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Location of eigenvalues of three-dimensional non-self-adjoint Dirac operators

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Abstract

We prove the absence of eigenvalues of the three-dimensional Dirac operator with non-Hermitian potentials in unbounded regions of the complex plane under smallness conditions on the potentials in Lebesgue spaces. Our sufficient conditions are quantitative and easily checkable.

Keywords Dirac operator · Complex potential · Non-self-adjoint perturbation · Pseudo-Friedrichs extension · Birman–Schwinger principle · Absence of eigenvalues

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1 Introduction

Let us consider a relativistic quantum particle of spin $\frac{1}{2}$ and mass $m \geq 0$ in $\mathbb{R}^3$, subject to an external electric field described by a potential V. The dynamics is governed by the Dirac Hamiltonian

$$H_V := -i\,\alpha \cdot \nabla + m\,\alpha_4 + V \tag{1}$$

acting in the Hilbert space of spinors $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$. Here $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ with α_μ being the usual 4×4 Hermitian Dirac matrices satisfying the anticommutation rules

$$\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu} I_{\mathbb{C}^4} \tag{2}$$

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for $\mu, \nu \in \{1, \ldots, 4\}$ and the dot denotes the scalar product in $\mathbb{R}^3$. Motivated by a growing interest in non-self-adjoint operators in quantum mechanics (*cf.* [1] for a mathematical overview), we proceed in a greater generality by allowing V to be a possibly non-Hermitian 4×4 matrix in (1).

In the traditional self-adjoint case (*i.e.* V is a real scalar multiple of the diagonal matrix), the literature on spectral properties of H_V is enormous and we limit ourselves to quoting the classical Thaller's monograph [17]. For V being matrix-valued and possibly non-Hermitian, a systematic study of the spectrum of H_V was initiated by the pioneering work of Cuenin, Laptev and Tretter [5] in the one-dimensional setting and followed by [2,6,8]. Some spectral aspects in the present three-dimensional situation are covered by [3,7,15].

In the field-free case (*i.e.* $V = 0$), it is well known that $\sigma(H_0) = (-\infty, -m] \cup [+m, +\infty)$ and that the spectrum is purely continuous. That is, the residual spectrum is empty and there are no eigenvalues. The objective of this paper is to derive quantitative smallness conditions on the potential V, which guarantee that the spectrum of H_V remains purely continuous, at least in certain regions of the complex plane. Denoting by $|V(x)|$ the operator norm of the matrix $V(x) \in \mathbb{C}^{4 \times 4}$ for almost every fixed $x \in \mathbb{R}^3$, the smallness is measured through Lebesgue norms of the real-valued function $|V|$.

We present two types of results in this paper. The first reads as follows:

Theorem 1 *Assume* $|V| \in L^3(\mathbb{R}^3)$. *If*

$$C f(\lambda, m) \, \||V|\|_{L^3(\mathbb{R}^3)} < 1 \tag{3}$$

with

$$C := \left(\frac{\pi}{2}\right)^{1/3} \sqrt{1 + e^{-1} + 2e^{-2}} \quad and \quad f(\lambda, m) := \sqrt{1 + \frac{(\Re\lambda)^2}{\left(\Re\sqrt{m^2 - \lambda^2}\right)^2}} \, ,$$

then $\lambda \notin \sigma_{\mathrm{p}}(H_V)$.

The hypothesis (3) is essentially a smallness requirement about V. In view of Kato's general smoothness theory [12] (see also [14, Sec. XIII.7]), it is not surprising that a result of this type should hold for weakly coupled potentials. For the Dirac operator in all dimensions, a similar result has been obtained by Cuenin [3, Thm. 6.1.b] (see also [4, Thm. 1.4.6]). The strength of our result lies in that the condition is explicit and easy to check in applications. Notice that $C \approx 1.5$ and that $f(\lambda, m)$ is finite if, and only if, $\lambda \notin (-\infty, -m] \cup [+m, +\infty)$. Consequently, the condition (3) is obeyed for such λ whenever the norm $\||V|\|_{L^3(\mathbb{R}^3)}$ is sufficiently small. Let us also remark that $f(\lambda, m) \sim |\lambda|/|\Im(\lambda)|$ as $|\Im(\lambda)| \to +\infty$ and in fact $f(\lambda, 0) = |\lambda|/|\Im(\lambda)|$ in the massless case. We illustrate the dependence of $f(\lambda, m)$ on λ in Fig. 1. We leave as an open problem whether the asymptotic double cone is optimal.

