, and it is also empty whenever $B = 0$. The only exception to this emptiness is over the intersection $A = B = 0$, where x_N is arbitrary and the fibre is the full $\mathbb{G}_m$. This fibration proves (3.24); equivalently, we can write it via $[\mathbb{V}(A \cdot B)] = [\mathbb{V}(A)] + [\mathbb{V}(B)] - [\mathbb{V}(A, B)]$ and $[\mathbb{G}_m] = \mathbb{L} - 1$ as

$$[\mathbb{G}_m^N \backslash \mathbb{V}(A + x_N B)] = [\mathbb{G}_m] \left([\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A)] + [\mathbb{G}_m^{N-1} \backslash \mathbb{V}(B)]\right)$$
$$- \mathbb{L} \cdot [\mathbb{G}_m^{N-1} \backslash \mathbb{V}(A \cdot B)].$$

Applying the Euler characteristic proves (3.25) due to $\chi(\mathbb{A}^1) = 1$ and $\chi(\mathbb{G}_m) = 0$. $\square$

Corollary 49 *Set $\widetilde{\mathcal{U}} := \mathcal{U}|_{x_N=1}$ and $\widetilde{\mathcal{F}} := \mathcal{F}|_{x_N=1}$. Then*

$$\chi\left(\mathbb{G}_m^N \mathbb{V}(\mathcal{G})\right) = \chi\left(\mathbb{G}_m^{N-1} \mathbb{V}(\widetilde{\mathcal{U}}, \widetilde{\mathcal{F}})\right) - \chi\left(\mathbb{G}_m^{N-1} \mathbb{V}(\widetilde{\mathcal{U}})\right) - \chi\left(\mathbb{G}_m^{N-1} \mathbb{V}(\widetilde{\mathcal{F}})\right)$$
$$(3.26)$$

$$= -\chi\left(\mathbb{G}_m^{N-1} \mathbb{V}(\widetilde{\mathcal{U}} \cdot \widetilde{\mathcal{F}})\right). \qquad (3.27)$$

Proof Recall that $\mathcal{U}$ and $\mathcal{F}$ are homogeneous of degrees L and $L + 1$, respectively (Corollary 63). Since multiplication with $x_N \in \mathbb{G}_m$ is invertible, we can rescale

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[26] Such *linear reductions* were first investigated by Stembridge [85] and have led, via the c_2-invariant [74] of Schnetz, to the discovery of graph hypersurfaces that are not of mixed Tate type [17].

Fig. 2 A 1-scale subgraph γ of G is replaced by a single edge in G'

all variables x_i with $i < N$ by x_N. This change of coordinates transforms $\mathcal{G}$ into $x_N^L(\widetilde{\mathcal{U}} + x_N\widetilde{\mathcal{F}})$. Since $x_N \neq 0$, this shows that $[\mathbb{G}_m^N \backslash \mathbb{V}(\mathcal{G})] = [\mathbb{G}_m^N \backslash \mathbb{V}(\widetilde{\mathcal{U}} + x_N\widetilde{\mathcal{F}})]$ such that the claim is just a special case of Lemma 48. $\square$

Example 50 Both graphs consisting of a pair of massless edges,

$$G_{\text{series}} = \text{—•—•—•—} \quad and \quad G_{\text{parallel}} = \text{—•⊂⊃•—}, \tag{3.28}$$

with the external momentum p such that $p^2 \neq 0$, have a single master integral $\mathfrak{C}(G) = 1$.

Proof According to (3.22), the graph polynomials of the graphs in (3.28) are $\mathcal{U}_{\text{series}} = 1$, $\mathcal{F}_{\text{series}} = -p^2(x_1 + x_2)$, $\mathcal{U}_{\text{parallel}} = x_1 + x_2$, $\mathcal{F}_{\text{parallel}} = -p^2 x_1 x_2$ such that

$$\mathcal{G}_{\text{series}} = 1 - p^2(x_1 + x_2) \quad \text{and} \quad \mathcal{G}_{\text{parallel}} = x_1 + x_2 - p^2 x_1 x_2.$$

In both cases, the number of master integrals (3.10) is $\mathfrak{C}(G) = -\chi(\mathbb{G}_m \backslash \mathbb{V}(\widetilde{\mathcal{U}}\widetilde{\mathcal{F}})) = \chi(\mathbb{G}_m \cap \mathbb{V}(\widetilde{\mathcal{U}}\widetilde{\mathcal{F}})) = \chi(\mathbb{G}_m \cap \mathbb{V}(1 + x_1)) = \chi(\{-1\}) = 1$ according to (3.27). $\square$

Much more on the Grothendieck ring calculus of graph hypersurfaces $\mathbb{V}(\mathcal{U})$ can be found, for example, in [1,17]. These techniques can be used to prove some general statements about the counts of master integrals. Let us give just one example:

Lemma 51 *Let G be a Feynman graph with a subgraph γ such that all propagators in γ are massless and γ has only two vertices which are connected to external legs or edges in $G\backslash\gamma$.*[27] *Write G' for the graph obtained from G by replacing γ with a single edge (see Fig. 2), then*

$$\mathfrak{C}(G) = \mathfrak{C}(\gamma) \cdot \mathfrak{C}(G'). \tag{3.29}$$

Proof Every spanning tree T of G restricts on γ either to a spanning tree or to a spanning two-forest. In the first case, $T\backslash\gamma$ is a spanning tree of G/γ (the graph where γ is contracted to a single vertex); in the second case, $T\backslash\gamma$ is a spanning tree of $G\backslash\gamma$. Note that the two-forests $T \cap \gamma$ in the second case determine $\mathcal{F}_\gamma$ from (3.22), since all propagators in γ are massless. Therefore, we find $\mathcal{U}_G = \mathcal{U}_\gamma \cdot \mathcal{U}_{G/\gamma} + \mathcal{F}'_\gamma \cdot \mathcal{U}_{G\backslash\gamma}$

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[27] Such a graph γ is called *massless propagator* or *p-integral*.

where we set $\mathcal{F}'_\gamma := \mathcal{F}_\gamma\big|_{p^2=-1}$. Going through the same considerations for $\mathcal{F}_G$ shows that[28]

$$\mathcal{G}_G = \mathcal{U}_\gamma \cdot \mathcal{G}_{G/\gamma} + \mathcal{F}'_\gamma \cdot \mathcal{G}_{G\setminus\gamma}.$$

Now label the edges in γ as $1, \ldots, N_\gamma$ and rescale all Schwinger parameters x_e with $2 \leq e \leq N_\gamma$ by x_1. Due to the homogeneity of $\mathcal{U}_\gamma$ and $\mathcal{F}_\gamma$ from Corollary 63, we see that $[\mathbb{G}_m^N \setminus \mathbb{V}(\mathcal{G}_G)] = [\mathbb{G}_m^N \setminus \mathbb{V}(A + x_1 B)]$ where $A = \widetilde{U}_\gamma \mathcal{G}_{G/\gamma}$ and $B = \widetilde{\mathcal{F}}'_\gamma \mathcal{G}_{G\setminus\gamma}$ in terms of $\widetilde{U}_\gamma := \mathcal{U}_\gamma\big|_{x_1=1}$ and $\widetilde{\mathcal{F}}'_\gamma := \mathcal{F}'_\gamma\big|_{x_1=1}$. Applying (3.25), we obtain a separation of variables:

$$\begin{aligned}
\chi\left(\mathbb{G}_m^N \setminus \mathbb{V}(\mathcal{G}_G)\right) &= -\chi\left(\mathbb{G}_m^{N-1} \setminus \mathbb{V}(\widetilde{U}_\gamma \widetilde{\mathcal{F}}'_\gamma \mathcal{G}_{G/\gamma} \mathcal{G}_{G\setminus\gamma})\right) \\
&= -\chi\left(\mathbb{G}_m^{N_\gamma-1} \setminus \mathbb{V}(\widetilde{U}_\gamma \widetilde{\mathcal{F}}'_\gamma)\right) \cdot \chi\left(\mathbb{G}_m^{N-N_\gamma} \setminus \mathbb{V}(\mathcal{G}_{G/\gamma} \mathcal{G}_{G\setminus\gamma})\right) \\
&= -\chi\left(\mathbb{G}_m^{N_\gamma} \setminus \mathbb{V}(\mathcal{G}_\gamma)\right) \cdot \chi\left(\mathbb{G}_m^{N-N_\gamma+1} \setminus \mathbb{V}(\mathcal{G}_{G'})\right).
\end{aligned}$$

In the last line we used (3.27), upon noting the contraction-deletion formula $\mathcal{G}_{G'} = \mathcal{G}_{G/\gamma} + x_0 \mathcal{G}_{G\setminus\gamma}$ in terms of the additional Schwinger parameter x_0 for the (massless) edge that replaces γ in G'. (This formula is easily checked by considering which spanning trees and forests contain this edge or not.) Note that for the subgraph γ, the value of p^2 does not matter for $\mathbb{V}(\widetilde{\mathcal{F}}_\gamma) = \mathbb{V}(\widetilde{\mathcal{F}}'_\gamma)$, as long as $p^2 \neq 0$. Finally, recall (3.10). □

Of course, this result is well known on the function level: If G has a 1-scale subgraph γ, then the Feynman integral of G factorizes into the product

$$\mathcal{I}_G(\nu) = \mathcal{I}_\gamma(\nu_\gamma)\big|_{p^2=-1} \cdot \mathcal{I}_{G'}(\omega_\gamma, \nu') \tag{3.30}$$

of the integrals of γ and G'. Here, we denote by ν_γ and ν' the indices corresponding to the edges in γ and outside γ, respectively, such that $\nu = (\nu_\gamma, \nu')$. Note that the edge replacing γ in G' (see Fig. 2) gets the index $\omega_\gamma = \sum_{e \in \gamma} \nu_e - L_\gamma \cdot (d/2)$ from (2.4), which depends on the indices of γ (L_γ denotes the loop number of γ).

Corollary 52 *Let G be a graph with a pair $\{e, f\}$ of massless edges in series or in parallel. Then, $\mathfrak{C}(G) = \mathfrak{C}(G')$ where G' is the graph obtained by replacing the pair with a single edge. In other words, repeated application of the series-parallel operations from Fig. 3 does not change the number of master integrals.*

Proof Combine Lemma 51 with Example 50. □

---

[28] Note that the intersection of a two-forest F of G with γ has at most two trees connected to external vertices of γ. Any further tree in $F \cap \gamma$ is thus necessarily a full component of F and forces its contribution to $\mathcal{F}_G$ to vanish by $p_F^2 = 0$ (due to the absence of external legs).
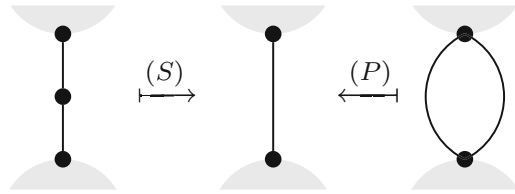
4 Tools and examples

The Euler characteristic of a singular hypersurface can be computed algorithmically via several methods.[29] In this section, we demonstrate how some of these techniques can be used to compute the number of master integrals in various examples.

We begin with methods based on fibrations. In particular, the Euler characteristic can be computed very easily for the class of *linearly reducible* graphs, see Sect. 4.1. However, the decomposition of the Euler characteristic of the total space E of a fibration $E \longrightarrow B$ with fibre F into the product

$$\chi(E) = \chi(B) \cdot \chi(F) \tag{4.1}$$

is true in general and not restricted to the linear case. In Sect. 4.2, we use a quadratic fibration in order to count the master integrals of all sunrise graphs.

Apart from these geometric approaches, which seem to work very well for Feynman graphs, there are general algorithms for the computation of de Rham cohomology and the Euler characteristic of hypersurfaces. In Sect. 4.3 we discuss some available implementations of these algorithms in computer algebra systems.

In final Sect. 4.4, we comment on the relation of our result to other approaches in the physics literature.

4.1 Linearly reducible graphs

If the polynomial $\mathbb{V}(f) = a + x_N b$ is linear in a variable x_N, we saw in Lemma 48 that we can easily eliminate this variable x_N in the computation of the Euler characteristic (or the class in the Grothendieck ring) of the hypersurface $\mathbb{V}(f)$ (or its complement). Analogous formulas also exist in the case of a variety $\mathbb{V}(f_1, \ldots, f_n)$ of higher codimension, given that all of the defining polynomials $f_i = a_i + x_N b_i$ are linear in x_N. Such *linear reductions* have been used heavily in the study of graph hypersurfaces and are straightforward to implement on a computer [74,85].

If such linear reductions can be applied repeatedly until all Schwinger parameters have been eliminated, the graph is called *linearly reducible* [15]. Linear reducibility is particularly common among graphs with massless propagators; we give some examples in Fig. 4.

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[29] We will not discuss Kouchnirenko's Theorem 44 here, because in most examples we found that it does not apply. It seems that coefficients of graph polynomials are often not sufficiently generic.
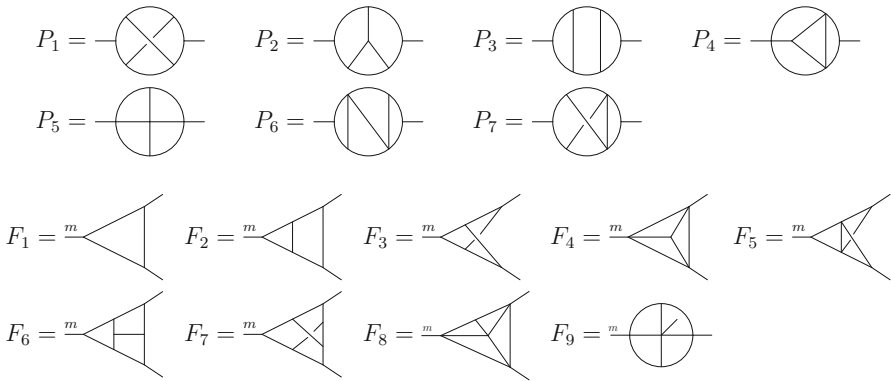
Fig. 4 Some linearly reducible propagators (P_i) and form factors (F_i). All internal edges are massless, and the form factors have two massless external legs ($p_1^2 = p_2^2 = 0$) and one massive leg $p_3^2 \neq 0$, indicated by the label m

Table 1 The number $\mathfrak{C}(G)$ of master integrals, computed as the Euler characteristic 3.10, for the graphs in Fig. 4

G	P_1	P_2	P_3	P_4	P_5	P_6	P_7	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
$\mathfrak{C}(G)$	16	10	10	10	10	15	22	1	4	5	4	5	20	24	12	13

At the end of a full linear reduction, the class of $\mathbb{G}_m^N \setminus \mathbb{V}(\mathcal{G})$ in the Grothendieck ring is expressed as a polynomial in the Lefschetz motive $\mathbb{L}$. To get the Euler characteristic, one then merely needs to substitute $\mathbb{L} = [\mathbb{A}^1] \mapsto \chi(\mathbb{A}^1) = 1$.

