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Covariance in the Batalin–Vilkovisky formalism and the Maurer–Cartan equation for curved Lie algebras

Ezra Getzler[1]

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Abstract We express covariance of the Batalin–Vilkovisky formalism in classical mechanics by means of the Maurer–Cartan equation in a curved Lie superalgebra, defined using the formal variational calculus and Sullivan's Thom–Whitney construction. We use this framework to construct a Batalin–Vilkovisky canonical transformation identifying the Batalin–Vilkovisky formulation of the spinning particle with an AKSZ field theory.

Keywords Spinning particle · Batalin–Vilkovisky field theory · Variational calculus · Supergravity · Thom–Whitney normalization

Mathematics Subject Classification 70S05 · 37K05

1 Introduction

In the Batalin–Vilkovisky formalism, a classical field theory is specified by a solution of the classical master equation

$$\tfrac{1}{2}\int(S,S)=0. \tag{1}$$

Alexandrov et al. [1] have studied a particularly important class of solutions of this equation, known as AKSZ field theories. An AKSZ field theory in dimension d is a non-linear sigma-model in which the target is a graded supermanifold M with a shifted symplectic form of ghost number $d-1$. There is a function W on M of ghost number d satisfying the Maurer–Cartan equation

✉ Ezra Getzler
getzler@northwestern.edu

1 Department of Mathematics, Northwestern University, Evanston, IL, USA

$$\tfrac{1}{2}\{W, W\} = 0.$$

Thus, W determines a Hamiltonian vector field $Q(f) = \{W, f\}$ on M of ghost number 1 and odd parity, which is cohomological:

$$Q^2 = \tfrac{1}{2}[Q, Q] = 0.$$

Important examples of AKSZ field theories are Chern–Simons theory (Axelrod and Singer [2]) and the Poisson sigma-model (Cattaneo and Felder [4,5]).

In this paper, we restrict attention to field theories with $d = 1$, in other words, classical mechanics. Our main constructions should have analogues in all dimensions, but our application, showing that the particle and spinning particle possess hidden AKSZ field theories, only requires the formalism in $d = 1$, and we will focus our attention on that case.

In Sect. 2 and 3, we recall some needed background results on curved Lie algebras and the formal variational calculus.

In Sect. 4, we show that an AKSZ field theory with $d = 1$, associated to a graded supermanifold M and an exact symplectic form $\omega = \mathrm{d}\nu$, gives rise to a Maurer–Cartan element for a certain curved Lie superalgebra: we call such Maurer–Cartan elements (classical) covariant field theories.

To incorporate covariant field theories with topological terms, where the symplectic form ω is no longer exact, we introduce the Thom–Whitney totalization for cosimplicial curved Lie superalgebras in Sect. 5. The Thom–Whitney totalization replaces the rather rigid homotopies of piecewise linear topology with the more flexible homotopies of de Rham theory. In this setting, we associate a covariant field theory to a graded supermanifold M together with the following data: a symplectic form $\omega \in \Omega^2(M)$, a cover $\mathscr{U} = \{U_\alpha\}$ of M, and one-forms $\nu_\alpha \in \Omega^1(U_\alpha)$ such that $d\nu_\alpha = \omega$.

Topological terms of this type do not occur in AKSZ models when $d > 1$, since the target symplectic form has ghost number $d - 1$, and hence is exact. Our motivation for introducing the Thom–Whitney totalization in Sect. 4 is the hope that using it, the superstring may be understood as a generalized AKSZ model, in the sense that it extends to a covariant field theory. In [12], we study this problem in the setting of the toy model of the superparticle (though admittedly still with $d = 1$).

After a Batalin–Vilkovisky canonical transformation, the covariant field theory for a particle moving in a curved spacetime may be identified with the AKSZ model introduced in [11]. In the introduction, we explain this in the special case of a particle moving in a flat background. The Lagrangian of this theory is as follows:

$$S_0 = p_\mu \partial x^\mu - \tfrac{1}{2}\eta^{\mu\nu} e p_\mu p_\nu. \tag{2}$$

The fields $(x^\mu, p_\mu)_{1\le\mu\le n}$ are the coordinates of the flat space in which the particle moves, and their conjugate momenta. The remaining (non-propagating) field of the theory is the graviton, a nowhere-vanishing one-form e on the world-line.

The solution to the Batalin–Vilkovisky master equation for the particle, extending the Lagrangian (2), incorporates an additional field, the ghost c. This is a fermionic field of ghost number one, transforming as a world-line vector field, and is associated to the covariance of the theory under diffeomorphism of the world-line. The corresponding antifield c^+ is a bosonic field of ghost number -2 transforming as a world-line quadratic differential. Introduce the expression

$$\mathsf{D} = x^+_\mu \partial x^\mu + p^{+\mu} \partial p_\mu - e\partial e^+ + c^+ \partial c,$$

of ghost number -1. Consider the Lagrangian $S = S_0 + S_1$, where $S_1 = c\mathsf{D}$. It is straightforward to check that the action $\int S \, \mathrm{d}t$ satisfies the classical master equation (1). Form the graded Lie superalgebra of polynomials in a variable u of degree 2 with coefficients in the Batalin–Vilkovisky graded Lie superalgebra. The element $u \int \mathsf{D}$ lies in the centre of this graded Lie algebra, and we may form the curved Lie superalgebra with the same underlying graded Lie algebra but with nonzero curvature $u \int \mathsf{D}$. Then

$$\int \mathsf{S} = \int (S + uc^+) \tag{3}$$

is a Maurer–Cartan element in this curved Lie algebra, in other words, a covariant field theory. This means that it satisfies the perturbation of the classical master Eq. (1)

$$\tfrac{1}{2} \int (S, S) = -u \int \mathsf{D}.$$

This field theory bears some resemblance to a Chern–Simons field theory. Recall that the Batalin–Vilkovisky extension of the Lagrangian for Chern–Simons theory (Axelrod and Singer [2]) may be expressed in terms of a composite field

$$\mathbf{A} = c + A + A^+ + c^+.$$

Here, A is the Chern–Simons field, a connection form on the 3-manifold M for the Lie algebra $\mathfrak{g}$, $c \in \Omega^0(M, \mathfrak{g})$ is the ghost field for local gauge transformations, and $A^+ \in \Omega^2(M, \mathfrak{g})$ and $c^+ \in \Omega^3(M, \mathfrak{g})$ are their respective antifields. The top degree component of the differential form

$$\tfrac{1}{2} \langle \mathbf{A}, \mathrm{d}\mathbf{A} \rangle + \tfrac{1}{6} \langle \mathbf{A}, [\mathbf{A}, \mathbf{A}] \rangle,$$

is the Batalin–Vilkovisky Lagrangian $S = S_0 + S_1$ of the Chern–Simons theory:

$$S_0 = \tfrac{1}{2} \langle A, \mathrm{d}A \rangle + \tfrac{1}{6} \langle A, [A, A] \rangle \qquad S_1 = \langle A^+, \mathrm{d}c + [A, c] \rangle + \tfrac{1}{2} \langle c^+, [c, c] \rangle.$$

In order to see that the complete action $\int S$ of the particle has a hidden AKSZ structure, we apply to it a sequence of canonical transformations. Consider the flow Φ_τ associated to the Hamiltonian $cx^+_\mu p^{+\mu}$: this is the solution to the ordinary differential equation

$$\frac{\mathrm{d}(\Phi^*_\tau f)}{\mathrm{d}\tau} = (cx^+_\mu p^{+\mu}, f),$$

and is given by the explicit formula

$$\begin{aligned} \Phi_\tau^*(x^\mu) &= x^\mu + \tau c p^{+\mu} & \Phi_\tau^*(x_\mu^+) &= x_\mu^+ \\ \Phi_\tau^*(p_\mu) &= p_\mu - \tau c x_\mu^+ & \Phi_\tau^*(p^{+\mu}) &= p^{+\mu} \\ \Phi_\tau^*(e) &= e & \Phi_\tau^*(e^+) &= e^+ \\ \Phi_\tau^*(c) &= c & \Phi_\tau^*(c^+) &= c^+ + \tau x_\mu^+ p^{+\mu} \end{aligned}$$

Under this flow, the densities S_0 and S_1 transform as follows:

$$\begin{aligned} \Phi_\tau^* \mathsf{S}_0 &= S_0 - \tau c(x_\mu^+ \partial x^\mu + p^{+\mu} \partial p_\mu) + \tau \partial (c p^{+\mu} p_\mu) \\ &\quad + \tau^2 c \partial c x_\mu^+ p^{+\mu} + \tau \eta^{\mu\nu} e c p_\mu x_\nu^+ \\ \Phi_\tau^* \mathsf{S}_1 &= S_1 - \tau c \partial c x_\mu^+ p^{+\mu}. \end{aligned}$$

Let Φ be the canonical transformation obtained by evaluating the flow Φ_τ at $\tau = 1$: we see that

$$\Phi^* S = S_0 + ec(\eta^{\mu\nu} p_\mu x_\nu^+ - \partial e^+) + c^+ c \partial c + \partial (c p_\mu p^{+\mu}).$$

Next, consider the canonical transformation Ψ which leaves the fields x^μ and p_μ and their antifields fixed, and acts on the remaining fields by the formulas

$$\Psi^* e = e \qquad \Psi^* e^+ = e^+ + e^{-1} c^+ c \qquad \Psi^* c = e^{-1} c \qquad \Psi^* c^+ = e c^+.$$

Formally, this is the value of the flow Ψ_t generated by the Hamiltonian $\log(e) c^+ c$ at $\tau = 1$. The canonical transformation $\Xi = \Phi \circ \Psi$ obtained by composing Φ and Ψ transforms the complete Lagrangian S as follows:

$$\Xi^* S = \Psi^* \Phi^* S = S_0 + c\big(\eta^{\mu\nu} p_\mu x_\nu^+ - \partial e^+\big) + \partial\big(c(p_\mu p^{+\mu} + e e^+)\big).$$

After this transformation, the Maurer–Cartan element (3) becomes

$$\textstyle\int \Xi^* \mathsf{S} = \int \big(p_\mu \partial x^\mu - \frac{1}{2} \eta^{\mu\nu} e p_\mu p_\nu + c\big(\eta^{\mu\nu} p_\mu x_\nu^+ - \partial e^+\big) + u(x_\mu^+ p^{+\mu} + e c^+)\big).$$

In terms of the composite fields

$$\begin{aligned} \mathbf{x}^\mu &= x^\mu + \mathrm{d}t\, p^{+\mu} & \mathbf{p}_\mu &= p_\mu - \mathrm{d}t\, x_\mu^+ \\ \mathbf{c} &= c - \mathrm{d}t\, e & \mathbf{b} &= e^+ + \mathrm{d}t\, c^+ \end{aligned}$$

we see that $\Xi^* S$ equals the coefficient of $\mathrm{d}t$ in the differential form

$$\mathbf{p}_\mu \mathrm{d}\mathbf{x}^\mu + \mathbf{c} d\mathbf{b} + \tfrac{1}{2} \eta^{\mu\nu} \mathbf{c} \mathbf{p}_\mu \mathbf{p}_\nu,$$

modulo total derivatives. In summary, the particle embeds, by an explicit canonical transformation, in an AKSZ field theory with fields $\{\mathbf{x}^\mu, \mathbf{p}_\mu, \mathbf{c}, \mathbf{b}\}$. A similar transformation, for three-dimensional gravity, has been studied by Cattaneo, Schiavina and Selliah [8].

In passing, we note that the canonical transformation Ξ is the value at $\tau = 1$ of the flow Ξ_τ associated to the Batalin–Vilkovisky Hamiltonian

$$\frac{\log(e)}{e-1}\, c\big((x_\mu^+ p^{+\mu} + ec^+) - c^+\big). \tag{4}$$

In Sect. 6, we will show that the above remarks may be generalized to a general covariant field theory coupled to the gravity multiplet (e, c). In particular, this includes the case of a particle in a curved spacetime with a background electromagnetic field.

In Sect. 7, we turn to the spinning particle, which we have previously studied in the Batalin–Vilkovisky formalism [11]. The spinning particle is a toy model for a supersymmetric sigma-model coupled to supergravity, in which the world-line (or spacetime) is reduced from two to one dimensions. (The corresponding quantum system has Hamiltonian the square of the Dirac operator.) The fields of this model, in addition to $\{x^\mu, p_\mu, e, c\}$, comprise fermionic fields ψ^μ and χ and the bosonic ghost γ, supersymmetric partners to x^μ, e and c, respectively. In a flat background, the Lagrangian of the spinning particle equals

$$S_0 = p\partial x + \tfrac{1}{2}\psi\partial\psi - \tfrac{1}{2}ep^2 + \chi p\psi,$$

and the associated solution to the classical master equation is

$$\begin{aligned} S = S_0 &+ c(x^+\partial x + p^+\partial p + \psi^+\partial\psi - e\partial e^+ + c^+\partial c - \chi\partial\chi^+ + \gamma^+\partial\gamma) \\ &- \gamma(\partial\chi^+ - p\psi^+ + \psi x^+ + 2\chi e^+) \\ &+ e^{-1}\gamma^2(c^+ - x^+p^+ - \tfrac{1}{2}\psi^+\psi^+ - \chi\gamma^+). \end{aligned} \tag{5}$$

Let Ξ_τ be the flow associated to the Batalin–Vilkovisky Hamiltonian

$$\frac{\log(e)}{e-1}\, c\big((x_\mu^+ p^{+\mu} + \tfrac{1}{2}\eta^{\mu\nu}\psi_\mu^+\psi_\nu^+ + ec^+ + \chi\gamma^+) - c^+\big)$$

generalizing (4), and let $\Xi = \Xi_{\tau=1}$ be the value of the flow at $\tau = 1$. After transformation by the Batalin–Vilkovisky canonical transformation Ξ, the Lagrangian (5) becomes the AKSZ field theory

$$\Xi^* S = S_0 - c(\partial e^+ - px^+) - \gamma(\partial\chi^+ - p\psi^+ + \psi x^+ + 2\chi e^+) + \gamma^2 c^+.$$

In addition to the previous composite fields $(\mathbf{x}, \mathbf{p}, \mathbf{c}, \mathbf{b})$ associated to the particle, we now have the additional composite fields

$$\boldsymbol{\psi}^\mu = \psi^\mu + \mathrm{d}t\,\eta^{\mu\nu}\psi_\nu^+ \qquad \boldsymbol{\gamma} = -\gamma + \mathrm{d}t\,\chi \qquad \boldsymbol{\beta} = \chi^+ + \mathrm{d}t\,\gamma^+.$$

Up to a total derivative, the Lagrangian $\Xi^* S$ equals the coefficient of dt in the expression

$$\mathbf{p}_\mu \mathrm{d}\mathbf{x}^\mu - \tfrac{1}{2}\eta_{\mu\nu}\boldsymbol{\psi}^\mu \mathrm{d}\boldsymbol{\psi}^\nu + \mathbf{c}d\mathbf{b} + \boldsymbol{\gamma}\mathrm{d}\boldsymbol{\beta} + \tfrac{1}{2}\eta^{\mu\nu}\mathbf{c}\mathbf{p}_\mu\mathbf{p}_\nu + \boldsymbol{\gamma}\mathbf{p}_\mu\boldsymbol{\psi}^\mu + \mathbf{b}\boldsymbol{\gamma}^2.$$

In Sect. 7, we carry out the above construction for a more general class of covariant field theories coupled to the supergravity multiplet (e, c, χ, γ): this class of theories includes the spinning particle with curved target, as in [11].

