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M-theory from the superpoint

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Abstract The "brane scan" classifies consistent Green–Schwarz strings and membranes in terms of the invariant cocycles on super Minkowski spacetimes. The "brane bouquet" generalizes this by consecutively forming the invariant higher central extensions induced by these cocycles, which yields the complete fundamental brane content of string/M-theory, including the D-branes and the M5-brane, as well as the various duality relations between these. This raises the question whether the super Minkowski spacetimes themselves arise as maximal invariant central extensions. Here, we prove that they do. Starting from the simplest possible super Minkowski spacetime, the superpoint, which has no Lorentz structure and no spinorial structure, we give a systematic process of consecutive "maximal invariant central extensions" and show that it discovers the super Minkowski spacetimes that contain superstrings, culminating in the 10- and 11-dimensional super Minkowski spacetimes of string/M-theory and leading directly to the brane bouquet.

Keywords M-theory · Rational homotopy theory · Cohomology of super Lie algebras

Mathematics Subject Classification 81T30 (String and superstring theories; other extended objects) · 55P62 (Rational homotopy theory) · 17B56 (Cohomology of Lie (super)algebras)

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1 Introduction

In his "vision talk" at the annual string theory conference in 2014, Greg Moore highlighted the following open question in string theory [42, Section 9]:

> Perhaps we need to understand the nature of time itself better. […] One natural way to approach that question would be to understand in what sense time itself is an emergent concept, and one natural way to make sense of such a notion is to understand how pseudo-Riemannian geometry can emerge from more fundamental and abstract notions such as categories of branes.

We are going to tell an origin story for spacetime, in which it emerges from the simplest kind of supermanifold: the superpoint, denoted $\mathbb{R}^{0|1}$. This is the supermanifold with no bosonic coordinates, and precisely one fermionic coordinate. From this minimal mathematical space, which has no Lorentz structure and no spinorial structure, we will give a systematic process to construct super Minkowski spacetimes up to dimension 11 and complete with their Lorentz structures and spinorial structures. Indeed, this is the same mathematical mechanism that makes, for instance, the M2-brane and then the M5-brane emerge from 11d spacetime. It is directly analogous to the D0-brane condensation by which 11d spacetime emerges out of the type IIA spacetime of dimension 10.

To make all this precise, first recall that the super p-branes of string theory and M-theory, in their incarnation as "fundamental branes" or "probe branes," are mathematically embodied in terms of what are called "κ-symmetric Green–Schwarz-type functionals." See Sorokin [50] for review and further pointers.

Not long after Green and Schwarz [32] discovered their celebrated action functional for the superstring, Henneaux and Mezincescu observed [33] that the previously somewhat mysterious term in the Green–Schwarz action, the one which ensures its κ-symmetry, is in a fact nothing but the WZW-type functional for super Minkowski spacetime regarded as a supergroup. This is mathematically noteworthy, because WZW-type functionals are a natural outgrowth of super Lie algebra cohomology [18,27]. This suggests that the theory of super p-branes is to some crucial extent a topic purely in super Lie theory, hence amenable to mathematical precision and classification tools.

Indeed, Azcárraga and Townsend [17] later showed (following Achúcarro et al. [1]) that it is the $\mathrm{Spin}(d-1,1)$-invariant super Lie algebra cohomology of super Minkowski spacetime which classifies the Green–Schwarz superstring [32], the

Table 1 The old brane scan

$d \backslash p$	1	2	3	4	5	6	7	8	9
11		⋆							
10	⋆				⋆				
9				⋆					
8			⋆						
7		⋆							
6	⋆		⋆						
5		⋆							
4	⋆	⋆							
3	⋆								

Green–Schwarz-type supermembrane [7], as well as all their double dimensional reductions [22] [29, Section 2], a fact now known as the "old brane scan" [19].[1]

For example, for minimal spacetime supersymmetry there is, up to rescaling, a single non-trivial invariant $(p+2)$-cocycle corresponding to a super p-brane in d dimensional spacetime, for just those pairs of (d, p) with $d \leq 11$ that are marked by an asterisk in Table 1.

Here, the entry at $d = 10$ and $p = 1$ corresponds to the Green–Schwarz superstring, the entry at $d = 10$ and $p = 5$ to the NS5-brane, and the entry at $d = 11$, $p = 2$ to the M2-brane of M-theory fame [21, Chapter II]. Moving down and to the left on the table corresponds to double dimensional reduction [22] [29, Section 2].

This result is striking in its achievement and its failure: On the one hand, it is remarkable that the existence of super p-brane species may be reduced to a mathematical classification of super Lie algebra cohomology. But on the other hand, it is disconcerting that this classification misses so many p-brane species that are thought to exist: The M5-brane in $d = 11$ and all the D-branes in $d = 10$ are absent from the old brane scan, as are all their double dimensional reductions.[2]

However, it turns out that this problem is not a shortcoming of super Lie theory as such, but only of the tacit restriction to ordinary super Lie algebras, as opposed to "higher" super Lie algebras, also called "super Lie n-algebras" or "super L_∞-algebras" [27,36].[3]

One way to think of super Lie n-algebras is as the answer to the following question: Since, by a classical textbook fact, 2-cocycles on a super Lie algebra classify its central extensions in the category of super Lie algebras, what do higher-degree cocycles

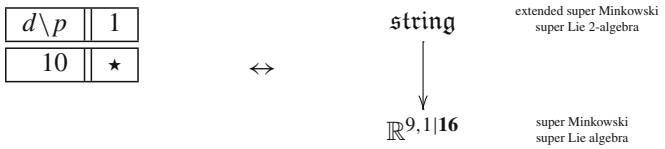
[1] The classification of these cocycles is also discussed by Movshev et al. [43] and Brandt [9–11]. A unified derivation of the cocycle conditions is given by Baez and Huerta [4,5]. See also Foot and Joshi [31].

[2] A partial completion of the old brane scan can be achieved by classifying superconformal structures that may appear in the near horizon geometry of "solitonic" or "black" p-branes [8,20].

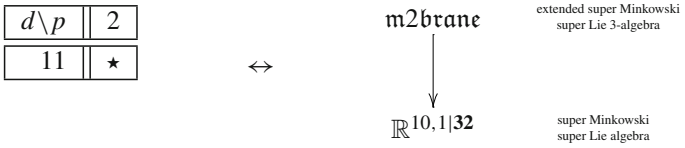
[3] Notice that these are Lie n-algebras in the sense of Stasheff [38,39,47] as originally found in string field theory by Zwiebach [54, Section 4] not "n-Lie algebras" in the sense of Filippov. However, the two notions are not unrelated. At least the Filippov 3-Lie algebras that appear in the Bagger–Lambert model for coincident solitonic M2-branes may naturally be understood as Stasheff Lie 2-algebras equipped with a metric form [44, Section 2].

classify? The answer ([27, Prop. 3.5] based on [25, Theorem 3.1.13] and [3, Theorem 57]) is that higher-degree cocycles classify precisely *higher* central extensions, formed in the homotopy theory of super L_∞-algebras. But in fact the Chevalley–Eilenberg algebras for the canonical models of these higher extensions are well known in parts of the supergravity literature, these are just the "free differential algebras"[4] or "FDA"s of D'Auria and Fré [16].

Hence, every entry in the "old brane scan," since it corresponds to a cocycle, gives a super Lie n-algebraic extension of super Minkowski spacetime. Notably, the 3-cocycles for the superstring give rise to super Lie 2-algebras and the 4-cocycles for the supermembrane give rise to super Lie 3-algebras. These are superalgebraic analogs of the *string Lie 2-algebra* [3] [26, appendix] which controls the Green–Schwarz anomaly cancelation of the heterotic string [48], and hence, they are called the *superstring Lie 2-algebra* [36], to be denoted $\mathfrak{string}$:



and the *supermembrane Lie 3-algebra*, denoted $\mathfrak{m2brane}$:



A discussion of these structures as objects in higher Lie theory appears in Huerta's thesis [36]. Note that $\mathfrak{string}$ comes in several variants, denoted $\mathfrak{string}_{\mathrm{IIA}}$, $\mathfrak{string}_{\mathrm{IIB}}$, and $\mathfrak{string}_{\mathrm{het}}$, corresponding to the type IIA, IIB, and heterotic variants of string theory. In their dual incarnation as "FDAs", the $\mathfrak{string}$ and $\mathfrak{m2brane}$ algebras are the extended super Minkowski spacetimes considered by Chryssomalakos et al. [13]. We follow their idea and call extensions of super Minkowski spacetime to super Lie n-algebras *extended super Minkowski spacetimes*.

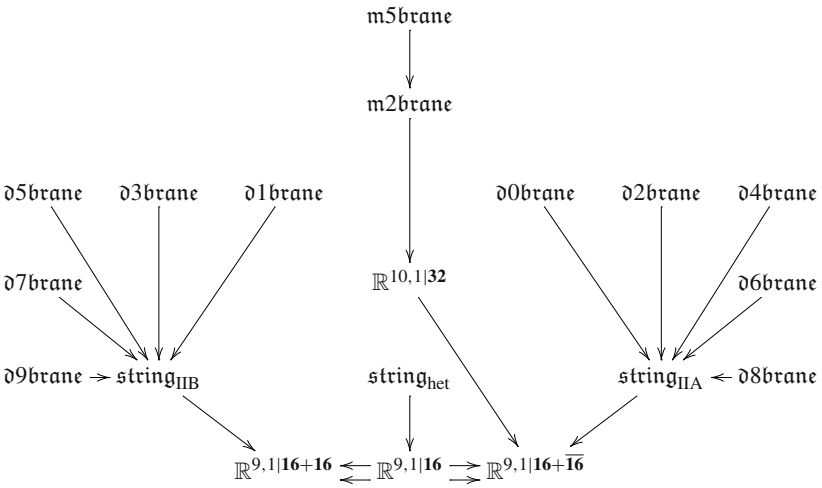
Now that each entry in the old brane scan is identified with a higher super Lie algebra in this way, something remarkable happens: *new* cocycles appear on these extended super Minkowski spacetimes, cocycles which do not show up on plain super Minkowski spacetime itself. (In homotopy theory, this is a familiar phenomenon: It is the hallmark of the construction of the "Whitehead tower" of a topological space.)

And indeed, it turns out that the new invariant cocycles thus found do correspond to the branes that were missing from the old brane scan [27]: On the super Lie 3-

[4] Unfortunately, the "free differential algebras" of D'Auria and Fré are not free. In the parlance of modern mathematics, they are differential graded commutative algebras, where the underlying graded commutative algebra is free, but the differential is not. We will thus refer to them as "FDA"s, with quotes.

algebra $\mathfrak{m2brane}$ there appears an invariant 7-cocycle, which corresponds to the M5-brane, on the super Lie 2-algebra $\mathfrak{string}_{\mathrm{IIA}}$ there appears a sequence of $(p+2)$-cocycles for $p \in \{0, 2, 4, 6, 8\}$, corresponding to the type IIA D-branes, and on the superstring Lie 2-algebra $\mathfrak{string}_{\mathrm{IIB}}$ there appears a sequence of $(p+2)$-cocycles for $p \in \{1, 3, 5, 7, 9\}$, corresponding to the type IIB D-branes. Under the identification of super Lie n-algebras with formal duals of "FDAs," the algebra behind this statement is in fact an old result: For the M5-brane and the type IIA D-branes, this is due to Chryssomalakos et al. [13], while for the type IIB D-branes this is due to Sakaguchi [46, Section 2]. In fact, the 7-cocycle on the supermembrane Lie 3-algebra that corresponds to the M5-brane [6] was already discovered in the 1982 paper by D'Auria and Fré [16, Equations (3.27) and (3.28)].