Remark 1 An analogous result holds (with an unspecified constant) provided that $L^3(\mathbb{R}^3)$ is replaced by the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$. Since this space in particular
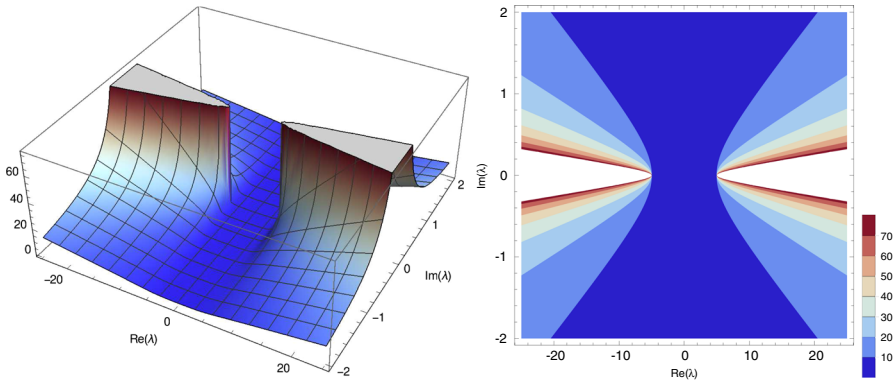
Fig. 1 Graph of the surface $\lambda \mapsto f(\lambda, m)$ and its contour plot for $m = 5$

contains the Coulomb potential $V_Z(x) := -Z/|x|$, which creates discrete eigenvalues in the gap $(-m, +m)$ whenever $Z > 0$, *cf.* [17, Sec. 7.4], we see that the presence of a λ-dependent function $f(\lambda, m)$ diverging as $\lambda \to \pm m$ is in fact unavoidable in (3).

To get a more uniform control over the spectrum and exclude possible eigenvalues embedded in the essential spectrum $(-\infty, -m] \cup [+m, +\infty)$, we have to strengthen the condition about the potential V.

Theorem 2 *Assume $|V| \in L^3(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$. If*

$$C \, \||V|\|_{L^3(\mathbb{R}^3)} + C' \, |\Re\lambda| \, \||V|\|_{L^{3/2}(\mathbb{R}^3)} < 1 \,, \tag{4}$$

where C is the same constant as in Theorem 1 and

$$C' := \frac{2^{17/6}}{3\pi^{2/3}} \,,$$

then $\lambda \notin \sigma_{\mathrm{p}}(H_V)$.

In this case, given a potential V with sufficiently small norm $\||V|\|_{L^3(\mathbb{R}^3)}$, the hypothesis (4) excludes the existence of eigenvalues in thin tubular neighbourhoods of the imaginary axis, with the thinness determined by the norm $\||V|\|_{L^{3/2}(\mathbb{R}^3)}$. Notice that $C' \approx 1.1$ and that the condition (4) is m-independent.

Of course, Theorems 1 and 2 can be combined to disprove the existence of eigenvalues in a union of the *unbounded* regions of the complex plane initially covered by the theorems separately. From this perspective, our theorems are an improvement upon Cuenin [3, Thm. 2.3], who disproves the existence of eigenvalues in *compact* regions only. Moreover, our results are much more explicit and quantitative. On the other hand, our method seems to be restricted to the three-dimensional situation, while the results of [3] are stated in all dimensions greater than or equal to two.

Remark 2 We note that V obeys (3) (respectively, (4)) if, and only if, the adjoint V^* does. Noticing additionally that $f(\bar{\lambda}, m) = f(\lambda, m)$, we see that any $\lambda \in \mathbb{C}$ satisfying (3) or (4) is not in the residual spectrum of H_V either. Consequently, λ is either in the continuous spectrum or in the resolvent set of H_V. Finally, let us notice that $\sigma_{\text{ess}}(H_V) = \sigma_{\text{ess}}(H_0) = (-\infty, -m] \cup [+m, +\infty)$ under the conditions stated in Theorems 1 and 2.

Our approach to establish Theorems 1 and 2 is based on two ingredients. The main technique is the Birman–Schwinger principle stating that if λ is in the point spectrum of the differential operator H_V, then -1 is in the spectrum of the (formal) integral operator

$$W^{1/2} (H_0 - \lambda)^{-1} U W^{1/2}, \tag{5}$$

where $V = U W$ is the polar decomposition of V (*i.e.* W is the absolute value of V and U is unitary). In this paper, we make an effort to justify the principle under the minimal regularity assumption that V is in a sense relatively form-bounded with respect to H_0. Moreover, we rigorously cover the embedded eigenvalues $\lambda \in (-\infty, -m] \cup [+m, +\infty)$. Once the Birman–Schwinger principle is established, it is enough to show that the norm of (5) is less than one in order to establish Theorems 1 and 2. To this aim, we use the second ingredient based on a careful estimate of the resolvent integral kernel of H_0. Putting these two together, we end up with the left-hand sides of the conditions (3) and (4) being precisely upper bounds to the norm of (5) obtained additionally with help of Sobolev-type inequalities.

The same approach has been recently employed in [9, Sect. 2] to disprove the existence of eigenvalues of Schrödinger operators with form-subordinated complex potentials (*cf.* [9, Thm. 1]). As an additional ingredient in comparison with [9, Sect. 2], here we also use the Hardy–Littlewood–Sobolev inequality [13, Thm. 4.3] to deal with the resolvent integral kernel of H_0 (see [10, Thm. 2] for a similar trick in the case of Schrödinger operators). In the present case of Dirac operator, however, the method does not seem to lead to uniform results in λ.

The rest of the paper consists of a precise definition of H_V as a pseudo-Friedrichs extension in Sect. 2, the analysis of the free resolvent in Sect. 3, the justification of the Birman–Schwinger principle in Sect. 4 and its application to the proofs of Theorems 1 and 2 in Sect. 5.

