Lett Math Phys (2018) 108:2303–2314
https://doi.org/10.1007/s11005-018-1084-x

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# Drinfeld–Sokolov reduction in quantum algebras: canonical form of generating matrices

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Received: 5 October 2017 / Revised: 27 March 2018 / Accepted: 31 March 2018 /
Published online: 11 April 2018


**Abstract** We define the second canonical forms for the generating matrices of the Reflection Equation algebras and the braided Yangians, associated with all even skew-invertible involutive and Hecke symmetries. By using the Cayley–Hamilton identities for these matrices, we show that they are similar to their canonical forms in the sense of Chervov and Talalaev (J Math Sci (NY) 158:904–911, 2008).

**Keywords** Reflection Equation algebra · Braided Yangian · Second canonical form · Quantum elementary symmetric functions · Quantum power sums · Cayley–Hamilton identity

**Mathematics Subject Classification** 81R50

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## 1 Introduction

In the seminal paper [2], it was shown that any connection operator $\partial_u - M(u)$, where $\partial_u = \frac{\mathrm{d}}{\mathrm{d}u}$ and $M(u)$ is a $N \times N$ matrix smoothly depending on the parameter $u$, can be reduced by means of the gauge transformations

$$\partial_u - M(u) \mapsto g(u)(\partial_u - M(u))g(u)^{-1} = \partial_u - g(u)M(u)g(u)^{-1} + g(u)\partial_u(g(u)^{-1})$$

to the form $\partial_u - M_{\text{can}}(u)$, where $M_{\text{can}}(u)$ looks like (3.2). Also, the authors of [2] found that the reduced Poisson structure can be identified with the second Gelfand–Dickey one. Lately, the reduced Poisson structure was also identified with $W$-algebras (see [3]).

Hereafter, matrices of the form (3.2) (or transposed to them) will be called matrices of *canonical* or, more precisely, *second canonical* form. The ground field is assumed to be $\mathbb{C}$.

A $q$-counterpart of the Drinfeld–Sokolov (DS) reduction was defined in [4,5]. There the operator $D_q f(x) = f(qx)$, $q \neq 1$ was considered instead of $\partial_u$, and the gauge transformations above were replaced by

$$D_q - M(u) \mapsto g(qu)(D_q - M(u))g(u)^{-1}.$$

It was shown that the operator $D_q - M(u)$ can be reduced to that $D_q - M_{\text{can}}(u)$, where $M_{\text{can}}(u)$ is similar to (3.2) [(mutatis mutandis, since the authors deal with $SL(N)$-valued functions $M(u)$].

Emphasize that all these results are "operator analogs" of the reduction procedure for numerical matrices (independent of parameters). According to a theorem by Frobenius, such a matrix $M$ can be reduced to the second canonical form by transformations $M \mapsto g\,M\,g^{-1}$.

However, if a matrix $M \in \text{Mat}(A)$ has entries belonging to a noncommutative algebra $A$, for instance, to the enveloping algebra $U(gl(N))$, such a reduction is not possible. Nevertheless, if $M \in \text{Mat}(U(gl(N))$ is a *specially chosen* matrix (namely, the generating matrix of this algebra, see below), its reduction to the second canonical form is possible in the sense of [1]. This reduction consists in the following: For a given matrix $M \in \text{Mat}(A)$, there is defined another matrix $M_{\text{can}}$ of the form (3.2) with entries expressed via these of $M$ with a subsequent demonstration of *similarity* of these two matrices. This similarity means that there exists a nontrivial matrix $C \in \text{Mat}(A)$, such that $M\,C = C\,M_{\text{can}}$.

A reduction of operators in the spirit of [1] is defined in the same way. Namely, in this sense, the operator $\partial_u - M(u)$, where $M(u)$ is the Lax matrix for the Gaudin model, has been reduced to the operator $\partial_u - M_{\text{can}}(u)$ in [1].

The main objective of the present note is to generalize this result to the generating matrices of some quantum algebras, namely the Reflection Equation (RE) algebras, associated with even skew-invertible involutive or Hecke symmetries (see the next section) and the braided Yangians, recently introduced by two of us [10]. As a by-product, we get a similar reduction for the generating matrices of the enveloping algebras $U(gl(N))$.