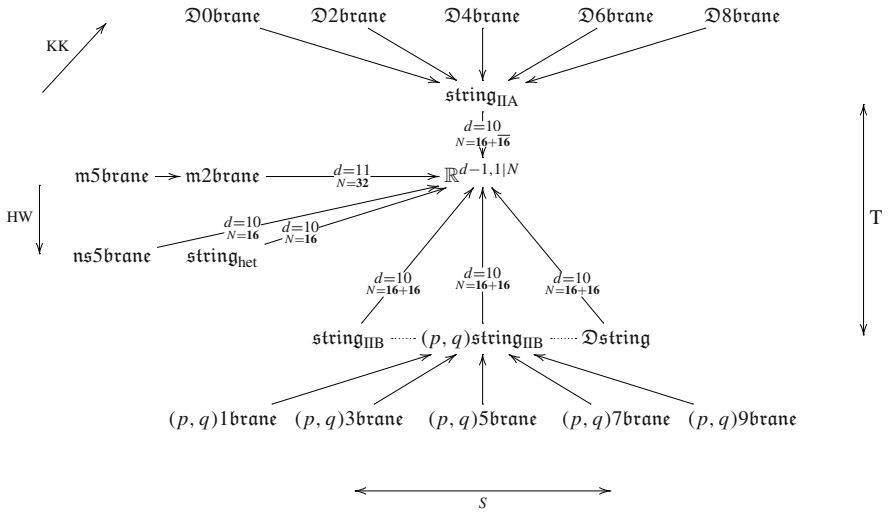
Each of these cocycles gives a super Lie n-algebra extension. If we name these extensions by the super p-brane species whose WZW term is given by the cocycle, then we obtain the following diagram in the category of super L_∞-algebras:



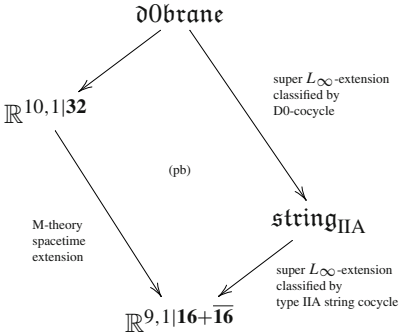
Hence, in the context of higher super Lie algebra, the "old brane scan" is completed to a tree of consecutive higher central extensions emanating out of the super Minkowski spacetimes, with one leaf for each brane species in string/M-theory and with one edge whenever one fundamental brane species may end on another, with its boundary sourcing a vector or tensor multiplet on the world volume of the other brane [27, Section 3]. This is the fundamental *brane bouquet* [27, Def. 3.9 and Section 4.5]. (The *black* branes and their more general intersection laws are obtained from this by passing to equivariant cohomology [37], but this will not concern us here.)

Interestingly, a fair bit of the story of string/M-theory is encoded in this purely super Lie-n-algebraic mathematical structure. This includes in particular the pertinent dualities: the KK-reduction between M-theory and type IIA theory, the HW-reduction between M-theory and heterotic string theory, the T-duality between type IIA and type IIB, the S-duality of type IIB, and the relation between type IIB and F-theory. All of these are reflected as equivalences of super Lie n-algebras obtained from the

brane bouquet [28,29,37]. The diagram of super L_∞-algebras that reflects these L_∞-equivalences looks like a candidate to fill Polchinski's famous schematic picture of M-theory [45, Figure 1] [53, Figure 4] with mathematical life:



Now note that not all of the super p-brane cocycles are of higher degree. One of them, the cocycle for the D0-brane, is an ordinary 2-cocycle. Accordingly, the extension that it classifies is an ordinary super Lie algebra extension. In fact, one finds that the D0-cocycle classifies the central extension of 10-dimensional type IIA super Minkowski spacetime to the 11-dimensional spacetime of M-theory. We can express these relationships by noting the following diagram of super Lie n-algebras is, in the sense of homotopy theory, a "homotopy pullback":



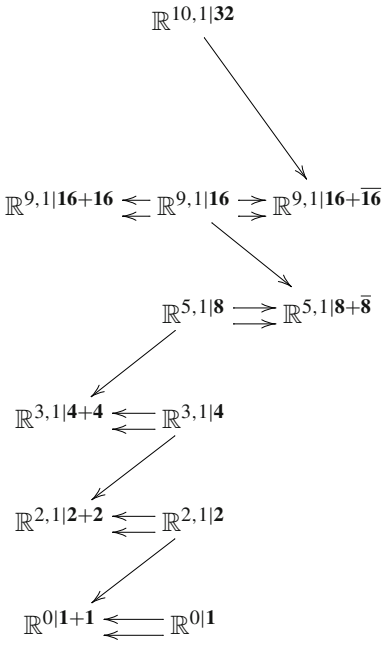
This is the precise way to say that the D0-brane cocycle on $\mathfrak{string}_{\mathrm{IIA}}$ comes from pulling back an ordinary 2-cocycle on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$, which in turn is extended to $\mathbb{R}^{10,1|\mathbf{32}}$ by the same 2-cocycle. We may think of this as a super L_∞-theoretic incarnation of the observation that D0-brane condensation in type IIA string theory leads to the growth of the 11th dimension of M-theory [27, Remark 4.6], as explained by Polchinski [45, Section 6].

This raises an evident question: Might there be a precise sense in which *all* dimensions of spacetime originate from the condensation of some kind of 0-branes in this way? Is the brane bouquet possibly rooted in the superpoint? Such that the ordinary super Minkowski spacetimes, not just extended super Minkowski spacetimes such as $\mathfrak{string}$ and $\mathfrak{m2brane}$, arise from a process of 0-brane condensation "from nothing"?

Since the brane bouquet proceeds at each stage by forming *maximal invariant* extensions, the mathematical version of this question is: Is there a sequence of *maximal invariant* central extensions that start at the superpoint and produce the super Minkowski spacetimes in which superstrings and supermembranes exist?

To appreciate the substance of this question, notice that it is clear that every super Minkowski spacetime is *some* central extension of a superpoint [13, Section 2.1]: the super 2 cocycle classifying this extension is just the superbracket that turns two supercharges into a translation generator. But there are many central extensions of superpoints that are nothing like super Minkowski spacetimes. The question is whether the simple principle of consecutively forming *maximal invariant* central extensions of super Lie algebras (as opposed to more general central extensions) discovers spacetime.

We shall prove that this is the case: This is our main result, Theorem 14. It says that in the following diagram of super Minkowski super Lie algebras, each diagonal morphism is singled out as being the maximal invariant central extension of the super Lie algebra that it points to:[5]



[5] The double arrows stand for the two different canonical inclusions of $\mathbb{R}^{d-1,1|N}$ into $\mathbb{R}^{d-1,1|N+N}$, being the identity on $\mathbb{R}^{d-1,1}$ and sending N identically either to the first or to the second copy in the direct sum $N + N$.

Note that we do *not* specify by hand the groups under which these extensions are to be invariant. Instead, these groups are being discovered stagewise, along with the spacetimes. Namely, we say (Definition 7) that an extension $\widehat{\mathfrak{g}} \to \mathfrak{g}$ is *invariant* if it is invariant with respect to the "simple external automorphisms" inside the automorphism group of $\mathfrak{g}$ (Definition 1). This is a completely intrinsic concept of invariance.

We show that for $\mathfrak{g}$ a super Minkowski spacetime, then this intrinsic group of simple external automorphisms is the spin group, the double cover of the connected Lorentz group in the corresponding dimension—this is Proposition 6, read at the Lie group level. This may essentially be folklore [24, p. 95], but it seems worthwhile to pinpoint this statement. For it says that as the extension process grows out of the superpoint, not only are the super Minkowski spacetimes being discovered as supertranslation supersymmetry groups, but also their Lorentzian metric structure is being discovered alongside.

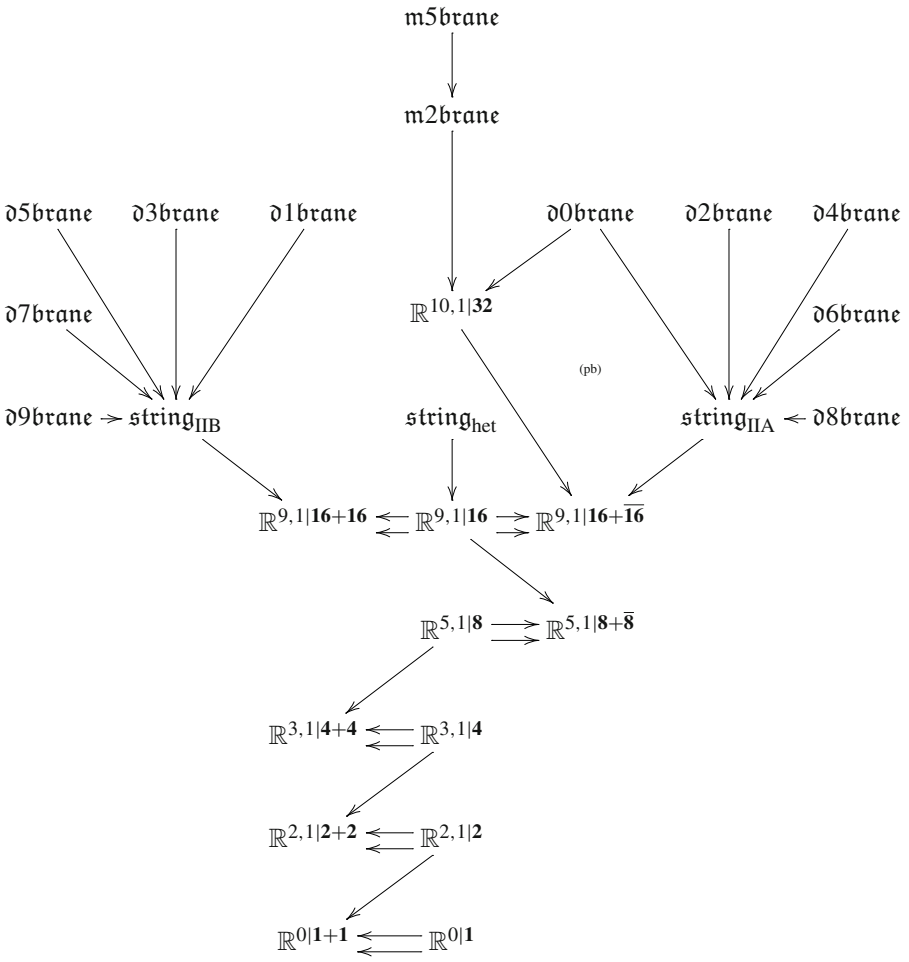
Super Minkowski super Lie algebra	Simple external automorphisms	Induced Cartan geometry	Torsion-freeness
$\mathbb{R}^{d-1,1\|N}$	$\mathrm{Spin}(d-1,1)$	Supergravity	In $d = 11$: Einstein's equations

To highlight this, observe that with every pair (V, G) consisting of a super vector space V and a subgroup $G \subset \mathrm{GL}(V)$ of its general linear supergroup, there is associated a type of geometry, namely the corresponding *Cartan geometry*: A (V, G)-*geometry* is a supermanifold with tangent spaces isomorphic to V and equipped with a reduction of the structure group of its super frame bundle from $\mathrm{GL}(V)$ to G [41].

Now for the pairs $(\mathbb{R}^{d-1,1|N}, \mathrm{Spin}(d-1,1))$ that emerge out of the superpoint according to Proposition 6 and Theorem 14, this is what encodes a field configuration of d-dimensional N-supersymmetric supergravity: Supermanifolds locally modeled on $\mathbb{R}^{d-1,1|N}$ are precisely what underlie the *superspace* formulation of supergravity, and the reduction of its structure group to the $\mathrm{Spin}(d-1,1)$-cover of the connected Lorentz group $\mathrm{SO}_0(d-1,1)$ is equivalently a choice of supervielbein field, which is a field configuration of supergravity.

Observe also that the mathematically most natural condition to demand from such a super Cartan geometry is that it be "torsion-free" [41]. In view of this, it is worthwhile to recall the remarkable theorem of Howe [35], based on Candiello and Lechner [12]: For $d = 11$, the equations of motion of supergravity are *implied* by the torsion-freeness of the supervielbein.