Example 53 Consider the 2-loop propagator WS_3' from Fig. 5. Linear reductions give

$$[\mathbb{G}_m^5 \setminus \mathbb{V}(\mathcal{G}_{\mathrm{WS}_3'})] = -\mathbb{L}^4 + 5\mathbb{L}^3 - 13\mathbb{L}^2 + 21\mathbb{L} - 15 \tag{4.2}$$

in the Grothendieck ring. Substituting $\mathbb{L} \mapsto 1$ shows that $\mathfrak{C}(\mathrm{WS}_3') = 3$ via (3.10).

In this way, we calculated the Euler characteristics for the graphs in Fig. 4, using an implementation of the linear reductions similar to the method of Stembridge [85]. Our results are listed in Table 1.

Beyond the computation of such results for individual graphs, it is possible to obtain results for some infinite families of linearly reducible graphs. In particular, efficient computations are possible for graphs of *vertex width* three [16]. For example, the class in the Grothendieck ring of $\mathbb{V}(\mathcal{U})$ was computed for all wheel graphs in [17]. It is possible to adapt such calculations to our setting (where the ambient space is $\mathbb{G}_m^N$ instead of $\mathbb{A}^N$). For example, we could prove

Proposition 54 *The number of master integrals of the massless propagators obtained by cutting a wheel* WS_L *with* L *loops, either at a rim or a spoke (see Fig. 5), is*

$$\mathfrak{C}(\mathrm{WS}_L') = \mathfrak{C}(\mathrm{WS}_L'') = \frac{L(L-1)}{2}. \tag{4.3}$$
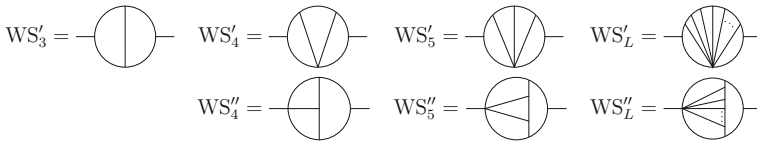
Fig. 5 The propagator graphs WS'_L (WS''_L) with $L-1$ loops are obtained from cutting a rim (spoke) of the wheel WS_L with L loops
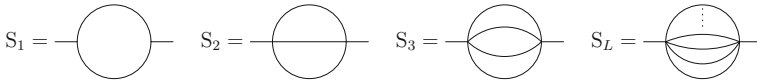

Fig. 6 The sunrise graphs S_L with L loops

The proof of this and related results will be presented elsewhere.

4.2 Sunrise graphs

In [43], the number of master integrals was computed for all sunrise integrals; based on a Mellin–Barnes representation and the *differential reduction* [19,42] of an explicit solution in terms of Lauricella hypergeometric functions. To our knowledge, this has hitherto been the only non-trivial[30] infinite family of Feynman integrals with explicitly known master integral counts. Their first result can be phrased as[31]

Proposition 55 *The L-loop sunrise graph* S_L *from Fig.* 6 *with* $L+1$ *nonzero masses (and non-exceptional external momentum) has* $\mathfrak{C}(S_L) = 2^{L+1} - 1$ *master integrals.*

We will now demonstrate that this result can be obtained from a straightforward computation of the Euler characteristic, according to Corollary 37.

Proof The graph polynomials (3.22) for the sunrise graph are

$$\mathcal{U} = \sum_{i=1}^{L+1} \prod_{j \neq i} x_j = \left(\prod_{i=1}^{L+1} x_i \right) \left(\sum_{i=1}^{L+1} \frac{1}{x_i} \right) \quad \text{and} \quad \mathcal{F} = (-p)^2 \prod_{i=1}^{L+1} x_i + \mathcal{U} \sum_{i=1}^{L+1} x_i m_i^2. \tag{4.4}$$

We note that for the first term in (3.26), we find that $\mathcal{U} = \mathcal{F} = 0$ imply $\prod_i x_i = 0$, which has no solutions in the torus—hence, this term contributes $\chi(\mathbb{G}_m^L) = 0$. We thus obtain

$$(-1)^L \mathfrak{C}(S_L) = \chi \left(\mathbb{G}_m^L \backslash \mathbb{V} \left(1 + \sum_{i=1}^{L} x_i^{-1} \right) \right) + \chi \left(\mathbb{G}_m^L \backslash X_{p^2}^L \right), \tag{4.5}$$

---

[30] We consider families that arise simply by duplication of massless propagators, like those shown in [19, Figure 2], as trivial (due to Corollary 52).

[31] Beware that the number $2^{L+1} - L - 2$ given in [43, equation (4.5)] counts only *irreducible* master integrals, which means that it discards the $L+1$ integrals associated to the subtopologies obtained by contracting any of the edges. Our conventions, however, do take these integrals into account.

where we introduced the notation

$$X^L_{p^2} := \mathbb{V}\left(-p^2 + \left[m^2_{L+1} + \sum_{i=1}^{L} m^2_i x_i\right] \cdot \left[1 + \sum_{i=1}^{L} x_i^{-1}\right]\right) \subset \mathbb{G}^L_m. \quad (4.6)$$

The first Euler characteristic in (4.5) is readily evaluated to $(-1)^L$ by applying (3.25) repeatedly (being on the torus, we may replace x_i^{-1} by x_i), so we conclude that

$$\mathfrak{C}(S_L) = 1 + (-1)^L \cdot \chi\left(\mathbb{G}^L_m \backslash X^L_{p^2}\right). \quad (4.7)$$

Now let us consider the projection $\pi : \mathbb{G}^L_m \longrightarrow \mathbb{G}^{L-1}_m$ that forgets x_L. Set $y := m^2_{L+1} + \sum_{i<L} m^2_i x_i$ and $z := 1 + \sum_{i<L} x_i^{-1}$, such that $X^L_{p^2} = \{x_L p^2 = (1 + z x_L)(y + m^2_L x_L)\} \subset \mathbb{G}^L_m$. We note that the discriminant D of this quadric in x_L factorizes into

$$D = (m^2_L - p^2 + yz)^2 - 4m^2_L yz = \left(yz - [p + m_L]^2\right) \cdot \left(yz - [p - m_L]^2\right), \quad (4.8)$$

such that $\mathbb{G}^{L-1}_m \subset \mathbb{V}(D) = X^{L-1}_{(p+m_L)^2} \dot{\cup} X^{L-1}_{(p-m_L)^2}$ is the disjoint union of two hypersurfaces.[32] Since the factors are related to the $(L-1)$-loop sunrise by (4.7), we find

$$\chi(\mathbb{V}(D)) = 2 \cdot (-1)^L \cdot (\mathfrak{C}(S_{L-1}) - 1). \quad (4.9)$$

Over a point $x' \in X^{L-1}_{p \pm m_L} \subset \mathbb{V}(D)$ in the discriminant, the fibre of $\pi^{-1}(x')$ has precisely one solution (x', x_L) in $X^L_{p^2}$, determined by $x_L = -y/[m_L(m_L \pm p)]$:

$$\left[(\mathbb{G}^L_m \backslash X^L_{p^2}) \cap \mathbb{V}(D)\right] = ([\mathbb{G}_m] - 1) \cdot [\mathbb{V}(D)]. \quad (4.10)$$

If $D(x') \neq 0$ is nonzero and also $yz \neq 0$, then the fibre $\pi^{-1}(x')$ has precisely two distinct solutions x_L in the quadric $X^L_{p^2}$. Hence, $\chi(\pi^{-1}(x')) = \chi(\mathbb{G}_m) - 2 = -2$ and thus

$$\chi\left((\mathbb{G}^L_m \backslash X^L_{p^2})\mathbb{V}(yzD)\right) = -2\chi\left(\mathbb{G}^{L-1}_m\mathbb{V}(yzD)\right) = 2\chi(\mathbb{V}(D))$$

$$+2\chi(\mathbb{V}(yz)), \quad (4.11)$$

where used that $\mathbb{V}(D) \cap \mathbb{V}(yz) = \emptyset$ for non-exceptional values of p^2, such that $(p \pm m_L)^2 \neq 0$ in (4.8). The reason that we need to exclude the case when $yz = 0$ in (4.11) is that for $y = 0$, one of the solutions of $X^L_{p^2} = \{x_L p^2 = (1 + z x_L)m^2_L x_L\}$ is $x_L = 0 \notin \mathbb{G}_m$; whereas for $z = 0$ the equation for $X^L_{p^2} = \{x_L p^2 = y + m^2_L x_L\}$

---

[32] We assume $p^2 \neq 0$ and $m^2_L \neq 0$, which guarantees that $(p + m_L)^2 \neq (p - m_L)^2$.

becomes linear. In both cases, there is only one solution in the fibre, and there is none
if both $y = z = 0$ vanish. (We assume $p^2 \neq m_L^2$):

$$[(\mathbb{G}_m^L \backslash X_{p^2}^L) \cap \mathbb{V}(yz)] = ([\mathbb{G}_m] - 2) \cdot [\mathbb{V}(yz)] + [\mathbb{V}(y)] + [\mathbb{V}(z)]. \quad (4.12)$$

We can now combine (4.10)–(4.12) via $[Y] = [Y \cap \mathbb{V}(D)] + [Y \cap \mathbb{V}(yz)] + [Y \backslash \mathbb{V}(D \cdot yz)]$ for $Y = \mathbb{G}_m^L \backslash X_{p^2}^L$ into the reduction formula

$$\chi(\mathbb{G}_m^L \backslash X_{p^2}^L) = \chi(\mathbb{V}(D)) + \chi(\mathbb{V}(y)) + \chi(\mathbb{V}(z)) = 2 \cdot (-1)^L \cdot \mathfrak{C}(S_{L-1}). \quad (4.13)$$

Here, we inserted (4.9) and used $\chi(\mathbb{V}(y)) = \chi(\mathbb{V}(z)) = -\chi(\mathbb{G}_m^{L-1} \backslash \mathbb{V}(z)) = (-1)^L$,
which follows from repeated application of (3.25)—just as above, when we computed
the first term in (4.5). According to (4.7), we can write the reduction as the recursion

$$\mathfrak{C}(S_L) = 2\mathfrak{C}(S_{L-1}) + 1,$$

which is obviously solved by the claimed $\mathfrak{C}(S_L) = 2^{L+1} - 1$. It merely remains to
verify the base case $L = 1$, and indeed, $\mathfrak{C}(S_1) = 1 + \chi(X_{p^2}^1) = 3$ follows easily from
(4.7) since $X_{p^2}^1 = \{x_1 p^2 = (1 + x_1)(m_2^2 + m_1^2 x_1)\} \subset \mathbb{G}_m$ consists of precisely two
points.   □

It should be clear that our calculation can be adapted to the situation when some masses
are zero. Let us demonstrate how to obtain another result of [43][33]:

Proposition 56 *The L-loop sunrise graph with $R \leq L$ nonzero masses, $L + 1 - R$
vanishing masses and non-exceptional external momentum, has $\mathfrak{C}(S_L) = 2^R$ master
integrals.*

Proof By Corollary 52, we may replace all massless edges by a single (massless) edge
without changing the number of master integrals; hence, we can assume $L = R \geq 1$.
(The totally massless case $R = 0$ reduces to the trivial case of a single edge.) Label
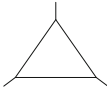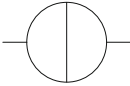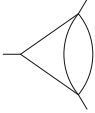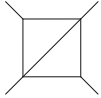the edges such that the massless edge is $m_{L+1} = 0$.

We can apply the exact same recursion as in the proof of Proposition 55; the only
difference to (4.13) is that now, $\chi(\mathbb{V}(y)) = 0$ vanishes because $y = \sum_{i<L} m_i^2 x_i$ has
become homogeneous in x such that $[\mathbb{V}(y)] = [\mathbb{G}_m] \cdot [\mathbb{V}(y) \cap \{x_1 = 1\}]$. Therefore,
(4.13) takes the form $\chi(\mathbb{G}_m^L \backslash X_{p^2}^L) = (-1)^L \cdot (2\mathfrak{C}(S_{L-1}) - 1)$ and yields, via (4.7),
the recursion

$$\mathfrak{C}(S_L) = 2\mathfrak{C}(S_{L-1}).$$

We are done after verifying the base case: Indeed, $\mathfrak{C}(S_1) = 1 + \chi(X_{p^2}^1) = 2$ from
(4.7) is clear since $X_{p^2}^1 = \{p^2 x_1 = (1 + x_1)m_1^2 x_1\}$ is the single point $x_1 = p^2/m_1^2 - 1$
in $\mathbb{G}_m$.   □

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[33] The additional term $-\delta_{0,L-R}$ in [43, Equation (4.13)] subtracts a *reducible* integral that can be attributed
to a subtopology. Our counting, however, accounts for all master integrals.

Table 2 Counts of master integrals according to (3.10) computed with *Macaulay2*'s Euler for some graphs for massless and massive internal propagators. (As no symmetries are regarded, all masses can be assumed to be different from each other.) All external momenta are assumed to be non-degenerate (nonzero and not on any internal mass shell) in both cases

Graph G					
$\mathfrak{C}(G)$ massless	4	11	3	4	20
$\mathfrak{C}(G)$ massive	7	15	30	19	55

4.3 General algorithms

The computer algebra system *Macaulay2* [34] provides the function Euler in the package CharacteristicClasses. It implements the algorithm of [38] for the computation of the Euler characteristic. This program requires projective varieties as input, so we need to homogenize $\mathcal{G}$ to $\tilde{\mathcal{G}} = x_0\mathcal{U} + \mathcal{F}$, and can then use one of

$$[\mathbb{A}^N \setminus \mathbb{V}(x_1 \cdots x_N \mathcal{G})] = [\mathbb{P}^N \setminus \mathbb{V}(x_1 \cdots x_N \tilde{\mathcal{G}})] - [\mathbb{P}^{N-1} \setminus \mathbb{V}(x_1 \cdots x_N \mathcal{F})] \quad (4.14)$$

$$= [\mathbb{P}^N \setminus \mathbb{V}(x_0 x_1 \cdots x_N \tilde{\mathcal{G}})] \quad (4.15)$$

to express the sought-after number of master integrals as the Euler characteristic of a projective hypersurface complement. We found that this algorithm performs well for small numbers of variables (edges): The examples in Table 2 require not more than a couple of minutes of runtime. For more variables, however, the computations tend to rapidly become much more time consuming and often impracticable. Apart from the results in Table 2, we also verified Proposition 55 for the sunrise graphs S_L using Euler for up to six loops.

Example 57 Consider the one-loop sunrise graph S_1 with $m_1^2 = m_2^2 = -p^2 = 1$, which is a non-degenerate kinematic configuration. According to Example 47, its Lee–Pomeransky polynomial is $\mathcal{G} = (x_1 + x_2)(x_1 + x_2 + 1) + x_1 x_2$. The *Macaulay2* script

```
load "CharacteristicClasses.m2"
R=QQ[x0,x1,x2]
I=ideal(x0*x1*x2*((x1+x2)*x0+(x1+x2)^2+x1*x2))
Euler(I)
```

computes the output 0 for $\chi(\mathbb{V}(x_0 x_1 x_2 \tilde{\mathcal{G}}) \cap \mathbb{P}^2)$. Using $\chi(\mathbb{P}^2) = 3$ and (4.15), we conclude $\mathfrak{C}(S_1) = 3 - 0 = 3$ in agreement with Proposition 55.