2 The pseudo-Friedrichs extension

We consider H_V as a perturbation of H_0, the latter being the operator that acts as (1) with $V = 0$ and has the domain

$$\mathsf{D}(H_0) := \left\{ \psi \in \mathcal{H} : \nabla \psi \in \mathcal{H}^3 \right\}.$$

It is well known that H_0 is self-adjoint and that $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$ is a core of H_0.

Notice that $H_0^2 = (-\Delta + m^2) I_{\mathbb{C}^4}$, where $-\Delta + m^2$ is the self-adjoint Schrödinger operator in $L^2(\mathbb{R}^3)$ with the usual domain $H^2(\mathbb{R}^3)$. The absolute value of H_0 thus

equals $(H_0^2)^{1/2} = \sqrt{-\Delta + m^2}\, I_{\mathbb{C}^4}$, which is again a self-adjoint operator when considered on the domain $H^1(\mathbb{R}^3; \mathbb{C}^4)$. The form domain of $\sqrt{-\Delta + m^2}$ equals the fractional Sobolev space $H^{1/2}(\mathbb{R}^3)$, cf. [13, Sect. 7.11]. Notice that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^{1/2}(\mathbb{R}^3)$, cf. [13, Sect. 7.14]. Clearly, $\sqrt{-\Delta + m^2} \geq \sqrt{-\Delta}$.

We always assume that $|V| \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and that V is relatively form-bounded with respect to the massless H_0 in the following sense: There exist numbers $a \in (0, 1)$ and $b \in \mathbb{R}$ such that, for all $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |V(x)||\psi(x)|^2\, \mathrm{d}x \leq a \int_{\mathbb{R}^3} |\sqrt[4]{-\Delta}\, \psi(x)|^2\, \mathrm{d}x + b \int_{\mathbb{R}^3} |\psi(x)|^2\, \mathrm{d}x. \qquad (6)$$

We recall that $|V(x)|$ denotes the operator norm of the matrix $V(x)$ in $\mathbb{C}^4$. Similarly, $|\psi(x)|$ denotes the norm of the vector $\psi(x) \in \mathbb{C}^4$. The double norm $\|\cdot\|_{\mathcal{X}}$ is reserved for functional spaces $\mathcal{X}$. For instance, $\|\psi\|_{\mathcal{H}} = \||\psi|\|_{L^2(\mathbb{R})}$. The symbol $(\cdot, \cdot)_{\mathcal{X}}$ stands for an inner product in a Hilbert space $\mathcal{X}$.

By the pseudo-Friedrichs extension [11, Thm. VI.3.11][1] (see also [18] for more recent developments), there exists a unique closed extension H_V of the operator sum $H_0 + V$, where V is understood as the multiplication operator by the matrix V in $\mathcal{H}$ with initial domain $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$. More specifically, the operator H_V satisfies

$$(\phi, H_V \psi)_{\mathcal{H}} = \left(G_0^{1/2}\phi, H_0 G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} + \int_{\mathbb{R}^3} \phi(x)^* V(x)\psi(x)\, \mathrm{d}x \qquad (7)$$

for all $\psi \in \mathsf{D}(H_V) \subset H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$ and $\phi \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$. Here $G_0 := (H_0^2)^{1/2} + (a^{-1}b + \delta)I_{\mathbb{C}^4}$ with any positive δ (which does not influence the definition of H_V). Notice that H_0 and G_0 commute (in the usual sense for unbounded operators), so the first term on the right-hand side of (7) equals $(\phi, H_0\psi)_{\mathcal{H}}$ if $\psi \in H^1(\mathbb{R}^3; \mathbb{C}^4)$ Consequently, H_V realises in a sense the form sum of H_0 and V. We have $\mathsf{D}(H_V{}^*) \subset H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$ and $i\eta \notin \sigma(H_V)$ for all real η with sufficiently large $|\eta|$.

Remark 3 By Kato's inequality $\sqrt{-\Delta} \geq (2/\pi)|x|^{-1}$ valid in the sense of quadratic forms in $L^2(\mathbb{R}^3)$, cf. [11, Rem. V.5.12], we see that the potentials V satisfying the pointwise inequality

$$|V(x)| \leq a\, \frac{2}{\pi} \frac{1}{|x|} + b$$

for almost every $x \in \mathbb{R}^3$ obey the hypothesis (6).

Another sufficient condition is given by the following proposition. (More optimal results can be obtained by using the Lorentz spaces.)

---

[1] Kato assumes $a < 1/2$ for an abstract non-symmetric operator V, but it is straightforward to check that his proof works with $a < 1$ (as in the symmetric case) in our special case when V is a multiplication by matrix and (6) is assumed.