In summary, Theorem 14 shows that the brane bouquet, and with it at least a fair chunk of the structure associated with the word "M-theory," has its mathematical root in the superpoint, and Proposition 6 adds that as the superspacetimes grow out of the superpoint, they consecutively discover their relevant Lorentzian metric structure and spinorial structure, and finally their supergravity equations of motion.



2 Automorphisms of super Minkowski spacetimes

For our main result, Theorem 14, we need to know the automorphisms (Definition 18) of the "super Minkowski super Lie algebras" $\mathbb{R}^{d-1,1|N}$. We give the precise definition of $\mathbb{R}^{d-1,1|N}$ as Definition 22, but for the reader's convenience, we quickly recall the idea. The super Minkowski super Lie algebra $\mathbb{R}^{d-1,1|N}$ is the version of Minkowski spacetime, $\mathbb{R}^{d-1,1}$, used when discussing supersymmetry. Unlike Minkowski spacetime, which is merely a vector space, super Minkowski is a super Lie algebra: It has an underlying vector space that is $\mathbb{Z}_2$-graded, with an even and odd part:

$$\mathbb{R}^{d-1,1|N}_{\text{even}} = \mathbb{R}^{d-1,1}, \quad \mathbb{R}^{d-1,1|N}_{\text{odd}} = N.$$

Here, $\mathbb{R}^{d-1,1}$ is ordinary Minkowski spacetime, while N is a spinor representation of $\mathrm{Spin}(d-1,1)$. The Lie group $\mathrm{Spin}(d-1,1)$ is the double cover of the connected

Lorentz group, $\mathrm{SO}_0(d-1,1)$, so it also acts on $\mathbb{R}^{d-1,1}$. The Lie bracket on this super Lie algebra is nonzero only on N and consists of a pairing turning spinors into vectors:

$$[-,-]\colon N \otimes N \to \mathbb{R}^{d-1,1}$$

which is required to be an equivariant map between representations of $\mathrm{Spin}(d-1,1)$.

Our key idea is that we can extract the Lorentz symmetries of $\mathbb{R}^{d-1,1|N}$ merely from its structure as a super Lie algebra, by looking at a particular piece of the automorphisms we call the "simple external automorphisms." This result may be folklore (see Evans [24, p. 95]), but since we did not find a full account in the literature, we provide a proof here. After some simple lemmas, the result is Proposition 6. To begin, we define the "simple external automorphisms" of a super Lie algebra.

Definition 1 (*External and internal automorphisms, admissible algebras*) Let $\mathfrak{g}$ be a super Lie algebra (Definition 15), and let $\mathfrak{aut}(\mathfrak{g})$ be the ordinary Lie algebra of infinitesimal automorphisms of $\mathfrak{g}$ which preserve the $\mathbb{Z}_2$-grading (Proposition 19). We define the Lie algebra $\mathfrak{int}(\mathfrak{g})$ of *internal automorphisms* of $\mathfrak{g}$ as the Lie subalgebra of $\mathfrak{aut}(\mathfrak{g})$ which acts trivially on the even part $\mathfrak{g}_{\mathrm{even}}$. In other words, it is the Lie subalgebra of even derivations of $\mathfrak{g}$ which vanish on $\mathfrak{g}_{\mathrm{even}}$. This is clearly an ideal, so that the quotient

$$\mathfrak{ext}(\mathfrak{g}) := \mathfrak{aut}(\mathfrak{g})/\mathfrak{int}(\mathfrak{g})$$

of all automorphisms by internal ones is again a Lie algebra, the Lie algebra of *external automorphisms* of $\mathfrak{g}$. We thus have a short exact sequence:

$$0 \to \mathfrak{int}(\mathfrak{g}) \to \mathfrak{aut}(\mathfrak{g}) \to \mathfrak{ext}(\mathfrak{g}) \to 0.$$

We will say that $\mathfrak{g}$ is *admissible* if this sequence splits and the external automorphism algebra $\mathfrak{ext}(\mathfrak{g})$ is reductive. For an admissible algebra $\mathfrak{g}$, we can thus view $\mathfrak{ext}(\mathfrak{g})$ as a subalgebra of $\mathfrak{aut}(\mathfrak{g})$. Moreover, because we demand $\mathfrak{ext}(\mathfrak{g})$ be reductive, $\mathfrak{ext}(\mathfrak{g})$ decomposes as a direct sum of its center and its maximal semisimple Lie subalgebra. We thus define the *simple external automorphisms*

$$\mathfrak{ext}_{\mathrm{simp}}(\mathfrak{g}) \hookrightarrow \mathfrak{ext}(\mathfrak{g}) \hookrightarrow \mathfrak{aut}(\mathfrak{g})$$

to be the semisimple part of $\mathfrak{ext}(\mathfrak{g})$.

Example 2 The internal automorphisms (Definition 1) of the super Minkowski super Lie algebra $\mathbb{R}^{d-1,1|N}$ are the "R-symmetries" from the physics literature [30, p. 56].

Because the super Minkowski super Lie algebra $\mathbb{R}^{d-1,1|N}$ is built from $\mathrm{Spin}(d-1,1)$ representations and $\mathrm{Spin}(d-1,1)$-equivariant maps, $\mathrm{Spin}(d-1,1)$ acts on this super Lie algebra by automorphism. It thus acts on the full automorphism group $\mathrm{Aut}(\mathbb{R}^{d-1,1|N})$ by conjugation and on the Lie algebra $\mathfrak{aut}(\mathbb{R}^{d-1,1|N})$ by the adjoint action. These facts are key for our first lemma.

Lemma 3 *Consider a super Minkowski super Lie algebra* $\mathbb{R}^{d-1,1|N}$ *(Definition 22) in any dimension* $d \geq 3$ *and for any real spinor representation* N *of* $\mathrm{Spin}(d-1,1)$. *Then, the automorphism Lie algebra* $\mathfrak{aut}(\mathbb{R}^{d-1,1|N})$ *(Proposition 19) is the graph of a surjective,* $\mathrm{Spin}(d-1,1)$*-equivariant Lie algebra homomorphism*

$$K : \mathfrak{g}_s \longrightarrow \mathfrak{g}_v,$$

where $\mathfrak{g}_s \subseteq \mathfrak{gl}(N)$ *and* $\mathfrak{g}_v \subseteq \mathfrak{gl}(\mathbb{R}^{d-1,1})$ *are the projections of* $\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \subseteq \mathfrak{gl}(N) \oplus \mathfrak{gl}(\mathbb{R}^{d-1,1})$ *onto the summands. Here,* K *is equivariant with respect to the adjoint action of* $\mathrm{Spin}(d-1,1)$ *restricted to* $\mathfrak{g}_s$ *and* $\mathfrak{g}_v$.

In particular, the kernel of K *is the internal automorphism algebra (Definition 1), also known as the R-symmetry algebra (Example 2):*

$$\ker(K) \simeq \mathrm{int}(\mathbb{R}^{d-1,1|N}).$$

Proof We will consider the corresponding inclusion at the level of groups

$$\mathrm{Aut}(\mathbb{R}^{d-1,1|N}) \hookrightarrow \mathrm{GL}(N) \times \mathrm{GL}(\mathbb{R}^{d-1,1})$$

with projections $G_s \subseteq \mathrm{GL}(N)$ and $G_v \subseteq \mathrm{GL}(\mathbb{R}^{d-1,1})$. The result will then follow by differentiation.

Note that the spinor-to-vector pairing

$$[-,-]\colon N \otimes N \to \mathbb{R}^{d-1,1}$$

is surjective, because it is a nonzero map of $\mathrm{Spin}(d-1,1)$-representations, and $\mathbb{R}^{d-1,1}$ is irreducible for $d \geq 3$. Hence, for every vector $v \in \mathbb{R}^{d-1,1}$, there is a pair of spinors $\psi, \phi \in N$ such that

$$v = [\psi, \phi].$$

It follows that for any automorphism $(f, g) \in \mathrm{Aut}(\mathbb{R}^{d-1,1|N}) \subseteq G_s \times G_v$, g is uniquely determined by f because (f, g) is an automorphism:

$$g(v) = [f(\psi), f(\phi)].$$

This determines a function $k : G_s \to G_v$ sending f to g. It is surjective by construction of G_v and is a group homomorphism because its graph $\mathrm{Aut}(\mathbb{R}^{d-1,1|N})$ is a group. Finally, conjugating (f, g) by an element of $\mathrm{Spin}(d-1,1)$, it is a quick calculation to check that k is $\mathrm{Spin}(d-1,1)$-equivariant, using the equivariance of the spinor-to-vector pairing $[-,-]$. □

Lemma 4 *Let* $\mathfrak{g}_s$ *be as in Lemma 3. Then,* $\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \simeq \mathfrak{g}_s$ *as Lie algebras.*

Proof Because $\mathfrak{aut}(\mathbb{R}^{d-1,1|N})$ is the graph of the homomorphism $K : \mathfrak{g}_s \to \mathfrak{g}_v$ from Lemma 3, it is isomorphic to the domain of this homomorphism, $\mathfrak{g}_s$. □

Lemma 5 *Let N be a real spinor representation of* $\mathrm{Spin}(d-1,1)$ *in some dimension* $d \geq 3$. *Then, the Lie algebra* $\mathfrak{g}_v$ *from Lemma 3 decomposes as a* $\mathrm{Spin}(d-1,1)$*-representation into the direct sum of the adjoint representation with the trivial representation:*

$$\mathfrak{g}_v \simeq \mathfrak{so}(d-1,1) \oplus \mathbb{R}.$$

Similarly, the Lie algebra $\mathfrak{g}_s$ *from Lemma 3 decomposes as a direct sum of exterior powers of the vector representation* $\mathbb{R}^{d-1,1}$*:*

$$\mathfrak{g}_s \simeq \bigoplus_i \Lambda^{n_i} \mathbb{R}^{d-1,1}.$$

Proof First assume that N is a Majorana spinor representation as in Example 25, and consider $\mathfrak{g}_s$. Since the Majorana representation N is a real subrepresentation of a complex Dirac representation $\mathbb{C}^{\dim_{\mathbb{R}}(N)}$, there is a canonical $\mathbb{R}$-linear inclusion

$$\mathrm{End}_{\mathbb{R}}(N) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\dim_{\mathbb{R}}(N)}).$$

Therefore, it is sufficient to note that the space of endomorphisms of the Dirac representation over the complex numbers decomposes into a direct sum of exterior powers of the vector representation. This is indeed so, thanks to the inclusion:

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\dim_{\mathbb{R}}(N)}) \hookrightarrow \mathrm{Cl}(\mathbb{R}^{d-1,1}) \otimes \mathbb{C}.$$

Explicitly, in terms of the Dirac Clifford basis of Example 25, the decomposition is given by the usual component formula:

$$\psi \otimes \overline{\phi} \;\mapsto\; \overline{\phi}\psi + \left(\overline{\phi}\Gamma_a \psi\right)\Gamma^a + \tfrac{1}{2}\left(\overline{\phi}\Gamma_{ab}\psi\right)\Gamma^{ab} + \tfrac{1}{3!}\left(\overline{\phi}\Gamma_{a_1a_2a_3}\psi\right)\Gamma^{a_1a_2a_3} + \cdots .$$

Now consider $\mathfrak{g}_v$. Recall that, by definition, the automorphism group of $\mathbb{R}^{d-1,1|N}$ is

$$\mathrm{Aut}(\mathbb{R}^{d-1,1|N}) := \Big\{ (f,g) \in \mathrm{GL}(N) \times \mathrm{GL}(\mathbb{R}^{d-1,1}) \\ : \; [f(\psi), f(\phi)] = g[\psi,\phi] \text{ for } \psi,\phi \in N \Big\}$$

and its Lie algebra is

$$\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) = \Big\{ (X,Y) \in \mathfrak{gl}(N) \oplus \mathfrak{gl}(\mathbb{R}^{d-1,1}) \\ : \; [X\psi,\phi] + [\psi, X\phi] = Y[\psi,\phi] \text{ for } \psi,\phi \in N \Big\}.$$