Recall that the number of master integrals depends on the kinematical configuration; in Table 2 we give the results both for massless and for massive internal propagators. In particular, note how the massless 2-loop propagator WS_3' from Example 53 with only $\mathfrak{C}(WS_3') = 3$ master integrals grows to carry $\mathfrak{C}(WS_3') = 30$ master integrals in the fully massive case.

Furthermore, *Macaulay2* also provides an implementation (the command `deRham`) of algorithm [66] of Oaku and Takayama for the computation of the individual de Rham cohomology groups. This uses *D*-modules and Gröbner bases and tends to demand more resources than the method discussed above.

Example 58 Consider again the massive one-loop sunrise from Example 47. The program

```
load "Dmodules.m2"
R=QQ[x1,x2]
f=x1*x2*(x1+x2+(x1+x2)^2+x1*x2)
deRham f
```

computes the following cohomology groups of $X = \mathbb{G}_m^2 \setminus \mathbb{V}(\mathcal{G})$: $H^0(X) \cong \mathbb{Q}$, $H^1(X) \cong \mathbb{Q}^3$ and $H^2(X) \cong \mathbb{Q}^5$. Hence, $\mathfrak{C}(S_1) = \chi(X) = 5 - 3 + 1 = 3$ as in Example 57.

The same functionality is provided by SINGULAR's `deRham.lib` library via the command `deRhamCohomology`. The SINGULAR analogue of Example 58 is

```
LIB "deRham.lib";
ring R = 0,(x1,x2),dp;
list L = (x1*x2*(x1+x2+(x1+x2)^2+x1*x2));
deRhamCohomology(L);
```

4.4 Comparison to other approaches

We successfully reproduced all of our results above (the wheels WS'_L with $L \leq 6$, the sunrises S_L with $L \leq 4$ loops and the graphs from Fig. 4 and Table 2) with the program AZURITE [32], which provides an implementation of Laporta's approach [50]. While it employs novel techniques to boost performance, in the end it solves linear systems of equations between integrals obtained from annihilators of the integrand of Baikov's representation (2.48) in order to count the number of master integrals.

The observed agreement with our results is to be expected, since the identification of integral relations with parametric annihilators that we elaborated on in Sect. 2.3 works equally for the Baikov representation, which can also be interpreted as a Mellin transform. Note, however, that we must use the options `Symmetry -> False` and `GlobalSymmetry -> False` for AZURITE in order to switch off the identification of integrals that differ by a permutation of the edges. The reason being that, in our approach, all edges e carry their own index ν_e and no relation between these indices for different edges is assumed.

Unfortunately, due to the way AZURITE treats subsectors, this can occasionally lead to an apparent mismatch. However, this is rather a technical nuisance than an actual disagreement.

Example 59 For the graph G in Fig. 7, the Euler characteristic gives $\mathfrak{C}(G) = 15$, whereas both `Reduze` [98] and AZURITE produce 16 master integrals. The problem arises from the *subsector* where the edges 1 and 2 are contracted: As shown in Fig. 7, it does have a remaining external momentum p_4, such that the momenta running through edges 3 and 4 are different—however, since $p_4^2 = 0$, the graph polynomials (and hence the Feynman integrals) are identical to those of the vacuum graph G' in Fig. 7. Since
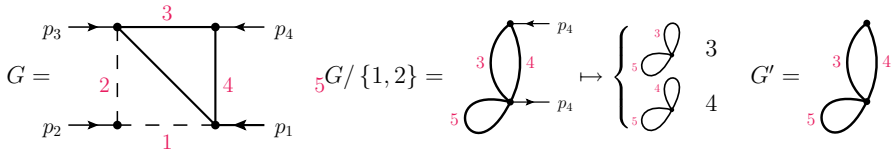
Fig. 7 A graph G with a massive loop (edges 3, 4 and 5), two massless propagators $\{1, 2\}$ and four external legs ($p_1^2 \neq 0$ massive and $p_2^2 = p_3^2 = p_4^2 = 0$ massless). Since the only momentum p_4 running through the graph after contracting 1 and 2 is null ($p_4^2 = 0$), the associated integral is the same as for G'

edges 3 and 4 in G' have the same mass, they can be combined and thus G' clearly has only a single master integral: the product of two tadpoles.

But AZURITE and Reduze instead consider the subsectors of $G/\{1, 2\}$ obtained by contracting a further edge (3 or 4), and obtain the two tadpoles (see Fig. 7) consisting only of edges $\{4, 5\}$ and $\{3, 5\}$, respectively, as master integrals. Of course, these would be recognized as identical if symmetries were allowed, but the point is that even without using symmetries, there is only a single master integral for G' (as computed by the Euler characteristic).

Our results are also consistent with the conclusions obtained within the differential reduction approach [42]; indeed, we demonstrated in Sect. 4.2 how the master integral counts of [43] for the sunrise graphs emerge directly from the computation of the Euler characteristic. Let us point out again, however, that some care is required for these comparisons, since those works refer to *irreducible* master integrals, which excludes integrals that can be expressed with gamma functions. In particular, the fact that the two-loop sunrise S_2 with one massless line has $\mathfrak{C}(S_2) = 4$ master integrals (see Proposition 56) is consistent with [41]. We are counting all master integrals and are not concerned here with the much more subtle question addressed by the observation that two of these integrals may be expressed with gamma functions.

Finally, let us note that also the work of Lee and Pomeransky [58] addresses a different problem: Considering only integer indices $\nu \in \mathbb{Z}^N$, how many *top-level* master integrals are there for a graph G? This means that integrals obtained from subsectors (graphs G/e with at least one edge e contracted) are discarded. Geometrically, the number of the remaining master integrals is identified with the dimension of the cohomology group $H^N(\mathbb{C}^N \setminus \mathbb{V}(\mathcal{G}))$.[34] In most cases, the program Mint computes this number correctly, which then agrees with the other mentioned methods.[35] We refer to [13, section 4] and [43, section 6] for detailed discussions of this comparison. Note that the dimension (and a basis) of the top cohomology group can also be computed with the command deRhamCohom from the SINGULAR library dmodapp.lib.

The concept of *top-level* integrals does not literally make sense in our setting of arbitrary, non-integer indices ν. Here, there is no relation at all between integrals of a quotient graph G/e and integrals of G. (The former do not depend on ν_e at all; the

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[34] Actually, they initially refer to a different, relative cohomology group; but in the description of their implementation in Mint they seem to work with this total cohomology group.

[35] Occasional mismatches are known, like for the graph F_9 from Fig. 4 that was addressed in [13, section 4.1]. These discrepancies are due to an error in the implementation of Mint that misses contributions from critical points at infinity. (We thank Yang Zhang for bringing this to our attention.)

latter do.) However, discarding integrals from quotient graphs suggests a definition of a number as follows.

Remark 60 Using the inclusion–exclusion principle, one might be tempted to define

$$\widehat{\mathfrak{C}}(G) := \sum_{\gamma \subseteq G} (-1)^{|\gamma|} \mathfrak{C}(G/\gamma) = \mathfrak{C}(G) - \sum_e \mathfrak{C}(G/e) + \sum_{e<f} \mathfrak{C}(G/\{e, f\}) - \cdots \tag{4.16}$$

as the number of *top-level* master integrals, since it subtracts from all master integrals $\mathfrak{C}(G)$ the integrals associated to subsectors (and corrects for double counting). Note that if γ contains a loop, the corresponding term in the sum should be set to zero. (We only consider contractions with the same loop number as G.) The reverse relation,

$$\mathfrak{C}(G) = \sum_{\gamma \subseteq G} \widehat{\mathfrak{C}}(G/\gamma), \tag{4.17}$$

is consistent with the intuition that the total set of master integrals is obtained as the union of all top-level masters. By $\mathcal{G}_{G/\gamma} = \mathcal{G}|_{x_e=0 \forall e \in \gamma}$, we find that

$$\widehat{\mathfrak{C}}(G) = (-1)^N \chi \left(\mathbb{A}^N \backslash \mathbb{V}(\mathcal{G}) \right) \tag{4.18}$$

is the Euler characteristic of the hypersurface complement inside affine space (as compared to the torus $\mathbb{G}_m^N$ as ambient space). We find that this number behaves exactly as expected and is consistent with our calculations in Sect. 4. However, we point out that this number can take negative values. For example, $\widehat{\mathfrak{C}}(G/\{1, 2\}) = -1$ is negative for the graph from Fig. 7. In fact, this is necessary for the consistency of sum (4.17) of all master integrals:

$$1 = \mathfrak{C}(G/\{1, 2\}) = \widehat{\mathfrak{C}}(G/\{1, 2\}) + \widehat{\mathfrak{C}}(G/\{1, 2, 3\}) + \widehat{\mathfrak{C}}(G/\{1, 2, 4\}) = -1 + 1 + 1.$$

Namely, as we discussed in Example 59, the two subtopologies $G/\{1, 2, 3\}$ and $G/\{1, 2, 4\}$ have one master integral each, if we consider them individually. However, they are embedded into the graph $G/\{1, 2\}$, which has only a single master integral, $\mathfrak{C}(G/\{1, 2\}) = 1$. This is sometimes referred to as a "relation between subtoplogies", and the negative value of $\widehat{\mathfrak{C}}(G/\{1, 2\}) = -1$ is precisely correcting the total counting.

Note that symmetries play no role in this discussion - the "extra" relation is detected by the Euler characteristic and thus corresponds to a parametric annihilator.[36]

5 Outlook

We have studied linear relations between Feynman integrals that arise from parametric annihilators of the integrand $\mathcal{G}^s$ in the Lee–Pomeransky representation. Seen as a

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[36] As a referee kindly pointed out, this relation also follows from momentum space IBPs.

multivariate (twisted) Mellin transform, the integration bijects these special partial differential operators with relations of various shifts (in the indices) of a Feynman integral. In particular, every classical IBP relation (derived in momentum space) is of this type.

The question whether *all* shift relations of Feynman integrals (equivalently, all parametric annihilators of $\mathcal{G}^s$) follow from momentum space relations remains open (see Question 24). We showed that the well-known lowering and raising operators with respect to the dimension are consequences of the classical IBPs. A next step would be to clarify if the same applies to the relations implied by the trivial annihilators (2.14). Similarly, Question 23 asking whether the annihilator $\mathrm{Ann}(\mathcal{G}^s) = \mathrm{Ann}^1(\mathcal{G}^s)$ is linearly generated, remains to be settled. A positive answer to either of these would imply that the labour-intensive computation of the parametric annihilators could be simplified considerably (Mom is known explicitly, and Ann^1 can be calculated through syzygies).

The main insight of this article is a statement on the number of master integrals, which we define as the dimension of the vector space of the corresponding family of Feynman integrals over the field of rational functions in the dimension and the indices. Since we treat all indices ν_a as independent variables, this definition does not account for symmetries (automorphisms) of the underlying graph. An important next step, in particular for practical applications, is to incorporate these symmetries into our set-up by studying the action of the corresponding permutation group. The widely used partition of master integrals into *top-level* and *subsector* integrals can be mimicked in our framework, as discussed in Remark 60.

Our result shows that the number of master integrals is not only finite, but identical to the Euler characteristic of the complement of the hyperspace $\{\mathcal{G} = 0\}$ determined by the Lee–Pomeransky polynomial $\mathcal{G}$. This statement follows from a theorem of Loeser and Sabbah. We exemplified several methods to compute this number and found agreement with other established methods. We expect that, combining the available tools for the computation of the Euler characteristic, it should be possible to compile a program for the efficient calculation of the number of master integrals for a wide range of Feynman graphs.

Let us conclude by emphasizing, once again, that the main objects of the approach elaborated here—the *s*-parametric annihilators generating the integral relations, and the Euler characteristic giving the number of master integrals—are well-studied objects in the theory of *D*-modules and furthermore algorithms for their automated computation are available in principle.

In particular, we hope that this parametric, *D*-module theoretical and geometrical approach can also shed light on the problems most relevant for perturbative calculations in QFT: the construction of a basis of master integrals and the actual reduction in arbitrary integrals to such a basis. For this perspective, we would like to point out that our approach of treating the indices ν_a as free variables, in particular not tied to take integer values, is desirable in order to deal with dimensionally regulated integrals in position space, and for the ability to integrate out one-scale subgraphs (both situations introduce non-integer indices). For a recent step into this direction, see [91].

A. Integral representations

In this appendix, we add technical details on the material of Sect. 2. We summarize the various well-known parametric representations, including their proofs, and the explicit relation to momentum space via Propositions 2 and 20. Furthermore, we give an alternative, algebraic proof for Corollary 63.

A.1. Momentum space and Schwinger parameters

As in Sect. 3.2, we consider a connected Feynman graph G with N internal edges, $E+1$ external legs and loop number L, which is related by Euler's formula $L = N - V + 1$ to the number V of vertices of G. Let us consider the case where each edge e of G is associated with a Feynman propagator,[37]

$$\frac{1}{\mathsf{D}_e} = \frac{1}{-k_e^2 + m_e^2 - \mathrm{i}\epsilon} \quad (1 \leq e \leq N),$$

which depends on the mass m_e of the particle e and the d-dimensional momentum $k_e \in \mathbb{R}^d$ flowing through this edge. Enforcing momentum conservation at each vertex fixes all k_e in terms of E independent external momenta $p_1, \ldots, p_E$ and L free loop momenta $\ell_1, \ldots, \ell_L$. Note that the actual number of external legs of G is $E + 1$, since overall momentum conservation $\sum_{i=1}^{E+1} p_i = 0$ imposes one relation among the external momenta. Taking only the inverse Feynman propagators D_e as denominators, Eq. (2.1) defines the Feynman integral associated to G.

Example 61 The graph in Fig. 8 has $V = 5$ vertices, $N = 6$ internal edges and $L = 2$ loops. It depends on two independent external momenta p_1 and p_2. A choice of loop momenta and the resulting momentum flow is depicted in Fig. 8. With all masses zero,

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[37] We use the signature $(1, -1, \ldots, -1)$ for the Minkowski metric.

Fig. 8 A two-loop Feynman graph with a choice of loop momenta and the resulting momentum flow
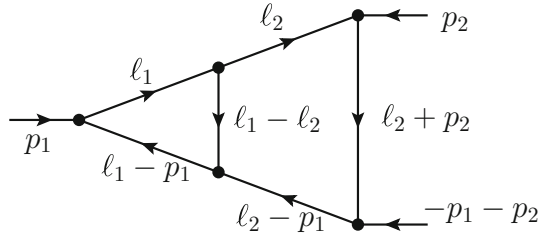
this yields

$$\mathcal{I}(\nu_1, \ldots, \nu_6) = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d \ell_1}{\mathrm{i}\pi^{d/2}} \int_{\mathbb{R}^d} \frac{\mathrm{d}^d \ell_2}{\mathrm{i}\pi^{d/2}} \frac{1}{[-\ell_1^2 - \mathrm{i}\epsilon]^{\nu_1}[-\ell_2^2 - \mathrm{i}\epsilon]^{\nu_2}[-(\ell_1 - \ell_2)^2 - \mathrm{i}\epsilon]^{\nu_3}}$$

$$\times \frac{1}{[-(\ell_2 + p_1)^2 - \mathrm{i}\epsilon]^{\nu_4}[-(\ell_2 - p_2)^2 - \mathrm{i}\epsilon]^{\nu_5}[-(\ell_1 - p_2)^2 - \mathrm{i}\epsilon]^{\nu_6}}.$$

Typically, the number $|\Theta| = L(L+1)/2 + LE$ of independent scalar products $s_{\{i,j\}}$ in (2.28) is larger than the number of edges in a graph G. We can then extend the initial set of denominators (given as the inverse propagators of the graph) by a suitable choice of additional quadratic (or linear) forms in the loop momenta, such that we reach a set of $|\Theta|$ denominators with the property that the matrix $\mathcal{A}$ defined by

$$\mathsf{D}_a = \sum_{\{i,j\} \in \Theta} \mathcal{A}_a^{\{i,j\}} s_{\{i,j\}} + \lambda_a$$

becomes invertible. This means that all loop-momentum-dependent scalar products can be written as linear combinations of the denominators, see (2.29). The additional denominators introduced in this way are called *irreducible scalar products*.