2 The Maurer–Cartan equation in a curved Lie superalgebra

Let $L^\bullet$ be a $\mathbb{Z}$-graded superspace: adopting the language of theoretical physics, we say that an element $x \in L^k$ has ghost number k, and write $gh(x) = k$. Furthermore, L^k has a $\mathbb{Z}/2$-grading, making it into a superspace: we call this grading the parity, and write $p(x) \in \{0, 1\}$. We will also say that x is even (respectively, odd) if $p(x) = 0$ (respectively, 1). A graded vector space is a special case of a graded superspace, in which the ghost number and parity are congruent modulo 2.

A 1-shifted curved Lie superalgebra is a graded superspace with the following data (all of which have odd parity):

1. an element $R \in L^1$ (the curvature);
2. a linear operation $d : L^k \to L^{k+1}$ (the differential);
3. a bilinear operation $(-, -) : L^k \times L^\ell \to L^{k+\ell+1}$ (the antibracket).

The axioms are as follows:

(a) (the Bianchi identity) $dR = 0$;
(b) (the curvature identity) for all $x \in L^\bullet$,

$$d^2 x = (R, x);$$

(c) (the Leibniz identity) for all $x, y \in L^\bullet$,

$$d(x, y) = (dx, y) + (-1)^{p(x)+1}(x, dy);$$

(d) (antisymmetry) for all $x, y \in L^\bullet$,

$$(y, x) = -(-1)^{(p(x)+1)(p(y)+1)}(x, y);$$

(e) (the Jacobi rule) for all $x, y, z \in L^\bullet$,

$$(x, (y, z)) = ((x, y), z) + (-1)^{p(x)+1}(y, (x, z)).$$

All curved Lie superalgebras considered in this paper are 1-shifted.

Let $L^\bullet$ be a curved Lie superalgebra. If $x \in L^k$, we denote the operation

$$y \mapsto (x, y) : L^\bullet \to L^{\bullet+k+1}$$

by ad(x). A curved Lie superalgebra L is nilpotent if, for every odd element $x \in L^\bullet$, the endomorphism ad(x) is nilpotent.

A Maurer–Cartan element in a curved Lie algebra is an even element $x \in L^0$ such that the following equation holds:

$$R + dx + \tfrac{1}{2}(x, x) = 0.$$

The set of all Maurer–Cartan elements is denoted MC(L). The importance of Maurer–Cartan elements stems from the following result.

Lemma 1 *If $x \in \mathrm{MC}(L)$, the operator $d + \mathrm{ad}(x) : L^\bullet \to L^{\bullet+1}$ is a differential (a graded derivation of square zero).*

Proof It is evident that $d + \mathrm{ad}(x)$ is a graded derivation. Moreover, we have

$$\begin{aligned}(d + \mathrm{ad}(x))^2 y &= (d + \mathrm{ad}(x))(\mathrm{d}y + (x, y))\\ &= d^2 y + \mathrm{d}(x, y) + (x, \mathrm{d}y) + (x, (x, y))\\ &= (R, y) + (\mathrm{d}x, y) + (x, (x, y)).\end{aligned}$$

The proof is completed by observing that $(x, (x, y)) = \frac{1}{2}((x, x), y)$. □

In the special case in which the curvature is zero, we recover the definition of Maurer–Cartan elements in a differential graded Lie superalgebra.

If L is a curved Lie superalgebra, the space of odd elements of L^{-1} form a Lie algebra. If L is nilpotent, there is a gauge action of this Lie algebra on the set of Maurer–Cartan elements, given by the equation

$$x \bullet y = x + \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}(y))^n (\mathrm{d}y + (x, y))}{(n+1)!}.$$

Informally, this formula expresses the conjugation of the differential $d + \mathrm{ad}(x)$ by the gauge transformation $e^{\mathrm{ad}(y)}$

$$d + \mathrm{ad}(x \bullet y) = e^{-\mathrm{ad}(y)} \circ (d + \mathrm{ad}(x)) \circ e^{\mathrm{ad}(y)}.$$

This explains why the action preserves solutions of the Maurer–Cartan equation. In particular, if $dy = 0$, then $x \bullet y = e^{-\mathrm{ad}(y)} x$.

In order to derive the formula for $x \bullet y$, one introduces a parameter s, and considers the ordinary differential equation

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{ad}(x \bullet sy)}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s} e^{-\mathrm{ad}(sy)} (d + \mathrm{ad}(x)) e^{\mathrm{ad}(sy)}\\ &= [d + \mathrm{ad}(x \bullet sy), \mathrm{ad}(y)]\\ &= \mathrm{ad}(dy + (x \bullet sy, y)).\end{aligned}$$

This leads to the consideration of the ordinary differential equation

$$\frac{d(x \bullet sy)}{ds} = dy + (x \bullet sy, y), \tag{6}$$

with initial condition $x \bullet sy = x$ at $s = 0$, whose solution is

$$x \bullet sy = x + s \sum_{n=0}^{\infty} \frac{(-s\,\mathrm{ad}(y))^n (dy + (x, y))}{(n+1)!}.$$

The Baker–Campbell–Hausdorff formula gives an expression for the composition of two gauge transformations. For a proof, see Tao [16, Section 1.2].

Proposition 1 *If y, z are odd elements of L^{-1} and $x \in \mathrm{MC}(L)$, we have $x \bullet y \bullet z = x \bullet (y * z)$, where*

$$y * z = y + \int_0^1 \frac{\mathrm{ad}(y * tz)}{1 - e^{-\mathrm{ad}(y*tz)}} z \,\mathrm{d}t$$

is the solution of the equation $e^{\mathrm{ad}(y*z)} = e^{\mathrm{ad}(y)} e^{\mathrm{ad}(z)}$.

3 Formal variational calculus and the classical Batalin–Vilkovisky master equation

Let M be a graded supermanifold, with coordinates $\{\xi^a\}_{a \in A}$, where ξ^a has ghost number $gh(\xi^a) \in \mathbb{Z}$ and parity $p(\xi^a) \in \mathbb{Z}/2$. Introduce the shifted cotangent bundle $T^*[-1]M$, whose coordinates are the coordinates $\{\xi^a\}_{a \in A}$ of M, and dual coordinates $\{\xi_a^+\}_{a \in A}$, of ghost number

$$gh(\xi_a^+) = gh(\xi^a) - 1,$$

and parity

$$p(\xi_a^+) = 1 - p(\xi^a).$$

In the Batalin–Vilkovisky formalism, the coordinates ξ^a are called fields, and the coordinates ξ_a^+ are called antifields. However, this division is somewhat arbitrary, since we may just as well exchange the rôles of field ξ^a and antifield ξ_a^+. In the work of Batalin and Vilkovisky, it was assumed that the fields have nonnegative ghost number and the antifields have negative ghost number, but this proves to be too restrictive in the setting of AKSZ field theories.

Let $\mathscr{A}(M)$ be the graded commutative superalgebra generated over $\mathscr{O}_\infty(M)$ by (graded) polynomials in the derivatives $\{\partial^k \xi_a^+\}_{k \geq 0}$ of the antifields. (In fact, one should take a certain completion of this algebra whereby we allow infinite sums of terms with decreasing ghost number, but we will be sloppy and neglect this subtlety here, as we did in [10,11]. Working with this completion would not affect the conclusions of those papers.) This is the graded commutative superalgebra of functions on the jet space $J_\infty T^*[-1]M$.

Let $\mathscr{O}(M)$ be the graded commutative superalgebra of functions on M (which may be polynomial, rational, analytic, or differentiable, depending on the setting).

Let $\mathscr{O}_\infty(M)$ be the graded superspace of all differential expressions in the fields and antifields, graded by total ghost number, that is, (graded) polynomials over $\mathscr{O}(M)$ in the formal derivatives $\{\partial^k \xi^a\}_{k>0}$ of the coordinates with respect to a formal parameter t. In other words, $\mathscr{O}_\infty(M)$ is the graded commutative superalgebra of functions on the jet space $J_\infty M$ of M. Note that for now we only consider expressions that carry no explicit dependence on the variable t.

Introduce the abbreviations

$$\partial_{k,a} = \frac{\partial}{\partial(\partial^k \xi^a)} : \mathscr{A}^j \to \mathscr{A}^{j-gh(\xi^a)}, \qquad \partial_k^a = \frac{\partial}{\partial(\partial^k \xi_a^+)} : \mathscr{A}^j \to \mathscr{A}^{j-gh(\xi_a^+)}.$$

Let ∂ be the total derivative with respect to t:

$$\partial = \sum_{k=0}^{\infty} \big((\partial^{k+1}\xi^a)\partial_{k,a} + (\partial^{k+1}\xi_a^+)\partial_k^a \big).$$

Let $\phi : M_0 \to M_1$ be an étale map (local embedding) of graded supermanifolds, where M_0 has coordinates $\{\xi^a\}_{a\in A}$ and M_1 has coordinates $\{\eta^b\}_{b\in B}$: such a map is determined by functions

$$y^b(\xi) \in \mathscr{O}(M_0)$$

such that $\phi^*\eta^b = y^b(\xi)$. This defines a morphism of algebras $\phi^* : \mathscr{O}(M_1) \to \mathscr{O}(M_0)$, which extends to a morphism

$$\phi^* : \mathscr{O}_\infty(M_1) \to \mathscr{O}_\infty(M_0) \tag{7}$$

by the requirement that $\partial\phi^* = \phi^*\partial$, so that

$$\phi^*\partial^k \eta^b = \partial^k y^b(\xi).$$

In particular,

$$\phi^*\partial\eta^b = J(\xi)^b_a \partial\xi^a,$$

where $J(\xi)^b_a$ is the Jacobian of ϕ,

$$J(\xi)^b_a = \frac{\partial y^b(\xi)}{\partial \xi^a}.$$

Since ϕ is étale, J is invertible. The morphism (7) extends to a morphism

$$\phi^* : \mathscr{A}(M_1) \to \mathscr{A}(M_0),$$

on setting $\phi^*\eta_b^+ = J^{-1}(\xi)^a_b \xi_a^+$, and

$$\phi^*\partial^k \eta_b^+ = \partial^k \big(J^{-1}(\xi)^a_b \xi_a^+ \big).$$

An evolutionary vector field is a graded derivation of the graded commutative superalgebra $\mathscr{A}(M)$ that commutes with ∂. In other words, it is a vector field of the form

$$\mathrm{pr}\big(X^a\partial_a + X_a\partial^a\big) = \sum_{k=0}^{\infty}\big((\partial^k X^a)\partial_{k,a} + (\partial^k X_a)\partial_k^a\big).$$

The evolutionary vector field associated to the expression $X^a\partial_a + X_a\partial^a$ by the above formula is called its prolongation.

The Soloviev antibracket on $\mathscr{A}(M)$ is defined by the formula

$$(\!(f, g)\!) = (-1)^{(p(f)+1)p(\xi^a)} \sum_{k,\ell=0}^{\infty} \big(\partial^\ell(\partial_{a,k} f)\,\partial^k(\partial_\ell^a g) + (-1)^{p(f)}\partial^\ell(\partial_k^a f)\,\partial^k(\partial_{a,\ell} g)\big). \tag{8}$$

It satisfies the axioms for a graded Lie superalgebra, is linear over ∂,

$$(\!(\partial f, g)\!) = (\!(f, \partial g)\!) = \partial(\!(f, g)\!),$$

and invariant under étale changes of coordinates [9, Theorem 4.1]:

$$(\!(\phi^* f, \phi^* g)\!) = \phi^*(\!(f, g)\!).$$

The superspace $\mathscr{F} = \mathscr{A}/\partial\mathscr{A}$ of functionals is the graded quotient of $\mathscr{A}$ by the subspace $\partial\mathscr{A}$ of total derivatives. Denote the image of $f \in \mathscr{A}$ in $\mathscr{F}$ by $\textstyle\int f$. The Soloviev antibracket $(\!(f, g)\!)$ descends to an antibracket

$$\textstyle\int (f, g)$$

on $\mathscr{F}$, called the Batalin–Vilkovisky antibracket. Denote by $\delta_a : \mathscr{F}^j \to \mathscr{A}^{j-gh(\xi^a)}$ and $\delta^a : \mathscr{F}^j \to \mathscr{A}^{j-gh(\xi_a^+)}$ the variational derivatives

$$\delta_a = \sum_{k=0}^{\infty}(-\partial)^k \circ \partial_{k,a} \qquad\qquad \delta^a = \sum_{k=0}^{\infty}(-\partial)^k \circ \partial_k^a.$$

Lemma 2 *The Batalin–Vilkovisky antibracket is given by the formula*

$$\textstyle\int (f, g) = (-1)^{(p(f)+1)p(\xi^a)} \displaystyle\int \big((\delta_a f)\,(\delta^a g) + (-1)^{p(f)}(\delta^a f)\,(\delta_a g)\big).$$

The (Batalin–Vilkovisky) Hamiltonian vector field associated to an element $\textstyle\int f \in \mathscr{F}$ is the evolutionary vector field given by the formula

$$\mathsf{H}_f = \sum_{k=0}^{\infty}(-1)^{(p(f)+1)p(\xi^a)}\big(\partial^k(\delta_a f)\,\partial_k^a + (-1)^{p(f)}\partial^k(\delta^a f)\,\partial_{k,a}\big).$$

Despite the notation, H_f only depends on f through its image $\textstyle\int f$ in $\mathscr{F}$.