Our method is based on the Cayley–Hamilton (CH) identity for the generating matrices of the RE algebras and braided Yangians, established in [6,10] respectively. Emphasize that this CH identity enables us to relate quantum analogs of some symmetric elements, namely the power sums and elementary symmetric polynomials. Note that the power sums in the RE algebras are defined by $\mathrm{Tr}_R L^k$, where $\mathrm{Tr}_R$ is the so-called $R$-trace associated with the initial involutive or Hecke symmetry $R$. In the braided Yangians, the quantum power sums are defined by similar formulae but with shifted arguments $u$. Namely, the use of the $R$-trace instead of the usual one is a specific feature of these algebras. Another feature is that the CH identities in these algebras are more similar to the classical ones.

Note that the mentioned quantum symmetric elements constitute commutative subalgebras in the braided Yangians (called Bethe subalgebras). This property enables us to generate quantum integrable systems. Their explicit construction will be published in a separate paper. To conclude Introduction, we emphasize that the present note deals with a quantum counterpart of the first step of the DS reduction. Namely, we present the canonical forms of generating matrices and establish the mentioned similarity of the corresponding operators. The main results are formulated in Theorems 7 and 8, Sect. 4. We do not consider a quantum version of the second step of this reduction. We plan to define a new version of the "$q$-W-algebras" based on the RE algebras and braided Yangians in our next publications.

## 2 Quantum matrix algebras

Let us briefly describe the quantum algebras under consideration. First, recall that by a current $R$-matrix, one usually means an operator $R(u, v)$ depending on parameters $u$ and $v$ and subject to the so-called quantum Yang–Baxter equation

$$R_{12}(u, v)R_{23}(u, w)R_{12}(v, w) = R_{23}(v, w)R_{12}(u, w)R_{23}(u, v),$$

where $R_{12}(u, v) = R(u, v) \otimes I$ and $R_{23}(u, v) = I \otimes R(u, v)$. If $R$ is independent of the parameters, it is also called *a braiding*. In this case, we shall assume the operator $R : V^{\otimes 2} \to V^{\otimes 2}$ to be either involutive $R^2 = I$ or to satisfy the Hecke condition

$$(q\, I - R)(q^{-1}\, I + R) = 0, \quad q \in \mathbb{C}, \quad q^2 \neq 1.$$

Here, $V$ is a vector space of the dimension $N$ (over the field $\mathbb{C}$). These braidings are, respectively, called *involutive and Hecke symmetries*.

The best known are the Hecke symmetries coming from the quantum groups $U_q(sl(N))$. However, there are known numerous examples of involutive and Hecke symmetries which are deformations neither of usual nor of super-flips (see [7] and the references therein).

All symmetries $R$, we are dealing with, are assumed to be even and *skew-invertible*. The term *even* means that there exists a natural $m > 0$ such that $R$ has the bi-rank $(m|0)$.

For the definitions of the notions “skew-invertible” and “bi-rank,” the reader is referred to [10,11]. We want only to mention that for a skew-invertible braiding $R$, an $R$-trace

$$\mathrm{Tr}_R : \mathrm{End}(V) \to \mathbb{C}$$

in a sense coordinated with $R$ can be defined. Moreover, for any matrix $A$, the expression $\mathrm{Tr}_R\, A$ is also well-defined. The properties of such an $R$-trace can be found in [7,16].

Let us, respectively, define two quantum matrix algebras by the following systems of relations on their generators

$$R\,T_1\,T_2 - T_1\,T_2\,R = 0, \quad T = \|t_i^j\|_{1\le i,j\le N} \tag{2.1}$$

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = 0, \quad L = \|l_i^j\|_{1\le i,j\le N} \tag{2.2}$$

where $R$ is assumed to be a skew-invertible involutive or Hecke symmetry. The former algebra is called an RTT one, and the latter one is called a Reflection Equation (RE) algebra. Their detailed consideration can be found in [7]. Here, we only observe that if $R$ is a deformation of the usual flip $P$, the dimensions of the homogenous components of both algebras are classical, i.e., equal to those in $\mathrm{Sym}(gl(N))$ (if $R$ is a Hecke symmetry, the value of $q$ is assumed to be generic).

Now, let us exhibit Yangian-like algebras associated with current $R$-matrices. First, observe that the current $R$-matrices, we are dealing with, are constructed from involutive or Hecke symmetries via the so-called Baxterization procedure described in [10,11]. This procedure results in the following current $R$-matrix

$$R(u,v) = R - g(u,v)I, \quad \text{where} \quad g(u,v) = \frac{1}{u-v} \quad \text{or} \quad g(u,v) = \frac{(q-q^{-1})u}{u-v}. \tag{2.3}$$

If $R$ is an involutive symmetry, $g(u,v)$ is defined by the former formula. If $R$ is a Hecke symmetry, $g(u,v)$ is defined by the latter one.