Example 62 We again consider the graph from Fig. 8 with 6 internal edges labelled as in Example 61. In the massless case, the inverse propagators are just $\mathsf{D}_e = -k_e^2$ and their explicit decomposition into the $|\Theta| = 7$ scalar products takes the form

$$\begin{pmatrix} \mathsf{D}_1 \\ \mathsf{D}_2 \\ \mathsf{D}_3 \\ \mathsf{D}_4 \\ \mathsf{D}_5 \\ \mathsf{D}_6 \end{pmatrix} = \begin{pmatrix} -\ell_1^2 \\ -\ell_2^2 \\ -(\ell_1 - \ell_2)^2 \\ -(\ell_2 + p_1)^2 \\ -(\ell_2 - p_2)^2 \\ -(\ell_1 - p_2)^2 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}}_{\mathcal{A}} \begin{pmatrix} \ell_1^2 \\ \ell_1 \ell_2 \\ \ell_2^2 \\ \ell_1 p_1 \\ \ell_1 p_2 \\ \ell_2 p_1 \\ \ell_2 p_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -p_1^2 \\ -p_2^2 \\ -p_2^2 \end{pmatrix}}_{\lambda}.$$

The matrix $\mathcal{A}$ has rank 6 and annihilates $(0, 0, 0, 1, 0, 0, 0)^\mathsf{T}$. Thus, we can choose $\mathsf{D}_7 = \ell_1 p_1$ as an irreducible scalar product to complete the basis of quadratic forms in

the loop momenta. The matrix $\mathcal{A}$ then acquires an additional row $(0, 0, 0, 1, 0, 0, 0)$ and becomes invertible.

Note that we assume that the inverse propagators of G (the initial set of denominators) are linearly independent (that is, the $N \times |\Theta|$ matrix $\mathcal{A}$ of the inverse propagators has full rank N) in order to be able to extend them to a basis of quadratic forms by choosing $|\Theta| - N$ irreducible scalar products.[38]

For each denominator D_a we introduce a scalar x_a, which is known as Schwinger-, Feynman- or α-parameter. In Definition 1 we have introduced the decomposition

$$\sum_{a=1}^{N} x_a \mathsf{D}_a = - \sum_{i,j=1}^{L} \Lambda_{ij} \ell_i \ell_j + \sum_{i=1}^{L} 2Q_i \ell_i + J, \tag{A.1}$$

which determines a symmetric $L \times L$ matrix Λ, a vector Q and a scalar J. With their help we defined the polynomials $\mathcal{U}$, $\mathcal{F}$ and $\mathcal{G} = \mathcal{U} + \mathcal{F}$ in (2.3). Explicitly, from Definition 1 and (2.26) we can read off that

$$\Lambda_{ij} = -\frac{1 + \delta_{ij}}{2} \sum_{a=1}^{N} x_a \mathcal{A}_a^{\{i,j\}} \quad \text{for } 1 \leq i, j \leq L, \tag{A.2}$$

$$Q_i = \frac{1}{2} \sum_{a=1}^{N} \sum_{j=L+1}^{M} x_a \mathcal{A}_a^{\{i,j\}} q_j \quad \text{for } 1 \leq i \leq L \text{ and} \tag{A.3}$$

$$J = \sum_{a=1}^{N} x_a \lambda_a. \tag{A.4}$$

Since Λ_{ij} is an $L \times L$ matrix with entries that are linear in the Schwinger parameters, the polynomial $\mathcal{U}$ is homogeneous of degree L. By Cramer's rule, $(\det \Lambda)\Lambda_{ij}^{-1}$ is homogeneous of degree $L - 1$ and the linearity of Q and J in the x_a implies

Corollary 63 $\mathcal{U}$ and $\mathcal{F}$ are homogeneous polynomials in the variables $x_1, \ldots, x_N$ with the degrees $\deg(\mathcal{U}) = L$ and $\deg(\mathcal{F}) = L + 1$. Hence, for $\mathcal{G} = \mathcal{U} + \mathcal{F}$, we have

$$\left(\sum_{a=1}^{N} x_a \partial_a \right) \mathcal{G} = L\mathcal{U} + (L+1)\mathcal{F} = (L+1)\mathcal{G} - \mathcal{U} = L\mathcal{G} + \mathcal{F}. \tag{A.5}$$

Let us now come to the proof of Proposition 2 following [64] and [58].

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[38] If there are linear dependencies between the inverse propagators, these relations imply that the Feynman integral can be expressed in terms of contracted graphs with linearly independent inverse propagators. For example, if $\alpha\mathsf{D}_1 + \beta\mathsf{D}_2 = 1$, then iterated use of $1/(\mathsf{D}_1\mathsf{D}_2) = \alpha/\mathsf{D}_2 + \beta/\mathsf{D}_1$ allows one to ultimately eliminate one of D_1 or D_2. Therefore, requesting linear independence is no restriction.

Proof of Proposition 2 We consider the Feynman integral defined in (2.1),

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{j=1}^{L} \int \frac{\mathrm{d}^d \ell_j}{\mathrm{i}\pi^{d/2}} \right) \prod_{i=1}^{N} \mathsf{D}_i^{-\nu_i}. \tag{A.6}$$

Using the Schwinger trick to exponentiate each denominator,[39]

$$\frac{1}{\mathsf{D}_a^{\nu_a}} = \frac{1}{\Gamma(\nu_a)} \int_0^\infty x_a^{\nu_a-1} e^{-x_a \mathsf{D}_a} \mathrm{d}x_a, \tag{A.7}$$

the integral in (A.6) turns into

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i-1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \left(\prod_{j=1}^{L} \int \frac{\mathrm{d}^d \ell_j}{\mathrm{i}\pi^{d/2}} \right) e^{-\sum_{a=1}^{N} x_a \mathsf{D}_a}.$$

According to (A.1) and (2.3), we can complete the square in the exponent

$$-\sum_{a=1}^{N} x_a \mathsf{D}_a = (\ell - \Lambda^{-1} Q)^\mathsf{T} \Lambda (\ell - \Lambda^{-1} Q) - \mathcal{F}/\mathcal{U}$$

to perform the Gaußian integrals over the shifted loop momenta $\ell' := \ell - \Lambda^{-1} Q$ as[40]

$$\left(\prod_{j=1}^{L} \int \frac{\mathrm{d}^d \ell'_j}{\mathrm{i}\pi^{d/2}} \right) e^{(\ell')^\mathsf{T} \Lambda \ell'} = (\det \Lambda)^{-d/2} = \mathcal{U}^{-d/2}.$$

In summary, we therefore arrive at the integral representation (2.5):

$$\mathcal{I}(\nu_1, \ldots, \nu_N) = \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i-1} \mathrm{d}x_i}{\Gamma(\nu_i)} \right) \frac{e^{-\mathcal{F}/\mathcal{U}}}{\mathcal{U}^{d/2}}.$$

We now multiply with $1 = \int_0^\infty \delta(\rho - \sum_{j=1}^{N} x_j) \mathrm{d}\rho$ and substitute $x_a \to \rho x_a$.[41] The Jacobian ρ^N, the monomials $x_i^{\nu_i-1}$ and $\delta(\rho - \sum_j x_j) \to \delta(1 - \sum_j x_j)/\rho$ contribute

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[39] Integral (A.7) converges only for $\mathrm{Re}(\nu_a) > 0$ and therefore restricts the domain of convergence for the parametric integral. However, this has no consequences for algebraic relations, see Remark 4.

[40] Recall that our metric has signature $(1, -1, \ldots, -1)$, so the integrations over the $d - 1$ spacelike components are Euclidean and give $\sqrt{\pi^L \mathcal{U}}$ each. The timelike integrations are understood as contour integrals and yield the same factor after rotating the integration contour to the imaginary axis, according to the Feynman iε-prescription.

[41] Much more generally, we could replace $\sum_{j=1}^{N} x_j$ in the δ-constraint with any other function as long as it is homogeneous of degree 1 and positive on $\mathbb{R}_+^N$.

the power $\rho^{|\nu|-1}$, whereas the homogeneity of $\mathcal{F}$ and $\mathcal{U}$ from Corollary 63 implies that $\mathcal{U} \to \rho^L \mathcal{U}$ and $\mathcal{F}/\mathcal{U} \to \rho \mathcal{F}/\mathcal{U}$. Overall, by realizing that the integral over ρ is

$$\int_0^\infty \rho^{\omega-1} e^{-\rho \mathcal{F}/\mathcal{U}} d\rho = \Gamma(\omega) \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^\omega,$$

we arrive at the first parametric formula (2.6). Similarly, we multiply the integrand of (2.7) with $1 = \int_0^\infty \delta(\rho - \sum_i x_i) d\rho$ and substitute $x_i \to \rho x_i$. Using $\mathcal{U} \to \rho^L \mathcal{U}$ and $\mathcal{F} \to \rho^{L+1} \mathcal{F}$ from Corollary 63, the integral over ρ becomes

$$\int_0^\infty \rho^{\omega-1} (\mathcal{U} + \rho \mathcal{F})^{-d/2} = \mathcal{U}^{-d/2} \left(\frac{\mathcal{U}}{\mathcal{F}}\right)^\omega \frac{\Gamma(\omega)\Gamma(\frac{d}{2} - \omega)}{\Gamma(\frac{d}{2})}$$

and combines with the prefactors in (2.7) to reproduce (2.7). $\qquad\square$

We conclude the section with the proof of Proposition 20 following Grozin [35]:

Proof of Proposition 20 The action of $\mathbf{o}_j^i$ on the integrand from (2.30) is

$$\mathbf{o}_j^i f = d\delta_{ij} f + f \sum_{a=1}^N \frac{-\nu_a}{\mathsf{D}_a} q_j \frac{\partial \mathsf{D}_a}{\partial q_i}.$$

According to (2.26), the chain rule gives

$$q_j \frac{\partial}{\partial q_i} \mathsf{D}_a = q_j \frac{\partial}{\partial q_i} \sum_{\{k,m\} \in \Theta} \mathcal{A}_a^{\{k,m\}} q_k q_m = \sum_{m=1}^M \mathcal{A}_a^{\{i,m\}} (1 + \delta_{i,m}) q_j q_m$$

and we can express the scalar products $q_j q_m$ with $\{j, m\} \in \Theta$ in terms of denominators using (2.29). The remaining terms with $j, m > L$ are products of external momenta, so

$$\mathbf{o}_j^i f = d\delta_{ij} f - f \sum_{a,b=1}^N C_{aj}^{bi} \frac{\nu_a}{\mathsf{D}_a} (\mathsf{D}_b - \lambda_b) \quad \text{for} \quad 1 \leq j \leq L \quad \text{and}$$

$$\mathbf{o}_j^i f = -f \sum_{a,b=1}^N C_{aj}^{bi} \frac{\nu_a}{\mathsf{D}_a} (\mathsf{D}_b - \lambda_b) - f \sum_{a=1}^N \sum_{m=L+1}^M \mathcal{A}_a^{\{i,m\}} q_j q_m \frac{\nu_a}{\mathsf{D}_a} \quad \text{if} \quad L < j \leq M.$$

We conclude by noticing that multiplying the integrand f with ν_a/D_a is equivalent to the action of the operator $\hat{\mathbf{a}}^+$ defined in (2.17), whereas multiplication with D_b lowers the index ν_b and corresponds to $\mathbf{b}^-$. $\qquad\square$

A.2. Algebraic proof for Corollary 21

With the proof of Corollary 21 we have shown that, for every momentum space IBP relation, there is a corresponding annihilator in Ann $\left(\mathcal{G}^{-d/2}\right)$. The proof rests on the inverse Mellin transform, which may be seen as a convenient but rather abstract argument. As a more direct alternative, we prove the statement in a purely algebraical way by use of properties of the graph polynomials.

Lemma 64 *The operators $\widetilde{\mathbf{O}}^i_j$ from (2.36) and (2.37) corresponding to the momentum space IBP relations annihilate the parametric integrand $\mathcal{G}^{-d/2}$:*

$$\widetilde{\mathbf{O}}^i_j \in \operatorname{Ann}\left(\mathcal{G}^{-d/2}\right) \quad \text{for all } 1 \le i \le L \text{ and } 1 \le j \le M.$$

Proof Let us first consider the case $j \le L$. After acting with $\widetilde{\mathbf{O}}^i_j$ from (2.36) on $\mathcal{G}^{-d/2}$ and dividing by $(d/2)\mathcal{G}^{-d/2-1}$, we are left to prove the vanishing of

$$2\mathcal{G}\delta_{i,j} - \sum_{a,b} C^{bi}_{aj} x_a \left(\partial_b \mathcal{G} - \lambda_b \left[L + 1 - \sum_c x_c \partial_c\right]\mathcal{G}\right)$$
$$= 2\mathcal{G}\delta_{i,j} - \sum_{a,b} C^{bi}_{aj} x_a \left(\partial_b \mathcal{G} - \lambda_b \mathcal{U}\right) \tag{A.8}$$

where we exploited the homogeneity from (A.5). Using (2.3), we note that

$$\partial_b \mathcal{G} - \lambda_b \mathcal{U} = \partial_b \left[\mathcal{U}\left(1 + J + Q^{\mathsf{T}}\Lambda^{-1}Q\right)\right] - \lambda_b \mathcal{U} = \mathcal{G}\frac{\partial_b \mathcal{U}}{\mathcal{U}} + \mathcal{U}\partial_b\left(Q^{\mathsf{T}}\Lambda^{-1}Q\right) \tag{A.9}$$

because $\partial_b J = \lambda_b$ according to (A.4). In order to evaluate $\partial_b \mathcal{U}$ with Jacobi's formula $(\partial_b \mathcal{U})/\mathcal{U} = \sum_{r,s=1}^L \Lambda^{-1}_{r,s}\partial_b \Lambda_{s,r}$, we use (A.2) to compute

$$\sum_b \mathcal{A}^b_{\{m,j\}}\partial_b \Lambda_{s,r} = -\frac{1 + \delta_{s,r}}{2}\sum_b \mathcal{A}^b_{\{m,j\}}\mathcal{A}^{\{s,r\}}_b = -\frac{1 + \delta_{s,r}}{2}\delta_{\{m,j\},\{r,s\}}$$
$$= -\frac{\delta_{m,r}\delta_{j,s} + \delta_{m,s}\delta_{j,r}}{2}$$

which restricts m to either r or s. So in particular, $m \le L$ and we can use (A.2) in

$$\sum_{a,b} C^{bi}_{aj} x_a \partial_b \Lambda_{r,s} = -\sum_{a,m} x_a \mathcal{A}^{\{i,m\}}_a \frac{1 + \delta_{i,m}}{2}\left(\delta_{m,r}\delta_{j,s} + \delta_{m,s}\delta_{j,r}\right)$$
$$= \Lambda_{i,r}\delta_{j,s} + \Lambda_{i,s}\delta_{j,r} \tag{A.10}$$

which proves that for arbitrary j (independent of whether $j \leq L$ or $j > L$)

$$\sum_{a,b} C^{bi}_{aj} x_a \frac{\partial_b \mathcal{U}}{\mathcal{U}} = \sum_{r,s=1}^{L} \Lambda^{-1}_{r,s} (\Lambda_{i,r} \delta_{j,s} + \Lambda_{i,s} \delta_{j,r}) = 2\delta_{i,j}. \qquad (A.11)$$