The following theorem is proved, though not in precisely these terms, in Olver [14], but we give here a simpler proof, taken from [11].

Theorem 1 *The map* $f \mapsto \mathsf{H}_f$ *is a morphism of graded Lie superalgebras from* $\mathscr{F}$ *to the evolutionary vector fields.*

Recall the higher Euler operators of Kruskal et al. [13]:

$$\delta_{k,a} = \sum_{\ell=0}^{\infty} \binom{k+\ell}{k} (-\partial)^{\ell} \partial_{k+\ell,a}, \qquad \delta_k^a = \sum_{\ell=0}^{\infty} \binom{k+\ell}{k} (-\partial)^{\ell} \partial_{k+\ell}^a.$$

When $k = 0$, $\delta_{0,a} = \delta_a$ and $\delta_0^a = \delta^a$ are the classical variational derivatives.

If $f \in \mathscr{A}$, the differential operator $\mathrm{ad}(f) = (\!(f, -)\!)$ associated to f by the Soloviev antibracket is given by the formula [11, Proposition 2.1]

$$\mathrm{ad}(f) = \sum_{k=0}^{\infty} \partial^k \mathsf{f}_{(k)}, \tag{9}$$

where $\mathsf{f}_{(k)}$ is the sequence of evolutionary vector fields

$$\mathsf{f}_{(k)} = (-1)^{(p(f)+1)\,p(\xi^a)}\, \mathrm{pr}\big(\big(\delta_{k,a} f\big)\partial^a + (-1)^{p(f)}\big(\delta_k^a f\big)\partial_a\big).$$

In particular, $\mathsf{f}_{(0)} = H_f$.

The proof of Theorem 1 relies on [11, Theorem 2.1], which we reformulate for convenience.

Lemma 3 *Let* t_k, $k \geq 0$, *be a sequence of evolutionary vector fields such that* $\mathsf{t}_k = 0$, $k \gg 0$, *and*

$$\sum_{k=0}^{\infty} \partial^k \mathsf{t}_k = 0.$$

Then $\mathsf{t}_k = 0$ *for all* $k \geq 0$.

Proof We prove by downward induction in k that the vector fields t_k vanish. Let K be the largest integer such that t_K is nonzero. Let ξ be one of the fields of the theory having even parity, and take the $(K+1)$-fold commutator of the left-hand side of (3) with ξ. We obtain

$$(K+1)!\,(\partial\xi)^K \mathsf{t}_K(\xi) = 0.$$

It follows that $\mathsf{t}_K(\xi) = 0$.

Next, we take the commutator with the antifield ξ^+ followed by the K-fold commutator with ξ: we obtain the equation

$$K!\,(\partial\xi)^{K-1}\big(\partial\xi\, \mathsf{t}_K(\xi^+) + K\, \partial\xi^+\, \mathsf{t}_K(\xi)\big) = K!\,(\partial\xi)^K \mathsf{t}_K(\xi^+) = 0.$$

We conclude that $\mathsf{t}_K(\xi^+) = 0$.

The vanishing of $\mathsf{t}_K(\xi)$ and $\mathsf{t}_K(\xi^+)$ may be proved for fields ξ of odd parity by exchanging the rôles of ξ and its antifield ξ^+ in the above argument. In this way, we see that $\mathsf{t}_K = 0$. Arguing by downward induction, we conclude that $\mathsf{t}_k = 0$ for all $k \geq 0$, proving the lemma. □

Proof (Proof of Theorem 1) If $(\!(f, g)\!) = h$, it follows from (9) that

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \partial^k [\mathsf{f}_{(\ell)}, \mathsf{g}_{(k-\ell)}] = \sum_{k=0}^{\infty} \partial^k \mathsf{h}_{(k)}.$$

Consider the evolutionary vector fields

$$\mathsf{t}_k = \sum_{\ell=0}^{k} [\mathsf{f}_{(\ell)}, \mathsf{g}_{(k-\ell)}] - \mathsf{h}_{(k)}.$$

We are in the situation of Lemma 3: it follows that $\mathsf{t}_k = 0$ for all $k \geq 0$, and in particular,

$$\mathsf{t}_0 = [\mathsf{H}_f, \mathsf{H}_g] - \mathsf{H}_{(\!(f,g)\!)} = 0.$$

Since $\int (\!(f, g)\!) = \int (f, g)$, we see that the map H is a morphism of graded Lie superalgebras. □

The following lemma shows that the kernel of the Hamiltonian map $f \mapsto \mathsf{H}_f$ vanishes except in ghost number 0, where it equals the constant multiples of 1.

Theorem 2 *If* $\mathsf{H}_f = 0$*, then* f *is the sum of a constant and a total derivative.*

Proof (Proof (Olver [14, Theorem 4.7])) We must show that if $\delta_a f = \delta^a f = 0$, then f is the sum of a constant and a total derivative. For $0 \leq s \leq 1$, let f_s be the rescaled quantity

$$f_s(\xi, \partial\xi, \dots) = f(s\xi, s\partial\xi, \dots).$$

It is an exercise in binomial coefficients to show that an evolutionary vector field may be written in terms of the higher Euler operators:

$$\mathrm{pr}\big(X^a \partial_a + X_a \partial^a\big) = \sum_{k=0}^{\infty} \partial^k \big(X^a \delta_{k,a} + X_a \delta_k^a\big).$$

It follows that

$$\begin{aligned} \frac{\mathrm{d} f_s}{\mathrm{d} s} &= \mathrm{pr}(\xi^a\, \partial_a + \xi_a^+\, \partial^a) f_s \\ &= \sum_{k=0}^{\infty} \partial^k (\xi^a\, \delta_{k,a} f + \xi_a^+\, \delta_k^a f)_s. \end{aligned}$$

Integrating over s from 0 to 1, we see that

$$f = f_0 + \sum_{k=0}^{\infty} \left(\int_0^1 \partial^k (\xi^a \, \delta_{k,a} f_s + \xi_a^+ \, \delta_k^a f_s) \, \mathrm{d}s \right).$$

In particular, if $\delta_a f = \delta^a f = 0$, we see that

$$f = f_0 + \sum_{k=1}^{\infty} \partial \left(\int_0^1 \partial^{k-1} (\xi^a \, \delta_{k,a} f_s + \xi_a^+ \, \delta_k^a f_s) \, \mathrm{d}s \right),$$

proving the lemma. □

A Maurer–Cartan element of $\mathscr{F}$ is a solution $\int S \in \mathscr{F}^0$ of the classical Batalin–Vilkovisky master equation (1). In the Batalin–Vilkovisky formalism, a Maurer–Cartan element $\int S$ determines a classical field theory.

There is a more precise formulation of the classical master equation, obtained by lifting a solution in the space of functionals $\mathscr{F}$ to a resolution of this space. We review the details of this construction, taken from [11] .

Introduce the quotient complex $\tilde{\mathscr{A}}$ of $\mathscr{A}$ by the constants:

$$\tilde{\mathscr{A}}^j = \begin{cases} \mathscr{A}^0/\mathbb{C}, & j = 0, \\ \mathscr{A}^j, & j \neq 0. \end{cases}$$

The space of functionals $\mathscr{F}$ has a resolution

$$\mathscr{B}^j = \mathscr{A}^j \oplus \tilde{\mathscr{A}}^{j+1} \varepsilon,$$

where the symbol ε is understood to have odd parity and ghost number -1, so that the parities of the superspace $\tilde{\mathscr{A}}^{j+1}$ are reversed in $\mathscr{B}^j$. The differential $d : \mathscr{B}^j \to \mathscr{B}^{j+1}$ equals

$$d\big(f + g\varepsilon\big) = (-1)^{p(g)} \, \partial g. \tag{10}$$

The Soloviev antibracket extends to $\mathscr{B}$ by the formula

$$(\!(f_0 + g_0\varepsilon, f_1 + g_1\varepsilon)\!) = (\!(f_0, f_1)\!) + (\!(f_0, g_1)\!) \, \varepsilon + (-1)^{p(f_1)+1} \, (\!(g_0, f_1)\!) \, \varepsilon. \tag{11}$$

We have

$$d(\!(a, b)\!) = (\!(da, b)\!) + (-1)^{p(a)+1} (\!(a, db)\!),$$

and the differential graded Lie superalgebra $(\mathscr{B}, d)$ is a resolution of the graded Lie superalgebra $\mathscr{F}$.

If $\int S$ is a solution of the classical master equation (1), there is an element $\tilde{S} \in \mathscr{A}^1$ of odd parity such that

$$\tfrac{1}{2} \, (\!(S, S)\!) = \partial \tilde{S}.$$

The classical master equation (1) may be recast as the Maurer–Cartan equation

$$d\mathsf{S} + \tfrac{1}{2}(\!(\mathsf{S}, \mathsf{S})\!) = 0 \tag{12}$$

in $\mathscr{B}$, where

$$\mathsf{S} = S + \tilde{S}\varepsilon \in \mathscr{B}^0.$$

This refinement of the classical master equation is closely related to the modified classical master equation of Cattaneo, Mnëv and Reshitikhin [6, Proposition 3.1].

Let s be the Hamiltonian vector field H_{S}: this is an evolutionary vector field of degree 1. By Theorem 1, we see that a solution of the classical master equation (12) yields an odd Hamiltonian vector field s, of degree 1, satisfying the relation $\mathsf{s}^2 = 0$. The differential graded Lie superalgebra $\mathscr{B}$ with differential $d + \mathsf{s}$ is a resolution of the differential graded Lie superalgebra $\mathscr{F}$, with differential $\int(S, -)$. The cohomology of this complex is the Batalin–Vilkovisky cohomology of the classical field theory S.

4 Covariant field theories in one dimension: local case

AKSZ field theories are a class of solutions of the classical master equation, introduced by Alexandrov et al. [1]. Here, we only consider the case of one-dimensional AKSZ field theories: these include the main model of interest to us in this paper, the spinning particle in a curved background. (The focus in [1] is rather on the two and three-dimensional cases.) An AKSZ field theory is associated with a symplectic form ω on the graded supermanifold whose coordinates are the fields of the theory.

In this section, we define a curved Lie superalgebra whose Maurer–Cartan elements consist of a solution of the master equation (12), together with additional structure that expresses covariance with respect to time translation. In the case of an AKSZ field theory, the additional structure involves the Poisson tensor $\pi = \omega^{-1}$, and thus incorporates the nondegeneracy of the symplectic form.

Let u be a variable of ghost number 2. We consider the graded Lie algebras of power series in u with coefficients in the graded Lie algebras $\mathscr{F}$ and $\mathscr{B}$, such that

$$\textstyle\int(uf, g) = \int(f, ug) = u\int(f, g),$$

respectively

$$(\!(uf, g)\!) = (\!(f, ug)\!) = u(\!(f, g)\!).$$

The element

$$\mathsf{D} = \xi_a^+ \partial \xi^a \in \mathscr{A}(M)$$

is invariant under changes of coordinates, and its image $\int \mathsf{D}$ in $\mathscr{F}$ lies in the centre, that is, $\operatorname{ad}(\int \mathsf{D}) = 0$. Consider the curved Lie superalgebra $\mathscr{F}[[u]]$ with vanishing differential and curvature $u\int \mathsf{D}$. A Maurer–Cartan element of $\mathscr{F}[[u]]$ is a solution $\int S_u \in \mathscr{F}[[u]]$, of ghost number 0, of the equation

$$\tfrac{1}{2}\textstyle\int(S_u, S_u) = -u\int \mathsf{D}. \tag{13}$$

Expand S_u in powers of u

$$S_u = S_0 + uS_1 + u^2 S_2 + \cdots$$

The Maurer–Cartan equation (13) is equivalent to the classical master equation (1) for $S = S_0$, the equation

$$\textstyle\int(S_0, S_1) = -\int \mathsf{D},$$

and, for $n > 1$, the sequence of equations

$$\sum_{k=0}^{n} \textstyle\int(S_k, S_{n-k}) = 0.$$

In [10,11], such a structure was found in the case of the spinning particle: in those papers, S_0 was called S, S_1 was called G, while S_n vanished for $n > 1$.

The operator $\mathrm{ad}(\mathsf{D}) : \mathscr{A}^\bullet \to \mathscr{A}^\bullet$ is given by the explicit formula

$$\mathrm{ad}(\mathsf{D}) = \partial\ \mathrm{pr}(\xi_a^+ \partial^a) - \partial.$$

Introduce the graded derivation ι on $\mathscr{B}$, of degree -1,

$$\iota\big(f + g\varepsilon\big) = (-1)^{p(f)}\big(\mathrm{pr}(\xi_a^+ \partial^a) f - f\big)\,\varepsilon.$$

It is easily seen that $d\iota + \iota d = \mathrm{ad}(\mathsf{D})$ and $\iota^2 = 0$. Let $\mathscr{B}[[u]]$ be the curved Lie superalgebra with graded derivation $d_u = d + u\iota$ and curvature $u\mathsf{D}$.

The following definition is central to this paper.

Definition 1 A (one-dimensional) covariant field theory is a Maurer–Cartan element $\mathsf{S}_u \in \mathscr{B}[[u]]$, that is, an element of ghost number 0 and even parity such that

$$d_u \mathsf{S}_u + \tfrac{1}{2}\,(\!(\mathsf{S}_u, \mathsf{S}_u)\!) = -u\mathsf{D}. \tag{14}$$

As in the case of the classical master equation, any solution of (13) gives rise to a solution of (14).