As we noted above, $\mathrm{Aut}(\mathbb{R}^{d-1,1|N})$ always contains $\mathrm{Spin}(d-1,1)$, acting canonically, since the spinor-to-vector pairing is $\mathrm{Spin}(d-1,1)$-equivariant. Another subgroup of automorphisms that exists generally is a copy of the multiplicative group of real

numbers $\mathbb{R}^\times$ where $t \in \mathbb{R}^\times$ acts on spinors ψ as rescaling by t and on vectors v as rescaling by t^2:

$$\psi \mapsto t\psi, \quad v \mapsto t^2 v.$$

The Lie algebra of this scaling action is the scaling derivations of Example 20. Hence, for all d and N we have the obvious Lie algebra inclusion

$$\mathfrak{so}(d-1,1) \oplus \mathbb{R} \hookrightarrow \mathfrak{aut}(\mathbb{R}^{d-1,1|N}).$$

This shows that there is an inclusion

$$\mathfrak{so}(d-1,1) \oplus \mathbb{R} \hookrightarrow \mathfrak{g}_v \hookrightarrow \mathfrak{gl}(\mathbb{R}^d).$$

Hence, it now only remains to see that there is no further summand in $\mathfrak{g}_v$. But we know that there is at most one further summand in $\mathfrak{gl}(\mathbb{R}^{d-1,1})$, since this decomposes in the form

$$\mathfrak{gl}(\mathbb{R}^{d-1,1}) \simeq \mathfrak{so}(d-1,1) \oplus \mathbb{R} \oplus \mathrm{Sym}^2_0(\mathbb{R}^{d-1,1}),$$

where $\mathrm{Sym}^2_0(\mathbb{R}^{d-1,1})$ denotes the space of traceless, symmetric $d \times d$ matrices. It follows that the only further summand that could appear in $\mathfrak{g}_v$ is $\mathrm{Sym}^2_0(\mathbb{R}^{d-1,1})$. But by Lemma 3, the homomorphism $K : \mathfrak{g}_s \to \mathfrak{g}_v$ is surjective, so its image $\mathfrak{g}_v$ must be a subset of the exterior powers appearing in $\mathfrak{g}_s$. Since the symmetric traceless matrices and the exterior powers $\Lambda^\bullet \mathbb{R}^d$ are distinct irreducible representations of $\mathrm{Spin}(d-1,1)$, we conclude $\mathrm{Sym}^2_0(\mathbb{R}^{d-1,1})$ is not a summand of $\mathfrak{g}_v$.

This concludes the proof for the case that N is a Majorana representation. The argument for N symplectic Majorana (Example 25) is similar. Finally, a general real spin representation is a direct multiple of N or a sum of multiples of the two Weyl representations $N \simeq N_- \oplus N_+$. We generalize to these cases in turn.

First, we consider nN, a direct multiple of N, for n some nonnegative integer. Since $\mathrm{End}_{\mathbb{R}}(nN) \simeq n^2 \mathrm{End}_{\mathbb{R}}(N)$, the left-hand side is indeed a sum of exterior powers.

Next, if N decomposes as $N_- \oplus N_+$, a general spin representation is of the form $n_- N_- \oplus n_+ N_+$, for n_- and n_+ nonnegative integers. We wish to show that

$$\begin{aligned}\mathrm{End}_{\mathbb{R}}(n_- N_- \oplus n_+ N_+) \simeq\ & n_-^2 \mathrm{End}_{\mathbb{R}}(N_-) \oplus n_- n_+ \mathrm{Hom}_{\mathbb{R}}(N_-, N_+) \\ & \oplus n_+ n_- \mathrm{Hom}_{\mathbb{R}}(N_+, N_-) \oplus n_+^2 \mathrm{End}_{\mathbb{R}}(N_+)\end{aligned}$$

is a sum of exterior powers. Yet we have already shown that

$$\mathrm{End}_{\mathbb{R}}(N_- \oplus N_+) \simeq \mathrm{End}_{\mathbb{R}}(N_-) \oplus \mathrm{Hom}_{\mathbb{R}}(N_-, N_+) \oplus \mathrm{Hom}_{\mathbb{R}}(N_+, N_-) \oplus \mathrm{End}_{\mathbb{R}}(N_+)$$

is a sum of exterior powers. Thus, every summand on the right-hand side is a sum of exterior powers, and it follows that $\mathrm{End}_{\mathbb{R}}(n_- N_- \oplus n_+ N_+)$ is also. □

Proposition 6 *For any dimension $d \geq 3$ and real spinor representation N of* $\mathrm{Spin}(d-1,1)$*, the super Minkowski super Lie algebra $\mathbb{R}^{d-1,1|N}$ (Definition 22) is admissible (Definition 1). Moreover, the Lie algebra of external automorphisms (Definition 1) of $\mathbb{R}^{d-1,1|N}$ is the direct sum:*

$$\mathfrak{ext}(\mathbb{R}^{d-1,1|N}) \simeq \mathfrak{so}(d-1,1) \oplus \mathbb{R},$$

where $\mathfrak{so}(d-1,1)$ acts in the canonical way on $\mathbb{R}^{d-1,1|N}$ (Definition 22) and $\mathbb{R}$ acts by the scaling action from Example 20.

Proof The admissibility of $\mathbb{R}^{d-1,1|N}$ will follow when we determine $\mathfrak{ext}(\mathbb{R}^{d-1,1|N})$ has the form claimed, since this form is reductive, and the action of $\mathfrak{ext}(\mathbb{R}^{d-1,1|N})$ on $\mathbb{R}^{d-1,1|N}$ described in the proposition gives the splitting. So we prove this form is correct. By Lemma 4, we have $\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \simeq \mathfrak{g}_s$, and by Lemma 5, we have a decomposition as $\mathrm{Spin}(d-1,1)$-representations

$$\mathfrak{aut}(\mathbb{R}^{d-1,1|N}) \simeq (\mathfrak{so}(d-1,1) \oplus \mathbb{R}) \oplus \underbrace{\ker(K)}_{=\mathfrak{int}(\mathbb{R}^{d-1,1|N})},$$

where the last summand is the algebra of internal automorphisms (Definition 1) and hence the R-symmetries (Example 2). Therefore, the claim follows by Definition 1. □

3 The maximal invariant central extensions of the superpoint

With the results from the previous section in hand, we have a way of talking about the $\mathfrak{so}(d-1,1)$ symmetries of a super Minkowski super Lie algebra $\mathbb{R}^{d-1,1|N}$ *purely in terms of its Lie bracket*: It is the algebra of simple external automorphisms of $\mathbb{R}^{d-1,1|N}$, by Proposition 6. This allows us to begin with a super Lie algebra that lacks any apparent relation to spacetime, and discover spacetime symmetries via the automorphisms. We make repeated use of this in our construction of super Minkowski spacetimes by central extension of the superpoint, $\mathbb{R}^{0|1}$. This is our main result, Theorem 14.

To be precise, we compute consecutive "maximal invariant central extensions" of the superpoint. First, we state the definition of the extension process:

Definition 7 (*maximal invariant central extensions*) Let $\mathfrak{g}$ be an admissible super Lie algebra (Definitions 1 and 15), let $\mathfrak{h} \hookrightarrow \mathfrak{aut}(\mathfrak{g})$ be a subalgebra of its automorphism Lie algebra (Proposition 19) and let

$$\begin{array}{ccc} V & \hookrightarrow & \widehat{\mathfrak{g}} \\ & & \downarrow \\ & & \mathfrak{g} \end{array}$$

be a central extension of $\mathfrak{g}$ by a vector space V in even degree. Then, we say that $\widehat{\mathfrak{g}}$ is

1. an $\mathfrak{h}$-*invariant central extension* if the even 2-cocycles that classify the extension, according to Example 17, are $\mathfrak{h}$-invariant 2-cocycles as in Definition 21;

2. an *invariant central extension* if it is $\mathfrak{h}$-invariant and $\mathfrak{h} = \mathfrak{ext}_{\text{simp}}(\mathfrak{g})$ is the semisimple part of its external automorphism Lie algebra (Definition 1);
3. a *maximal* $\mathfrak{h}$*-invariant central extension* if it is an $\mathfrak{h}$-invariant central extension such that the n-tuple of $\mathfrak{h}$-invariant even 2-cocycles that classifies it (according to Example 17) is a linear basis for the even $\mathfrak{h}$-invariant cohomology $H^2_{\text{even}}(\mathfrak{g}, \mathbb{R})^{\mathfrak{h}}$ (Definition 21).

When the central extension $\widehat{\mathfrak{g}}$ is both maximal and invariant, we say it is a *maximal invariant central extension* and distinguish it with the symbol $\star$ on the projection map $\widehat{\mathfrak{g}} \to \mathfrak{g}$, like this:

$$\begin{array}{ccc} V & \hookrightarrow & \widehat{\mathfrak{g}} \\ & & \downarrow \star \\ & & \mathfrak{g} \end{array}$$

We now begin to climb the tower of maximal invariant central extensions, beginning with the superpoint. But first we must note that our starting point is admissible.

Lemma 8 *The superpoint* $\mathbb{R}^{0|N}$ *(Definition 24) is admissible (Definition 1) for any natural number* $N \in \mathbb{N}$.

Proof All automorphisms of $\mathbb{R}^{0|N}$ are internal (Definition 1), so the external automorphisms are trivial. Hence, they are trivially reductive, and trivially a subalgebra of $\mathfrak{aut}(\mathbb{R}^{0|N})$, which is what it means to be admissible. □

In our next proposition, we see spacetime appear by extending a superpoint.

Proposition 9 *The maximal invariant central extension (Definition 7) of the superpoint* $\mathbb{R}^{0|2}$ *(Definition 24) is the three-dimensional super Minkowski super Lie algebra* $\mathbb{R}^{2,1|\mathbf{2}}$ *as in Definition 22:*

$$\begin{array}{ccc} \mathbb{R}^3 & \hookrightarrow & \mathbb{R}^{2,1|\mathbf{2}} \\ & & \downarrow \star \\ & & \mathbb{R}^{0|2} \end{array}$$

with $N = \mathbf{2}$ *the unique irreducible real spinor representation of* $\mathrm{Spin}(2,1)$ *from Proposition 32.*

Proof Since $\mathbb{R}^{0|2}$ is concentrated in odd degree, the external automorphisms are trivial: $\mathfrak{ext}(\mathbb{R}^{0|2}) = 0$. Thus, every central extension is invariant (Definition 7).

According to Example 17, the maximal central extension is the one induced by the all of the even super Lie algebra 2-cocycles on $\mathbb{R}^{0|2}$. Since $\mathbb{R}^{0|2}$ is concentrated in odd degree and has trivial Lie bracket, an even 2-cocycle in this case is given by a symmetric bilinear form on $\mathbb{R}^2$. There is a three-dimensional real vector space of these. This shows that the underlying supervector space of the maximal central extension is $\mathbb{R}^{3|2}$. It remains to check that the Lie bracket is that of 3d super Minkowski.

If we let $\{d\theta_1, d\theta_2\}$ denote the canonical basis of the dual space $\mathbb{R}^{0|2*}$, then the space of even 2-cocycles is spanned by:

$$d\theta^1 \wedge d\theta^1 \quad d\theta^1 \wedge d\theta^2$$
$$d\theta^2 \wedge d\theta^2 \ ,$$

where the wedge product is symmetric between these odd elements. By the formula for the central extension from Example 17, this means that the super Lie bracket is given on the spinors $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ by

$$[\psi, \phi] = \begin{pmatrix} \psi_1\phi_1 & \frac{1}{2}(\psi_1\phi_2 + \phi_1\psi_2) \\ \frac{1}{2}(\psi_1\phi_2 + \phi_1\psi_2) & \psi_2\phi_2 \end{pmatrix} \quad = \quad \tfrac{1}{2}\left(\psi\phi^\dagger + \psi\phi^\dagger\right),$$

Comparing this formula to Proposition 32, we see this is indeed the spinor-to-vector pairing for the real representation **2** of Spin(2, 1). □

To deduce the maximal invariant central extensions of $\mathbb{R}^{2,1|2}$, we use the representation of spinors via the normed division algebras as a key tool. We give all the details in Proposition 32 of our appendix, but for the reader's convenience, we quickly summarize the idea.