Proposition 1 *Let* $|V| = v_1 + v_2$ *with* $v_1 \in L^3(\mathbb{R}^3)$ *and* $v_2 \in L^\infty(\mathbb{R}^3)$ *and assume that*

$$\|v_1\|_{L^3(\mathbb{R}^3)} < (2\pi^2)^{1/3}.$$

Then (6) *holds true.*

Proof For every $\psi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\left\|v_1^{1/2}\psi\right\|_{L^2(\mathbb{R}^3)}^2 \le \|v_1\|_{L^3(\mathbb{R}^3)} \|\psi\|_{L^3(\mathbb{R}^3)}^2 \le (2\pi^2)^{-1/3} \|v_1\|_{L^3(\mathbb{R}^3)} \left\|\sqrt[4]{-\Delta}\,\psi\right\|_{L^2(\mathbb{R}^3)}^2,$$

where the first estimate follows by Hölder inequality and the second inequality quantifies the Sobolev-type embedding $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, *cf.* [13, Thm. 8.4]. At the same time, $\|v_2^{1/2}\psi\|_{L^2(\mathbb{R}^3)}^2 \le b\,\|\psi\|_{L^2(\mathbb{R}^3)}^2$ with some non-negative constant b. $\square$

Remark 4 Since (6) allows V to be merely defined through a quadratic form, one can in principle introduce H_V under the weaker hypothesis $|V| \in L^1_{\text{loc}}(\mathbb{R}^3)$.

3 The free resolvent

Using the well-known trick

$$(H_0 - z)^{-1} = (H_0 + z)(H_0^2 - z^2)^{-1}$$

and the knowledge of the integral kernel of the free Schrödinger operator $-\Delta + m^2$, the integral kernel of the free resolvent $(H_0 - z)^{-1}$ can be written down explicitly:

$$
\begin{aligned}
&(H_0 - z)^{-1}(x, x') \\
&= \frac{e^{-\sqrt{m^2-z^2}\,|x-x'|}}{4\pi\,|x-x'|}\left(\frac{i\alpha\cdot(x-x')}{|x-x'|^2} + \sqrt{m^2-z^2}\,\frac{i\alpha\cdot(x-x')}{|x-x'|} + m\alpha_4 + z\right)
\end{aligned}
\tag{8}
$$

for every $z \notin \sigma(H_0)$. Here and in the sequel, we choose the principal branch of the square root. From now on, we also usually suppress writing the identity operators in the formulae; therefore, we simply write z instead of $zI_\mathcal{H}$ (respectively, $zI_{\mathbb{C}^4}$) on the left-hand (respectively, right-hand) side of (8) and elsewhere.

Given any matrix $M \in \mathbb{C}^{4\times 4}$, we use the notation $|M|_{\text{HS}} := \sqrt{\text{tr}(M^*M)}$ for the Hilbert–Schmidt (or Frobenius) norm of M. Recall that $|M| \le |M|_{\text{HS}}$.

Lemma 1 (Hilbert–Schmidt norm) *For almost every* $x, x' \in \mathbb{R}^3$, *one has*

$$
\begin{aligned}
&\left|(H_0 - z)^{-1}(x, x')\right|_{\text{HS}}^2 \\
&= 4\,\frac{e^{-2\Re\sqrt{m^2-z^2}\,|x-x'|}}{(4\pi)^2|x-x'|^4}\left(1 + 2\Re\sqrt{m^2-z^2}\,|x-x'|\right. \\
&\quad\left. + 2\left[(\Re\sqrt{m^2-z^2})^2 + (\Re z)^2\right]|x-x'|^2\right).
\end{aligned}
$$

Proof Writing

$$\left|(H_0 - z)^{-1}(x, x')\right|^2_{HS} = \frac{e^{-2\Re\sqrt{m^2-z^2}\,|x-x'|}}{(4\pi)^2|x-x'|^6}\,\mathrm{tr}(A^*A),$$

where

$$A := a \cdot \alpha + a_4\alpha_4 + a_0 I_{\mathbb{C}^4} \tag{9}$$

with

$$a := i\left(1 + \sqrt{m^2 - z^2}\,|x - x'|\right)(x - x'), \quad a_4 := m\,|x - x'|^2, \quad a_0 := z\,|x - x'|^2,$$

the proof reduces to straightforward manipulations with the Dirac matrices. Using the anticommutation relations (2), we immediately arrive at

$$A^*A = (|a|^2 + a_4^2 + |a_0|^2)I_{\mathbb{C}^4} + B,$$

where

$$B := b \cdot \alpha + b_4\,\alpha_4 + c \cdot i\alpha\alpha_4$$

with

$$
\begin{aligned}
b &:= 2\Re(\bar{a}a_0)(x - x') \\
&= 2\left[\Im z - \left(\Im z\,\Re\sqrt{m^2 - z^2} + \Re z\,\Im\sqrt{m^2 - z^2}\right)|x - x'|\right]|x - x'|^2(x - x'), \\
b_4 &:= 2a_4\Re(a_0) = 2m\,\Re z\,|x - x'|^4, \\
c &:= -2\Im(a)a_4(x - x') = -2m\left(1 + \Re\sqrt{m^2 - z^2}\,|x - x'|\right)|x - x'|^2(x - x').
\end{aligned}
$$

Since $\mathrm{tr}(\alpha_\mu) = 0$ for $\mu \in \{1, \ldots, 4\}$ as well as $\mathrm{tr}(\alpha_4\alpha_k) = 0$ for $k \in \{1, \ldots, 3\}$ and $\mathrm{tr}(I_{\mathbb{C}^4}) = 4$, we see that also the matrix B is traceless. Consequently, using in addition the explicit expressions for a_1, a_2 and a_3, we obtain

$$
\begin{aligned}
\frac{1}{4}\,\mathrm{tr}(A^*A) &= (|z|^2 + m^2)|x - x'|^4 + \left|1 + \sqrt{m^2 - z^2}\,|x - x'|\right|^2 |x - x'|^2 \\
&= \left(|z|^2 + m^2 + |m^2 - z^2|\right)|x - x'|^4 + 2\Re\sqrt{m^2 - z^2}\,|x - x'|^3 + |x - x'|^2.
\end{aligned}
$$

It remains to use the identity $|z|^2 + m^2 + |m^2 - z^2| = 2\left[\left(\Re\sqrt{m^2 - z^2}\right)^2 + (\Re z)^2\right]$.