Proposition 2 *Let $\int S_u \in \mathscr{F}[[u]]$ be a solution of* (13)*, and choose a lift of $\int S_u$ to an element $S_u \in \mathscr{A}[[u]]$. Let $\tilde{S}_u \in \tilde{\mathscr{A}}[[u]]$ be the element determined by the equation*

$$\tfrac{1}{2}(\!(S_u, S_u)\!) = -u\mathsf{D} + \partial \tilde{S}_u.$$

Then

$$\mathsf{S}_u = S_u + \tilde{S}_u\,\varepsilon \in \mathscr{B}[[u]]$$

is a solution of (14).

Proof We have

$$
\begin{aligned}
u\mathsf{D} + d_u\mathsf{S}_u + \tfrac{1}{2}(\!(\mathsf{S}_u, \mathsf{S}_u)\!) &= u\mathsf{D} - \partial \tilde{S}_u + u\iota(S_u) + \tfrac{1}{2}(\!(S_u + \tilde{S}_u\varepsilon, S_u + \tilde{S}_u\varepsilon)\!) \\
&= u\iota(S_u) + (\!(S_u, \tilde{S}_u)\!)\varepsilon.
\end{aligned}
$$

We would like to show that the right-hand side has the form a constant times ε: it suffices to verify that on applying ∂ to it, we obtain zero. But we have

$$
\begin{aligned}
u\partial\iota(S_u) + \partial(\!(S_u, \tilde{S}_u)\!)\varepsilon &= \big((\!(S_u, -u\mathsf{D} + \partial\tilde{S}_u)\!)\big)\varepsilon \\
&= \tfrac{1}{2}(\!(S_u, (\!(\mathsf{S}_u, \mathsf{S}_u)\!))\!)\varepsilon = 0.
\end{aligned}
$$

It follows that $u\iota(S_u) + (\!(S_u, \tilde{S}_u)\!)\varepsilon$ represents zero in $\mathscr{B}[[u]]$. □

Let S_u be a covariant field theory. Expanding S_u in powers of u, we obtain a series of elements $\mathsf{S}_n \in \mathscr{B}^{-2n}$:

$$\mathsf{S}_u = \sum_{n=0}^{\infty} u^n \mathsf{S}_n.$$

Let s_n be the Hamiltonian vector field $\mathsf{H}_{\mathsf{S}_n}$: this is an evolutionary vector field of degree $1-2n$. By Theorem 1, we see that a covariant field theory yields a sequence of Hamiltonian vector fields s_n, of degree $1-2n$, such that s_0 is the Batalin–Vilkovisky differential, satisfying the relation $\mathsf{s}_0^2 = 0$, s_1 is a homotopy for the operator ∂, in the sense that

$$[\mathsf{s}_0, \mathsf{s}_1] = -\partial,$$

and for $n > 1$,

$$\sum_{k=0}^{n} [\mathsf{s}_k, \mathsf{s}_{n-k}] = 0.$$

All of the examples considered in this paper satisfy $\mathsf{S}_n = 0$, $n > 1$; in particular, $\mathsf{s}_1^2 = 0$.

An odd element $H \in \mathscr{B}[[u]]$ of ghost number -1 generates a flow on the space of covariant field theories by gauge action on the curved Lie superalgebra:

$$\mathsf{S}_{u,\tau} = \mathsf{S}_u \bullet \tau H.$$

We may also consider twists of covariant field theories, by which we mean the flow associated to a Hamiltonian in $\mathscr{B}(\!(u)\!) = \mathscr{B}[[u]][u^{-1}]$ such that the Maurer–Cartan element $\mathsf{S}_u \bullet \tau H$ remains in $\mathscr{B}[[u]]$. The class of twists discussed in the following proposition are the ones of importance to the study of AKSZ field theories.

If S_u is a covariant field theory, the operator

$$d = d_u + \operatorname{ad}(\mathsf{S}_u) \tag{15}$$

is a differential on $\mathscr{B}[[u]]$.

Proposition 3 *Consider an element $W \in \mathscr{O}(M)$ of ghost number* 1 *and odd parity such that* $\mathsf{d}W$ *is divisible by* u *and* $(\!(\mathsf{d}W, W)\!) = 0$. *Then the twist* $\mathsf{S}_u \bullet u^{-1}W$ *of* S_u *by* $u^{-1}W$ *is a covariant field theory, given by the formula*

$$\mathsf{S}_u \bullet u^{-1}W = \mathsf{S}_u + u^{-1}\mathsf{d}W.$$

Let us now show how these formulas capture AKSZ field theories in the one-dimensional case. The de Rham complex $\Omega^\bullet(M)$ of the graded supermanifold is generated over $\mathscr{O}(M)$ by the one-forms $\mathrm{d}f$, $f \in \mathscr{O}(M)$, of parity $p(f)+1$, subject to the Leibniz relation

$$\mathrm{d}(fg) = \mathrm{d}f\, g + (-1)^{p(f)} f\, \mathrm{d}g.$$

We adopt the sign convention that one-forms graded commute:

Let

$$\nu = \nu_a(\xi)\, \mathrm{d}\xi^a \in \Omega^1(M)$$

be a one-form on M of ghost number 0 and odd parity; in other words, $gh(\nu_a) = -\,gh(\xi^a)$ and $p(\nu_a) = p(\xi^a)$. The two-form $\omega = d\nu$ equals

$$\begin{aligned}\omega &= \tfrac{1}{2}\, \mathrm{d}\xi^a\, \omega_{ab}(\xi)\, \mathrm{d}\xi^b \in \Omega^2(M)\\ &= \tfrac{1}{2}(-1)^{(p(\xi^a)+1)\,p(\xi^b)}\, \omega_{ab}(\xi)\, \mathrm{d}\xi^a\, \mathrm{d}\xi^b,\end{aligned}$$

where

$$\omega_{ab} = \partial_a \nu_b - (-1)^{p(\xi^a)\,p(\xi^b)} \partial_b \nu_a. \tag{16}$$

In particular, $gh(\omega_{ab}) = -\,gh(\xi^a) - gh(\xi^b)$ and $p(\omega_{ab}) = p(\xi^a) + p(\xi^b)$.

Denote the frame of the tangent bundle TM dual to the frame $\mathrm{d}\xi^a$ of the cotangent bundle by $\tau_a = \partial/\partial\xi^a$. The two-form ω induces a morphism of vector bundles $TM \to T^*M$, which is denoted $X \mapsto X^\flat = X \lrcorner\, \omega$, or in terms of the frames $\{\tau_a\}$ and $\{\mathsf{d}\xi^a\}$,

$$\tau_a^\flat = \omega_{ab}\, \mathrm{d}\xi^b.$$

Likewise, a bivector field π on M induces a morphism of vector bundles $T^*M \to TM$, denoted $\theta \mapsto \theta^\sharp = \pi \lrcorner\, \theta$. The two-form ω is symplectic if there is a bivector field π such that $(X^\flat)^\sharp = X$. Expanding the bivector field in the local frame $\{\tau_a\}$,

$$\begin{aligned}\pi &= \tfrac{1}{2}\, \tau_a\, \pi^{ab}(\xi)\, \tau_b \in \Gamma(M, \mathrm{Sym}^2(T[-1]M))\\ &= \tfrac{1}{2}\, (-1)^{(p(\xi^a)+1)\,p(\xi^b)}\, \pi^{ab}(\xi)\, \tau_a\, \tau_b,\end{aligned}$$

the relationship between ω and τ becomes

$$(-1)^{p(\xi^a)}\, \pi^{ab} \omega_{bc} = \delta^a_c. \tag{17}$$

Note that the coefficients π^{ab} possess the same symmetry as ω_{ab}, namely

$$\pi^{ab} = -(-1)^{p(\xi^a)\,p(\xi^b)}\, \pi^{ba}.$$

Lemma 4 *If ω and π are inverse to each other in the above sense, then ω is closed if and only if the bivector field π is a Poisson tensor:*

$$\pi^{ab}\,\partial_b\pi^{cd} = (-1)^{p(\xi^a)\,p(\xi^c)}\,\pi^{cb}\,\partial_b\pi^{ad}. \tag{18}$$

Let M be a graded supermanifold, and let ν be a one-form of ghost number 0 and odd parity such that $\omega = \mathrm{d}\nu$ is a symplectic form. The following theorem follows by a lengthy but straightforward calculation based on (16), (17) and (18).

Theorem 3 *The elements of $\mathscr{B}[[u]]$*

$$\begin{aligned}
\mathsf{S}_0 &= (-1)^{p(\xi^a)}\,\nu_a(\xi)\,\partial\xi^a,\\
\mathsf{S}_1 &= \tfrac{1}{2}\,\xi_a^+\,\pi^{ab}(\xi)\,\xi_b^+ + (-1)^{p(\xi^a)}\,\nu_a(\xi)\,\pi^{ab}(\xi)\,\xi_b^+\varepsilon\\
&= \tfrac{1}{2}(\xi_a^+ - \nu_a\varepsilon)\pi^{ab}(\xi_b^+ - \nu_b\varepsilon),
\end{aligned}$$

of ghost number 0 *and* -2*, respectively, and even parity, are independent of the coordinate system* $\{\xi^a\}$*. Their sum*

$$\mathsf{S}_u = \mathsf{S}_0 + u\mathsf{S}_1 \in \mathscr{B}[[u]],$$

satisfies (14)*, and hence defines a covariant field theory.*

Let ν and ν' be two one-forms such that

$$\mathrm{d}\nu = \mathrm{d}\nu' = \omega,$$

and in particular, $\nu - \nu'$ is closed. If M is simply connected, then $\nu - \nu'$ is exact: there is a function $\mu \in \mathscr{O}(M)$ such that

$$\nu - \nu' = \mathrm{d}\mu.$$

It follows that

$$\begin{aligned}
\mathsf{S}_0 - \mathsf{S}_0' &= (-1)^{p(\xi^a)}\,\partial_a\mu(\xi)\,\partial\xi^a\\
&= \partial\mu(\xi),
\end{aligned}$$

and in particular,

$$\textstyle\int\mathsf{S}_0 = \int\mathsf{S}_0'$$

and

$$\textstyle\int\mathsf{S}_u = \int\mathsf{S}_u'.$$

Thus, locally in M, the choice of ν is unimportant in the definition of the field theory: it is only when the world-line has nonempty boundary (or M has nonzero first homology) that this ambiguity comes into play. This is one of the reasons that we have introduced the resolution $\mathscr{B}$ of $\mathscr{F}$

The Poisson bracket associated to the symplectic form ω is the bilinear form on $\mathscr{O}(M)$ given by the formula

$$\begin{aligned} u\{f,g\} &= (-1)^{p(f)} (\!(\mathsf{d}f, g)\!) \\ &= (-1)^{(p(f)+p(\xi^a))\,p(\xi^b)} u\,\pi^{ab} \partial_a f\, \partial_b g, \end{aligned}$$

where d is the differential introduced in (15).

Lemma 5 *The Poisson bracket satisfies the graded symmetry condition*

$$\{f,g\} = -(-1)^{p(f)\,p(g)} \{g,f\}.$$

Proof We have $(\!(f,g)\!) = 0$, hence

$$\begin{aligned} 0 = \mathsf{d}(\!(f,g)\!) &= (\!(\mathsf{d}f, g)\!) + (-1)^{p(f)+1} (\!(f, \mathsf{d}g)\!) \\ &= (\!(\mathsf{d}f, g)\!) - (-1)^{(p(f)+1)(p(g)+1)} (\!(\mathsf{d}g, f)\!). \end{aligned}$$

□

The following lemma generalizes to graded supermanifolds the proof of the Jacobi rule for the Poisson bracket associated to a Poisson tensor.

Lemma 6 *The Poisson bracket satisfies the graded Jacobi identity*

$$\{f,\{g,h\}\} = \{\{f,g\},h\} + (-1)^{p(f)\,p(g)} \{g,\{f,h\}\}.$$

Proof We have

$$\begin{aligned} u^2\{\{f,g\},h\} &= (-1)^{p(g)} (\!(\mathsf{d}(\!(\mathsf{d}f, g)\!), h)\!) \\ &= (-1)^{p(f)+p(g)} (\!((\!(\mathsf{d}f, \mathsf{d}g)\!), h)\!) \\ &= (-1)^{p(f)+p(g)} \big((\!(\mathsf{d}f, (\!(\mathsf{d}g, h)\!))\!) - (-1)^{p(f)\,p(g)} (\!(\mathsf{d}g, (\!(\mathsf{d}f, h)\!))\!) \big) \\ &= u^2 \big(\{f,\{g,h\}\} - (-1)^{p(f)\,p(g)} \{g,\{f,h\}\} \big). \end{aligned}$$

□

If $W \in \mathscr{O}(M)$ is a function on M of ghost number 1 and odd parity, then

$$\mathsf{d}W = u\big(-\xi_a^+ \,\pi^{ab}\, \partial_b W + (-1)^{p(\xi^a)}\, \nu_a\, \pi^{ab}\, \partial_b W \varepsilon + W\varepsilon \big)$$

is divisible by u, and $(\!(\mathsf{d}W, W)\!) = u\{W,W\}$ vanishes if and only if

$$\{W,W\} = 0,$$

in other words, precisely when the Hamiltonian vector field associated to W is cohomological. Applying Proposition 3, we obtain the following result.

Theorem 4 *Let M be a graded supermanifold, and let $\nu \in \Omega^1(M)$ be a one-form of ghost number 0 and odd parity such that $\omega = \mathrm{d}\nu$ is a symplectic form. Let π be the Poisson tensor associated to ω.*

Let $W \in \mathscr{O}(M)$ be a function on M of ghost number 1 and odd parity, such that $\{W, W\} = 0$. Then the twist $\mathsf{S}_u \bullet u^{-1}W$ of the covariant field theory S_u by $u^{-1}W$, given by the formula

$$\begin{aligned}\mathsf{S}_u + u^{-1}\mathsf{d}W &= (-1)^{p(\xi^a)}\nu_a\,\partial\xi^a + W\varepsilon \\ &\quad + \tfrac{1}{2}u(\xi_a^+ - u^{-1}\partial_a W - \nu_a\varepsilon)\pi^{ab}(\xi_b^+ - u^{-1}\partial_b W - \nu_b\varepsilon),\end{aligned}$$

is a covariant field theory.