The corresponding Yangian-like algebras introduced in [10,11] are also of two types. They are, respectively, defined by the following systems:

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(v)\,T_2(u)\,R(u,v), \tag{2.4}$$

$$R(u,v)\,L_1(u)\,R\,L_1(v) = L_1(v)\,R\,L_1(u)\,R(u,v), \tag{2.5}$$

where the generating matrices $T(u)$ and $L(u)$ are assumed to be series

$$T(u) = \sum_{k\ge 0} \frac{T[k]}{u^k}, \qquad L(u) = I + \sum_{k>0} \frac{L[k]}{u^k}.$$

We call the former (resp., latter) algebra *a Yangian of RTT type* (resp., *a braided Yangian*). Note that the braided Yangians are defined similarly to the RE algebras, but with the current $R$-matrices at the outside positions.

Let us introduce the following notation:

$$L_{\overline{1}}(u) = L_1(u), \quad L_{\overline{k}}(u) = R_{k-1} L_{\overline{k-1}}(u) R_{k-1}^{-1}, \quad k \geq 2, \tag{2.6}$$

where we write $R_i$ instead of $R_{i\,i+1}$ (recall that $R_{i\,i+1}$ stands for the operator $R$ acting at the $i$-th and $i+1$-th positions in the tensor product $V^{\otimes k}$, $i+1 \leq k$). In the RE algebras, we employ the same notation for the generating matrix $L$ which is independent of parameters.

By using this notation, we can present the defining relations of RE algebras under the form

$$R_1 \, L_{\overline{1}} \, L_{\overline{2}} - L_{\overline{1}} \, L_{\overline{2}} \, R_1 = 0,$$

which looks like that in RTT algebras.

Moreover, in this algebra, the following holds:

$$R_k \, L_{\overline{k}} \, L_{\overline{k+1}} - L_{\overline{k}} \, L_{\overline{k+1}} \, R_k = 0, \quad \forall \, k \geq 1.$$

This notation enables us to define analogs of symmetric polynomials in the RTT and RE algebras in a uniform way. Thus, the *power sums* are, respectively, defined as follows:

$$\begin{aligned} p_k(T) &= \mathrm{Tr}_{R(12\ldots k)} R_{k-1} R_{k-2} \ldots R_2 R_1 T_1 \, T_2 \ldots T_k, \\ p_k(L) &= \mathrm{Tr}_{R(12\ldots k)} R_{k-1} R_{k-2} \ldots R_2 R_1 L_{\overline{1}} L_{\overline{2}} \ldots L_{\overline{k}}. \end{aligned} \tag{2.7}$$

Here, the notation $\mathrm{Tr}_{R(12\ldots k)}$ means that the $R$-traces are applied at the positions $1, 2, \ldots, k$. Note that in the RE algebra, the formula (2.7) can be simplified to $p_k(L) = \mathrm{Tr}_R L^k$, whereas for the power sums $p_k(T)$ in the RTT algebra, such a simplification is not possible.

In a similar manner, the "quantum powers" of the matrices $T$ and $L$ can be defined:

$$\begin{aligned} T^{[k]} &:= \mathrm{Tr}_{R(2\ldots k)} R_{k-1} R_{k-2} R_2 \ldots R_1 T_1 T_2 \ldots T_k, \\ L^{[k]} &:= \mathrm{Tr}_{R(2\ldots k)} R_{k-1} R_{k-2} \ldots R_2 R_1 L_{\overline{1}} L_{\overline{2}} \ldots L_{\overline{k}}. \end{aligned}$$

However, if the former formula cannot be simplified, the latter one can be reduced to the usual one: $L^{[k]} = L^k$.