There are four real normed division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. They have dimensions 1, 2, 4, and 8, respectively. It is a famous fact that the octonions $\mathbb{O}$ are not associative, while $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are. For $\mathbb{K}$ a normed division algebra of dimension k, we can construct spinors for spacetime of dimension $k + 2$. More precisely, we can cook up two irreducible, real spinor representations of the spin group $\mathrm{Spin}(k + 1, 1)$, the double cover of the connected Lorentz group $\mathrm{SO}_0(k + 1, 1)$. Both of these spinor representations are defined on the vector space $\mathbb{K}^2$, but they differ in the action of $\mathrm{Spin}(k + 1, 1)$:

$$N_+ = \mathbb{K}^2, \quad N_- = \mathbb{K}^2.$$

We can also define Minkowski spacetime itself in terms of $\mathbb{K}$, as the space of 2×2 hermitian matrices over $\mathbb{K}$:

$$\mathbb{R}^{k+1,1} := \left\{ \begin{bmatrix} t + x & \overline{y} \\ y & t - x \end{bmatrix} : t, x \in \mathbb{R},\ y \in \mathbb{K} \right\},$$

where $\overline{y} \in \mathbb{K}$ is denotes the conjugate of $y \in \mathbb{K}$. For the more details, see Proposition 32.

Our next lemma relates the construction of spinors from $\mathbb{K}$ and from the 'Cayley–Dickson double', $\mathbb{K}_{\mathrm{dbl}}$ (Definition 26). Roughly, the Cayley–Dickson double takes a normed division algebra $\mathbb{K}$ of dimension k and gives the "next" normed division algebra $\mathbb{K}_{\mathrm{dbl}}$ of dimension $2k$:

$$\mathbb{R}_{\mathrm{dbl}} = \mathbb{C}, \quad \mathbb{C}_{\mathrm{dbl}} = \mathbb{H}, \quad \mathbb{H}_{\mathrm{dbl}} = \mathbb{O}.$$

This process breaks down for $\mathbb{K} = \mathbb{O}$, when $\mathbb{O}_{\text{dbl}}$ fails to be a division algebra.

In any case, the Cayley–Dickson double contains the original algebra as a subalgebra, $\mathbb{K} \subseteq \mathbb{K}_{\text{dbl}}$. This means that the 2×2 hermitian matrices over $\mathbb{K}$ are a subset of those over $\mathbb{K}_{\text{dbl}}$, and hence, there is an inclusion of spacetimes $\mathbb{R}^{k+1,1} \subseteq \mathbb{R}^{2k+1,1}$, and a corresponding inclusion of spin groups $\text{Spin}(k+1,1) \subseteq \text{Spin}(2k+1,1)$. By restricting along this inclusion, spinor representations of $\text{Spin}(2k+1,1)$ become representations of $\text{Spin}(k+1,1)$. The next lemma tells us precisely which representations we obtain in this way.

Lemma 10 *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ be an associative normed division algebra (Example 27) of dimension k, and let $\mathbb{K}_{\text{dbl}} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be its Cayley–Dickson double (Definition 26) of dimension $2k$. Let N_+ and N_- be the real spinor representations defined in terms of $\mathbb{K}$, and let N_{dbl} denote either of the real spinor representations defined in terms of $\mathbb{K}_{\text{dbl}}$, as in Proposition 32. Consider the inclusion of spin groups $\text{Spin}(k+1,1) \subseteq \text{Spin}(2k+1,1)$ induced by the inclusion of normed division algebras $\mathbb{K} \subseteq \mathbb{K}_{\text{dbl}}$. Restricting along this inclusion, the irreducible real $\text{Spin}(2k+1,1)$-representation N_{dbl} branches into the direct sum of the two irreducible real $\text{Spin}(k+1,1)$-representations N_+, N_-:*

$$N_{\text{dbl}} \simeq N_+ \oplus N_-.$$

Proof We will prove the result for $N_{\text{dbl}+}$, as the argument for $N_{\text{dbl}-}$ will be similar. By Proposition 32, the spin representation $N_{\text{dbl}+}$ is defined on the real vector space $\mathbb{K}^2_{\text{dbl}}$. By Cayley–Dickson doubling (Definition 26), this is given in terms of $\mathbb{K}$ as the direct sum

$$\mathbb{K}^2_{\text{dbl}} \simeq \mathbb{K}^2 \oplus \mathbb{K}^2 \ell.$$

This makes it immediate that the first summand $\mathbb{K}^2$ is N_+ as a representation of $\text{Spin}(k+1,1)$. We need to show that the second summand is isomorphic to N_-.

To that end observe, by the relations in the Cayley–Dickson construction (Definition 26), that for $\psi \in \mathbb{K}^2$ and $A \in \mathfrak{h}_2(\mathbb{K})$ a 2×2 hermitian matrix, we have the following identity:

$$\begin{aligned} A(\psi\ell) &= A(\ell\overline{\psi}) \\ &= \ell(\overline{A}\,\overline{\psi}) \\ &= \ell(\overline{A}_L\,\overline{\psi}) \\ &= \ell(\overline{A_R\psi}) \\ &= (A_R\psi)\ell, \end{aligned}$$

where A_L and A_R denote the right and left actions of the matrix A, respectively (Definition 28), and we have used Proposition 30 to relate left and right actions under conjugation.

Recall from Proposition 32 that $\text{Spin}(k+1,1)$ is the subgroup of the Clifford algebra generated by products of pairs of unit vectors of the same sign:

$$\mathrm{Spin}(k+1,1) = \langle AB \in \mathcal{C}\ell(k+1,1) :\ A, B \in \mathbb{R}^{k+1,1},\ \eta(A,A) = \eta(B,B) = \pm 1\rangle.$$

It follows from our above calculation that the action of a generator $AB \in \mathrm{Spin}(k+1,1)$ on the summand $\mathbb{K}^2\ell$ is the composition of right actions on $\mathbb{K}^2$:

$$\begin{aligned}\tilde{A}_L B_L(\psi\ell) &= \tilde{A}(B(\psi\ell)) \\ &= \left(\tilde{A}_R B_R(\psi)\right)\ell.\end{aligned}$$

Therefore, we are now reduced to showing that this action of $\mathrm{Spin}(k+1,1)$ on $\mathbb{K}^2$:

$$\psi \mapsto \tilde{A}_R B_R(\psi) \quad \text{for} \quad \psi \in \mathbb{K}^2$$

is isomorphic to the action of $\mathrm{Spin}(k+1,1)$ on N_-, which also has the underlying vector space $\mathbb{K}^2$:

$$\psi \mapsto A_L \tilde{B}_L(\psi) \text{ for } \psi \in \mathbb{K}^2.$$

We claim there is an isomorphism given by

$$F\colon \psi \mapsto J\overline{\psi},$$

where J is the matrix:

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A quick calculation shows that J satisfies the matrix identity:

$$J\overline{A} = -\tilde{A}J$$

for any $A \in \mathfrak{h}_2(\mathbb{K})$ a 2×2 hermitian matrix. We use this to show that F is indeed an isomorphism:

$$\begin{aligned}F(\tilde{A}_R B_R(\psi)) &= J\,\overline{\tilde{A}_R B_R(\psi)} \\ &= J\,\overline{\tilde{A}}_L\,\overline{B}_L(\overline{\psi}) \\ &= J\,\overline{\tilde{A}}(\overline{B}\,\overline{\psi}) \\ &= A(\tilde{B}J\overline{\psi}) \\ &= A_L\tilde{B}_L(F(\psi)).\end{aligned}$$

□

Next, for our spinor representation $N_\pm$ constructed from a normed division algebra, we need to know certain invariants of the spin group action.

Lemma 11 *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ *be a normed division algebra (Example 27) of dimension k, and let* $N_\pm$ *be the real* $\mathrm{Spin}(k+1, 1)$ *spinor representations from Proposition 32. Then,*

1. $(\mathrm{End}(N_\pm))^{\mathrm{Spin}} \simeq \begin{cases} \mathbb{K} & \text{if} \quad \mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\} \\ \mathbb{R} & \text{if} \quad \mathbb{K} = \mathbb{O} \end{cases}$
2. $\left(\mathrm{Sym}^2(N_\pm)\right)^{\mathrm{Spin}} \simeq 0$

where the superscript Spin *denotes the subspace left invariant by* $\mathrm{Spin}(k+1, 1)$.

Proof For part 1, the algebra of $\mathrm{Spin}(k+1, 1)$-equivariant real linear endomorphisms of $N_\pm$:

$$\mathrm{End}_{\mathrm{Spin}(k+1,1)}(N_\pm) = (\mathrm{End}(N_\pm))^{\mathrm{Spin}}$$

is called the *commutant* of $N_\pm$. For an irreducible representation such as $N_\pm$, Schur's lemma tells us the commutant must be an associative division algebra. By the Frobenius theorem, the only associative real division algebras are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. We must now determine which case occurs, but this is done by Varadarajan [52, Theorem 6.4.2].

For part 2, recall from Proposition 32 that we have an invariant pairing

$$\langle -, - \rangle \colon N_+ \otimes N_- \to \mathbb{R}.$$

Thus, $N_\pm \simeq N_\mp^*$, and in particular, $\mathrm{Sym}^2 N_\pm \simeq \mathrm{Sym}^2 N_\mp^*$. But the latter is the space of symmetric pairings:

$$\mathrm{Sym}^2 N_\mp \to \mathbb{R},$$

which is a subspace of the space of all pairings on $N_\mp$. The invariant elements of the space of all pairings are tabulated according to dimension and signature mod 8 by Varadarajan [52, Theorem 6.5.10]. In particular, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ where $N_\mp = \mathbb{K}^2$ are the spinors in signature (2, 1) and (3, 1), respectively, the nonzero invariant pairings are antisymmetric, so $\left(\mathrm{Sym}^2(N_\pm)\right)^{\mathrm{Spin}} = 0$. For $\mathbb{K} = \mathbb{H}, \mathbb{O}$, where $N_\mp = \mathbb{K}^2$ is the space of spinors in signature (5, 1) and (9, 1), respectively, $N_\mp$ is not self-dual, so there are no nonzero invariant pairings, and again we conclude $\left(\mathrm{Sym}^2(N_\pm)\right)^{\mathrm{Spin}} = 0$. □

Combining the previous two lemmas, we can prove a surprising relationship between the Cayley–Dickson double and maximal invariant central extension: They are in essence the same! More precisely, the maximal invariant central extension of $\mathbb{R}^{k+1,1|N_+ \oplus N_-}$, constructed from the normed division algebra $\mathbb{K}$, is given by $\mathbb{R}^{2k+1,1|N_{\mathrm{dbl}}}$, constructed from the Cayley–Dickson double $\mathbb{K}_{\mathrm{dbl}}$.

Proposition 12 *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ *be an associative normed division algebra (Example 27) of dimension k, and let* $\mathbb{K}_{\mathrm{dbl}} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ *be its Cayley–Dickson double (Definition 26) of dimension* $2k$. *Then, the maximal invariant central extension of* $\mathbb{R}^{k+1,1|N_+ \oplus N_-}$, *with* $N_\pm$ *the irreducible real spinor representations constructed from* $\mathbb{K}$ *as in Proposition 32, is* $\mathbb{R}^{2k+1,1|N_{\mathrm{dbl}}}$.

$$\begin{array}{ccc} \mathbb{K} & \hookrightarrow & \mathbb{R}^{2k+1,1|N_{\mathrm{dbl}}} \\ & & \downarrow \star \\ & & \mathbb{R}^{k+1,1|N_+ \oplus N_-}, \end{array}$$

for N_{dbl} either of the irreducible real spinor representations induced by the Cayley–Dickson double $\mathbb{K}_{\mathrm{dbl}}$.