5 Covariant field theories in one dimension: global case

The formalism of the last section only applies when the symplectic form ω on the graded supermanifold M is exact. When this condition is not satisfied, the best we can do is to choose a cover

$$\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$$

of M, where each U_α is an open subspace of the graded supermanifold M, such that the restriction of ω to each U_α is exact:

$$\omega|_{U_\alpha} = \mathrm{d}\nu_\alpha.$$

The nerve $N_k\mathscr{U}$ of the cover is the sequence of graded supermanifolds indexed by $k \geq 0$

$$N_k\mathscr{U} = \bigsqcup_{\alpha_0\ldots\alpha_k\in I^{k+1}} U_{\alpha_0\ldots\alpha_k},$$

where

$$U_{\alpha_0\ldots\alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}.$$

Denote by $\epsilon : N_0\mathscr{U} \to M$ the map which on each summand U_α equals the inclusion $U \hookrightarrow M$.

The collection $\nu = \{\nu_\alpha\}_{\alpha\in I}$ gives a one-form on $N_0\mathscr{U}$, such that

$$\mathrm{d}\nu = \epsilon^*\omega.$$

For all $\alpha_0, \alpha_1 \in I$, the one-form

$$\nu_{\alpha_0}|_{U_{\alpha_0\alpha_1}} - \nu_{\alpha_1}|_{U_{\alpha_0\alpha_1}} \in \Omega^1(U_{\alpha_0\alpha_1})$$

is closed. Assume the cover $\mathscr{U}$ is chosen such that this form is exact for all (α_0, α_1): there exists functions $\mu_{\alpha_0\alpha_1} \in \Omega^0(U_{\alpha_0\alpha_1})$ such that

$$\mathrm{d}\mu_{\alpha_0\alpha_1} = \nu_{\alpha_0}|_{U_{\alpha_0\alpha_1}} - \nu_{\alpha_1}|_{U_{\alpha_0\alpha_1}}.$$

Assemble the functions $\{\mu_{\alpha_0\alpha_1}\}_{\alpha_0\alpha_1\in I}$ into a single function μ on $N_1\mathscr{U}$. Let $\delta_0, \delta_1 : N_1\mathscr{U} \to N_0\mathscr{U}$ be the morphisms which on $U_{\alpha_0\alpha_1}$ are, respectively, the inclusions into U_{α_1} and U_{α_0} (sic). We have

$$\mathrm{d}\mu = \delta_1^*\nu - \delta_0^*\nu.$$

The collection of differential forms

$$(\nu, \mu) \in \Omega^1(N_0\mathscr{U}) \times \Omega^0(N_1\mathscr{U})$$

will serve as a replacement for the non-existent one-form $\nu \in \Omega^1(M)$ solving the equation $\omega = \mathrm{d}\nu$. In order to repeat the discussion of the last section, we must extend the definition of the Soloviev bracket, and the curved Lie algebra $\mathscr{B}[[u]]$, from graded supermanifolds M to sequences of graded supermanifolds of the form $\{N_k\mathscr{U}\}$. Since we will use the formalism of simplicial and cosimplicial objects in our discussion, we now review their definition.

Let Δ be the category whose objects are the totally ordered sets

$$[k] = (0 < \cdots < k), \quad k \in \mathbb{N},$$

and whose morphisms are the order-preserving functions. A simplicial graded supermanifold $M_\bullet$ is a contravariant functor from Δ to the category of graded supermanifolds. (We leave open here whether we are working in the smooth, analytic or algebraic setting.) Here, M_k is the value of $M_\bullet$ at the object $[k]$, and $f^* : M_\ell \to M_k$ is the action of the arrow $f : [k] \to [\ell]$ of Δ. The arrow $d_i : [k] \to [k+1]$ which takes $j < i$ to j and $j \geq i$ to $j+1$ is known as a face map, while the arrow $s_i : [k] \to [k-1]$ which takes $j \leq i$ to j and $j > i$ to $j-1$ is known as a degeneracy map.

The simplicial graded supermanifolds used in this paper are the Čech nerves $N_\bullet\mathscr{U}$ of covers $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$. The face map $\delta_i = d_i^* : N_{k+1}\mathscr{U} \to N_k\mathscr{U}$ corresponds to the inclusion of the open subspace

$$U_{\alpha_0\ldots\alpha_{k+1}} \subset N_{k+1}\mathscr{U}$$

into the open subspace

$$U_{\alpha_0\ldots\widehat{\alpha}_i\ldots\alpha_{k+1}} \subset N_k\mathscr{U},$$

and the degeneracy map $\sigma_i = s_i^* : N_{k-1}U \to N_kU$ corresponds to the identification of the open subspace

$$U_{\alpha_0\ldots\alpha_{k-1}} \subset N_{k-1}\mathscr{U}$$

with the open subspace

$$U_{\alpha_0\ldots\alpha_i\alpha_i\ldots\alpha_k} \subset N_k\mathscr{U}.$$

Any simplicial map $f^* : M_\ell \to M_k$ is the composition of a sequence of face maps followed by a sequence of degeneracy maps. In particular, we see that in the case $M_\bullet = N_\bullet\mathscr{U}$ of the nerve of a cover, all of these maps are étale.

A covariant functor $X^\bullet$ from Δ to a category $\mathscr{C}$ is called a cosimplicial object of $\mathscr{C}$. These arise as the result of applying a contravariant functor to a simplicial space: for example, applying the de Rham functor $\Omega^\bullet(-)$ to the simplicial graded supermanifold $N_\bullet\mathscr{U}$, we obtain the cosimplicial differential graded commutative superalgebra $\Omega^\bullet(N_\bullet\mathscr{U})$. We will also be interested in the cosimplicial graded Lie superalgebra

$$\mathscr{F}(N_\bullet\mathscr{U})$$

with the Batalin–Vilkovisky antibracket, the cosimplicial differential graded Lie superalgebra

$$\mathscr{B}(N_\bullet\mathscr{U})$$

with the Soloviev antibracket and differential d, and the cosimplicial curved Lie superalgebra

$$\mathscr{B}(N_\bullet\mathscr{U})[[u]]$$

with the Soloviev antibracket, differential d_u and curvature $u\mathsf{D}$.

Associated to a cosimplicial superspace $V^\bullet$ is a graded superspace $N^\bullet(V)$, called the normalized cochain complex, defined as follows:

$$N^k(V) = \bigcap_{i=0}^{k-1} \ker\bigl(s^i : V^k \to V^{k-1}\bigr).$$

This is a complex, with differential

$$\delta = \sum_{i=0}^{k+1}(-1)^i d^i : N^k(V) \to N^{k+1}(V).$$

If the superspaces V^k making up the cosimplicial superspace are themselves complexes $V^k = V^{\bullet k}$, with differential $d : V^{jk} \to V^{j+1,k}$, we obtain a double complex, with external differential $\delta : N^k(V^j) \to N^{k+1}(V^j)$ and internal differential $d : N^k(V^j) \to N^k(V^{j+1})$: the totalization of $V^{\bullet\bullet}$ is the graded superspace

$$|V|^n = \prod_{k=0}^{\infty} N^k(V^{n-k}),$$

with differential $d_{\mathrm{Tot}} = \delta + (-1)^k d$.

The de Rham complex of the simplicial graded supermanifold $N_\bullet\mathscr{U}$ is the totalization $|\Omega^\bullet(N_\bullet\mathscr{U})|$ of the de Rham complex of $N_\bullet\mathscr{U}$. There is a morphism $N_\bullet\mathscr{U} \to M$ of simplicial graded supermanifolds from $N_\bullet\mathscr{U}$ to the constant simplicial graded supermanifold M, which induces a quasi-isomorphism of complexes

$$\Omega^\bullet(M) \to |\Omega^\bullet(N_\bullet\mathscr{U})|.$$

We have $d\nu = \epsilon^*\omega$ and $\delta\nu + d\mu = 0$. Let $[\omega] \in \Omega^0(N_2\mathscr{U})$ be the Čech differential of $\mu \in \Omega^0(N_1\mathscr{U})$:

$$[\omega]_{\alpha_0\alpha_1\alpha_2} = \mu_{\alpha_0\alpha_1}|_{U_{\alpha_0\alpha_1\alpha_2}} - \mu_{\alpha_0\alpha_2}|_{U_{\alpha_0\alpha_1\alpha_2}} + \mu_{\alpha_1\alpha_2}|_{U_{\alpha_0\alpha_1\alpha_2}}.$$

We have $d[\omega] = d\delta\mu = \delta d\mu = -\delta^2\mu = 0$, hence $[\omega]$ is a locally constant Čech 2-cocycle. By the formula

$$d_{\mathrm{Tot}}(\nu + \mu) = \epsilon^*\omega - [\omega],$$

we see that $[\omega]$ represents the cohomology class of ω in the Čech complex.

The cocycle $[\omega]$ is irrelevant in classical mechanics, since being locally constant, it does not contribute to the Euler–Lagrange equations. It assumes great importance in quantum mechanics, since it measures shifts in the phase of the Feynman integrand.

The construction of $|\Omega^\bullet(N_\bullet\mathscr{U})|$ behaves well under refinement of covers. A refinement $\mathscr{V} = \{V_\beta\}_{\beta\in J}$ of a cover $\mathscr{U} = \{U_\alpha\}_{\alpha\in I}$ is determined by a function of indexing sets $\phi : J \to I$, such that for all $\beta \in J$, V_β is a subset of $U_{\phi(\beta)}$. There is a morphism of cosimplicial differential graded superalgebras $\Phi^* : \Omega^\bullet(N_\bullet\mathscr{U}) \to \Omega^\bullet(N_\bullet\mathscr{V})$, obtained by restricting differential forms on $U_{\phi(\alpha_0)\dots\phi(\alpha_k)}$ to differential forms on $V_{\alpha_0\dots\alpha_k}$. Applying the totalization functor, we obtain a morphism of complexes

$$\Phi^* : |\Omega^\bullet(N_\bullet\mathscr{U})| \to |\Omega^\bullet(N_\bullet\mathscr{V})|.$$

If we have a further refinement $\mathscr{W} = \{W_\gamma\}_{\gamma\in K}$ of $\mathscr{V} = \{V_\beta\}_{\beta\in J}$ with $\psi : K \to J$, we may define a composition of these refinements $\phi\psi : K \to I$, and we obtain a commuting triangle of morphisms of complexes

$$\begin{array}{ccccc} & & |\Omega^\bullet(N_\bullet\mathscr{U})| & & \\ & \swarrow^{\Phi^*} & & \searrow^{\Psi^*\Phi^*} & \\ |\Omega^\bullet(N_\bullet\mathscr{V})| & & \xrightarrow[\Psi^*]{} & & |\Omega^\bullet(N_\bullet\mathscr{W})| \end{array}$$

In the special case where the cover $\mathscr{U} = \{M\}$ has just one element, the whole space M, we obtain the commutative diagram

$$\begin{array}{ccccc} & & \Omega^\bullet(M) & & \\ & \swarrow & & \searrow & \\ |\Omega^\bullet(N_\bullet\mathscr{V})| & & \xrightarrow[\Psi^*]{} & & |\Omega^\bullet(N_\bullet\mathscr{W})| \end{array}$$

We now generalize the classical master equation of Batalin–Vilkovisky theory to a Maurer–Cartan equation for the cosimplicial graded Lie superalgebra $\mathscr{F}(N_\bullet\mathscr{U})$. We might expect this generalization to simply be the Maurer–Cartan equation for the totalization $|\mathscr{F}(N_\bullet\mathscr{U})|$, but $|\mathscr{F}(N_\bullet\mathscr{U})|$ is not a differential graded Lie superalgebra. (This problem is related to the absence of a natural graded commutative product on the singular cochains of a topological space.) To circumvent this difficulty, we use a technique introduced in rational homotopy theory by Sullivan [15] (see also Bousfield and Guggenheim [3]), the Thom–Whitney normalization.

Let Ω_k be the free graded commutative algebra with generators t_i of degree 0 and $\mathrm{d}t_i$ of degree 1, and relations

$$t_0 + \cdots + t_k = 1$$

and $\mathrm{d}t_0 + \cdots + \mathrm{d}t_k = 0$. There is a unique differential δ on Ω_k such that $\delta(t_i) = \mathrm{d}t_i$, and $\delta(\mathrm{d}t_i) = 0$.