Also, exhibit analogs of elementary symmetric polynomials in both algebras

$$\begin{aligned} e_0(T) &= 1, \qquad e_k(T) := \mathrm{Tr}_{R(1\ldots k)}(\mathcal{A}^{(k)} T_1 \, T_2 \ldots T_k), \quad k \geq 1, \\ e_0(L) &= 1, \qquad e_k(L) := \mathrm{Tr}_{R(1\ldots k)}(\mathcal{A}^{(k)} L_{\overline{1}} \, L_{\overline{2}} \ldots L_{\overline{k}}), \quad k \geq 1. \end{aligned} \tag{2.8}$$

Here, $\mathcal{A}^{(k)} : V^{\otimes k} \to V^{\otimes k}$, $k \geq 1$ are the skew-symmetrizers (i.e., the projectors of skew-symmetrization) which are recursively defined by

$$\mathcal{A}^{(1)} = I, \quad \mathcal{A}^{(k)} = \frac{1}{k_q} \mathcal{A}^{(k-1)} \left( q^{k-1} I - (k-1)_q R_{k-1} \right) \mathcal{A}^{(k-1)}, \quad k \geq 2, \tag{2.9}$$

Hereafter, we use the standard notation: $k_q = \frac{q^k - q^{-k}}{q - q^{-1}}$.

Note that if $R$ is even of bi-rank $(m|0)$, $m \geq 2$, the skew-symmetrizers $\mathcal{A}^{(k)}$ are trivial for $k > m$ and the rank of the skew-symmetrizer $\mathcal{A}^{(m)}$ is equal to 1.

Analogs of matrix powers, power sums, and elementary symmetric polynomials can be also defined for the generalized Yangians of both classes, see [10,11] (below, we define these objects in the braided Yangians). Besides, in all these algebras, there exist quantum analogs of the CH identity. However, in the RTT algebras and Yangians of RTT type, the CH identity is not similar to its classical form. This prevents us from performing a reduction of the generating matrices of these algebras in the spirit of [1].

In [14], there are introduced more general matrix algebras, which are associated with couples of braidings. If one of these braidings is a symmetry (involutive or Hecke), quantum analogs of elementary symmetric polynomials and power sums can be also defined. They also commute with each other. Moreover, similar quantum symmetric elements can be defined in the so-called half-quantum algebras as defined in [13]. However, in general, the corresponding quantum symmetric elements do not commute with each other.

## 3 Reduction in RE algebras

Let $R$ be again an even skew-invertible involutive or Hecke symmetry of bi-rank $(m|0)$. Denote the RE (2.2) by $\mathcal{L}(R)$. As was shown in [6], the generating matrix $L$ of the algebra $\mathcal{L}(R)$ meets the *quantum CH identity* $Q(L) = 0$, where the *characteristic polynomial* $Q(t)$ reads

$$Q(t) = t^m - qt^{m-1}e_1(L) + q^2t^{m-2}e_2(L) + \cdots + (-q)^{m-1}te_{m-1}(L) \\ + (-q)^m e_m(L) = 0. \tag{3.1}$$

Here, the factors $e_k(L)$ are the quantum elementary symmetric polynomials defined by (2.8).

*Remark 1* We stress a very important property of the polynomial (3.1): Its coefficients belong to the center $Z(\mathcal{L}(R))$ of the algebra $\mathcal{L}(R)$. Let us introduce "eigenvalues" $\{\mu_i\}_{1\leq i\leq m}$ of the matrix $L$ in a natural way

$$e_1(L) = \mu_1 + \cdots + \mu_m, \ \ldots, e_m(L) = \mu_1 \cdot \cdots \cdot \mu_m.$$

These "eigenvalues" are elements of an algebraic extension of the center $Z(\mathcal{L}(R))$. Consider the quotient algebra

$$\mathcal{L}(R)/\langle e_1(L) - \alpha_1, \ldots, e_m(L) - \alpha_m\rangle, \qquad \alpha_i \in \mathbb{C},$$

where $< I >$ stands for the ideal generated by a set $I \subset \mathcal{L}(R)$. This quotient is a quantum analog of an orbit (or a union of orbits) in the coadjoint representation of the group $GL(N)$. In [9], there was considered the problem: for which values of $\alpha_i$, this quotient is an analog of a regular orbit. If it is so, we introduce the diagonal matrix

$\mathrm{diag}(\mu_1, \ldots, \mu_m)$, where the elements $\mu_i$ solve the system $e_1(L) = \alpha_1, \ldots, e_m(L) = \alpha_m$, and treat this matrix as the *first canonical form* of the generating matrix $L$.