Proof By Proposition 6, we need to compute the even $\mathfrak{so}(k+1,1)$-invariant cohomology of $\mathbb{R}^{k+1,1|N_+\oplus N_-}$ in degree 2. Such even Lorentz-invariant 2-cocycles need to pair two spinors in $N_+ \oplus N_-$: There are no even pairing between spinors in $N_+ \oplus N_-$ and vectors in $\mathbb{R}^{k+1,1}$ and no antisymmetric Lorentz-invariant pairing between vectors in $\mathbb{R}^{k+1,1}$. Due to the simple nature of the Lie bracket on super Minkowski spacetime, this means that we need to compute the space of $\mathfrak{so}(k+1,1)$-invariant symmetric bilinear forms on $N_+ \oplus N_-$, because every symmetric bilinear form on $N_+ \oplus N_-$ is an even 2-cocycle.

We now apply Lemma 10 to produce these Lorentz-invariant 2-cocycles. Namely, let $v \in \mathbb{R}^{2k+1,1}$ be any vector in the orthogonal complement of $\mathbb{R}^{k+1,1}$. Then, the symmetric pairing

$$\begin{array}{rcl} N_{\mathrm{dbl}} \otimes N_{\mathrm{dbl}} \to & & \mathbb{R} \\ \psi \otimes \phi & \mapsto & \eta(v, [\psi, \phi]) \end{array}$$

is clearly $\mathrm{Spin}(k+1,1)$-invariant, by the equivariance of the spinor pairing (Proposition 32) and the assumption on v. But by Lemma 10, N_{dbl} is $N_+ \oplus N_-$ as a $\mathrm{Spin}(k+1,1)$-representation. Therefore, this construction yields a k-dimensional space of Spin-invariant symmetric bilinear pairings on $N_+ \oplus N_-$. Moreover, by the definition of the pairing above, it follows that the central extension classified by these pairings, regarded as 2-cocycles, is $\mathbb{R}^{2k+1,1|N_{\mathrm{dbl}}}$.

To conclude the proof, it remains to show this invariant extension is maximal, hence that the dimension of the space of all invariant symmetric pairings on $N_+ \oplus N_-$ is k. The space of all symmetric pairings, invariant or not, is:

$$\mathrm{Sym}^2(N_+ \oplus N_-) \simeq \mathrm{Sym}^2(N_+) \oplus N_+ \otimes N_- \oplus \mathrm{Sym}^2(N_-).$$

So, we seek the invariant elements of the latter space. By Lemma 11, the invariant subspaces of $\mathrm{Sym}^2(N_\pm)$ vanish. Therefore, the space of invariant 2-cocycles is the space of invariant elements in $N_+ \otimes N_-$. By the spinor-to-scalar pairing from Proposition 32, the two spaces N_+ and N_- are dual to each other as $\mathrm{Spin}(\mathrm{k}+1,1)$-representations. Therefore, the invariant elements in $N_+ \otimes N_-$ are equivalently the equivariant linear endomorphisms of N_+:

$$N_+ \to N_+.$$

By Lemma 11, this space of invariant endomorphisms is identified with $\mathbb{K}$

$$(\mathrm{End}(N_+))^{\mathrm{Spin}} \simeq \mathbb{K}.$$

Hence, the dimension of this space is k, which concludes the proof. □

Proposition 13 *The maximal invariant central extension (Definition 7) of the type IIA super Minkowski spacetime* $\mathbb{R}^{9,1|\mathbf{16}\oplus\overline{\mathbf{16}}}$ *is* $\mathbb{R}^{10,1|\mathbf{32}}$*:*

$$\begin{array}{ccc} \mathbb{R} & \hookrightarrow & \mathbb{R}^{10,1|\mathbf{32}} \\ & & \downarrow \star \\ & & \mathbb{R}^{9,1|\mathbf{16}\oplus\overline{\mathbf{16}}} \end{array}$$

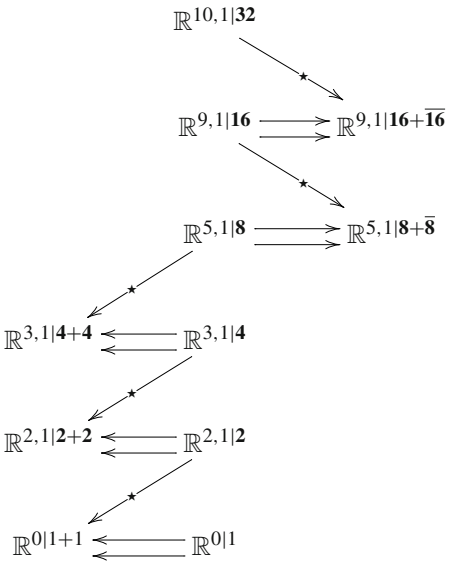
Proof By Proposition 6, we seek even $\mathfrak{so}(9,1)$-invariant 2-cocycles. Since the extension in question is clearly $\mathfrak{so}(9,1)$-invariant, it is sufficient to show that the space of all $\mathfrak{so}(9,1)$-invariant 2-cocycles on $\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}$ is one dimensional. As in the proof of Proposition 12, that space is equivalently the space of $\mathfrak{so}(9,1)$-invariant elements in

$$\mathrm{Sym}^2(N_+ \oplus N_-) \simeq \mathrm{Sym}^2(N_+) \ \oplus \ N_+ \otimes N_- \ \oplus \ \mathrm{Sym}^2(N_-).$$

By Lemma 11, the invariants in $\mathrm{Sym}^2(N_\pm)$ vanish and the space of invariants in $N_+ \otimes N_-$ is one dimensional. □

Putting together our results in this section, we prove our main theorem.

Theorem 14 *The process that starts with the superpoint* $\mathbb{R}^{0|1}$ *and then consecutively doubles the supersymmetries and forms the maximal invariant central extension according to Definition 7 discovers the super Minkowski super Lie algebras* $\mathbb{R}^{d-1,1|N}$ *from Definition 22 in dimensions* $d \in \{2, 3, 6, 10, 11\}$ *for* $N = 1$ *and* $N = 2$ *supersymmetry: There is a diagram of super Lie algebras of the following form*



where each single arrow $\xrightarrow{\star}$ *denotes a maximal invariant central extension according to Definition 7 and where each double arrow denotes the two evident injections (Remark 23).*

Proof This is the joint statement of Propositions 9, 12, and 13. Here, we use in Proposition 12 that for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ a commutative division algebra, the two representations $N_\pm$ from Proposition 32 are in fact isomorphic. □

4 Outlook

In view of the brane bouquet [27], Theorem 14 is suggestive of phenomena still to be uncovered. Further corners of M-theory, currently less well understood, might be found by following the process of maximal invariant central extensions in other directions. Indeed, note that Theorem 14 only exhibits some maximal invariant central extensions. It does not claim that there are no further maximal central extensions.

For instance, the $N = 1$ superpoint $\mathbb{R}^{0|1}$ also has a maximal central extension, namely the superline $\mathbb{R}^{1|1} = \mathbb{R}^{1,0|1}$

$$\begin{array}{c} \mathbb{R}^{1|1} \\ \big\downarrow{\scriptstyle \star} \\ \mathbb{R}^{0|1} \end{array}$$

This follows immediately with the same argument as in Proposition 9.

The natural next question is, what is the bouquet of maximal central extensions emerging out of $\mathbb{R}^{0|3}$? It is clear that the first step yields $\mathbb{R}^{6|3}$, with the underlying even six-dimensional vector space canonically identified with the 3×3 symmetric matrices with entries in the real numbers. Now if an analog of Proposition 12 continued to hold in this case, then the further consecutive maximal invariant extensions might involve the 3×3 hermitian matrices with coefficients in the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and finally the octonions $\mathbb{O}$. The last of these, denoted $\mathfrak{h}_3(\mathbb{O})$ for the 3×3 hermitian matrices over $\mathbb{O}$, is the famous exceptional Jordan algebra. Just as $\mathfrak{h}_2(\mathbb{O})$, the 2×2 hermitian matrices over $\mathbb{O}$, is isomorphic to Minkowski spacetime $\mathbb{R}^{9,1}$, so $\mathfrak{h}_3(\mathbb{O})$ is isomorphic to the 27-dimensional direct sum $\mathbb{R}^{9,1} \oplus \mathbf{16} \oplus \mathbb{R}$ consisting of 10d spacetime, one copy of the real 10d spinors and a scalar [2, Section 3.4]. This kind of data is naturally associated with heterotic M-theory, and grouping its spinors together with the vectors and the scalar to a 27-dimensional bosonic space is reminiscent of the speculations about bosonic M-theory [34]. Therefore, should the bouquet of maximal invariant extensions truly include $\mathfrak{h}_3(\mathbb{O})$, this might help to better understand the nature of the bosonic or heterotic corners of M-theory.

In a similar vein, we ought to ask how the tower of steps in Theorem 14 continues beyond dimension 11, and what the resulting structures mean.

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A Background

For reference, we briefly recall some definitions and facts that we use in the main text.

A.1 Super Lie algebra cohomology

We recall the definition of super Lie algebras and their cohomology. All our vector spaces and algebras are over $\mathbb{R}$, and they are all finite dimensional. We write *even* for $0 \in \mathbb{Z}_2$ and *odd* for $1 \in \mathbb{Z}_2$.

Definition 15 The *tensor category of supervector spaces* is the category of $\mathbb{Z}_2$-graded vector spaces and grade-preserving linear maps, equipped with the unique non-trivial braiding, τ^{super}. For any two supervector space V and W, τ^{super} is the isomorphism

$$\begin{aligned} \tau^{\text{super}} \colon V \otimes W &\to W \otimes V \\ v_1 \otimes v_2 &\mapsto (-1)^{\sigma_1 \sigma_2}\, v_2 \otimes v_1, \end{aligned}$$

for elements $v_1 \in V$, $v_2 \in W$ of homogeneous degree $\sigma_i \in \mathbb{Z}_2$.

A *super Lie algebra* is a Lie algebra internal to supervector spaces. That is, it is a supervector space

$$\mathfrak{g} = \mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}}$$

equipped with a bilinear map, called the *Lie bracket*:

$$[-,-] \colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

which is graded skew symmetric:

$$[v_1, v_2] = -(-1)^{\sigma_1 \sigma_2} [v_2, v_1]$$

and which satisfies the graded Jacobi identity:

$$[v_1, [v_2, v_3]] = [[v_1, v_2], v_3] + (-1)^{\sigma_1 \sigma_2} [v_2, [v_1, v_3]].$$

A homomorphism of super Lie algebras $\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a linear map preserving the $\mathbb{Z}_2$-grading and the bracket.

Definition 16 (*super Lie algebra cohomology*) Let V be a finite-dimensional supervector space. Then, the *super Grassmann algebra* $\Lambda^\bullet V^*$ is the $\mathbb{Z} \times \mathbb{Z}_2$-bigraded-commutative associative algebra freely generated by V^* in degree $1 \in \mathbb{Z}$. That is to say, it is generated by the elements in V^*_{even} regarded as being in bidegree (1, even),

and the elements in V^*_{odd} regarded as being in bidegree (1, odd), subject to the relation that for α_i two elements of homogeneous bidegree (n_i, σ_i), then

$$\alpha_1 \wedge \alpha_2 \;=\; (-1)^{n_1 n_2}(-1)^{\sigma_1\sigma_2}\,\alpha_2 \wedge \alpha_1.$$

In particular, this relation says that elements of bidegree (1, even) anticommute with each other, those of bidegree (1, odd) commute with each other, while an element of bidegree (1, even) anticommutes with an element of bidegree (1, odd).