The differential graded commutative algebras Ω_k are the components of a simplicial differential graded commutative algebra $\Omega_\bullet$ (that is, contravariant functor from Δ to the category of differential graded commutative algebras): the arrow $f : [k] \to [\ell]$ in Δ acts by the formula

$$f^* t_i = \sum_{f(j)=i} t_j, \quad 0 \le i \le n.$$

The Thom–Whitney normalization of a cosimplicial superspace is an example of the categorical construction called an end:

$$N^\bullet_{\mathrm{TW}}(V) = \int_\Delta \Omega_\bullet \otimes V^\bullet.$$

In other words, $N^\bullet_{\mathrm{TW}}(V)$ is the equalizer of the maps

$$\prod_{k=0}^\infty \Omega_k \otimes V^k \underset{1\otimes f_*}{\overset{f^*\otimes 1}{\rightrightarrows}} \prod_{k,\ell=0}^\infty \prod_{f:[k]\to[\ell]} \Omega_k \otimes V^\ell$$

In [17], Whitney defines an injective morphism between the two normalizations

$$\mathsf{w} : N(V) \to N_{\mathrm{TW}}(V)$$

compatible with the differentials. The Whitney map takes a Čech k-cochain $(\nu_{\alpha_0\dots\alpha_k})$ to

$$\mathsf{w}(\nu) = \frac{1}{k+1} \sum_{\alpha_0,\dots,\alpha_k\in I} \sum_{i=0}^k (-1)^i t_{\alpha_i} \mathrm{d}t_{\alpha_0} \dots \widehat{\mathrm{d}t}_{\alpha_i} \dots \mathrm{d}t_{\alpha_k} \otimes \nu_{\alpha_0\dots\alpha_k}.$$

The differential $\delta\nu$ is taken by this map to

$$\mathsf{w}(\delta\nu) = \frac{1}{k+2} \sum_{\alpha_0,\dots,\alpha_{k+1}\in I} \sum_{j=0}^{k+1} \sum_{i=0}^{k+1} (-1)^{i+j} t_{\alpha_i} \mathrm{d}t_{\alpha_0} \dots \widehat{\mathrm{d}t}_{\alpha_i} \dots \mathrm{d}t_{\alpha_{k+1}} \otimes \nu_{\alpha_0\dots\widehat{\alpha}_j\dots\alpha_{k+1}}.$$

Only the terms with $i = j$ contribute, and we obtain

$$\mathsf{w}(\delta\nu) = \sum_{\alpha_1,\ldots,\alpha_{k+1}\in I} \mathrm{d}t_{\alpha_1}\ldots\mathrm{d}t_{\alpha_{k+1}}\,\nu_{\alpha_1\ldots\alpha_{k+1}}.$$

On the other hand, we have

$$\delta\mathsf{w}(\nu) = \sum_{\alpha_0,\ldots,\alpha_k\in I} \mathrm{d}t_{\alpha_0}\ldots\mathrm{d}t_{\alpha_k}\otimes\nu_{\alpha_0\ldots\alpha_k},$$

and we conclude that

$$\delta\mathsf{w}(\nu) = \mathsf{w}(\delta\nu). \tag{19}$$

If the superspaces V^k making up the cosimplicial superspace are themselves graded $V^k = V^{\bullet k}$, with differential $d : V^{jk} \to V^{j+1,k}$, we obtain a double complex, with external differential $\delta : N^k_{\mathrm{TW}}(V^j) \to N^{k+1}_{\mathrm{TW}}(V^j)$ and internal differential $d : N^k_{\mathrm{TW}}(V^j) \to N^k_{\mathrm{TW}}(V^{j+1})$: the Thom–Whitney totalization of $V^{\bullet\bullet}$ is the graded superspace

$$\|V\|^n = \prod_{k=0}^{\infty} N^k_{\mathrm{TW}}(V^{n-k}),$$

with differential $d_{\mathrm{TW}} = \delta + (-1)^k d$. The Whitney map w induces an injective morphism of graded superspaces

$$\mathsf{w} : |V|^\bullet \to \|V\|^\bullet.$$

By (19), this is a morphism of complexes. For the cosimplicial superspaces which we consider in this paper, w is a quasi-isomorphism. (This is proved using a spectral sequence, and we must impose additional hypotheses in order for the spectral sequence to converge. It is sufficient to assume that $V^\bullet$ is the (graded super)space of sections of a sheaf over the Čech nerve of a cover $\mathscr{U}$ of bounded dimension; that is, $U_{\alpha_0\ldots\alpha_k}$ is empty if the cardinality of the set of indices $\{\alpha_0,\ldots,\alpha_k\}$ is sufficiently large. In particular, this condition holds if the cover is finite.)

Applying this construction to the cosimplicial complex $\Omega^\bullet(N_\bullet\mathscr{U})$, we obtain the Thom–Whitney totalization $\|\Omega^\bullet(N_\bullet\mathscr{U})\|$, and an injective morphism of complexes

$$|\Omega^\bullet(N_\bullet\mathscr{U})| \hookrightarrow \|\Omega^\bullet(N_\bullet\mathscr{U})\|.$$

The advantage of Sullivan's Thom–Whitney normalization is that it takes cosimplicial differential graded commutative superalgebras to differential graded commutative superalgebras. (Its disadvantage is that its use is restricted to characteristic zero.) The reason is very simple: if V^k is a differential graded commutative superalgebra, then so is $\Omega_k \otimes V^k$. The differential on $\Omega_k \otimes V^k$ is

$$d(\alpha\otimes v) = \delta\alpha\otimes v + (-1)^i\alpha\otimes dv, \quad \alpha\in\Omega^i_k,\, v\in V^{jk},$$

and the product is

$$(\alpha_1 \otimes v_1)(\alpha_2 \otimes v_2) = (-1)^{i_2 p(v_1)} \alpha_1\alpha_2 v_1 v_2,$$

where $\alpha_\ell \in \Omega_k^{i_\ell}$ and $v_\ell \in V^{j_\ell k}$. The Thom–Whitney totalization $\|V\|$ is a subspace of the product of differential graded commutative superalgebras $\Omega_k \otimes V^k$, and this subspace is preserved by the differential and by the product. In this way, we see that by expanding $|\Omega^\bullet(N_\bullet\mathscr{U})|$ to the larger complex $\|\Omega^\bullet(N_\bullet\mathscr{U})\|$, we obtain a construction which associates to the cover $\mathscr{U}$ a differential graded commutative superalgebra.

The Thom–Whitney totalization also takes cosimplicial curved Lie superalgebras to curved Lie superalgebras. In particular, if $L^\bullet$ is a cosimplicial curved Lie superalgebra, the antibracket on $\|L\|$ is given by the formula

$$(\alpha_1 \otimes v_1, \alpha_2 \otimes v_2) = (-1)^{j_2(p(v_1)+1)} \alpha_1\alpha_2 (v_1, v_2).$$

The Thom–Whitney totalization $\|V\|$ is a subspace of the product curved Lie superalgebra $\prod_k \Omega_k \otimes V^k$, which is preserved by the antibracket. In the curved case, the curvatures of the curved Lie superalgebras V^k assemble to an element of degree 1 in $|V| \subset \|V\|$, which is easily seen to be a curvature element for the Thom–Whitney totalization.

In particular, the Thom–Whitney totalizations $\|\mathscr{F}(N_\bullet\mathscr{U})\|$ and

$$\|\mathscr{B}(N_\bullet\mathscr{U})\|$$

are differential graded Lie superalgebras. The differential of $\|\mathscr{F}(N_\bullet\mathscr{U})\|$ is induced by the differentials δ of the algebras Ω_k, while the differential of $\|\mathscr{B}(N_\bullet\mathscr{U})\|$ also involves the internal differential of $\mathscr{B}(N_\bullet\mathscr{U})$. The antibracket of $\|\mathscr{F}(N_\bullet\mathscr{U})\|$ is induced by the Batalin–Vilkovisky antibracket on $\mathscr{F}(N_\bullet\mathscr{U})$, while the antibracket of $\|\mathscr{B}(N_\bullet\mathscr{U}))$ is induced by the Soloviev antibracket. Similarly, the Thom–Whitney totalization

$$\|\mathscr{B}(N_\bullet\mathscr{U})[[u]]\| \cong \|\mathscr{B}(N_\bullet\mathscr{U})\|[[u]]$$

is a curved Lie superalgebra, whose differential is the sum the differential d_u of $\mathscr{B}(N_\bullet\mathscr{U})[[u]]$, with curvature D, and δ, and whose antibracket is induced by the Soloviev antibracket.

Given a refinement $\mathscr{V}$ of a cover $\mathscr{U}$, and a refinement $\mathscr{W}$ of $\mathscr{V}$, we obtain a commuting diagram of Thom–Whitney totalizations

$$\begin{array}{ccc} & \|\mathscr{F}(N_\bullet\mathscr{U})\| & \\ {}^{\Phi^*}\swarrow & & \searrow^{\Psi^*\Phi^*} \\ \|\mathscr{F}(N_\bullet\mathscr{V})\| & \xrightarrow[\Psi^*]{} & \|\mathscr{F}(N_\bullet\mathscr{W})\| \end{array}$$

The arrows in this diagram are morphisms of differential graded Lie algebras. There are also commuting triangles of differential graded Lie superalgebras

$$\begin{array}{ccc} & \|\mathscr{B}(N_\bullet\mathscr{U})\| & \\ \swarrow{\scriptstyle \Phi^*} & & \searrow{\scriptstyle \Psi^*\Phi^*} \\ \|\mathscr{B}(N_\bullet\mathscr{V})\| & \xrightarrow[\Psi^*]{} & \|\mathscr{B}(N_\bullet\mathscr{W})\| \end{array}$$

and of curved Lie superalgebras

$$\begin{array}{ccc} & \|\mathscr{B}(N_\bullet\mathscr{U})[[u]]\| & \\ \swarrow{\scriptstyle \Phi^*} & & \searrow{\scriptstyle \Psi^*\Phi^*} \\ \|\mathscr{B}(N_\bullet\mathscr{V})[[u]]\| & \xrightarrow[\Psi^*]{} & \|\mathscr{B}(N_\bullet\mathscr{W})[[u]]\| \end{array}$$

The analogue of the classical master equation (12) in the global setting is the Maurer–Cartan equation for the differential graded superalgebra $\|\mathscr{B}(N_\bullet\mathscr{U})\|$:

$$d_{\mathrm{TW}}\mathsf{S} + \tfrac{1}{2}(\!(\mathsf{S},\mathsf{S})\!) = 0.$$

Here, S is a collection of elements $\mathsf{S}^j_{\alpha_0\ldots\alpha_k} \in \Omega^j_k \otimes \mathscr{B}^{-j}(U_{\alpha_0\ldots\alpha_k})$ of total degree 0, simplicial, in the sense that for each $f : [k] \to [\ell]$,

$$(f^* \otimes 1)\mathsf{S}^j_\ell = (1 \otimes f_*)\mathsf{S}^j_k,$$

which satisfies the sequence of Maurer–Cartan equations

$$\delta\mathsf{S}^{j-1} + (-1)^j d\mathsf{S}^j + \frac{1}{2}\sum_{i=0}^{j}(\!(\mathsf{S}^i,\mathsf{S}^{j-i})\!) = 0.$$

This makes the following definition natural. The graded derivation on the curved Lie superalgebra $\|\mathscr{B}(N_\bullet\mathscr{U})[[u]]\|$ is

$$d_{\mathrm{TW},u} = d_{\mathrm{TW}} + u\iota.$$

Definition 2 Let M be a graded supermanifold M. A global covariant field theory for M is a solution of the Maurer–Cartan equation for the curved Lie superalgebra $\|\mathscr{B}(N_\bullet\mathscr{U})[[u]]\|$, where $\mathscr{U}$ is a cover of M:

$$d_{\mathrm{TW},u}\mathsf{S}_u + \tfrac{1}{2}(\!(\mathsf{S}_u,\mathsf{S}_u)\!) = -u\mathsf{D}.$$

Since the Maurer–Cartan set $\mathrm{MC}(L)$ is a functor on the category of curved Lie superalgebras, there is a commutative diagram of sets

$$\begin{array}{ccc} & \mathrm{MC}(\|\mathscr{B}(N_\bullet\mathscr{U})[[u]]\|) & \\ \swarrow^{\Phi^*} & & \searrow^{\Psi^*\Phi^*} \\ \mathrm{MC}(\|\mathscr{B}(N_\bullet\mathscr{V})[[u]]\|) & \xrightarrow[\Psi^*]{\qquad\qquad} & \mathrm{MC}(\|\mathscr{B}(N_\bullet\mathscr{W})[[u]]\|) \end{array}$$

We see that if S_u is a covariant field theory with respect to a cover $\mathscr{U}$ of M, then it induces a covariant field theory with respect to any refinement of $\mathscr{U}$.

We now come to the main result of this section.

Theorem 5 *Let M be a graded supermanifold with symplectic form ω. Let $\mathscr{U}$ be a cover of M, and let $(\nu, \mu) \in |\Omega^\bullet(N_\bullet\mathscr{U})|$ be a one-form such that $\mathrm{d}\nu = \epsilon^*\omega$ and $\delta\nu = \mathrm{d}\mu$.*

Let $\mathsf{S}_u \in \check{C}^0(\mathscr{U}, \mathscr{B}[[u]])$ be the Čech cochain which over U_α equals the local covariant field theory $\mathsf{S}_{\alpha,u}$ associated to the one-form ν_α.

The element $\mathbb{S}_u = \mathsf{w}(\mathsf{S}_u + \mu\varepsilon)$ is a global covariant field theory, that is, a Maurer–Cartan element in the curved Lie superalgebra $\|\mathscr{B}(N_\bullet\mathscr{U})[[u]]\|$.

This result is proved by lengthy calculation. One subtle point is that

$$\delta\mathsf{w}(\mu\varepsilon) = \mathsf{w}(\delta\mu\varepsilon) \in \check{C}^2(\mathscr{U}, \mathscr{B}[[u]])$$

vanishes. Indeed,

$$\delta\mu \in \check{C}^2(\mathscr{U}, \mathscr{O})$$

is locally constant and by definition constant multiples of ε vanish in the sheaf $\mathscr{B}$.

Over the set $U_{\alpha_0\ldots\alpha_k}$, the covariant field theory $\mathbb{S}_u$ is given by the explicit formula

$$\mathbb{S}_u = \sum_{i=0}^{k} t_{\alpha_i} \otimes \mathsf{S}_{\alpha_i,u} \big|_{U_{\alpha_0\ldots\alpha_k}} + \frac{1}{2} \sum_{i,j=0}^{k} (t_{\alpha_i}\mathrm{d}t_{\alpha_j} - t_{\alpha_j}\mathrm{d}t_{\alpha_i}) \otimes \mu_{\alpha_i\alpha_j} \big|_{U_{\alpha_0\ldots\alpha_k}} \varepsilon.$$

Let $W \in \mathscr{O}(M)$ be a function on M of degree 1 and odd parity such that $\{W, W\} = 0$. As in the local case, we may twist this global covariant field theory by $u^{-1}W$:

$$\begin{aligned} \mathbb{S}_u \bullet u^{-1}W &= \mathbb{S}_u + u^{-1}\big(d_u W + (\!(\mathbb{S}_u, W)\!)\big) \\ &= \mathbb{S}_u + W\varepsilon + u^{-1} \sum_{i=0}^{k} t_{\alpha_i} \otimes (\!(\mathsf{S}_{\alpha_i,u}, W)\!)\big) \big|_{U_{\alpha_0\ldots\alpha_k}} . \end{aligned}$$

More generally, W might be any Čech cocycle $W \in \check{C}^\bullet(\mathscr{U}, \mathscr{O})$ of degree 1 and odd parity, such that $\{\mathsf{w}(W), \mathsf{w}(W)\} = 0$.