Let us define the *second canonical form* of the matrix $L$:

$$L_{\mathrm{can}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ a_m & a_{m-1} & a_{m-2} & \ldots & a_1 \end{pmatrix}, \tag{3.2}$$

where

$$a_k = -(-q)^k e_k(L).$$

Following [1], we show that the matrices $L$ and $L_{can}$ are in a sense similar. Let $v \in V$ be an arbitrary nontrivial vector written as a one-row matrix. Then, we introduce the following $N \times N$ matrix

$$C = \begin{pmatrix} v \\ vL \\ \ldots \\ vL^{m-1} \end{pmatrix}. \tag{3.3}$$

**Proposition 2** *The matrices $L$ and $L_{\mathrm{can}}$ are similar in the following sense:*

$$CL = L_{can}C. \tag{3.4}$$

*Proof* is straightforward and is left to the reader. □

*Remark 3* In order to justify the term "similar," it would be desirable to show that at least for some vectors $v$, the matrix $C$ is invertible in the skew-field of the algebra $\mathcal{L}(R)$. A similar problem is also open for the braided Yangians considered below.

Along with the RE algebra $\mathcal{L}(R)$, define its quadratic–linear deformation $\widehat{\mathcal{L}}(R)$ by the following system

$$R\,\hat{L}_1\,R\,\hat{L}_1 - \hat{L}_1\,R\,\hat{L}_1\,R = R\,\hat{L}_1 - \hat{L}_1\,R, \qquad \hat{L} = \|\hat{l}_i^j\|_{1\le i,j\le N}. \tag{3.5}$$

The algebra $\mathcal{L}(R)$ is a braided analog of the algebra $\mathrm{Sym}(gl(N))$, whereas $\widehat{\mathcal{L}}(R)$ is a braided analog of the enveloping algebra $U(gl(N))$. More precisely, if $R$ is a Hecke symmetry which is a deformation of the usual flip (for instance, that coming from the quantum group $U_q(sl(N))$), the algebras $\mathcal{L}(R)$ and $\widehat{\mathcal{L}}(R)$ turn into $\mathrm{Sym}(gl(N))$ and $U(gl(N))$, respectively, as $q \to 1$. By using the fact that the algebras $\mathcal{L}(R)$ and $\widehat{\mathcal{L}}(R)$ are isomorphic to each other provided that $q \neq \pm 1$, it is possible to get a characteristic polynomial $\hat{Q}(t)$ for the generating matrix $\hat{L}$ of the algebra $\widehat{\mathcal{L}}(R)$. Namely, we have (see [12])

$$\hat{Q}(t) = q^m \,\mathrm{Tr}_{R(1\dots m)} \Big(\mathcal{A}^{(m)}(tI - \hat{L}_{\overline{1}})((q^2 t - q)I - \hat{L}_{\overline{2}}) \dots \\ ((q^{2(m-1)}t - q^{m-1}(m-1)_q)I - \hat{L}_{\overline{m}})\Big).$$

Note that the polynomial $\hat{Q}(t)$ is monic.

**Proposition 4** *In the algebra $\widehat{\mathcal{L}}$, the matrix identity $\hat{Q}(L) = 0$ takes place.*

Passing to the limit $q \to 1$, we get the characteristic polynomial for the generating matrix[1] $M$ of the algebra $U(gl(N))$ (here $N = m$)

$$\mathcal{Q}(t) = \mathrm{Tr}_{(1\dots N)} \Big(\mathcal{A}^{(N)}(tI - M_1)((t-1)I - M_2)\dots((t-N+1)I - M_N)\Big),$$

where $\mathcal{A}^{(N)}$ is the usual skew-symmetrizer in $V^{\otimes N}$ and Tr is the usual trace.

**Proposition 5** *In the algebra $U(gl(N))$, the following relation holds $\mathcal{Q}(M) = 0$.*

Note that the famous Capelli determinant is defined by a similar formula. The same claim is valid for any algebra $\widehat{\mathcal{L}}$ provided $R$ be an involutive symmetry, which can be approximated by Hecke symmetries.

Besides, it is possible to introduce the second canonical forms $\hat{L}_{\mathrm{can}}$ and $M_{\mathrm{can}}$ for the matrices $\hat{L}$ and $M$, respectively, generating the algebras $\widehat{\mathcal{L}}(R)$ and $U(gl(N))$, and to perform a reduction of the matrices $\hat{L}$ and $M$ in a way suggested in [1]. Upon replacing the matrix $L$ in (3.3) by $\hat{L}$ and $M$, respectively, we get formulae similar to (3.4).