$\square$

The present paper extensively uses the following explicit bound on the Hilbert–Schmidt norm of the free resolvent.

Lemma 2 *For almost every $x, x' \in \mathbb{R}^3$, one has*

$$\left|(H_0 - z)^{-1}(x, x')\right|^2_{HS} \leq c_1^2\,\frac{1}{|x - x'|^4} + c_2^2\,(\Re z)^2\,\frac{e^{-2\Re\sqrt{m^2-z^2}\,|x-x'|}}{|x - x'|^2},$$

where

$$c_1 := \frac{\sqrt{1 + e^{-1} + 2e^{-2}}}{2\pi}, \qquad c_2 := \frac{\sqrt{2}}{2\pi}.$$

Proof The estimate follows from Lemma 1 by using the elementary bounds $re^{-r} \leq e^{-1}$ and $r^2 e^{-r} \leq 4e^{-2}$ valid for every $r \geq 0$.   □

4 The Birman–Schwinger principle

For almost every $x \in \mathbb{R}^3$, let us introduce the matrix $W(x) := (V(x)^* V(x))^{1/2}$, the absolute value of $V(x)$. Using the polar decomposition for matrices, we have

$$V(x) = U(x)W(x) = U(x)W(x)^{1/2}W(x)^{1/2},$$

where $U(x)$ is a unitary matrix on $\mathbb{C}^4$. Notice that $|V(x)| = |W(x)| = |W(x)^{1/2}|^2$, where we recall that $|\cdot|$ stands for the operator norm of a matrix.

Given any $z \notin \sigma(H_0)$, we introduce the *Birman–Schwinger operator*

$$K_z := \left[W^{1/2} G_0^{-1/2} \right] \left[G_0(H_0 - z)^{-1} \right] \left[(UW^{1/2})^* G_0^{-1/2} \right]^*, \tag{10}$$

where G_0 is the shifted absolute value of H_0 introduced in (7). With an abuse of notation, we denote by the same symbols $W^{1/2}$ and U the maximal multiplication operators in $\mathcal{H}$ generated by the matrix-valued function $x \mapsto W(x)^{1/2}$ and $x \mapsto U(x)$, respectively. It follows from (6) that, for every $\psi \in \mathcal{H}$,

$$\|W^{1/2}\psi\|_{\mathcal{H}}^2 \leq a \, \|G_0^{1/2}\psi\|_{\mathcal{H}}^2 \qquad \text{and} \qquad \|(UW^{1/2})^*\psi\|_{\mathcal{H}}^2 \leq a \, \|G_0^{1/2}\psi\|_{\mathcal{H}}^2.$$

Consequently, $\|W^{1/2} G_0^{-1/2}\|_{\mathcal{H} \to \mathcal{H}}^2 \leq a$ and $\|(UW^{1/2})^* G_0^{-1/2}\|_{\mathcal{H} \to \mathcal{H}}^2 \leq a$. Hence, K_z is a well-defined bounded operator on $\mathcal{H}$ (as a composition of three bounded operators) and one has the rough bound

$$\|K_z\|_{\mathcal{H} \to \mathcal{H}} \leq a \, \|G_0(H_0 - z)^{-1}\|_{\mathcal{H} \to \mathcal{H}} \leq a \sup_{\xi \in (-\infty, -m) \cup (+m, \infty)} \frac{|\xi| + a^{-1}b + \delta}{|\xi - z|}. \tag{11}$$

Using the commutativity of H_0 and G_0, the operator K_z admits the familiar form (5) provided that V is bounded. But we insist working under the minimal regularity assumption (6) in this section.

By $\varphi \in L_0^2(\mathbb{R}^3; \mathbb{C}^4)$ in the lemma below we mean $\varphi \in L^2(\mathbb{R}^3; \mathbb{C}^4)$ and that $\operatorname{supp} \varphi$ is compact.

Lemma 3 (Birman–Schwinger principle) *Assume* (6). *Let* $H_V \psi = \lambda \psi$ *with some* $\lambda \in \mathbb{C}$ *and* $\psi \in D(H_V)$, *and set* $\phi := W^{1/2}\psi$. *One has* $\phi \in \mathcal{H}$ *and* $\phi \neq 0$ *if* $\psi \neq 0$.

(i) *If $\lambda \notin \sigma(H_0)$, then $K_\lambda \phi = -\phi$.*
(ii) *If $\lambda \in \sigma(H_0)$, then $\lim_{\varepsilon \to 0^\pm} (\varphi, K_{\lambda+i\varepsilon}\phi)_{\mathcal{H}} = -(\varphi, \phi)_{\mathcal{H}}$ for every $\varphi \in L_0^2(\mathbb{R}^3; \mathbb{C}^4)$.*

Proof By (6), $\phi \in \mathcal{H}$ if $\psi \in D(H_V) \subset H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$. From (7) we deduce that $H_V \psi = H_0 \psi$ provided that $\phi = 0$. But then λ is an eigenvalue of H_0 (which is impossible) unless $\psi = 0$.