We close this section with a discussion of how the global covariant field theory associated to the class (ν, μ) changes under an equivalence of theories. The type of equivalence we have in mind is a homotopy of the form $\tilde{\nu} \in \check{C}^0(\mathscr{U}, \Omega^0)$ between a pair of classes (ν_i, μ_i), $i = 0, 1$:

$$\nu_1 - \nu_0 = \mathrm{d}\tilde{\nu} \qquad \mu_1 - \mu_0 = \delta\tilde{\nu}.$$

Proposition 4 *Let $\mathbb{S}_{i,u}$ be the global covariant field theory associated to (ν_i, μ_i). Then $\mathbb{S}_{0,u}$ and $\mathbb{S}_{1,u}$ are gauge equivalent:*

$$\mathbb{S}_{0,u} \bullet \mathsf{W}(\tilde{\nu}\varepsilon) = \mathbb{S}_{1,u}.$$

Proof We have

$$\begin{aligned} d_{\mathrm{TW},u}\mathsf{W}(\tilde{\nu}\varepsilon) &= \mathsf{W}(\delta\tilde{\nu}\varepsilon) + \partial\mathsf{W}(\tilde{\nu}) \\ &= \mathsf{W}(\mu_1 - \mu_0) + (-1)^{p(\xi^a)}\mathsf{W}((\nu_{1,a} - \nu_{0,a})\partial\xi^a) \end{aligned}$$

and, for $i = 0, 1$,

$$\begin{aligned} (\!(\mathbb{S}_{i,u}, \mathsf{W}(\tilde{\nu}\varepsilon))\!) &= -u\mathsf{W}(\xi_a^+\pi^{ab}\partial_b\tilde{\nu}\varepsilon) \\ &= -u\mathsf{W}(\xi_a^+\pi^{ab}(\nu_{1,b} - \nu_{0,b})\varepsilon). \end{aligned}$$

Adding these two equations, we see that

$$d_{\mathrm{TW},u}\mathsf{W}(\tilde{\nu}\varepsilon) + (\!(\mathbb{S}_{i,u}, \mathsf{W}(\tilde{\nu}\varepsilon))\!) = \mathbb{S}_{1,u} - \mathbb{S}_{0,u}.$$

Taking the difference of these two equations, we see that

$$(\!(\mathbb{S}_{1,u} - \mathbb{S}_{0,u}, \mathsf{W}(\tilde{\nu}\varepsilon))\!)) = 0.$$

We see that

$$\mathbb{S}_{0,u} \bullet \mathsf{W}(\tilde{\nu}\varepsilon) = \mathbb{S}_{0,u} + \big(\mathbb{S}_{1,u} - \mathbb{S}_{0,u}\big) = \mathbb{S}_{1,u},$$

proving the result. □

It follows from this proposition that the global covariant field theory $\mathbb{S}_u$ is invariantly associated, up to refinement of the cover $\mathscr{U}$ over which it is defined and a gauge transformation, to an element of the Deligne cohomology group

$$\check{\mathbb{H}}^2(\mathscr{U}, \mathbb{R} \to \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1).$$

Quantization of this model requires lifting the two-cocycle $\delta\mu \in \check{C}^2(\mathscr{U}, \mathbb{R})$ to $\check{C}^2(\mathscr{U}, \mathbb{Z})$. With this constraint, the global covariant field theory is classified by an element of the Deligne cohomology group

$$\check{\mathbb{H}}^2(\mathscr{U}, \mathbb{Z} \to \Omega^0 \xrightarrow{\mathrm{d}} \Omega^1)$$

which classifies Hermitian line bundles with connection on M which are trivialized on restriction to the cover $\mathscr{U}$.

6 The particle as a covariant field theory

In this section, we couple certain covariant field theories to gravity on the world-line. Of course, one-dimensional gravity carries no propagating fields: instead, the effect of coupling to gravity is to render the covariant field theory generally covariant.

The gravitational field in one dimension consists of a nowhere-vanishing one-form e on the world-line, whose square is the metric along the world-line. Its antifield e^+ is a fermionic scalar field of ghost number -1. In addition, there is a ghost field c, which is a fermionic field that transforms as a vector field on the world-line, and has ghost number 1: its antifield c^+ is a bosonic field that transforms as a quadratic differential on the world-line, and has ghost number -2.

Consider the graded manifold $T^*\mathbb{R}[-1] \cong \mathbb{R}[-1] \times \mathbb{R}[1]$, with fermionic coordinates b and c, respectively, of ghost number -1 and 1. We consider the covariant field theory X_u associated to the one-form $\nu = -c\,db$, given by the explicit formula

$$\mathsf{X}_u = \mathsf{X}_0 + u\mathsf{X}_1 = c\partial b + u(b^+c^+ + c^+c\varepsilon).$$

Consider the Batalin–Vilkovisky Hamiltonian flow generated by the Hamiltonian

$$\log(b^+)c^+c,$$

defined in a neighbourhood of the locus where $b^+ = 1$. The covariant field theory X_u flows to

$$\mathsf{X}_u \bullet \tau \log(b^+)c^+c = (b^+)^{\tau-1}c(b^+\partial b + \tau c^+\partial c) + u(b^+)^{\tau-1}c^+ + (1-\tau)uc^+c\varepsilon,$$

and, setting $\tau = 1$, we see that

$$\mathsf{X}_u \bullet \log(b^+)c^+c = c(b^+\partial b + c^+\partial c) + uc^+.$$

We may identify b^+ as the gravitational field e. The antifield e^+ of e is the field $-b$, and the action in these coordinates becomes

$$\mathsf{X}_u \bullet \log(b^+)c^+c = c(-e\partial e^+ + c^+\partial c) + uc^+.$$

Let $\mathsf{S}_u = \mathsf{S}_0 + u\mathsf{S}_1$ be a covariant field theory with $\mathsf{S}_i = 0$, $i > 1$; denote by $u\mathsf{D}$ its curvature. The product of the covariant field theories S_u and X_u is associated to the symplectic graded manifold $M \times T^*\mathbb{R}[-1]$:

$$\mathsf{S}_u + \mathsf{X}_u = \mathsf{S}_0 + c\partial b + u(\mathsf{S}_1 + b^+c^+ + c^+c\varepsilon).$$

The following theorem shows that after a further gauge transformation, generated by the Hamiltonian $c\mathsf{S}_1$, this model is transformed into a theory minimally coupled to the background gravitational field.

Theorem 6 *Let M be a graded supermanifold, and let ν be a one-form on M such that $\mathrm{d}\nu$ is a symplectic form. Let S_u be the associated covariant field theory. Then we have*

$$(\mathsf{S}_u + \mathsf{X}_u) \bullet \log(b^+)c^+c \bullet c\mathsf{S}_1 = \mathsf{S}_0 + c(\mathsf{D} + b^+\partial b + c^+\partial c) + c\iota\mathsf{S}_0 + uc^+.$$

Corollary 1 *Let $V \in \mathscr{O}(M)$ be a function on M of ghost number 0 and even parity, and let $W = cV$. After successively applying the gauge transformations generated by $\log(b^+)c^+c$ and $c\mathsf{S}_1$, the twisted covariant field theory*

$$(\mathsf{S}_u + \mathsf{X}_u) \bullet u^{-1}W$$

is transformed into the covariant field theory

$$(\mathsf{S}_u + \mathsf{X}_u) \bullet u^{-1}W \bullet \log(b^+)c^+c \bullet c\mathsf{S}_1 = \mathsf{S}_0 - b^+V + c(\mathsf{D} + b^+\partial b + c^+\partial c) + c\iota\mathsf{S}_0 + uc^+.$$

Proof We have $d_u(u^{-1}W) = cV\varepsilon$,

$$(\!(\mathsf{S}_u, u^{-1}W)\!) = (\!(\mathsf{S}_1, cV)\!) = -(\!(c\mathsf{S}_1, V)\!)$$

and

$$(\!(\mathsf{X}_u, u^{-1}W)\!) = -b^+V - cV\varepsilon.$$

It follows that

$$(\mathsf{S}_u + \mathsf{X}_u) \bullet u^{-1}W = \mathsf{S}_u + \mathsf{X}_u - b^+V - (\!(c\mathsf{S}_1, V)\!)$$

and that

$$\begin{aligned}(\mathsf{S}_u + \mathsf{X}_u) \bullet u^{-1}W \bullet \log(b^+)c^+c &= \big(\mathsf{S}_u + \mathsf{X}_u - b^+V - (\!(c\mathsf{S}_1, V)\!)\big) \bullet \log(b^+)c^+c \\ &= \mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+ \\ &\quad - b^+V - (\!(c\mathsf{S}_1, b^+V)\!).\end{aligned}$$

We see that

$$\begin{aligned}(\mathsf{S}_u + \mathsf{X}_u) \bullet u^{-1}W \bullet \log(b^+)c^+c \bullet c\mathsf{S}_1 &= (\mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+) \bullet c\mathsf{S}_1 \\ &\quad - e^{-\mathrm{ad}(c\mathsf{S}_1)}(b^+V + (\!(c\mathsf{S}_1, b^+V)\!)).\end{aligned}$$

Since $b^+V + (\!(c\mathsf{S}_1, b^+C)\!) = e^{\mathrm{ad}(c\mathsf{S}_1)}b^+V$, the corollary follows. □

Remark 1 Theorem 6 generalizes to the global case without any difficulties: if $\mathbb{S}_u$ satisfies the hypotheses of Theorem 5, we have

$$(\mathbb{S}_u + \mathsf{X}_u) \bullet \log(b^+)c^+c \bullet c\mathbb{S}_1 = \mathbb{S}_0 + c(\mathsf{D} + b^+\partial b + c^+\partial c) + c\iota\mathbb{S}_0 + uc^+.$$

Remark 2 After coupling to gravity, the covariant field theory S_u, which is only defined if the two-form $\omega = d\nu$ is symplectic, is seen to be equivalent to a covariant field theory which is defined for any one-form ν on M, without any condition that $d\nu$ is nondegenerate.

Theorem 6 may be restated in the following suggestive way.

Proposition 5 *We have*

$$(\mathsf{S}_u + \mathsf{X}_u) \bullet \log(b^+)c^+c \bullet c\mathsf{S}_1 = (\mathsf{S}_u + \mathsf{X}_u) \bullet (\log(b^+)c^+c * c\mathsf{S}_1),$$

where

$$\log(b^+)c^+c * c\mathsf{S}_1 = \frac{\log(b^+)}{b^+ - 1}\big(c(\mathsf{S}_1 + \mathsf{X}_1) - c^+c\big).$$

Proof Suppose that for all $n \geq 0$, we have

$$\mathrm{ad}(z)\mathrm{ad}(y)^n z = 0. \tag{20}$$

It follows that

$$e^{-t\mathrm{ad}(z)}e^{-\mathrm{ad}(y)}z = e^{-\mathrm{ad}(y)}z,$$

and hence that

$$\mathrm{ad}(y * tz)z = \mathrm{ad}(y)z.$$

By Proposition 1, we see that

$$\begin{aligned} y * z &= y + \int_0^1 \frac{\mathrm{ad}(y * tz)}{1 - e^{-t\mathrm{ad}(z)}e^{-\mathrm{ad}(y)}} z\, dt \\ &= y + \int_0^1 \frac{\mathrm{ad}(y)}{1 - e^{-\mathrm{ad}(y)}} z\, dt \\ &= y + \frac{\mathrm{ad}(y)}{1 - e^{-\mathrm{ad}(y)}} z. \end{aligned}$$

Let $y = \log(b^+)c^+c$ and $z = c\mathsf{S}_1$. We have

$$\mathrm{ad}(\log(b^+)c^+c)^n c\mathsf{S}_1 = (-\log(b^+))^n c\mathsf{S}_1,$$

and the hypothesis (20) is satisfied. Thus, we have

$$\begin{aligned} \log(b^+)c^+c * c\mathsf{S}_1 &= \log(b^+)c^+c + \frac{\mathrm{ad}(\log(b^+)c^+c)}{e^{\mathrm{ad}(\log(b^+)c^+c)} - 1} c\mathsf{S}_1 \\ &= \log(b^+)c^+c + \frac{\log(b^+)c\mathsf{S}_1}{b^+ - 1}, \end{aligned}$$

and the lemma follows. □

Using Corollary 1, we can generalize the discussion of the particle in a flat spacetime discussed in the introduction, allowing a curved target with a magnetic field. Let U be an open subset of $\mathbb{R}^n$, with coordinates $\{x^\mu\}_{1\leq\mu\leq n}$. Let

$$g = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

be a pseudo-Riemannian metric on U, and let $A = A_\mu \mathrm{d}x^\mu$ be an electromagnetic potential on U. Let $M = T^*U$ be the manifold with coordinates $\{x^\mu\}$ and $\{p_\mu\}$. We consider the covariant field theory S_u associated to the one-form

$$\nu = (p_\mu + A_\mu)\mathrm{d}x^\mu .$$

The corresponding symplectic form $\omega = \mathrm{d}\nu$ equals

$$\omega = \mathrm{d}p_\mu\, \mathrm{d}x^\mu + \tfrac{1}{2}F_{\mu\nu}(x)\mathrm{d}x^\mu\, \mathrm{d}x^\nu ,$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

is the electromagnetic field. The associated Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} + F_{\mu\nu}(x)\frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial p_\nu},$$

and the covariant field theory, after coupling to gravity, equals

$$\begin{aligned}\mathsf{S}_u + \mathsf{X}_u &= (p_\mu + A_\mu)\partial x^\mu + b\partial c \\ &\quad + u\big(x^+_\mu p^{+\mu} + \tfrac{1}{2}F_{\mu\nu}p^{+\mu}p^{+\nu} + b^+c^+\big) + u\big((p_\mu + A_\mu)p^{+\mu} + c^+c\big)\varepsilon .\end{aligned}$$