Via (A.9), this identity reduces the proof of (A.8) to showing that

$$\sum_{a,b} C^{bi}_{aj} x_a \partial_b \left(Q^\mathsf{T} \Lambda^{-1} Q \right) = \sum_{a,b} C^{bi}_{aj} x_a \left[2(\partial_b Q)^\mathsf{T} \Lambda^{-1} Q - Q^\mathsf{T} \Lambda^{-1} (\partial_b \Lambda) \Lambda^{-1} Q \right] \qquad (A.12)$$

vanishes. The last term is easily evaluated with (A.10) and gives

$$\sum_{a,b} C^{bi}_{aj} x_a Q^\mathsf{T} \Lambda^{-1} (\partial_b \Lambda) \Lambda^{-1} Q = 2 \sum_{r,s=1}^{L} (\Lambda^{-1} Q)_r (\Lambda^{-1} Q)_s \Lambda_{i,r} \delta_{j,s}$$
$$= 2 Q_i (\Lambda^{-1} Q)_j, \qquad (A.13)$$

whereas the derivative $\partial_b Q$ can be read off from (A.3) and the sum over b yields

$$\sum_b \mathcal{A}^b_{\{m,j\}} (2\partial_b Q_s) = \sum_b \mathcal{A}^b_{\{m,j\}} \sum_{r>L} \mathcal{A}^{\{s,r\}}_b q_r = \sum_{r>L} q_r \delta_{\{m,j\},\{s,r\}} = \sum_{r>L} q_r \delta_{m,r} \delta_{s,j} \qquad (A.14)$$

because $j \leq L < r$ excludes the possibility that $m = s$ and $r = j$. Thus with (A.3),

$$\sum_{a,b} C^{bi}_{aj} x_a (2\partial_b Q)^\mathsf{T} \Lambda^{-1} Q = (\Lambda^{-1} Q)_j \sum_a \sum_{m>L} x_a \mathcal{A}^{\{i,m\}}_a q_m = 2 Q_i (\Lambda^{-1} Q)_j \qquad (A.15)$$

cancels the contribution from (A.13) in (A.12) and finishes the proof in the case $j \leq L$. If instead we have $j > L$, then we must replace $\delta_{\{m,j\},\{s,r\}} = \delta_{j,r} \delta_{m,s}$ in (A.14) such that

$$\sum_{a,b} C^{bi}_{aj} x_a (2\partial_b Q)^\mathsf{T} \Lambda^{-1} Q = \sum_{s=1}^{L} \sum_a \mathcal{A}^{\{i,s\}}_a (1 + \delta_{i,s}) q_j (\Lambda^{-1} Q)_s$$
$$= -2 \sum_{s=1}^{L} \Lambda_{i,s} (\Lambda^{-1} Q)_s = -2 Q_i q_j$$

where we used (A.2) once more. Now recall that (A.11) remains true and becomes zero for $j > L$ because $\delta_{i,j} = 0$ since $i \leq L$. For the same reason, $\delta_{j,s} = 0$ in (A.13) and therefore, using (A.9),

$$-\sum_{a,b} C_{aj}^{bi} x_a (\partial_b \mathcal{G} - \lambda_b \mathcal{U}) = -\mathcal{U} \sum_{a,b} C_{aj}^{bi} x_a (2\partial_b Q)^{\mathsf{T}} \Lambda^{-1} Q = 2\mathcal{U} Q_i q_j.$$

This is precisely cancelled by the additional contribution to (A.8) coming from $\widetilde{\mathbf{O}}_j^i$ in (2.37) in the case $j > L$: The additional term acts on $\mathcal{G}^{-d/2}$ as

$$-\sum_a \sum_{m>L} \mathcal{A}_a^{\{i,m\}} q_j q_m x_a \Big[L + 1 - \sum_c x_c \partial_c \Big] \mathcal{G} = -\mathcal{U} q_j \sum_a \sum_{m>L} \mathcal{A}_a^{\{i,m\}} x_a q_m$$
$$= -2\mathcal{U} q_j Q_i$$

after dividing by $(d/2)\mathcal{G}^{-d/2-1}$. Note that here we used (A.5) and (A.3). $\qquad\square$

A.3. The Baikov representation

In this section, we discuss the representation of Feynman integrals suggested by Baikov in [7], whose complete form (2.48) was given by Lee in [53,54]. We will give some details on the derivation of this formula (see also [35, section 9]), which was presented in [53] and applied in our discussion of the lowering dimension shift in Sect. 2.5.

Assume that $q_1, \ldots, q_M$ are vectors in a Euclidean vector space and write

$$V_n := \begin{pmatrix} q_n \cdot q_n & \cdots & q_n \cdot q_M \\ \vdots & \ddots & \vdots \\ q_M \cdot q_n & \cdots & q_M \cdot q_M \end{pmatrix} = (q_i \cdot q_j)_{n \le i, j \le M} \quad \text{and} \quad G_n := \det V_n \text{ (A.16)}$$

for their Gram matrices and determinants. Note that

$$V_n = \begin{pmatrix} q_n^2 & q_\bullet \cdot q_n \\ q_n \cdot q_\bullet & V_{n+1} \end{pmatrix} \quad \text{where} \quad q_\bullet \cdot q_n := \begin{pmatrix} q_{n+1} \cdot q_n \\ \vdots \\ q_M \cdot q_n \end{pmatrix}, \quad q_n \cdot q_\bullet := (q_\bullet \cdot q_n)^{\mathsf{T}}$$

and thus, by adding $-(p_n \cdot p_\bullet) V_{n+1}^{-1}$ times the lower $M - n$ rows to the first row,

$$\frac{G_n}{G_{n+1}} = q_n^2 - \left\| \mathrm{pr}_{\mathrm{lin}\{q_{n+1},\ldots,q_M\}}^{\perp}(q_n) \right\|^2 = \left\| \mathrm{pr}_{\mathrm{lin}\{q_{n+1},\ldots,q_M\}^{\perp}}(q_n) \right\|^2. \quad \text{(A.17)}$$

Indeed, formula $\mathrm{pr}_{\mathrm{lin}\{q_{n+1},\ldots,q_M\}}^{\perp}(v) = \sum_{i,j=n+1}^{M} q_i \left(V_{n+1}^{-1} \right)_{i,j} (q_j \cdot v)$ for the orthogonal projection of v onto the space spanned by $q_{n+1}, \ldots, q_M$ shows that

$$\left\| \mathrm{pr}_{\mathrm{lin}\{q_{n+1},\ldots,q_M\}}^{\perp}(q_n) \right\|^2 = \sum_{i,j,k,l=n+1}^{M} \underbrace{(q_i \cdot q_k)}_{(V_{n+1})_{i,k}} \left(V_{n+1}^{-1} \right)_{i,j} (q_j \cdot q_n) \left(V_{n+1}^{-1} \right)_{k,l} (q_k \cdot q_n)$$

$$= \sum_{j,k,l=n+1}^{M} \delta_{k,j}(q_j \cdot q_n) \left(V_{n+1}^{-1}\right)_{k,l} (q_l \cdot q_n)$$

$$= \sum_{k,l=n+1}^{M} (q_k \cdot q_n) \left(V_{n+1}^{-1}\right)_{k,l} (q_l \cdot q_n)$$

$$= (q_n \cdot q_\bullet) V_{n+1}^{-1} (q_\bullet \cdot q_n).$$

Now assume our integrand f only depends on the scalar products $s_{i,j} = q_i \cdot q_j$, and we want to integrate out the first loop momentum q_1. Let us decompose $q_1 = q_\perp + q_\parallel$ into the component $q_\parallel \in \text{lin}\{q_2, \ldots, q_M\}$ that lies in the space spanned by the other momenta, and the component $q_\perp$ in its orthogonal complement. According to (A.17), $G_n^{1/2}$ is the volume of the parallelotope spanned by $q_n, \ldots, q_M$. Hence, changing coordinates from $q_\parallel$ to $(s_{1,2}, \ldots, s_{1,M})$ yields

$$\int_{\mathbb{R}} ds_{1,2} \cdots \int_{\mathbb{R}} ds_{1,M} = \sqrt{G_2} \int_{\mathbb{R}^{M-1}} d^{M-1} q_\parallel.$$

The integral over the orthogonal component is, due to $s_{1,1} = q_1^2 = q_\perp^2 + q_\parallel^2$, given by

$$\int_{\mathbb{R}^{d-M+1}} d^{d-M+1} q_\perp = \frac{\pi^{(d-M+1)/2}}{\Gamma\left(\frac{d-M+1}{2}\right)} \int_0^\infty dq_\perp^2 \left(q_\perp^2\right)^{(d-M-1)/2}$$

$$= \frac{\pi^{(d-M+1)/2}}{\Gamma\left(\frac{d-M+1}{2}\right)} \int_{q_\parallel^2}^\infty ds_{1,1} \left(\frac{G_1}{G_2}\right)^{(d-M-1)/2}.$$

Note that the lower boundary $s_{1,1} = q_\parallel^2$ corresponds to $0 = q_\perp^2 = G_1/G_2$. Altogether,

$$\int_{\mathbb{R}^d} \frac{d^d q_1}{\pi^{d/2}} f(s) = \frac{\pi^{(1-M)/2}}{\Gamma\left(\frac{d-M+1}{2}\right)} \int_{G_1/G_2>0} d^M s_{1,\bullet} \ f(s) \frac{G_1^{(d-M-1)/2}}{G_2^{(d-M)/2}}. \quad \text{(A.18)}$$

Transforming the remaining loop integrations analogously, all but two of the Gram determinants cancel, and we conclude that

$$\prod_{i=1}^L \int \frac{d^d \ell_i}{\pi^{d/2}} f(s) = \frac{\pi^{-LE/2 - L(L-1)/4}}{\Gamma\left(\frac{d-L-E+1}{2}\right) \cdots \Gamma\left(\frac{d-E}{2}\right)} \int d^N s_{\bullet,\bullet} \ f(s) \cdot \frac{G_1^{(d-M-1)/2}}{G_{L+1}^{(d-E-1)/2}} \quad \text{(A.19)}$$

where $M = L + E$ is the sum of the number L of loops and the number E of linearly independent external momenta. Note that $G_{L+1} = \det(p_i \cdot p_j)_{1 \le i, j \le E}$ is the Gram determinant of the external momenta and independent of the integration variables.

Proof of Theorem 29 Since every denominator D_e is a linear combination of $s_{i,j}$ (and some loop momentum independent constant λ_e) according to (2.26), an affine change

of variables allows us to integrate over the values of D_e's instead of $s_{i,j}$'s. This transformation only introduces a constant Jacobian $c' = \det(\mathcal{A}^a_{\{i,j\}})$. We set $f(s) = \prod_e \mathsf{D}_e^{-\nu_e}$ in formula (A.19) above. The (Euclidean) integration domain is determined, according to (A.17), by $0 < \|\mathrm{pr}^{\perp}_{\mathrm{lin}\{q_{n+1},\ldots,q_M\}^{\perp}}(q_n)\|^2 = G_n/G_{n+1}$ for $1 \leq n \leq L$. Therefore, a point on the boundary of the integration domain is determined by $G_n = 0$ for some $1 \leq n \leq L$, which is equivalent to a linear dependence $q_n \in \mathrm{lin}\{q_{n+1},\ldots,q_M\}$ and hence implies $G_1 = 0$.

Note that we have to analytically continue (A.19) from Euclidean to Minkowski space in order to obtain the Feynman integral (2.1). As Wick rotation turns $\int \mathrm{d}^d \ell_k/(\mathrm{i}\pi^{d/2})$ exactly into the measure $\int \mathrm{d}^d \ell_k/\pi^{d/2}$ on the left-hand side of (A.19), we only have to remember that, due to our mostly-minus signature $(1, -1, \ldots, -1)$ of the Minkowski metric, the Euclidean scalar products on the right-hand side of (A.19) receive a factor (-1). For example, the $M \times M$ determinant G_1 turns into $(-1)^M \mathsf{Gr}_1$; similarly, G_{L+1} becomes $(-1)^E \mathsf{Gr}$. Overall, analytic continuation gives an additional factor of

$$(-1)^N \cdot \frac{(-1)^{M(d-M-1)/2}}{(-1)^{E(d-E-1)/2}} = (-1)^{Ld/2} \cdot (-1)^{N+M(M-1)/2-E(E-1)/2}.$$

We absorb the last factor, together with the Jacobian c', into the constant prefactor c, and have thus finally arrived at (2.48). $\qquad\square$

B. The theory of Loeser–Sabbah

This section is devoted to Theorem 35, which was first stated in [59]. Beware that the original argument is flawed; a correct (but terse) proof was given in [61]. Our aim here is to provide a simplified and more detailed derivation.