If $\lambda \notin \sigma(H_0)$, then

$$
\begin{aligned}
(\varphi, K_\lambda \phi)_{\mathcal{H}} &= \left([W^{1/2}G_0^{-1/2}]^*\varphi, [G_0(H_0 - \lambda)^{-1}][(UW^{1/2})^*G_0^{-1/2}]^*\phi\right)_{\mathcal{H}} \\
&= \left([UW^{1/2})^*G_0^{-1/2}][G_0(H_0 - \lambda)^{-1}]^*[W^{1/2}G_0^{-1/2}]^*\varphi, W^{1/2}\psi\right)_{\mathcal{H}} \\
&= \int_{\mathbb{R}^3} \eta(x)^* V(x)\psi(x)\, dx
\end{aligned}
$$

for every $\varphi \in \mathcal{H}$ with $\eta := G_0^{-1/2}[G_0(H_0-\lambda)^{-1}]^*[W^{1/2}G_0^{-1/2}]^*\varphi \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$. Using (7), it follows that

$$
\begin{aligned}
(\varphi, K_\lambda \phi)_{\mathcal{H}} &= (\eta, H_V \psi)_{\mathcal{H}} - \left(G_0^{1/2}\eta, H_0 G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= \lambda(\eta, \psi)_{\mathcal{H}} - \left(G_0^{1/2}\eta, H_0 G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= \lambda\left(G_0^{1/2}\eta, G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} - \left(G_0^{1/2}\eta, H_0 G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= -\left(G_0^{1/2}\eta, (H_0 - \lambda)G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= -\left([W^{1/2}G_0^{-1/2}]^*\varphi, G_0(H_0 - \lambda)^{-1}(H_0 - \lambda)G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= -\left([W^{1/2}G_0^{-1/2}]^*\varphi, G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= -\left(\varphi, W^{1/2}\psi\right)_{\mathcal{H}} = -(\varphi, \phi)_{\mathcal{H}}
\end{aligned}
$$

for every $\varphi \in \mathcal{H}$. This proves that $K_\lambda \phi = -\phi$ and therefore (i).

If $\lambda \in \sigma(H_0)$, then there exists $\varepsilon_0 > 0$ such that $\lambda + \varepsilon \notin \sigma(H_0)$ for all real ε satisfying $0 < |\varepsilon| < \varepsilon_0$. Now let us assume that $\varphi \in L_0^2(\mathbb{R}^3; \mathbb{C}^4)$. As above, we have

$$
\begin{aligned}
(\varphi, K_{\lambda+i\varepsilon}\phi)_{\mathcal{H}} &= -\left([W^{1/2}G_0^{-1/2}]^*\varphi, G_0(H_0 - \lambda - i\varepsilon)^{-1}(H_0 - \lambda)G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= -(\varphi, \phi)_{\mathcal{H}} - i\varepsilon\left([W^{1/2}G_0^{-1/2}]^*\varphi, G_0(H_0 - \lambda - i\varepsilon)^{-1}G_0^{-1} G_0^{1/2}\psi\right)_{\mathcal{H}} \\
&= -(\varphi, \phi)_{\mathcal{H}} - i\varepsilon\left(\varphi, W^{1/2}(H_0 - \lambda - i\varepsilon)^{-1}\psi\right)_{\mathcal{H}}
\end{aligned}
$$

for every $\varphi \in \mathcal{H}$. Let us show that the last inner product vanishes as $\varepsilon \to 0$. Using Lemma 2, we have

$$
\left|(\varphi, W^{1/2}(H_0 - \lambda - i\varepsilon)^{-1}\psi)_{\mathcal{H}}\right| \le c_1 I_1 + c_2 |\lambda| I_2(\varepsilon),
$$

where

$$I_1 := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)| \, |W(x)|^{1/2} \, |\psi(x')|}{|x - x'|^2} \, \mathrm{d}x \, \mathrm{d}x' \,,$$

$$I_2(\varepsilon) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\varphi(x)| \, |W(x)|^{1/2} \, \frac{e^{-\Re \sqrt{m^2 - (\lambda + i\varepsilon)^2} \, |x - x'|}}{|x - x'|} \, |\psi(x')| \, \mathrm{d}x \, \mathrm{d}x' \,.$$