The field theory describing the particle is obtained by twisting this field theory by $u^{-1}cV$, where

$$V = \tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu .$$

The proof of Theorem 6 occupies the remainder of the section. We use the following formulas, whose proofs are similar to the proof that S_u is a covariant field theory:

$$d_u(c\mathsf{S}_1) + (\!(\mathsf{S}_u, c\mathsf{S}_1)\!) = c(\mathsf{D} + \iota\mathsf{S}_u) \tag{21}$$

and

$$(\!(d_u(c\mathsf{S}_1) + (\!(\mathsf{S}_u, c\mathsf{S}_1)\!), c\mathsf{S}_1)\!) = 2c\partial c\ . \tag{22}$$

Let us calculate $(\mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+) \bullet \tau c\mathsf{S}_1$. We have

$$\begin{aligned}(\mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+) \bullet \tau c\mathsf{S}_1 \\ = \mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+ + \sum_{n=0}^{\infty} \frac{(-\tau)^{n+1}}{(n+1)!} \operatorname{ad}(c\mathsf{S}_1)^n \mathsf{d}(c\mathsf{S}_1),\end{aligned}$$

where

$$\mathsf{d}(c\mathsf{S}_1) = d_u(c\mathsf{S}_1) + (\!(\mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+, c\mathsf{S}_1)\!).$$

It follows from (21) that

$$\mathsf{d}(c\mathsf{S}_1) = c(\mathsf{D} + \iota\mathsf{S}_u) - c\partial c\mathsf{S}_1 - u\mathsf{S}_1.$$

Since $c^2 = 0$, we see that both $(\!(c\partial c\mathsf{S}_1, c\mathsf{S}_1)\!)$ and $(\!(c\iota\mathsf{S}_1, c\mathsf{S}_1)\!)$ vanish. By (22), we see that

$$\begin{aligned}(\!(\mathsf{d}(c\mathsf{S}_1), c\mathsf{S}_1)\!) &= (\!(c(\mathsf{D} + \iota\mathsf{S}_u), c\mathsf{S}_1)\!) - u(\!(\mathsf{S}_1, c\mathsf{S}_1)\!) \\ &= 2c\partial c\mathsf{S}_1 - 2uc\iota\mathsf{S}_1.\end{aligned}$$

It is clear that

$$(\!((\!(\mathsf{d}(c\mathsf{S}_1), c\mathsf{S}_1)\!), c\mathsf{S}_1)\!) = 0.$$

In summary, we have

$$\begin{aligned}&(\mathsf{S}_u + c(b^+\partial b + c^+\partial c) + uc^+) \bullet \tau c\mathsf{S}_1 \\ &\quad = \mathsf{S}_0 + c(b^+\partial b + c^+\partial c) + u(\mathsf{S}_1 + c^+) \\ &\qquad + \tau\big(c(\mathsf{D} + \iota\mathsf{S}_u) - c\partial c\mathsf{S}_1 - u\mathsf{S}_1\big) + \frac{\tau^2}{2}\big(2c\partial c\mathsf{S}_1 - 2uc\iota\mathsf{S}_1\big) \\ &\quad = \mathsf{S}_0 + c(\tau\mathsf{D} + b^+\partial b + c^+\partial c) + \tau c\iota\mathsf{S}_0 + uc^+ \\ &\qquad + (1-\tau)\Big(u\mathsf{S}_1 + \tau uc\iota\mathsf{S}_1 - \tau c\partial c\mathsf{S}_1\Big).\end{aligned}$$

Setting $\tau = 1$, we obtain Theorem 6.

7 The spinning particle as a covariant field theory

In this section, we study a supersymmetric version of the results of the last section. Supergravity in one dimension has as its fields the graviton e and a fermionic field χ, the gravitino, which like e is transforms as a one-form on the world-line. In addition to the ghost c, there is also a superghost γ, which is a bosonic field of ghost number 1 that transforms as a world-line scalar.

Consider the graded manifold $T^*\Pi\mathbb{R}[-1] \cong \Pi\mathbb{R}[-1] \times \Pi\mathbb{R}[1]$, with bosonic coordinates β and γ, respectively, of ghost number -1 and 1. Consider the covariant field theory Ξ_u associated to the one-form $\nu = \gamma\, d\beta$:

$$\Xi_u = \Xi_0 + u\Xi_1 = \gamma\partial\beta + u(\beta^+\gamma^+ + \gamma^+\gamma\varepsilon).$$

Theorem 6, applied to the graded supermanifold $M \times T^*\Pi\mathbb{R}[-1]$, shows that

$$\begin{aligned}&(\mathsf{S}_u + \Xi_u + \mathsf{X}_u) \bullet \log(b^+)c^+c \bullet c(\mathsf{S}_1 + \Xi_1) = \mathsf{S}_0\\ &\quad + c(\mathsf{D} + \beta^+\partial\beta + \gamma^+\partial\gamma + b^+\partial b + c^+\partial c + \iota(\mathsf{S}_0 + \gamma\partial\beta)) + \gamma\partial\beta + uc^+.\end{aligned}$$

In order to obtain the supersymmetric analogue of the particle, called the spinning particle, we choose a function $Q \in \mathscr{O}(M)$ of ghost number 0 and odd parity. We twist the covariant field theory $\mathsf{S}_u + \Xi_u + \mathsf{X}_u$ by an element $u^{-1}W$, where W is given by the formula (cf. Cattaneo and Schiavina [7, Section 6.3], and [11, Eq. (5)])

$$W = \tfrac{1}{2}c\{Q, Q\} + \gamma Q - b\gamma^2. \tag{23}$$

The term linear in b is chosen in such a way as to guarantee that $\{W, W\} = 0$.

Applying the twist $u^{-1}W$ to the covariant field theory $\mathsf{S}_u + \Xi_u + \mathsf{X}_u$ gives the twisted covariant field theory

$$\begin{aligned}(\mathsf{S}_u + \Xi_u + \mathsf{X}_u) \bullet u^{-1}W &= \mathsf{S}_u + \Xi_u + \mathsf{X}_u\\ &\quad - \tfrac{1}{2}b^+\{Q, Q\} + \tfrac{1}{2}(\!(\{Q, Q\}, c\mathsf{S}_1)\!) - \beta^+Q + \gamma(\!(\mathsf{S}_1, Q)\!)\\ &\quad + c^+\gamma^2 - 2b\beta^+\gamma + b\gamma^2\varepsilon.\end{aligned}$$

Gauging by $\log(b^+)c^+c$ followed by $c\Xi_1$, we obtain

$$\begin{aligned}&(\mathsf{S}_u + \Xi_u + \mathsf{X}_u) \bullet u^{-1}W \bullet \log(b^+)c^+c \bullet c\Xi_1\\ &\quad = \mathsf{S}_u + c(\beta^+\partial\beta + \gamma^+\partial\gamma + b^+\partial b + c^+\partial c - \gamma\partial\beta\varepsilon) + uc^+\\ &\qquad - \tfrac{1}{2}b^+\{Q, Q\} + \tfrac{1}{2}(\!(b^+\{Q, Q\}, c\mathsf{S}_1)\!) - \beta^+Q + (\gamma + c\beta^+ - c\gamma\varepsilon)(\!(\mathsf{S}_1, Q)\!)\\ &\qquad + (b^+)^{-1}(c^+ - \Xi_1)\gamma^2 - 2b\beta^+\gamma - (b^+)^{-1}(c^+ - \Xi_1)c\gamma^2\varepsilon.\end{aligned}$$

Gauging by $c\mathsf{S}_1$, and observing that $c\Xi_1 * c\mathsf{S}_1 = c(\mathsf{S}_1 + \Xi_1)$, we obtain

$$\begin{aligned}&(\mathsf{S}_u + \Xi_u + \mathsf{X}_u) \bullet u^{-1}W \bullet \log(b^+)c^+c \bullet c(\mathsf{S}_1 + \Xi_1)\\ &\quad = \mathsf{S}_0 + c(\mathsf{D} + \beta^+\partial\beta + \gamma^+\partial\gamma + b^+\partial b + c^+\partial c + \iota\mathsf{S}_0 - \gamma\partial\beta\varepsilon) + uc^+\\ &\qquad - \tfrac{1}{2}b^+\{Q, Q\} - \beta^+Q + \gamma(\!(\mathsf{S}_1, Q)\!)\\ &\qquad + (b^+)^{-1}(c^+ - \mathsf{S}_1 - \Xi_1)\gamma^2(1 - c\varepsilon) - 2b\beta^+\gamma.\end{aligned}$$

Substituting the graviton field e for b^+ and the gravitino field χ for β^+, and projecting to $\mathscr{F}$, we obtain the action of the spinning particle:

$$\begin{aligned}&\textstyle\int(S_u + \Xi_u + \mathsf{X}_u) \bullet u^{-1}W \bullet \log(e)c^+c \bullet c(S_1 + \chi\gamma^+)\\ &\quad = \textstyle\int\big((-1)^{p(\xi^a)}\nu_a\partial\xi^a - \tfrac{1}{2}e\{Q, Q\} - \chi Q\\ &\qquad + c(\mathsf{D} - \chi\partial\chi^+ + \gamma^+\partial\gamma - e\partial e^+ + c^+\partial c)\\ &\qquad + \gamma\big(-\xi_a^+\pi^{ab}\partial_b Q + 2e^+\chi) + e^{-1}\gamma^2(c^+ + \gamma\partial\chi^+ - \tfrac{1}{2}\xi_a^+\pi^{ab}\xi_b^+)\big).\end{aligned} \tag{24}$$

Solutions of the classical master equation (1) are classified by their rank, that is, their degree as functions of the antifields. Gauge theories whose symmetries close off-shell have rank 1, but the action of the spinning particle (24) has rank 2, owing to the presence of term

$$-\tfrac{1}{2}\int e^{-1}\gamma^2\xi_a^+\pi^{ab}\xi_b^+.$$

It is possible that by adjoining auxiliary fields, this covariant field theory may be shown to be equivalent to a covariant field theory whose action is of rank 1.

We close this section by showing how, as an application of the above formulas, the spinning particle may be expressed as a covariant field theory. We discuss only the local case, leaving its globalization, which uses Theorem 5, to the reader.

We work, as in the last section, with an open subset U of $\mathbb{R}^n$, with coordinates $\{x^\mu\}$, pseudo-Riemannian metric

$$g = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu,$$

and electromagnetic field $A = A_\mu\,\mathrm{d}x^\mu$.

Denote the basis of $V = \mathbb{R}^n$ by $\{e_a\}$, and let $\eta_{ab} = \eta(e_a, e_b)$ be an inner product on V, of the same signature as the metric $g_{\mu\nu}$ on U. Let

$$\theta^a = \theta^a_\mu\,\mathrm{d}x^\mu \in \Omega^1(U, V)$$

be a moving frame, that is, a one-form defining an isometry between T_xU and V at each point $x \in U$, so that

$$g_{\mu\nu} = \eta_{ab}\theta^a_\mu\theta^b_\nu.$$

Denote by θ^μ_a the inverse of θ^a_μ, in the sense that

$$\theta^a_\mu\theta^\mu_b = \delta^a_b.$$

The connection one-form $\omega^a{}_b = \omega_\mu{}^a{}_b\,\mathrm{d}x^\mu \in \Omega^1(U, \mathrm{End}(V))$ is the antisymmetric matrix of one-forms on U characterized by two conditions: it is compatible with the metric η,

$$\omega^b{}_a = -\eta_{a\tilde{a}}\,\eta^{b\tilde{b}}\omega^{\tilde{a}}{}_{\tilde{b}},$$

and satisfies the first Cartan structure equation

$$\mathrm{d}\theta^a + \omega^a{}_b\,\theta^b = 0.$$

Let M be the supermanifold $T^*U \times \Pi V$, with coordinates (x^μ, p_μ, ψ^a), and consider the one-form

$$\nu = (p_\mu + A_\mu)\mathrm{d}x^\mu + \tfrac{1}{2}\eta_{ab}\psi^a\mathrm{d}\psi^b.$$

The symplectic form $\mathrm{d}\nu$ on M equals

$$\mathrm{d}\nu = \mathrm{d}p_\mu\,\mathrm{d}x^\mu + \tfrac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \tfrac{1}{2}\eta_{ab}\mathrm{d}\psi^a\mathrm{d}\psi^b.$$

The associated Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} + F_{\mu\nu}(x)\frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial p_\nu} + (-1)^{p(f)}\eta^{ab}\frac{\partial f}{\partial \psi^a}\frac{\partial g}{\partial \psi^b},$$

and the covariant field theory, after coupling to supergravity, equals

$$\begin{aligned} \mathsf{S}_u + \Xi_u + \mathsf{X}_u &= (p_\mu + A_\mu)\partial x^\mu - \tfrac{1}{2}\eta_{ab}\psi^a\partial\psi^b + \beta\partial\gamma + b\partial c \\ &\quad + u\big(x^+_\mu p^{+\mu} + \tfrac{1}{2}F_{\mu\nu}p^{+\mu}p^{+\nu} + \tfrac{1}{2}\eta^{ab}\psi^+_a\psi^+_b + \beta^+\gamma^+ + b^+c^+\big) \\ &\quad + u\big((p_\mu + A_\mu)p^{+\mu} + c^+c\big)\varepsilon. \end{aligned}$$

Let

$$\tilde{p}_\mu = p_\mu + \tfrac{1}{2}\omega_{\mu ab}\psi^a\psi^b.$$

The spinning particle in a curved background [11] is the twisted covariant field theory

$$(\mathsf{S}_u + \Xi_u + \mathsf{X}_u) \bullet u^{-1}(\tfrac{1}{2}c\{Q, Q\} + \gamma Q + b\gamma^2),$$

where $Q = \theta^\mu_a\psi^a\tilde{p}_\mu$. Observe that the quantization of Q is is the Dirac operator on U. The proof of Lichnerowicz's formula for the square of the Dirac operator shows that

$$\{\mathsf{D}, \mathsf{D}\} = \theta^\mu_a(x)\theta^\nu_b(x)\big(\eta^{ab}\tilde{p}_\mu\tilde{p}_\nu - \tfrac{1}{2}F_{\mu\nu}(x)\psi^a\psi^b\big).$$

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