Now let $(\mathfrak{g}, [-,-])$ be a finite-dimensional super Lie algebra. Then, its *Chevalley–Eilenberg algebra* $\text{CE}(\mathfrak{g})$ is the super Grassmann algebra $\Lambda^\bullet \mathfrak{g}^*$ equipped with the differential d_{CE} defined as follows. On the generators $\mathfrak{g}^*$, d_{CE} acts as the linear dual of the Lie bracket:

$$[-,-]^* \colon \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*.$$

The action of d_{CE} on generators is then extended to all of $\Lambda^\bullet \mathfrak{g}^*$ as a derivation, bigraded of bidegree (1, even). This makes $\text{CE}(\mathfrak{g})$ into a differential graded algebra. A calculation shows $d^2_{\text{CE}} = 0$, so $\text{CE}(\mathfrak{g})$ is also a cochain complex.

For $p \in \mathbb{N}$, we say that a $(p+2)$-*cocycle* on $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is a d_{CE}-closed element in $\Lambda^{p+2}\mathfrak{g}^*$. We say that cocycle is *even* if its degree in $\mathbb{Z}_2$ is even, and *odd* if it is odd. The *super Lie algebra cohomology* of $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is the cohomology of its Chevalley–Eilenberg algebra, regarded as a cochain complex:

$$H^\bullet(\mathfrak{g}, \mathbb{R}) := H^\bullet(\text{CE}(\mathfrak{g})).$$

The $\mathbb{Z}_2$-grading on $\text{CE}(\mathfrak{g})$ makes $H^p(\mathfrak{g}, \mathbb{R})$ into a supervector space for each p. We will be interested in its even part, $H^p_{\text{even}}(\mathfrak{g}, \mathbb{R})$

Example 17 Let $\mathfrak{g}$ be a finite-dimensional super Lie algebra, and let $\omega \in \Lambda^2\mathfrak{g}^*$ be an even 2-cocycle as in Definition 16. Then, there is a new super Lie algebra $\widehat{\mathfrak{g}}$ whose underlying supervector space is

$$\widehat{\mathfrak{g}} := \underbrace{\mathfrak{g}_{\text{even}} \oplus \mathbb{R}}_{\text{even}} \oplus \underbrace{\mathfrak{g}_{\text{odd}}}_{\text{odd}}$$

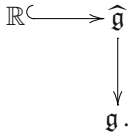
and with super Lie bracket given by

$$[(x_1, c_1), (x_2, c_2)] \;=\; ([x_1, x_2], \omega(x_1, x_2)).$$

We have a short exact sequence giving $\widehat{\mathfrak{g}}$ as a central extension of $\mathfrak{g}$:

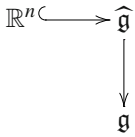
$$0 \to \mathbb{R} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

In the paper, we will often write this short exact sequence as follows; in the style, an algebraic topologist might use to write down a fibration:



Just as for ordinary Lie algebras, this construction establishes a natural equivalence between central extensions of $\mathfrak{g}$ by $\mathbb{R}$ (in even degree) and even super Lie algebra 2-cocycles on $\mathfrak{g}$.

More generally, a central extension in even degree is by some vector space $V \simeq \mathbb{R}^n$



which is equivalently the result of extending by n even 2-cocycles, one after the other, in any order.

We will be interested not in the full super Lie algebra cohomology, but in the *invariant* cohomology with respect to some action:

Definition 18 For $\mathfrak{g}$ a super Lie algebra (Definition 15), its *automorphism group* is the Lie subgroup

$$\mathrm{Aut}(\mathfrak{g}) \hookrightarrow \mathrm{GL}(\mathfrak{g}_{\mathrm{even}}) \times \mathrm{GL}(\mathfrak{g}_{\mathrm{odd}})$$

of the group of degree-preserving linear isomorphisms on the underlying vector space, consisting of those elements which are super Lie algebra isomorphisms.

Proposition 19 *For $\mathfrak{g}$ a super Lie algebra, the Lie algebra of its automorphism Lie group (Definition 18)*

$$\mathfrak{aut}(\mathfrak{g})$$

is called the automorphism Lie algebra *of $\mathfrak{g}$. It is the Lie algebra of those linear maps $\Delta : \mathfrak{g} \to \mathfrak{g}$ which preserve the degree and satisfy the derivation property:*

$$\Delta[X, Y] = [\Delta X, Y] + [X, \Delta Y]$$

for all $X, Y \in \mathfrak{g}$. The Lie bracket on $\mathfrak{aut}(\mathfrak{g})$ is the commutator:

$$[\Delta_1, \Delta_2] := \Delta_1\Delta_2 - \Delta_2\Delta_1 .$$

We caution the reader that, even though $\mathfrak{g}$ is a super Lie algebra, its automorphism algebra $\mathfrak{aut}(\mathfrak{g})$ is merely a Lie algebra. This is because we want elements of $\mathfrak{aut}(\mathfrak{g})$ to preserve the degree on $\mathfrak{g}$.

Example 20 The super Minkowski super Lie algebras $\mathbb{R}^{d-1,1|N}$ from Definition 22 all carry an automorphism action of the abelian Lie algebra $\mathbb{R}$ which is spanned by the *scaling derivation* that acts on vectors $v \in \mathbb{R}^{d-1,1}$ by

$$v \mapsto 2v$$

and on spinors $\psi \in N$ by

$$\psi \mapsto \psi.$$

Definition 21 Let $\mathfrak{g}$ be a super Lie algebra (Definition 15). Clearly, every automorphism of $\mathfrak{g}$ will induce an automorphism of the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$ (Definition 16). Explicitly, this works as follows. Let $\Delta \in \mathfrak{aut}(\mathfrak{g})$ be an infinitesimal automorphism (Proposition 19). The induced automorphism $\Delta_{\mathrm{CE}} \colon \mathrm{CE}(\mathfrak{g}) \to \mathrm{CE}(\mathfrak{g})$ acts on the generators $\mathfrak{g}^*$ of $\mathrm{CE}(\mathfrak{g})$ as the linear dual Δ^*:

$$\Delta_{\mathrm{CE}} \colon \mathfrak{g}^* \xrightarrow{\Delta^*} \mathfrak{g}^*.$$

This is then extended to all of $\mathrm{CE}(\mathfrak{g})$ as a derivation of bidegree $(0, \text{even})$. The fact that Δ_{CE} commutes with d_{CE} is equivalent to the fact that Δ is a derivation of $\mathfrak{g}$.

Now let $\mathfrak{h} \hookrightarrow \mathfrak{aut}(\mathfrak{g})$ be a Lie subalgebra of its automorphism Lie algebra. The elements of $\mathrm{CE}(\mathfrak{g})$ which are annihilated by Δ_{CE} for all $\Delta \in \mathfrak{h}$ form a differential graded subalgebra of $\mathrm{CE}(\mathfrak{g})$:

$$\mathrm{CE}(\mathfrak{g})^{\mathfrak{h}} \hookrightarrow \mathrm{CE}(\mathfrak{g}).$$

We say an $\mathfrak{h}$-*invariant* $(p+2)$-*cocycle* on $\mathfrak{g}$ is an element in $\mathrm{CE}(\mathfrak{g})^{\mathfrak{h}}$ which is d_{CE}-closed and the $\mathfrak{h}$-*invariant cohomology* of $\mathfrak{g}$ with coefficients in $\mathbb{R}$ is the cochain cohomology of this subcomplex:

$$H^\bullet(\mathfrak{g}, \mathbb{R})^{\mathfrak{h}} := H^\bullet(\mathrm{CE}(\mathfrak{g})^{\mathfrak{h}}).$$

We define even and odd invariant cocycles as before. The vector space $H^p(\mathfrak{g}, \mathbb{R})^{\mathfrak{h}}$ is $\mathbb{Z}_2$-graded for each p, and our focus will be on its even part, $H^p_{\mathrm{even}}(\mathfrak{g}, \mathbb{R})^{\mathfrak{h}}$.

A.2 Super Minkowski spacetimes

We recall the definition of "super Minkowski super Lie algebras" (Definition 22) as well as their construction, on the one hand via Majorana or symplectic Majorana spinors (Example 25) and on the other hand via the four normed division algebras (Proposition 32). We freely use basic facts about spinors, as may be found in the book of Lawson and Michelsohn [40].

Definition 22 (*super Minkowski Lie algebras*) Let $d \in \mathbb{N}$ (spacetime dimension) and let N be a real spinor representation of $\mathrm{Spin}(d-1,1)$, the double cover of the connected Lorentz group $\mathrm{SO}_0(d-1,1)$. Then, d-*dimensional* N-*supersymmetric super*

Minkowski spacetime $\mathbb{R}^{d-1,1|N}$ is the super Lie algebra (Definition 15) whose underlying supervector space is

$$\mathbb{R}^{d-1,1|N} := \underbrace{\mathbb{R}^{d-1,1}}_{\text{even}} \oplus \underbrace{N}_{\text{odd}}.$$

The Lie bracket is nonzero only on N and is a choice of symmetric, bilinear, $\mathrm{Spin}(d-1,1)$-equivariant map:

$$[-,-]\colon N \otimes N \longrightarrow \mathbb{R}^{d-1,1}.$$

Such a map is always available in spacetime signature $(d-1,1)$ [30], though there may be more than one choice [52].

There is a canonical action of $\mathrm{Spin}(d-1,1)$ on $\mathbb{R}^{d-1,1|N}$ by Lie algebra automorphisms, and the corresponding semidirect product Lie algebra is the *super Poincaré super Lie algebra*

$$\mathfrak{iso}(\mathbb{R}^{d-1,1|N}) = \mathbb{R}^{d-1,1|N} \rtimes \mathfrak{so}(d-1,1).$$

It is also called the *supersymmetry algebra*.

Remark 23 (number of supersymmetries) In the physics literature, the choice of real spin representation in Definition 22 is often referred to as the "number of supersymmetries." While this is imprecise, it fits well with the convention of labeling irreducible representations by their dimension in boldface. For example, when $d = 10$ there are two irreducible real spinor representations, both of real dimension 16, but of opposite chirality, and hence traditionally denoted $\mathbf{16}$ and $\overline{\mathbf{16}}$. Hence, we may speak of $N = \mathbf{16}$ supersymmetry (also called $N = 1$, *type I* or *heterotic*) and $N = \mathbf{16} \oplus \overline{\mathbf{16}}$ supersymmetry (also called $N = (1,1)$ or *type IIA*) and $N = \mathbf{16} \oplus \mathbf{16}$ supersymmetry (also called $N = (2,0)$ or *type IIB*).

In Sect. 3, the generalization of the last of these cases plays a central role, where for any given real spin representation N we pass to the doubled supersymmetry $N \oplus N$. Observe that the two canonical linear injections $N \to N \oplus N$ into the direct sum induce two super Lie algebra homomorphisms

$$\mathbb{R}^{d-1,1|N} \rightrightarrows \mathbb{R}^{d-1,1|N\oplus N}.$$

The following degenerate variation of super Minkowski spacetime will play a key role:

Definition 24 (*Superpoint*) A *superpoint* is the super Lie algebra

$$\mathbb{R}^{0|N},$$

which has zero Lie bracket, and whose underlying supervector space is concentrated in odd degree, where it is of dimension N.

We will use two different ways of constructing real spinor representation, and hence super Minkowski spacetimes: via "Majorana" or "symplectic Majorana" spinors (Example 25) and via real normed division algebras (Proposition 32).

Example 25 (Majorana representations) For $d = 2\nu$ or $2\nu + 1$, there exists a complex representation of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{d-1,1}) \otimes \mathbb{C}$, hence of the spin group $\mathrm{Spin}(d-1,1)$ on $\mathbb{C}^{2^\nu}$ such that

1. all skew-symmetrized products of $p \geq 1$ Clifford elements $\Gamma_{a_1 \cdots a_p}$ are traceless;
2. $\Gamma_0^\dagger = \Gamma_0$ and $\Gamma_i^\dagger = -\Gamma_i$, for $1 \leq i \leq d-1$.