The first integral is estimated as follows:

$$
\begin{aligned}
I_1 &\leq (2\pi)^{2/3} \||\varphi| |W|^{1/2} \|_{L^1(\mathbb{R}^3)} \||\psi|\|_{L^3(\mathbb{R}^3)} \\
&\leq (2\pi)^{2/3} \||\varphi| |W|^{1/2} \|_{L^1(\mathbb{R}^3)} (2\pi^2)^{-1/6} \big\| \sqrt[4]{-\Delta} \, |\psi| \big\|_{L^2(\mathbb{R}^3)} \\
&\leq (2\pi)^{2/3} \||\varphi|\|_{L^2(\mathbb{R}^3)} \|\chi_\Omega |V|\|_{L^1(\mathbb{R}^3)}^{1/2} (2\pi^2)^{-1/6} \big\| \sqrt[4]{-\Delta} \, |\psi| \big\|_{L^2(\mathbb{R}^3)}
\end{aligned}
$$

with $\Omega := \operatorname{supp} \varphi$. Here, the first bound is due to the Hardy–Littlewood–Sobolev inequality [13, Thm. 4.3], the second estimate is a Sobolev-type inequality [13, Thm. 8.4] quantifying the embedding $\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ and the last bound is the Schwarz inequality. Notice that the integrals on the last line are all finite, in particular because $\psi \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4)$ and $|V| \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$. To estimate the second integral, we proceed as in [9, proof of Lem. 2]:

$$I_2(\varepsilon) \leq \||\varphi|\|_{L^2(\mathbb{R}^3)} \|M_\varepsilon\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \||\psi|\|_{L^2(\mathbb{R}^3)} \,,$$

where M_ε is the integral operator with kernel

$$M_\varepsilon(x, x') := \chi_\Omega(x) \, |W(x)|^{1/2} \, \frac{e^{-\Re \sqrt{m^2 - (\lambda + i\varepsilon)^2} \, |x - x'|}}{|x - x'|} \,.$$

Using the Hilbert–Schmidt norm, we have

$$
\begin{aligned}
\|M_\varepsilon\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)}^2 &\leq \|M_\varepsilon\|_{\mathrm{HS}}^2 = \int_{\Omega \times \mathbb{R}^3} |V(x)| \, \frac{e^{-2\Re \sqrt{m^2 - (\lambda + i\varepsilon)^2} \, |x - x'|}}{|x - x'|^2} \, \mathrm{d}x \, \mathrm{d}x' \\
&= \frac{2\pi}{\Re \sqrt{m^2 - (\lambda + i\varepsilon)^2}} \int_\Omega |V(x)| \, \mathrm{d}x \,,
\end{aligned}
$$

where the integral is again finite because of the local integrability of V. It remains to realise that

$$\Re \sqrt{m^2 - (\lambda + i\varepsilon)^2} \sim \begin{cases} |\varepsilon|^{1/2} & \text{if } \lambda^2 = m^2 \ \& \ m \neq 0 \,, \\ |\varepsilon| & \text{otherwise.} \end{cases}$$

Hence, $\varepsilon I_2(\varepsilon) = O(|\varepsilon|^{1/2})$ as $\varepsilon \to 0$. Putting the estimates together, we have proved

$$\frac{|(\varphi, K_{\lambda+i\varepsilon}\phi + \phi)_\mathcal{H}|}{\|\varphi\|_\mathcal{H}}$$

$$\leq \varepsilon \, \|\chi_\Omega |V|\|_{L^1(\mathbb{R}^3)}^{1/2} \, \||\psi\||_{H^{1/2}(\mathbb{R}^3)} \left(c_1 2^{1/2} \pi^{1/3} + \frac{c_2 |\lambda| (2\pi)^{1/2}}{\left(\Re\sqrt{m^2 - (\lambda + i\varepsilon)^2}\right)^{1/2}} \right) \xrightarrow[\varepsilon \to 0]{} 0$$

for every given $\varphi \in L_0^2(\mathbb{R}^3; \mathbb{C}^4)$. This shows (ii) and concludes the proof of the lemma.
$\square$

The preceding lemma is a precise statement of one side of the Birman–Schwinger principle under the minimal regularity assumption (6). It says that if λ is an eigenvalue of H_V, then -1 is an eigenvalue of an integral equation related to K_λ. If $\lambda \notin \sigma(H_0)$ the converse implication also holds, but it is not generally true if $\lambda \in \sigma(H_0)$, *cf.* [16, Sec. III.2], and it is not needed for the purpose of this paper. In fact, we exclusively use the following corollary of Lemma 3.

Corollary 1 *Assume* (6) *and let* $\lambda \in \sigma_p(H_V)$.

(i) *If* $\lambda \notin \sigma(H_0)$, *then* $\|K_\lambda\| \geq 1$.
(ii) *If* $\lambda \in \sigma(H_0)$, *then* $\liminf_{\varepsilon \to 0^\pm} \|K_{\lambda+i\varepsilon}\|_{\mathcal{H} \to \mathcal{H}} \geq 1$.

Proof Let $\lambda \in \sigma_p(H_V)$, let ψ be a corresponding eigenfunction and set $\phi = W^{1/2} \psi \neq 0$.

If $\lambda \notin \sigma(H_0)$, then the statement (i) of Lemma 3 implies

$$\|\phi\|_\mathcal{H}^2 \|K_\lambda\|_{\mathcal{H} \to \mathcal{H}} \geq |(\phi, K_\lambda \phi)_\mathcal{H}| = \|\phi\|_\mathcal{H}^2,$$

from which the claim (i) immediately follows.