Throughout we will consider modules $\mathscr{M}$ over the algebra $\mathcal{D}_k^N = A_k^N[x^{-1}]$ of differential operators (3.4) on the torus in some number N of variables x_i, over some field k of characteristic zero. To lighten the notation, let us abbreviate $\theta_i := x_i\partial_i$ and set

$$\mathscr{M}(\theta_i,\ldots,\theta_N) := \mathscr{M} \otimes_{k[\theta_i,\ldots,\theta_N]} k(\theta_i,\ldots,\theta_N) \tag{B.1}$$

for the *algebraic Mellin transform* [60, Section 1.2]. We begin with the finite-dimensionality, which was proven in [60, Lemma 1.2.2]:

Lemma 65 *Let $\mathscr{M}$ denote a holonomic $\mathcal{D}_k^N$-module. Then, for any $1 \leq i \leq N$, its algebraic Mellin transform $\mathscr{M}(\theta_i,\ldots,\theta_N)$ is a holonomic $\mathcal{D}_{k(\theta_i,\ldots,\theta_N)}^{i-1}$-module.*

Corollary 66 *The full Mellin transform $\mathscr{M}(\theta_1,\ldots,\theta_N)$ is a finite-dimensional vector space over the field $k(\theta_1,\ldots,\theta_N)$.*

Proof of Lemma 65 Since $\mathscr{M}(\theta_i,\ldots,\theta_N) = [\mathscr{M}(\theta_{i+1},\ldots,\theta_N)](\theta_i)$, it suffices (by induction over i) to consider the case $i = N$. Introducing a new indeterminate ν, we

extend the scalars from k to $k[\nu]$ to obtain a $\mathcal{D}^N_{k[\nu]}$-module $\mathcal{M}[\nu] := \mathcal{M} \otimes_k k[\nu]$. It sits in an exact sequence

$$0 \longrightarrow \mathcal{M}[\nu] \xrightarrow{\partial_N + \nu/x_N} \mathcal{M}[\nu] \xrightarrow{\sum_j \nu^j m_j \mapsto \sum_j (-1-x_N\partial_N)^j m_j} \mathfrak{M} \longrightarrow 0$$

of $\mathcal{D}^{N-1}_{k[\nu]}$-modules, where $\mathfrak{M} = \mathcal{M}$ denotes the initial module $\mathcal{M}$ with the action of ν defined as $\nu m := -x_N \partial_N m$. Since $k(\nu)$ is flat, this sequence remains exact after tensoring with $k(\nu)$ over $k[\nu]$. Through identification of ν with $-\theta_N - 1$, we conclude that

$$\frac{\mathcal{M} \otimes_k k(\nu)}{(\partial_N + \nu/x_N)(\mathcal{M} \otimes_k k(\nu))} \cong \mathfrak{M} \otimes_{k[\nu]} k(\nu) \cong \mathcal{M} \otimes_{k[\theta_N]} k(\theta_N) = \mathcal{M}(\theta_N)$$

are isomorphic as $\mathcal{D}^{N-1}_{k(\nu)}$-modules. The left-hand side is the quotient $\mathcal{M}x_N^\nu / \partial_N \mathcal{M}x_N^\nu$ of the $\mathcal{D}^N_{k(\nu)}$-module $\mathcal{M}x_N^\nu := \mathcal{M} \otimes_k k(\nu)$ defined by the original action of $\mathcal{D}^{N-1}_k$ and $x_N^{\pm 1}$ on $\mathcal{M}$, but twisting the operator ∂_N to act like $\partial_N + \nu/x_N$.[42] The holonomicity of $\mathcal{M}$ implies that $\mathcal{M}x_N^\nu$ is also holonomic,[43] and hence its push-forward $\pi_*(\mathcal{M}x_N^\nu) = \mathcal{M}x_N^\nu / \partial_N(\mathcal{M}x_N^\nu) \cong \mathcal{M}(\theta_N)$, with respect the projection $\pi \colon \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^{N-1}$ that forgets the last coordinate, is also holonomic. $\qquad\square$

Now we want to relate this dimension to the de Rham complex $\mathrm{DR}(\mathcal{M}) = K(\mathcal{M}; \partial)$, which is a special case of the Koszul complex:

Definition 67 For commuting k-linear endomorphisms $s = (s_1, \dots, s_N)$ of a k-vector space $\mathcal{M}$, let

$$K(\mathcal{M}; s_1, \dots, s_N) := \left(\Lambda^\bullet k \otimes_k \mathcal{M}[N], \mathrm{d}\right) \tag{B.2}$$

denote the Koszul complex with r-forms sitting in degree $r - N$. The $r - N$ cochains

$$K^{r-N}(\mathcal{M}; s) = \Lambda^r \mathcal{M} = \bigoplus_{|I|=r} \mathbf{e}_I \otimes \mathcal{M}$$

have natural coordinates with respect to the basis $\mathbf{e}_I := \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r} \in \Lambda^r k$ indexed by r-sets $I = \{i_1 < \cdots < i_r\}$. The cochain map reads $\mathrm{d}(\mathbf{e}_I \otimes m) = \sum_{i \notin I}(\mathbf{e}_i \wedge \mathbf{e}_I) \otimes s_i m$.

Remark 68 Since $\theta_j \left(\prod_{i \in I} x_i\right) m = \left(\prod_{i \in I \cup \{j\}} x_i\right) \partial_j m$ (for $j \notin I$), the rule

$$K(\mathcal{M}; \partial) \longrightarrow K(\mathcal{M}; \theta), \quad \mathbf{e}_I \otimes m \mapsto \mathbf{e}_I \otimes x^I m \tag{B.3}$$

---

[42] This just encodes the natural action $\partial_N x_N^\nu m = x_N^\nu(\partial_N + \nu/x_N)m$ on products of elements m of $\mathcal{M}$ with the function x_N^ν—hence the suggestive notation $\mathcal{M}x_N^\nu$.

[43] Given a good filtration $\Gamma_\bullet$ of $\mathcal{M}$, $\Gamma'_j(\mathcal{M}x_N^\nu) := (x_N^{-j}\Gamma_{2j}\mathcal{M}) \otimes_k k(\nu)$ defines a filtration of $\mathcal{M}x_N^\nu$ with $\dim_{k(\nu)} \Gamma'_j(\mathcal{M}x_N^\nu) \leq \dim_k \Gamma_{2j}\mathcal{M} \leq c \cdot (2j)^N$ for some $c < \infty$.

defines a cochain map. It has an inverse, defined by $e_I \otimes m \mapsto e_I \otimes x^{-I} m$. We thus conclude that $DR(\mathcal{M}) = K(\mathcal{M}; \partial)$ and $K(\mathcal{M}; \theta)$ are quasi-isomorphic and therefore share the same Euler characteristic.

We prove Theorem 35 by an induction over the number of variables. The base case is

Theorem 69 ([59, Théorème 1]) *If $\mathcal{M}$ denotes a holonomic $\mathcal{D}_k^1$-module, then*

$$\dim_{k(\theta_1)} \mathcal{M}(\theta_1) = \chi(\mathcal{M}) := \dim_k \frac{\mathcal{M}}{\partial_1 \mathcal{M}} - \dim_k \ker(\partial_1). \qquad (B.4)$$

Proof We can pick a generator of $\mathcal{M}$ (by holonomicity, $\mathcal{M}$ is cyclic as a $\mathcal{D}_k^1$-module) and extend it to a (finite) basis of $\mathcal{M}(\theta_1)$ as a vector space over $k(\theta_1)$, due to Corollary 66. Let $\mathcal{N} \subset \mathcal{M}$ denote the $k[\theta_1]$-module generated by such a basis, hence $\mathcal{N}(\theta_1) = \mathcal{M}(\theta_1)$. By construction, $\mathcal{M} = A_k^1 \mathcal{N} = \sum_{j \in \mathbb{Z}} x_1^j \mathcal{N}$ is exhaustively filtered by the finitely generated $k[\theta_1]$-modules $\mathcal{N}_j := \sum_{i=-j}^{j} x_1^i \mathcal{N}$.[44]

Since $\mathcal{M}(\theta_1) = \mathcal{N}(\theta_1) = \mathcal{N}_1(\theta_1)$ is finitely generated, there is a nonzero polynomial $b(\theta_1) \in k[\theta_1]$ such that $b(\theta_1) \mathcal{N}_1 \subseteq \mathcal{N}$. Therefore, using $(\theta_1 - 1)x_1 = x_1 \theta_1$,

$$b(\theta_1 \mp j) x_1^{\pm(j+1)} \mathcal{N} = x_1^{\pm j} b(\theta_1) x_1^{\pm 1} \mathcal{N} \subseteq x_1^{\pm j} b(\theta_1) \mathcal{N}_1 \subseteq x_1^{\pm j} \mathcal{N} \subseteq \mathcal{N}_j$$

shows that the polynomials $b_{j+1}(\theta_1) := b(\theta_1 + j) b(\theta_1 - j) \in k[\theta_1]$ have the property $b_{j+1}(\theta_1) \mathcal{N}_{j+1} \subseteq \mathcal{N}_j$. Let $Z = b^{-1}(0)$ denote the zeroes of $b_1 = b^2$, then note that the zeroes of b_{j+1} are $(Z + j) \cup (Z - j)$ and get pushed away from zero for increasing j. In particular, there exists some $j_0 \in \mathbb{N}$ such that $b_j(0) \neq 0$ for all $j > j_0$. For each such value of j, we can find $u_j, v_j \in k[\theta_1]$ such that $1 = u_j(\theta_1) b_j(\theta_1) + v_j(\theta_1) \theta_1$; then

$$m = 1 \cdot m = u_j(\theta_1) b_j(\theta_1) m + v_j(\theta_1) \theta_1 m \in \mathcal{N}_{j-1} + v_j(\theta_1) \theta_1 m$$

holds for every $m \in \mathcal{N}_j$. This proves $\ker(\theta_1) \cap \mathcal{N}_j \subseteq \mathcal{N}_{j-1}$ for all $j > j_0$, and therefore $\ker(\theta_1) \subseteq \mathcal{N}_{j_0}$. Similarly, we conclude $\mathcal{M}/(\theta_1 \mathcal{M}) \cong \mathcal{N}_{j_0}/(\mathcal{N}_{j_0} \cap \theta_1 \mathcal{M})$. But given some $m = \theta_1 x \in \mathcal{N}_{j_0}$ with $x \in \mathcal{N}_j$, $x = u_j(\theta_1) b_j(\theta_1) x + v_j(\theta_1) m \in \mathcal{N}_{j-1} + \mathcal{N}_{j_0}$ proves that $\mathcal{N}_{j_0} \cap \theta_1(\mathcal{N}_j) = \mathcal{N}_{j_0} \cap \theta_1(\mathcal{N}_{j-1})$ for all $j > j_0$. In consequence, we have proven that

$$\ker(\partial_1) = \ker(\theta_1) = \ker\left(\theta_1|_{\mathcal{N}_{j_0}}\right) \quad \text{and} \quad \frac{\mathcal{M}}{\partial_1(\mathcal{M})} \cong \frac{\mathcal{M}}{\theta_1(\mathcal{M})} \cong \frac{\mathcal{N}_{j_0}}{\theta_1(\mathcal{N}_{j_0})};$$

in other words, the Koszul complexes $DR(\mathcal{M}) = K(\mathcal{M}; \partial_1)$ and $K(\mathcal{N}_{j_0}; \theta_1)$ are quasi-isomorphic (see Remark 68). The statement of the theorem thus reduces to the identity

$$\dim_{k(\theta_1)} \mathcal{N}_{j_0}(\theta_1) = \chi\left(K(\mathcal{N}_{j_0}; \theta_1)\right)$$

---

[44] Due to $\theta_1 x_1^i \mathcal{N} = x_1^i (\theta_1 + i) \mathcal{N} \subseteq x_1^i \mathcal{N} \subseteq \mathcal{N}_j$, indeed $\mathcal{N}_j$ is a $k[\theta_1]$-module.

for a finitely generated $k[\theta_1]$-module $\mathcal{N}_{j_0}$. Since both sides are additive under short exact sequences, this claim reduces (via a finite free resolution) to the case of a free rank one $k[\theta_1]$-module, i.e. $k[\theta_1]$, which is clear: $k[\theta_1](\theta_1) = k(\theta_1)$ is of dimension one over $k(\theta_1)$, while $\ker(\theta_1) = \{0\}$ is trivial and $k[\theta_1]/(\theta_1 k[\theta_1]) = k$ is one-dimensional. □

With this starting point, we can now prove Theorem 35 by induction. In fact, the higher-dimensional case can be seen as a straightforward corollary of the univariate case above. In contrast to [61], our demonstration avoids any reference to higher-dimensional lattices.

Proof of Theorem 35 Let $\mathcal{M}$ denote a holonomic $\mathcal{D}_k^N$-module, and suppose we have proven Theorem 35 for all holonomic modules in less than N variables. In particular, we may invoke the claim for the $\mathcal{D}_k^{N-1}$-modules $\ker \partial_N$ and $\mathcal{M}/\partial_N\mathcal{M}$, as these are holonomic because they are the cohomologies of the complex

$$0 \longrightarrow \mathcal{M} \xrightarrow{\partial_N} \mathcal{M} \longrightarrow 0$$

which computes the push-forward of $\mathcal{M}$ along the projection $\pi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^{N-1}$ that forgets the last coordinate. So we already know that

$$\chi\left(\frac{\mathcal{M}}{\partial_N\mathcal{M}}\right) = \dim_{k'}\left(\frac{\mathcal{M}}{\partial_N\mathcal{M}}(\theta')\right) = \dim_{k'}\frac{\mathcal{M}'}{\partial_N\mathcal{M}'}$$

where $k' := k(\theta')$ and $\mathcal{M}' := \mathcal{M}(\theta')$ with $\theta' := (\theta_1, \ldots, \theta_{N-1})$. Analogously, $\chi(\ker \partial_N) = \dim_{k'} \ker(\partial'_N)$, where ∂'_N denotes the action of ∂_N on $\mathcal{M}'$. In conclusion, we know that

$$\chi\left(\mathcal{M}/(\partial_N\mathcal{M})\right) - \chi\left(\ker \partial_N\right) = \dim_{k'}\mathcal{M}'/(\partial_N\mathcal{M}') - \dim_{k'}\ker(\partial'_N) = \chi(\mathcal{M}')$$
$$= \dim_{k'(\theta_N)}\mathcal{M}'(\theta_N) = \dim_{k(\theta)}\mathcal{M}(\theta),$$

where we recognized the first line as the Euler characteristic of the de Rham complex of the $\mathcal{D}_{k'}^1$-module $\mathcal{M}'$ and applied Theorem 69 to get to the last line ($\mathcal{M}' = \mathcal{M}(\theta')$ is holonomic by Lemma 65). So we only need to show that the left-hand side is equal to $\chi(\mathcal{M})$.