This is the *Dirac representation*, a complex representation of $\mathrm{Spin}(d-1,1)$. For $d = 2\nu$, this is the direct sum of two subrepresentations on $\mathbb{C}^{2^{\nu-1}}$, the *Weyl representations*.

For $d \in \{1, 2, 3, 4, 8, 9, 10, 11\}$, there exists a real structure J on the complex Dirac representation, restricting to the Weyl representations for $d = 2$ or $d = 10$. This is a $\mathrm{Spin}(d-1,1)$-equivariant antilinear endomorphism $J : S \to S$ which squares to the identity: $J^2 = +1$. It carves out a real representation called the *Majorana representation* $N := \mathrm{Eig}(J, +1)$, the eigenspace of J of eigenvalue +1, whose elements are called the *Majorana spinors*. In this case, the *Dirac conjugation* $\psi \mapsto \psi^\dagger \Gamma_0$ on elements $\psi \in \mathbb{C}^{2^\nu}$ restricts to N and is called the *Majorana conjugation*. We write it as simply $\psi \mapsto \overline{\psi}$. In terms of this matrix representation, then the spinor bilinear pairing that appears in Definition 22 is given by the following matrix product expression:

$$[\psi, \phi] \;=\; \left(\overline{\psi}\Gamma^a \phi\right)_{a=0}^{d-1}.$$

Similarly, for $d \in \{5, 6, 7\}$ there exists a quaternionic structure on the Dirac representation. This is a $\mathrm{Spin}(d-1,1)$-equivariant antilinear endomorphism $\tilde{J}$ which squares to minus the identity, $\tilde{J}^2 = -1$. It follows that

$$J := \begin{pmatrix} 0 & -\tilde{J} \\ \tilde{J} & 0 \end{pmatrix}$$

is a real structure on the direct sum of the Dirac representation with itself. Hence as before $N := \mathrm{Eig}(J, +1)$ is a real subrepresentation, called the *symplectic Majorana representation*. The spinor-to-vector bilinear pairing for symplectic Majorana spinors is similar to the case of Majorana spinors.

Definition 26 (*Cayley–Dickson double* [2, Section 2.2]) Let $\mathbb{K}$ be a real $*$-algebra. This is a real, not necessarily associative algebra $\mathbb{K}$ equipped with a *conjugation* $\overline{(-)} : \mathbb{K} \to \mathbb{K}$, satisfying:

$$\overline{a+b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b}\,\overline{a}, \quad \overline{\overline{a}} = a,$$

for any $a, b \in \mathbb{K}$. Then, the *Cayley–Dickson double* $\mathbb{K}_{\mathrm{dbl}}$ of $\mathbb{K}$ is the real $*$-algebra obtained from $\mathbb{K}$ by adjoining one element ℓ such that $\ell^2 = -1$ and such that the following relations hold, for all $a, b \in \mathbb{K}$:

$$a(\ell b) = \ell(\overline{a}b), \quad (a\ell)b = (a\overline{b})\ell, \quad (\ell a)(b\ell) = -\overline{(ab)}.$$

Finally, the conjugation $\overline{(-)}$ on $\mathbb{K}_{\text{dbl}}$ acts on elements of $\mathbb{K}$ by the conjugation on $\mathbb{K}$, and sends the new generator ℓ to $-\ell$.

Example 27 Consider $\mathbb{R}$ the real numbers regarded as a $*$-algebra with trivial conjugation $\overline{a} = a$. Then, its Cayley–Dickson double (Definition 26) is the complex numbers $\mathbb{C}$ with the usual conjugation, the Cayley–Dickson double of $\mathbb{C}$ is the quaternions $\mathbb{H}$, and the Cayley–Dickson double of $\mathbb{H}$ is the octonions $\mathbb{O}$.

By a classical result of Hurwitz, these four algebras are the only normed division algebras over the real numbers, as reviewed by Baez [2].

In the next proposition and elsewhere in the text, we will use $n \times n$ matrices over $\mathbb{K}$ to describe real linear operators on $\mathbb{K}^n$. We will write $\mathbb{K}[n]$ for the set of all $n \times n$ matrices with entries in $\mathbb{K}$. For any such matrix, there are two natural ways for it to induce a linear operator, one using left multiplication in $\mathbb{K}$ and the other right multiplication.

Definition 28 (*Matrices over* $\mathbb{K}$ *as linear operators*) Let $\mathbb{K}$ be a normed division algebra. Any element of $a \in \mathbb{K}$ induces a linear endomorphism on $\mathbb{K}$ by left or right multiplication, which we will denote by a_L or a_R, respectively:

$$\begin{aligned} a_L \colon \mathbb{K} &\to \mathbb{K} \\ x &\mapsto ax \end{aligned}, \qquad \begin{aligned} a_R \colon \mathbb{K} &\to \mathbb{K} \\ x &\mapsto xa. \end{aligned}$$

More generally, any $n \times n$ matrix $A \in \mathbb{K}[n]$ induces a linear endomorphism on $\mathbb{K}^n$ via either left multiplication or right multiplication:

$$\begin{aligned} A_L \colon \mathbb{K}^n &\to \mathbb{K}^n \\ x &\mapsto \sum a_{ij} x_j \end{aligned}, \qquad \begin{aligned} A_R \colon \mathbb{K}^n &\to \mathbb{K}^n \\ x &\mapsto \sum x_j a_{ij} \end{aligned}$$

where we are using the subscript x_j to denote the jth coordinate of $x \in \mathbb{K}^n$, and $A = (a_{ij})$. In other words, A_L and A_R are the linear maps on $\mathbb{K}^n$ with components $\big((a_{ij})_L\big)$ and $\big((a_{ij})_R\big)$, respectively. We say that A_L is the *left action* of A and A_R is the *right action* of A. We caution that because $\mathbb{K}$ is non-associative, $(AB)_L \neq A_L B_L$ in general, and because $\mathbb{K}$ is non-associative and non-commutative, $(AB)_R \neq A_R B_R$ in general.

Remark 29 The left action of a matrix A by A_L is just the usual matrix multiplication, so we will sometimes write:

$$A_L x = Ax.$$

The utility of defining the linear transformation A_L is that the composition of linear transformations is associative, so we do not need to worry about the non-associativity of $\mathbb{K}$ when we compose them. For example:

$$A_L B_L C_L x = A(B(Cx)).$$

Since $\mathbb{K}$ comes with a conjugation, we can define the conjugate of any matrix in $\mathbb{K}[n]$ by taking the conjugate of each entry, and the conjugate of any element of $\mathbb{K}^n$ by taking the conjugate of each coordinate. It is then an elementary calculation to show that the action of a matrix A by A_L and A_R is related by conjugation:

Proposition 30 *Let $A \in \mathbb{K}[n]$ be an $n \times n$ matrix over the normed division algebra $\mathbb{K}$ (as defined in Example 27). Then,*

$$\overline{A_L x} = (\overline{A})_R \, \overline{x} \text{ and } \overline{A_R x} = (\overline{A})_L \, \overline{x}$$

for all $x \in \mathbb{K}^n$.

The next definition is straightforward, but is central to realizing spin representations via normed division algebras.

Definition 31 [49] For $A \in \mathfrak{h}_2(\mathbb{K})$, a hermitian matrix with coefficients in one of the four real normed division algebras from Example 27. Then, its *trace reversal* is

$$\widetilde{A} := A - \mathrm{tr}(A) \cdot \mathbf{1}.$$

Proposition 32 *[4] Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ be one of the normed division algebras as in Example 27. Write $\mathfrak{h}_2(\mathbb{K})$ for the real vector space of 2×2 hermitian matrices with coefficients in $\mathbb{K}$, and k for the dimension of $\mathbb{K}$.*

Then:

1. *There is an isomorphism of inner product spaces ("forming Pauli matrices over $\mathbb{K}$")*

$$(\mathbb{R}^{k+1,1}, \eta) \xrightarrow{\simeq} (\mathfrak{h}_2(\mathbb{K}), -\det)$$

identifying $\mathbb{R}^{k+1,1}$ equipped with its Minkowski inner product

$$\eta(A, B) := -A^0 B^0 + A^1 B^1 + \cdots + A^{k+1} B^{k+1}, \text{ for } A, B \in \mathbb{R}^{k+1,1}$$

with the space of hermitian matrices equipped with the negative of the determinant operation.

2. *Let N_+ and N_- both denote the vector space $\mathbb{K}^2$. Then, $N_+ \oplus N_-$ is a module of the Clifford algebra $\mathcal{C}\ell(k+1, 1)$, with the action of a vector in $A \in \mathbb{R}^{k+1,1}$ given by*

$$\Gamma(A)(\psi, \phi) = (\tilde{A}_L \phi, A_L \psi)$$

for any element $(\psi, \phi) \in N_+ \oplus N_-$, where we are using the identification of vectors with 2×2 hermitian matrices. Here, $\widetilde{(-)}$ is the trace reversal operation from Definition 31, and $(-)_L$ denotes the linear map given by left multiplication as in Definition 28

3. *Realizing the spin group* $\mathrm{Spin}(k+1,1)$ *inside the Clifford algebra* $\mathcal{C}\ell(k+1,1)$ *by the standard construction, this induces irreducible representations* $\rho_\pm$ *of* $\mathrm{Spin}(k+1,1)$ *on* $N_\pm$. *Explicitly, recall that* $\mathrm{Spin}(k+1,1)$ *is the subgroup of the Clifford algebra generated by products of pairs of unit vectors of the same sign:*

$$\mathrm{Spin}(k+1,1) = \langle AB \in \mathcal{C}\ell(k+1,1) \,:\, A, B \in \mathbb{R}^{k+1,1},\ \eta(A,A) = \eta(B,B) = \pm 1 \rangle.$$

Then, restricting the Clifford action to these elements, a generator AB *of* $\mathrm{Spin}(k+1,1)$ *acts as*

$$\rho_+(AB) = \tilde{A}_L B_L \text{ on } N_+$$

and as

$$\rho_-(AB) = A_L \tilde{B}_L \text{ on } N_-,$$

where again $\widetilde{(-)}$ *is the trace reversal operation from Definition 31 and where* $(-)_L$ *denotes the linear map given by left multiplication as in Definition 28.*
For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, *then these two representations are in fact isomorphic and are the* Majorana representation *of* $\mathrm{Spin}(2,1)$ *and* $\mathrm{Spin}(3,1)$, *respectively, while for* $\mathbb{K} \in \{\mathbb{H}, \mathbb{O}\}$ *they are the two non-isomorphic* symplectic Majorana representations *of* $\mathrm{Spin}(5,1)$ *and* Majorana–Weyl representations *of* $\mathrm{Spin}(9,1)$, *respectively.*

4. *Under the above identifications, the symmetric bilinear* $\mathrm{Spin}(k+1,1)$*-equivariant spinor-to-vector pairings are given by*

$$\begin{aligned} [-,-] \colon N_+ \otimes N_+ &\to \mathbb{R}^{k+1,1} \\ \psi \otimes \phi &\mapsto \tfrac{1}{2}\left(\psi \widetilde{\phi^\dagger + \phi \psi^\dagger}\right) \end{aligned}$$

and

$$\begin{aligned} [-,-] \colon N_- \otimes N_- &\to \mathbb{R}^{k+1,1} \\ \psi \otimes \phi &\mapsto \tfrac{1}{2}\left(\psi \phi^\dagger + \phi \psi^\dagger\right) \end{aligned}$$

5. *There is a bilinear symmetric, non-degenerate and* $\mathrm{Spin}(k+1,1)$*-invariant spinor-to-scalar pairing given by*

$$\begin{aligned} \langle -,- \rangle \colon N_\pm \otimes N_\mp &\to \mathbb{R} \\ \psi \otimes \phi &\mapsto \mathrm{Re}(\psi^\dagger \phi). \end{aligned}$$

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