If $\lambda \in \sigma(H_0)$, we set $\phi_n := \xi_n \phi$ for every positive n, where $\xi_n(x) := \xi(x/n)$ and $\xi \in C_0^\infty(\mathbb{R}^3)$ is a usual cut-off function satisfying $\xi(x) = 1$ for $|x| \leq 1$ and $\xi(x) = 0$ for $|x| \geq 2$. As above, we write

$$\|\phi_n\|_\mathcal{H} \|\phi\|_\mathcal{H} \|K_{\lambda+i\varepsilon}\|_{\mathcal{H} \to \mathcal{H}} \geq |(\phi_n, K_{\lambda+i\varepsilon}\phi)_\mathcal{H}|,$$

Taking the limit $\varepsilon \to 0^\pm$, the statement (ii) of Lemma 3 implies

$$\|\phi_n\|_\mathcal{H} \|\phi\|_\mathcal{H} \liminf_{\varepsilon \to 0^\pm} \|K_{\lambda+i\varepsilon}\|_{\mathcal{H} \to \mathcal{H}} \geq |(\phi_n, \phi)_\mathcal{H}|.$$

The desired claim (ii) then follows by taking the limit $n \to \infty$.
$\square$

To disprove the existence of eigenvalues in a complex region, it is thus enough to show that the norm of the Birman–Schwinger operator is strictly less than one there.

5 Proofs

To implement the above idea, we first strengthen our hypotheses about V and establish a more precise estimate on the norm of K_z as compared to (11).

Lemma 4 *Let* $|V| \in L^3(\mathbb{R}^3)$. *For every* $z \notin \sigma(H_0)$,

$$\|K_z\|_{\mathcal{H} \to \mathcal{H}} \leq \left(\frac{\pi}{2}\right)^{1/3} \sqrt{1 + e^{-1} + e^{-2}} \sqrt{1 + \frac{(\Re z)^2}{\left(\Re \sqrt{m^2 - z^2}\right)^2}} \, \||V|\|_{L^3(\mathbb{R}^3)} \,.$$

Proof By the elementary bound $r^2 e^{-r} \leq 4e^{-2}$ valid for every $r \geq 0$, we get from Lemma 2 the estimate

$$\left|(H_0 - z)^{-1}(x, x')\right|_{\mathrm{HS}}^2 \leq \left(c_1^2 + \tilde{c}_2^2 \frac{(\Re z)^2}{\left(\Re \sqrt{m^2 - z^2}\right)^2}\right) \frac{1}{|x - x'|^4}$$

for almost every $x, x' \in \mathbb{R}^3$ with $\tilde{c}_2 := c_2 e^{-2} = \sqrt{2e^{-2}}/(2\pi)$. Consequently, for every $\phi, \psi \in \mathcal{H}$,

$$|(\phi, K_z \psi)_{\mathcal{H}}|$$
$$\leq \sqrt{c_1^2 + \tilde{c}_2^2 \frac{(\Re z)^2}{\left(\Re \sqrt{m^2 - z^2}\right)^2}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|V(x)|^{1/2} |\phi(x)| |V(x')|^{1/2} |\psi(x')|}{|x - x'|^2} \, \mathrm{d}x \, \mathrm{d}x'$$
$$\leq \sqrt{c_1^2 + \tilde{c}_2^2 \frac{(\Re z)^2}{\left(\Re \sqrt{m^2 - z^2}\right)^2}} \, 2^{2/3} \pi^{4/3} \||V|\|_{L^3(\mathbb{R}^3)} \|\phi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \,,$$

where the second estimate is due to the Hardy–Littlewood–Sobolev inequality with optimal constants [13, Thm. 4.3] together with the Hölder inequality. We eventually use $\tilde{c}_2 \leq c_1$, just to make the final result look more elegant. $\square$

Further strengthening hypothesis about V, we can also get a bound which is independent of the mass m and has a different behaviour in z.

Lemma 5 *Let* $|V| \in L^3(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$. *For every* $z \notin \sigma(H_0)$,

$$\|K_z\|_{\mathcal{H} \to \mathcal{H}} \leq \left(\frac{\pi}{2}\right)^{1/3} \sqrt{1 + e^{-1} + e^{-2}} \, \||V|\|_{L^3(\mathbb{R}^3)} + \frac{2^{17/6}}{3\pi^{2/3}} |\Re z| \, \||V|\|_{L^{3/2}(\mathbb{R}^3)} \,.$$

Proof For every $\phi, \psi \in \mathcal{H}$, we have

$$|(\phi, K_z \psi)_{\mathcal{H}}|$$
$$\leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} |V(x)|^{1/2} |\phi(x)| \left(\frac{c_1}{|x - x'|^2} + \frac{c_2 |\Re z|}{|x - x'|}\right) |V(x')|^{1/2} |\psi(x')| \, \mathrm{d}x \, \mathrm{d}x'$$
$$\leq \left(c_1 2^{2/3} \pi^{4/3} \||V|\|_{L^3(\mathbb{R}^3)} + c_2 |\Re z| \frac{2^{10/3}}{3\pi^{1/3}} \||V|\|_{L^{3/2}(\mathbb{R}^3)}\right) \|\phi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} \,.$$

Here, the first inequality follows by Lemma 2 (where the exponential is estimated by 1) and the second estimate is due to the Hardy–Littlewood–Sobolev inequality with optimal constants [13, Thm. 4.3] together with the Hölder inequality. □

Theorem 1 (respectively, Theorem 2) follows as a consequence of Corollary 1 and Lemma 4 (respectively, Lemma 5).

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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