This is well known and follows from the Grothendieck spectral sequence.[45] Alternatively, an elementary way to obtain the identity $\chi(\mathcal{M}/(\partial_N\mathcal{M})) - \chi(\ker \partial_N) = \chi(\mathcal{M})$ is given by the long exact sequence

$$\cdots \rightarrow H^{i+1}(\mathrm{DR}(\ker \partial_N)) \rightarrow H^i(\mathrm{DR}(\mathcal{M})) \rightarrow H^i(\mathrm{DR}(\mathcal{M}/\partial_N\mathcal{M}))$$
$$\rightarrow H^{i+2}(\mathrm{DR}(\ker \partial_N)) \rightarrow \cdots \qquad (\mathrm{B.5})$$

---

[45] Let $\pi^N : \mathbb{G}_m^N \longrightarrow \{\mathrm{pt}\}$ denote the projection to a point, such that $\pi^N = \pi^{N-1} \circ \pi$. The identity $\pi_+^N = \pi_+^{N-1} \circ \pi_+$ of the corresponding push-forwards in the derived category of $\mathcal{D}_k^N$-modules implies that $\chi(\mathrm{DR}(\mathcal{M})) = \chi(\pi_+^N(\mathcal{M})) = \sum_i (-1)^i \chi(H^i(\pi_+\mathcal{M}))$, where $H^{-1}(\pi_+\mathcal{M}) = \ker \partial_N$ and $H^0(\pi_+\mathcal{M}) = \mathcal{M}/(\partial_N\mathcal{M})$.

in de Rham cohomology [12, Ch. 2, Proposition 4.13]. □

For completeness, let us demonstrate (B.5), by following the standard construction in the proof of Kashiwara's theorem: First note that

$$\mathcal{N} := \left\{ m \in \mathcal{M} : \partial_N^k m = 0 \quad \text{for some} \quad k > 0 \right\} \subseteq \mathcal{M}$$

defines a $\mathcal{D}_k^N$-submodule of $\mathcal{M}$.[46] The exact sequence $0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{M}/\mathcal{N} \to 0$ of holonomic $\mathcal{D}_k^N$-modules induces a sequence of de Rham complexes, $0 \to \mathrm{DR}(\mathcal{N}) \to \mathrm{DR}(\mathcal{M}) \to \mathrm{DR}(\mathcal{M}/\mathcal{N}) \to 0$, which is also exact ($\mathcal{M} \mapsto \mathrm{DR}(\mathcal{M}) = \Lambda^\bullet k \otimes_k \mathcal{M}$ is exact by flatness of $\Lambda^\bullet k$). Hence, we get a long exact sequence in cohomology:

$$\cdots \to H^i(\mathrm{DR}(\mathcal{N})) \to H^i(\mathrm{DR}(\mathcal{M})) \to H^i(\mathrm{DR}(\mathcal{M}/\mathcal{N}))$$
$$\to H^{i+1}(\mathrm{DR}(\mathcal{N})) \to \cdots \tag{B.6}$$

The key observation now is that ∂_N is injective on $\mathcal{M}/\mathcal{N}$ and surjective on $\mathcal{N}$.[47] We thus get short exact sequences

$$0 \longrightarrow \ker(\partial_N) \hookrightarrow \mathcal{N} \xrightarrow{\partial_N} \mathcal{N} \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \mathcal{M}/\mathcal{N} \xrightarrow{\partial_N} \mathcal{M}/\mathcal{N} \twoheadrightarrow \mathcal{M}/\partial_N\mathcal{M} \longrightarrow 0$$

of $\mathcal{D}_k^{N-1}$-modules. The induced short exact sequences of de Rham complexes provide quasi-isomorphisms $\mathrm{DR}(\ker \partial_N) \simeq \mathrm{DR}(\mathcal{N})[1]$ and $\mathrm{DR}(\mathcal{M}/\partial_N\mathcal{M}) \simeq \mathrm{DR}(\mathcal{M}/\mathcal{N})$, because the de Rham complex of a $\mathcal{D}_k^N$-module $\mathcal{N}$ is the mapping cone of the map $\partial_N : K(\mathcal{N}; \partial') \longrightarrow K(\mathcal{N}; \partial')$. Hence we obtain (B.5) from (B.6) due to

$$H^i(\mathrm{DR}(\mathcal{N})) \cong H^{i+1}(\mathrm{DR}(\ker \partial_N)) \quad \text{and}$$
$$H^i(\mathrm{DR}(\mathcal{M}/\mathcal{N})) \cong H^i(\mathrm{DR}(\mathcal{M}/(\partial_N\mathcal{M}))).$$

To clarify this final step, first note that separating ∂_N from $\partial' := (\partial_1, \ldots, \partial_{N-1})$ yields an isomorphism of k-vector spaces

$$\Phi : K^{\bullet - N}(\mathcal{N}; \partial') \oplus K^{\bullet - 1 - N}(\mathcal{N}; \partial') \xrightarrow{\cong} K^{\bullet - N}(\mathcal{N}; \partial)$$
$$= \bigoplus_{N \notin I} e_I \otimes \mathcal{N} \oplus \bigoplus_{N \in I} e_I \otimes \mathcal{N}$$
$$x \oplus y \qquad\qquad \mapsto \quad x \oplus (e_N \wedge y).$$

---

[46] One only needs to check that $x_N^{\pm 1} \mathcal{N} \subseteq \mathcal{N}$, which follows from $\partial_N^{k+1} x_N^{\pm 1} m = \left[(k+1)(\partial_N x_N^{\pm 1}) + x_N^{\pm 1} \partial_N \right] \partial_N^k m = 0$ whenever $\partial_N^k m = 0$.

[47] The first statement is clear since $\ker \partial_N \subseteq \mathcal{N}$. The second claim follows from the identity $x^k \partial^k = (\partial x - k) x^{k-1} \partial^{k-1} = \cdots = \prod_{i=1}^k (\partial x - i)$, which implies $0 = x_N^k \partial_N^k m = (-1)^k (k!) m \mod \partial_N \mathcal{N}$ whenever $\partial_N^k m = 0$.

In this representation, the differential is given by

$$\Phi^{-1}\left(\mathrm{d}\,\Phi(x \oplus y)\right) = \mathrm{d}'(x) \oplus (\partial_N x - \mathrm{d}'(y)),$$

which is known as the mapping cone of $\partial_N \colon K(\mathcal{N}; \partial') \longrightarrow K(\mathcal{N}; \partial')$. Here, we denote by d' the differential of $K(\mathcal{N}; \partial')$.

If ∂_N is surjective, we can find x with $\partial_N x = y$ and hence $\mathrm{d}\,\Phi(x \oplus 0) = \Phi(\mathrm{d}'x \oplus y)$ for every y. Therefore, every element of $K(\mathcal{N}; \partial)$ has a representative of the form $\Phi(x \oplus 0)$, modulo exact forms. But such a form is closed, $\mathrm{d}\,\Phi(x \oplus 0) = 0$, if and only if $x \in \ker \partial_N \cap \ker \mathrm{d}'$.

Corollary 70 *If ∂_N is surjective, then $K(\mathcal{N}; \partial)$ and $K(\ker \partial_N; \partial')$ [1] are quasi-isomorphic. If ∂_N is injective, then $K(\mathcal{N}; \partial)$ and $K(\mathcal{N}/\partial_N \mathcal{N}; \partial')$ are quasi-isomorphic.*

The proof of the second statement is very similar to the surjective case and left as a straightforward exercise.

C. A two-loop example

We demonstrate some main points of this article by a pedagogical example. The complete results of this calculation can be obtained from https://doi.org/10.5287/bodleian:2RkGjPNG0, "Annihilators of the two-loop master integral". Consider the massless two-loop two-point graph with five propagators, graph WS_3' in Fig. 5. To this graph, we associate the family of integrals

$$
\begin{aligned}
&\mathcal{I}(\nu_1, \ldots, \nu_5) \\
&= \int \frac{\mathrm{d}^d l_1}{\mathrm{i}\pi^{d/2}} \int \frac{\mathrm{d}^d l_2}{\mathrm{i}\pi^{d/2}} \frac{1}{[-l_1^2]^{\nu_1}[-l_2^2]^{\nu_2}[-(l_2 - p)^2]^{\nu_3}[-(l_1 - p)^2]^{\nu_4}[-(l_1 - l_2)^2]^{\nu_5}}
\end{aligned}
\tag{C.1}
$$

with two-loop momenta $q_1 = l_1$, $q_2 = l_2$ and one external momentum $q_3 = p$. We normalize to $-p^2 = 1$. The graph polynomial $\mathcal{G} = \mathcal{U} + \mathcal{F}$ is given by the Symanzik polynomials

$$
\begin{aligned}
\mathcal{U} &= (x_1 + x_4)(x_2 + x_3) + x_5(x_1 + x_2 + x_3 + x_4) \quad \text{and} \\
\mathcal{F} &= x_1 x_2 (x_3 + x_4) + x_3 x_4 (x_1 + x_2) + x_5(x_1 + x_2)(x_3 + x_4).
\end{aligned}
$$

By Definition 6 the modified Feynman integral

$$\widetilde{\mathcal{I}}(\nu_1, \ldots, \nu_5) = \mathcal{M}\left\{\mathcal{G}^s\right\} \tag{C.2}$$

is related to the Feynman integral by

$$\mathcal{I}(\nu_1, \ldots, \nu_5) = \frac{\Gamma(-s)}{\Gamma(-s - \omega)} \widetilde{\mathcal{I}}(\nu_1, \ldots, \nu_5) \tag{C.3}$$

with $\omega = \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + 2s$ and $s = -d/2$.

C.1. From annihilators to integral relations

A set of generators of the annihilator ideal $\mathrm{Ann}_{A_5[s]}(\mathcal{G}^s)$ can be derived in SINGULAR [25] with algorithms introduced in [4]. Using the command `SannfsBM`, we obtain a set of 13 generators. To give an impression, the first five of them read

$$
\begin{aligned}
P_1 &= \partial_1 - \partial_2 + (x_3 + x_4 + 1)(\partial_3 - \partial_4) + (x_3 - x_4)\partial_5, \\
P_2 &= (x_2 - 1)\partial_1 + (-2x_1 - 3x_2 - 1)\partial_2 + (-x_4 + 1)\partial_3 \\
&\quad + (2x_3 + 3x_4 + 1)\partial_4 + (2x_1 - x_2 \\
&\quad - 2x_3 + x_4)\partial_5, \\
P_3 &= (x_1 + 2)\partial_1 + (x_1 + 2x_2)\partial_2 + x_4\partial_3 \\
&\quad + (-2x_3 - 3x_4 - 2)\partial_4 + (-x_1 + 2x_3 - x_4)\partial_5, \\
P_4 &= \partial_1 x_1 + (x_2 + 1)\partial_2 + (-x_3 - x_4 - 1)\partial_3 + (x_4 + x_5)\partial_5 - s, \\
P_5 &= (x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_1 + x_2 + x_3 + x_4)\partial_4 + (-x_1 x_2 - x_1 x_3 - x_1 x_5 \\
&\quad - x_2 x_3 - x_2 x_5 - x_2 - x_3 - x_5)\partial_5.
\end{aligned}
$$

We notice that this set includes one generator which is quadratic in the differential operators, reading

$$
\begin{aligned}
P_{13} &= (x_4 + x_5 - 2)\partial_1^2 + (2x_2 + x_5)\partial_2^2 \\
&\quad + (-2x_3^2 + 2x_3 x_4 - x_3 x_5 + x_4 x_5)\partial_5^2 - x_3 x_4 \partial_3 \partial_5 \\
&\quad + (2x_3^2 + x_3 x_4 - 2x_4^2 + 2x_3 - 2x_4)\partial_4 \partial_5 + (x_3 - x_4 + (-2x_2 - x_4 - 2x_5 + 2)\partial_2 \\
&\quad + (x_4^2 - x_4)\partial_3 + (-x_4^2 + 2x_3 + 3x_4 + 2)\partial_4 \\
&\quad + (x_3 x_4 + x_3 x_5 - x_4^2 - x_4 x_5 - 4x_3 + 2x_4 - x_5)\partial_5)\partial_1 \\
&\quad + (x_3 x_4 \partial_3 - x_3 + x_4 + (-x_3 x_4 - 2x_3 - 2x_4 - 2)\partial_4 + (-2x_2 x_3 + 2x_2 x_4 \\
&\quad - x_3 x_5 + x_4 x_5 + 2x_3 + x_5)\partial_5 + 2)\partial_2 + (-x_3 + x_4)\partial_5.
\end{aligned}
$$

Every operator in $\mathrm{Ann}_{A_5[s]}(\mathcal{G}^s)$ gives rise to an integral relation. According to Lemma 7, we just need to replace each x_i by $\hat{\mathbf{i}}^+$ and each ∂_i by $-\mathbf{i}^-$ to obtain a shift relation between modified Feynman integrals. For example, for the generator P_1, we obtain the shift operator

$$
\mathcal{M}\{P_1\} = -\mathbf{1}^- + \mathbf{2}^- - (\hat{\mathbf{3}}^+ + \hat{\mathbf{4}}^+ + 1)(\mathbf{3}^- - \mathbf{4}^-) - (\hat{\mathbf{3}}^+ - \hat{\mathbf{4}}^+)\mathbf{5}^-
$$

satisfying

$$
\mathcal{M}\{P_1\}\widetilde{\mathcal{I}}(\nu_1, \ldots, \nu_5) = 0.
$$

For the (unmodified) Feynman integral $\mathcal{I}(\nu_1, \ldots, \nu_5)$ we obtain the corresponding shift relation via (2.22). From P_1 we obtain the operator

$$O_1 = \left(\sum_{i=1}^{5} \mathbf{n}_i + 3s\right)(\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^- - \mathbf{4}^-)$$
$$-(\hat{\mathbf{3}}^+ + \hat{\mathbf{4}}^+)(\mathbf{3}^- - \mathbf{4}^-) - (\hat{\mathbf{3}}^+ - \hat{\mathbf{4}}^+)\mathbf{5}^- \tag{C.4}$$

which satisfies

$$O_1 \mathcal{I}(\nu_1, \ldots, \nu_5) = 0. \tag{C.5}$$

C.2. Linear annihilators

It is useful to consider the linear annihilators $\mathrm{Ann}^1_{A^N[s]}(\mathcal{G}^s) \subseteq \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$. Recall that they are the annihilators which are linear differential operators, being of the form $P = q + \sum_{i=1}^{N} p_i \partial_i$ with $p, q_1, \ldots, q_N \in \mathbb{Q}[s, x_1, \ldots, x_N]$. We compute the generators of $\mathrm{Ann}^1(\mathcal{G}^s)$ as generators of the Syzygy-module of $(\mathcal{G}, \partial_1\mathcal{G}, \ldots, \partial_N\mathcal{G})$, using the SINGULAR command syz. For our example we obtain the following 8 generators:

$$L_1 = \partial_1 - \partial_2 + (x_3 + x_4 + 1)\partial_3 + (-x_3 - x_4 - 1)\partial_4 + (x_3 - x_4)\partial_5,$$
$$L_2 = (-x_2 - 1)\partial_1 + (x_2 + 1)\partial_2 + (-x_4 - 1)\partial_3$$
$$\quad + (x_4 + 1)\partial_4 + (x_2 + x_4 + 2x_5)\partial_5 - 2s,$$
$$L_3 = (2x_1 + x_2 + 1)\partial_1 + (x_2 + 1)\partial_2 + (-2x_3 - x_4 - 1)\partial_3 + (-x_4 - 1)\partial_4$$
$$\quad + (-x_2 + x_4)\partial_5,$$
$$L_4 = (x_1 + x_2 + 1)\partial_1 + (-x_1 - x_2 - 1)\partial_2 + \partial_3 - \partial_4 + (x_1 - x_2)\partial_5,$$
$$L_5 = -2sx_4 + (x_4 + x_5)\partial_1 + (-2x_2 - x_4 - x_5 - 2)\partial_2$$
$$\quad + (2x_3x_4 + x_4^2 + 2x_3 + 3x_4 + 2)\partial_3$$
$$\quad + (x_4^2 + x_4)\partial_4 + (-x_4^2 - 2x_4 - x_5)\partial_5,$$
$$L_6 = (-x_2x_3 + x_2x_4 - x_2 - x_3 + 2x_4 + 2x_5)\partial_1$$
$$\quad + (-2x_1x_3 + 2x_1x_4 - x_2x_3 + x_2x_4 + 2x_1$$
$$\quad - 9x_2 - x_3 - 2x_5 - 6)\partial_2$$
$$\quad + (-2x_1x_3 - 2x_1x_4 - x_2x_3 - x_2x_4 - 2x_3^2 - 4x_3x_5 + 2x_4^2$$
$$\quad - 4x_4x_5 - 2x_1 - x_2 + 3x_3 + 6x_4 - 4x_5 + 6)\partial_3$$
$$\quad + (2x_1x_3 + 2x_1x_4 + x_2x_3 + x_2x_4$$
$$\quad + 4x_3x_4 + 4x_3x_5 + 4x_4^2 + 4x_4x_5 + 2x_1 + x_2 + x_3 + 6x_4 + 4x_5)\partial_4$$
$$\quad + (-2x_4^2 - 6x_4$$
$$\quad - 4x_5)\partial_5 + s(-4x_4 + 2),$$
$$L_7 = (2x_1x_2 + 2x_2^2 + 2x_1 + 4x_2 + 2)\partial_1$$
$$\quad + (x_2x_4 + 2x_2 + x_5)\partial_3 + (-x_2x_4 - 2x_2 - 2x_4$$

$$-x_5 - 2)\partial_4 + (-2x_2^2 - x_2x_4 - 2x_2x_5 - 2x_2 - x_5)\partial_5,$$

$$L_8 = (x_4 + x_5)\partial_1 + (2x_1x_2 + 2x_1x_4 + 2x_1x_5$$
$$+ 2x_2^2 + 2x_2x_4 + 2x_2x_5 + 2x_1 + 2x_2 + x_4$$
$$+ x_5)\partial_2 + (-2x_1x_3 - 2x_1x_4 - 2x_3x_5 + x_4^2 - 2x_1 + 2x_3 - x_4)\partial_3$$
$$+ (-x_4^2 - 2x_4x_5 - x_4$$
$$- 2x_5)\partial_4 + (-x_4^2 - 2x_4x_5 - 2x_5^2 + x_5)\partial_5 + s(-2x_2 + 2x_5 - 2),$$