*Remark 6* There are other matrices with entries from the algebras under consideration for which quantum analogs of the CH identity exist. First, consider the enveloping algebra $U(gl(N))$. Its generating matrix $M$ belongs to the so-called Kirillov's quantum family algebra (see [15])

$$(U(gl(N)) \otimes \mathrm{End}(V))^{gl(N)}. \tag{3.6}$$

Upon replacing $V$ by other irreducible $U(gl(N))$-modules, we get other quantum family algebras. For their generating matrices, characteristic polynomials can be also found. Note that in the algebra $\widehat{\mathcal{L}}(R)$, a $q$-analog of (3.6) is

$$(\widehat{\mathcal{L}} \otimes \mathrm{End}(V))^{U_q(sl(N))}.$$

[1] Note that the defining relations on the generators $m_i^j$ of the algebra $U(gl(N))$

$$m_i^j\, m_k^l - m_k^l\, m_i^j = m_i^l \delta_k^j - m_k^j \delta_i^l,$$

can be written in a matrix form with the use of the generating $N \times N$ matrix $M = \|m_i^j\|$:

$$P\, M_1\, P\, M_1 - M_1\, P\, M_1\, P = P\, M_1 - M_1\, P.$$

Changing $\widehat{\mathcal{L}}$ for $\mathcal{L}$, we get a "$q$-family algebra" for the algebra $\mathcal{L}(R)$.

## 4 DS reduction in braided Yangians

Now, consider a braided Yangian (2.5). Using notation (2.6), we can represent the defining system of this algebra as follows:

$$R(u,v)\,L_{\overline{1}}(u)\,L_{\overline{2}}(v) = L_{\overline{1}}(v)\,L_{\overline{2}}(u)\,R(u,v).$$

First, we assume $R$ to be an involutive symmetry of bi-rank $(m|0)$ and the corresponding $R$-matrix $R(u,v)$ to be given by the first formula (2.3).

In this case, we get the recurrence formula defining the skew-symmetrizers $\mathcal{A}^{(k)}$ by putting $q = 1$ in (2.9). Thus, we have

$$\mathcal{A}^{(k)} = \frac{1}{k}\,\mathcal{A}^{(k-1)}\,(I - (k-1)R_{k-1})\,\mathcal{A}^{(k-1)}.$$

Let us, respectively, introduce the *quantum elementary symmetric* elements and the *quantum matrix powers* of $L(u)$ as follows:

$$e_0(u) = 1, \quad e_k(u) = \mathrm{Tr}_{R(1\dots k)}\left(\mathcal{A}^{(k)} L_{\overline{1}}(u) L_{\overline{2}}(u-1)\dots L_{\overline{k}}(u-(k-1))\right), \quad k \geq 1,$$
$$L^{[0]}(u) = I, \quad L^{[k]}(u) = L(u-(k-1))\dots L(u-1)L(u), \quad k \geq 1.$$

Then, the following CH identity is valid for the generating matrix $L(u)$ (see [10,11])

$$\sum_{k=0}^{m}(-1)^k L^{[m-k]}(u-k)e_k(u) = 0. \tag{4.1}$$

Consider the operator $L(u)e^{\partial_u}$, where $\partial_u = \frac{\mathrm{d}}{\mathrm{d}u}$. Our current aim is to give an explicit canonical form $L_{\mathrm{can}}(u)e^{\partial_u}$ of this operator and to show that the operators $L(u)e^{\partial_u}$ and $L_{\mathrm{can}}(u)e^{\partial_u}$ are similar in the sense of formula (3.4).

Note that as $L_{\mathrm{can}}(u)$, we use the matrix transposed to (3.2). This is motivated by the fact that the coefficients $e_p(u)$ in the CH identity (4.1) are on the right-hand side from the factors $L^{[k]}(u)$. (Note that in the RE algebras, these canonical forms are equivalent due to the centrality of the coefficients $\sigma_k(L)$.)

So, the matrix $L_{\mathrm{can}}(u)$ reads

$$L_{\mathrm{can}}(u) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_m(u) \\ 1 & 0 & \dots & 0 & 0 & a_{m-1}(u) \\ 0 & 1 & \dots & 0 & 0 & a_{m-2}(u) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 & a_2(u) \\ 0 & 0 & \dots & 0 & 1 & a_1(u) \end{pmatrix}. \tag{4.2}$$

Let again $v \in V$ be an arbitrary nontrivial vector. Constitute an $N \times N$ matrix

$$C(u) = \Big(v, L(u)v, L^{[2]}(u+1)v \dots L^{[k]}(u+k-1))v \dots L^{[m-1]}(u+m-2)v\Big)$$

where $v$ stands for the corresponding column.

**Theorem 7** *The following relation holds true*

$$L(u)e^{\partial_u} C(u) = C(u) L_{\mathrm{can}}(u) e^{\partial_u}, \tag{4.3}$$

*provided the entries $a_k(u)$ are of the form:*

$$a_k(u) = (-1)^{k+1}\, e_k(u+m-1).$$

*Proof* By using the evident relation

$$e^{\partial_u} f(u) = f(u+1)e^{\partial_u},$$

and canceling the operator $e^{\partial_u}$, we can present (4.3) in the form

$$L(u)C(u+1) = C(u)L_{\mathrm{can}}(u). \tag{4.4}$$

The equality of the corresponding matrix columns in (4.4), except for the last ones, immediately follows from the relation

$$L(u)L^{[k]}(u+k) = L^{[k+1]}(u+k). \tag{4.5}$$

As for the last columns, their equality follows from the CH identity (4.1). □

Now, assume $R(u, v)$ to be defined by the second formula (2.3). Let us, respectively, define the *quantum elementary symmetric* elements and *quantum matrix powers* by

$$e_0(u) = 1, \quad e_k(u) = \mathrm{Tr}_{R(1\dots k)}\Big(\mathcal{A}^{(k)} L_{\overline{1}}(u) L_{\overline{2}}(q^{-2}u) \dots L_{\overline{k}}(q^{-2(k-1)}u)\Big), \quad k \geq 1,$$
$$L^{[0]}(u) = I, \quad L^{[k]}(u) = L(q^{-2(k-1)}u)L(q^{-2(k-2)}u) \dots L(u), \quad k \geq 1.$$

Then, the following form of the CH identity is valid for the generating matrix $L(u)$ (see [10,11])

$$\sum_{k=0}^{m} (-q)^k L^{[m-k]}(q^{-2k}u) e_k(u) = 0. \tag{4.6}$$

Consider the operator $L(u)q^{2u\partial_u}$ and constitute the following matrix:

$$C(u) = \Big(v, L(u)v, L^{[2]}(q^2u)v \dots L^{[k]}(q^{2(k-1)}u)v \dots L^{[m-1]}(q^{2(m-2)}u)v\Big), \tag{4.7}$$

where $v$ stands for the corresponding column.

**Theorem 8** *The following identity holds*

$$L(u)q^{2u\partial_u}C(u) = C(u)L_{\text{can}}(u)q^{2u\partial_u}, \tag{4.8}$$

*where the matrix* $L_{\text{can}}(u)$ *is given by (4.2) provided the entries* $a_k(u)$ *are defined by*

$$a_k(u) = -(-q)^k\, e_k(q^{2(m-1)}u).$$

*Proof* Now, we use the following relation:

$$q^{2u\partial_u} f(u) = f(q^2u)q^{2u\partial_u}.$$

Then, upon canceling the factor $q^{2u\partial_u}$, we rewrite (4.8) as

$$L(u)C(q^2u) = C(u)L_{\text{can}}(u). \tag{4.9}$$

Also, instead of (4.5), we use the relation

$$L(u)L^{[k]}(q^{2k}u) = L^{[k+1]}(q^{2k}u).$$

This relation entails the equality of the corresponding matrix columns in (4.9), except for the last ones. The equality of the last columns follows from (4.6). □

In conclusion, we want to remark that if $R$ is an involutive or Hecke symmetry of bi-rank $(m|n)$ with $n \neq 0$, the quantum analogs of all symmetric elements can also be defined and the CH identities can be found in the corresponding RTT and RE algebras (see [8]). The reduction formulae for the generating matrices of the RE algebras [analogous to (3.4)] can be also established. The peculiarity of bi-rank $(m|n)$ case with $n \neq 0$ is that the coefficients of the canonical matrix $L_{can}$ (3.2) are *ratios* of the symmetric Schur polynomials. As for the braided Yangians connected with such symmetries $R$, the corresponding Schur polynomials are not constructed yet. This fact is the main obstacle to constructing the corresponding CH identity and DS reduction.

**Acknowledgements** The work of D.T. was partially supported by the grant of the Simons foundation and the RFBR Grant 17-01-00366 A. The work of P.S. has been funded by the Russian Academic Excellence Project '5-100' and was also partially supported by the RFBR Grant 16-01-00562.

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