Using SINGULAR we find that for our two-loop example every generator P_i can be expressed as a linear combination of the linear generators L_i over $\mathrm{Ann}_{A^5[s]}(\mathcal{G}^s)$. For instance, the first five annihilators satisfy

$$P_1 = L_1, \quad P_2 = -2L_1 - L_3 + 2L_4, \quad P_3 = 2L_1 + L_3 - L_4, \quad 2P_4 = L_2 + L_3,$$
$$2P_5 = -2(x_1 + x_2 + 1)L_1 + (1 + x_2 - x_4 - x_5)(L_2 - L_3)$$
$$- 2(x_2 + x_4 + x_5)L_4 + L_5 + 2L_7 - L_8.$$

We emphasize that such a relation exists for all the P_i. In particular such a relation also exists for the quadratic P_{13}, which however is too long to be shown here. As a consequence we can view $\mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ as generated by the linear L_i, which will simplify the discussion in Sect. C.4.

C.3. From IBP relations to annihilators

Going in the other direction, we can derive annihilators from momentum space IBP relations. In the usual way, inserting the differential operators

$$\mathbf{o}_j^i = \frac{\partial}{\partial q_i} q_j \quad \text{for} \quad i \in \{1, 2\} \quad \text{and} \quad j \in \{1, 2, 3\} \tag{C.6}$$

we obtain six IBP relations $\mathbf{O}_j^i \mathcal{I} = 0$ with the shift operators

$$\mathbf{O}_1^1 = -\hat{\mathbf{4}}^+\mathbf{1}^- - \hat{\mathbf{5}}^+\mathbf{1}^- + \hat{\mathbf{5}}^+\mathbf{2}^- + \hat{\mathbf{4}}^+ - 2\mathbf{n}_1 - \mathbf{n}_4 - \mathbf{n}_5,$$
$$\mathbf{O}_2^1 = -\hat{\mathbf{1}}^+\mathbf{2}^- + \hat{\mathbf{1}}^+\mathbf{5}^- - \hat{\mathbf{4}}^+\mathbf{1}^- - \hat{\mathbf{4}}^+\mathbf{3}^- + \hat{\mathbf{4}}^+\mathbf{5}^- - \hat{\mathbf{5}}^+\mathbf{1}^- + \hat{\mathbf{5}}^+\mathbf{2}^- + \hat{\mathbf{4}}^+ - \mathbf{n}_1 + \mathbf{n}_5,$$
$$\mathbf{O}_3^1 = -s\hat{\mathbf{1}}^+ + s\hat{\mathbf{4}}^+ - \hat{\mathbf{4}}^+\mathbf{1}^- - \hat{\mathbf{5}}^+\mathbf{1}^- + \hat{\mathbf{5}}^+\mathbf{2}^- - \hat{\mathbf{5}}^+\mathbf{3}^- + \hat{\mathbf{1}}^+\mathbf{4}^- + \hat{\mathbf{5}}^+\mathbf{4}^- - \mathbf{n}_1 + \mathbf{n}_4,$$
$$\mathbf{O}_1^2 = -\hat{\mathbf{2}}^+\mathbf{1}^- + \hat{\mathbf{2}}^+\mathbf{5}^- - \hat{\mathbf{3}}^+\mathbf{2}^- - \hat{\mathbf{3}}^+\mathbf{4}^- + \hat{\mathbf{3}}^+\mathbf{5}^- + \hat{\mathbf{5}}^+\mathbf{1}^- - \hat{\mathbf{5}}^+\mathbf{2}^- + \hat{\mathbf{3}}^+ - \mathbf{n}_2 + \mathbf{n}_5,$$
$$\mathbf{O}_2^2 = -\hat{\mathbf{3}}^+\mathbf{2}^- + \hat{\mathbf{5}}^+\mathbf{1}^- - \hat{\mathbf{5}}^+\mathbf{2}^- + \hat{\mathbf{3}}^+ - 2s - 2\mathbf{n}_2 - \mathbf{n}_3 - \mathbf{n}_5$$
$$\mathbf{O}_3^2 = \hat{\mathbf{2}}^+\mathbf{3}^- - \hat{\mathbf{3}}^+\mathbf{2}^- + \hat{\mathbf{5}}^+\mathbf{1}^- - \hat{\mathbf{5}}^+\mathbf{2}^- + \hat{\mathbf{5}}^+\mathbf{3}^- - \hat{\mathbf{5}}^+\mathbf{4}^- - \hat{\mathbf{2}}^+ - \hat{\mathbf{3}}^+ - \mathbf{n}_2 + \mathbf{n}_3.$$

Following the steps in the proof of Corollary 21, we derive for each shift operator $\mathbf{O}^i_j$ a parametric annihilator $\widetilde{\mathbf{O}}^i_j$. We obtain

$$
\begin{aligned}
\widetilde{\mathbf{O}}^1_1 = {} & (x_1 x_4 + 2x_1 + x_4 + x_5)\partial_1 + (x_2 x_4 - x_5)\partial_2 \\
& + x_3 x_4 \partial_3 - 3s x_4 + (x_4^2 + x_4)\partial_4 + (x_4 x_5 + x_5)\partial_5 - 2s,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{O}}^1_2 = {} & -3s x_4 + (x_1 x_4 + x_1 + x_4 + x_5)\partial_1 \\
& + (x_2 x_4 + x_1 - x_5)\partial_2 + (x_3 x_4 + x_4)\partial_3 + x_4^2 \partial_4 \\
& + (x_4 x_5 - x_1 - x_4 - x_5)\partial_5,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{O}}^1_3 = {} & 3s x_1 - 3s x_4 + (-x_1^2 + x_1 x_4 + x_1 + x_4 + x_5)\partial_1 \\
& + (-x_1 x_2 + x_2 x_4 - x_5)\partial_2 + (-x_1 x_3 \\
& + x_3 x_4 + x_5)\partial_3 + (-x_1 x_4 + x_4^2 - x_1 - x_4 - x_5)\partial_4 + (-x_1 x_5 + x_4 x_5)\partial_5,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{O}}^2_1 = {} & -3s x_3 + (x_1 x_3 + x_2 - x_5)\partial_1 \\
& + (x_2 x_3 + x_2 + x_3 + x_5)\partial_2 + x_3^2 \partial_3 + (x_3 x_4 + x_3)\partial_4 \\
& + (x_3 x_5 - x_2 - x_3 - x_5)\partial_5,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{O}}^2_2 = {} & -3s x_3 + (x_1 x_3 - x_5)\partial_1 + (x_2 x_3 + 2x_2 + x_3 + x_5)\partial_2 \\
& + (x_3^2 + x_3)\partial_3 + x_3 x_4 \partial_4 + (x_3 x_5 \\
& + x_5)\partial_5 - 2s,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{O}}^2_3 = {} & 3s x_2 - 3s x_3 + (-x_1 x_2 + x_1 x_3 - x_5)\partial_1 \\
& + (-x_2^2 + x_2 x_3 + x_2 + x_3 + x_5)\partial_2 + (-x_2 x_3 \\
& + x_3^2 - x_2 - x_3 - x_5)\partial_3 + (-x_2 x_4 + x_3 x_4 + x_5)\partial_4 + (-x_2 x_5 + x_3 x_5)\partial_5.
\end{aligned}
$$

These operators are useful to compare both approaches as discussed next.

C.4. Comparing annihilators and IBP operators

According to Corollary 21, every momentum space IBP relation corresponds to a parametric annihilator. For our two-loop example, this is given by the fact that

$$
\widetilde{\mathbf{O}}^i_j \in \mathrm{Ann}_{A^5[s]}\left(\mathcal{G}^s\right) \quad \text{for } i \in \{1, 2\} \quad \text{and } j \in \{1, 2, 3\}.
$$

We may furthermore ask if the reverse is true: Can every annihilator of $\mathcal{G}$ be derived from IBP relations? If the answer would be no, the approach via parametric annihilators would provide new integral identities. While this question remains open for the general case, we can test it for simple Feynman graphs such as the present two-loop example.

In a first attempt, we could consider the shift relations obtained from the generators $P_1, \ldots, P_{13}$ and try to confirm that they are combined IBP relations. If we use one of the well-known implementations of Laporta's algorithm to reproduce e.g. Eq. (C.5), we have to fix the values of $\nu_1, \ldots, \nu_5$ and do not answer the question for arbitrary values

of the ν_i. We therefore approach the problem on the level of parametric differential operators instead.

We find that actually not all parametric annihilators are contained in Mom; however, they turn out to still be consequences of the momentum space IBP relations in the following sense: While we checked that $P_1 \notin$ Mom, we can find a polynomial $q_1 \in \mathbb{Q}[s, x_1\partial_1, \ldots, x_5\partial_5]$ such that $q_1 P_1 \in$ Mom. Recall that, under the Mellin transform, such a q_1 corresponds to a polynomial in the dimension and in the ν_e. The interesting question then is if we can find a polynomial $q \in \mathbb{Q}[s, x_1\partial_1, \ldots, x_N\partial_N]$ for every $P \in \mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ such that $q P \in$ Mom. If we can find such a q_i for every generator P_i, we can express every annihilator in terms of the $\widetilde{\mathbf{O}}_j^i$. The q_i are the denominators of the coefficients in such a linear combination.

In Sect. 1 we have seen for our example that $\mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ is generated by the linear annihilators L_i. As a consequence, it is sufficient to show that for each L_i there is a $\tilde{q}_i$ such that $\tilde{q}_i L_i \in$ Mom for $i = 1, \ldots, 8$. Indeed we can construct such $q_1, \ldots, q_8$ by an explicit Ansatz. For example we obtain the identity

$$\left(2\sum_{i=1}^{5} x_i\partial_i - 6s\right) L_1 = c_1\widetilde{\mathbf{O}}_1^1 + c_2\widetilde{\mathbf{O}}_1^2 + c_3\widetilde{\mathbf{O}}_2^1 + c_4\widetilde{\mathbf{O}}_2^2 + c_5\widetilde{\mathbf{O}}_3^1 + c_6\widetilde{\mathbf{O}}_3^2$$

with

$$
\begin{aligned}
c_1 = {}&-(-2\partial_1 + \partial_2 - \partial_3 + 2\partial_4 + x_5(\partial_1(s+1) + \partial_2(-s-1) + \partial_5 s)\\
&+ x_4\partial_4 + x_3(\partial_2(-s-1)\\
&- \partial_4 s + \partial_5 s + 3s^2 + 3s + x_5\partial_5(-s-1) + x_4\partial_4(-s-1))\\
&+ x_3^2\partial_3(-s-1) + x_2(-\partial_1 s\\
&+ \partial_2(-s-1) + \partial_5 s + x_3\partial_2(-s-1)) + x_1(\partial_1 + x_3\partial_1(-s-1))),\\
c_2 = {}&-(-\partial_2 + \partial_3 - 2s^2 - 2s + x_5\partial_5(2s+1)\\
&+ x_4(-\partial_3 s + \partial_4(s+1) + \partial_5 s) + x_3\partial_3 + x_2\partial_2\\
&+ x_1(\partial_1(s+1) - \partial_2 s + \partial_5 s)),\\
c_3 = {}&-(\partial_1 + 1 - \partial_4 + s + x_5(\partial_1(-s-1) + \partial_2(s+1) - \partial_5 s) - x_4\partial_4\\
&+ x_3(\partial_2(s+1) + \partial_4 s\\
&- \partial_5 s - 3s^2 - 3s + x_5\partial_5(s+1) + x_4\partial_4(s+1))\\
&+ x_3^2\partial_3(s+1) + x_2(\partial_1 s + \partial_2(s+1)\\
&- \partial_5 s + x_3\partial_2(s+1)) + x_1(-\partial_1 + x_3\partial_1(s+1))),\\
c_4 = {}&-(-\partial_1 + 2\partial_2 - 2\partial_3 + \partial_4 + x_5\partial_5\\
&+ x_4(-\partial_3 + \partial_5) - x_3\partial_3 - x_2\partial_2 + x_1(-\partial_2 + \partial_5)),\\
c_5 = {}&-(\partial_1 - \partial_2 + \partial_5)\\
c_6 = {}&-(\partial_1 - \partial_2 - \partial_5).
\end{aligned}
$$

Using these results and the expressions for the generators of $\mathrm{Ann}_{A^5[s]}(\mathcal{G}^s)$ in terms of the L_i, we can derive every annihilator from the IBP operators. We have done this computation with the same conclusion for several further graphs of low loop-order. These computations support our conjectures phrased in Questions 23 and 24.

C.5. The number of master integrals

The result of the previous subsection implies that the parametric approach and momentum space IBP lead to the same number of master integrals for this example. Indeed, as mentioned in Sect. 4.1, we compute the Euler characteristic

$$\mathfrak{C}(\mathcal{G}) = 3$$

and obtain the same number of master integrals with AZURITE. The three master integrals suggested by AZURITE are

$$I_1 = \mathcal{I}(1, 1, 1, 1, 0), \quad I_2 = \mathcal{I}(0, 1, 0, 1, 1) \quad \text{and} \quad I_3 = \mathcal{I}(1, 0, 1, 0, 1).$$

Notice that symmetries of the graph were not taken into account here, which in AZU-RITE is assured by setting `Symmetry -> False` and `GlobalSymmetry -> False`. For an integral reduction in practice, one would of course make use of the symmetry

$$\mathcal{I}(v_1, v_2, v_3, v_4, v_5) = \mathcal{I}(v_2, v_1, v_4, v_3, v_5)$$

and compute with one of the sets $\{I_1, I_2\}, \{I_1, I_3\}$.

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