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Sextonions, Zorn matrices, and $\mathbf{e}_{7\frac{1}{2}}$

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Abstract By exploiting suitably constrained Zorn matrices, we present a new construction of the algebra of sextonions (over the algebraically closed field $\mathbb{C}$). This allows for an explicit construction, in terms of Jordan pairs, of the non-semisimple Lie algebra $\mathbf{e}_{7\frac{1}{2}}$, intermediate between $\mathbf{e}_7$ and $\mathbf{e}_8$, as well as of all Lie algebras occurring in the sextonionic row and column of the extended Freudenthal Magic Square.

Keywords Exceptional Lie algebras · Intermediate algebras · Sextonions · Zorn matrices

Mathematics Subject Classification 17B10 · 17B25 · 17B45

1 Introduction

The field of composition, non-associative algebras, and related Lie algebras, underwent a series of interesting developments in recent times.

In [1] Deligne proposed dimension formulas for the exceptional series of complex simple Lie algebras, whose parametrization in terms of the dual Coxeter number was exploited further in [2] by Cohen and de Man (see also [3]). Landsberg and

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Manivel subsequently pointed out the relation between the dimension formulas and the dimensions of the composition algebras themselves in [4]. In [1,2] it was observed that all parameter values determining integer outputs in the dimension formulas were already accounted for by the known normed division algebras, with essentially one exception, intriguingly corresponding to a would be composition algebra of dimension six, sitting between the quaternions and octonions.

This algebra, whose elements were named *sextonions*, was recently studied by Westbury in [5], who pointed out the related existence of a whole new row in the Freudenthal Magic Square. Actually, the six-dimensional algebra of sextonions had been observed earlier as a curiosity; indeed, it was explicitly constructed in [6]. Moreover, it was used in [7] to study the conjugacy classes in the smallest exceptional Lie algebra $\mathbf{g}_2$ in characteristics other than 2 or 3. The sextonions were also constructed in [8] (*cfr.* Th. 5 therein), and proved to be a maximal subalgebra of the split octonions.

In [9], Landsberg and Manivel "filled in the hole" in the exceptional series of Lie algebras, observed by Cvitanovic, Deligne, Cohen and de Man, showing that sextonions, through the *triality* construction of [4], give rise to a non-simple *intermediate* exceptional Lie algebra, named $\mathbf{e}_{\mathbf{7}\frac{1}{2}}$, between $\mathbf{e}_\mathbf{7}$ and $\mathbf{e}_\mathbf{8}$, satisfying some of the decomposition and dimension formulas of the exceptional simple Lie algebras [1–4,10].

More recently, such a 190-dimensional Lie algebra $\mathbf{e}_{\mathbf{7}\frac{1}{2}}$ was also found by Mkrtchyan in the study of the Vogel plane [11], in the context of the analysis of the *universal* Vogel Lie algebra [12].

By the Hurwitz Theorem [13], the real normed division algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathfrak{C}$ (Cayley numbers). Each algebra can be constructed from the previous one by the so-called *Cayley-Dickson doubling procedure* [14,15].

All these algebras can be complexified to give complex algebras. These complex algebras, respectively, are $\mathbb{R} \otimes \mathbb{C} = \mathbb{C}$, $\mathbb{C} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, $\mathbb{H} \otimes \mathbb{C} = M_2(\mathbb{C})$, $\mathfrak{C} \otimes \mathbb{C}$ (M_2 denoting a 2×2 matrix). The three complex algebras other than $\mathbb{C}$ have a second real form, denoted $\mathbb{C}_s$, $\mathbb{H}_s$ and $\mathfrak{C}_s$, with the following isomorphisms holding: $\mathbb{C}_s = \mathbb{R} \oplus \mathbb{R}$ and $\mathbb{H}_s = M_2(\mathbb{R})$. The normed division algebras are called the *compact* forms and the aforementioned second real form is called the *split* real form. It is worth pointing out that split real forms are composition algebras but they are not division algebras.

On the field $\mathbb{R}$, the sextonions only exist in split form $\mathbb{S}_s$, and they are intermediate between the split quaternions $\mathbb{H}_s$ and the split octonions $\mathfrak{C}_s$:

$$\mathbb{H}_s \subset \mathbb{S}_s \subset \mathfrak{C}_s. \tag{1.1}$$

Note that $\mathbb{S}_s$ does not contain the divisional quaternions $\mathbb{H}$; see "Appendix A."

Nowadays, exceptional Lie algebras have a long-standing history of applications to physics (see, e.g., [16–25] for a partial list of results and Refs.). The relevance of compact exceptional Lie algebras (and groups) in realizing grand unification gauge theories and consistent string theories is well recognized. Similarly, the relevance of non-compact real forms for the construction of locally supersymmetric theories of gravity is well appreciated. Other frameworks include sigma models based on quotients of exceptional Lie groups, which are of interest for string theory and conformal field theories, as well. It is here worth pointing out that that the analysis of quantum criticality in Ising chains and the structure of magnetic materials such as Cobalt Niobate has also recently (and strikingly) turned out to be related to exceptional Lie algebras of type E (see, e.g., [26,27], respectively). Moreover, exceptional Lie algebras occur in models of confinement in non-Abelian gauge theories (for instance, *cfr.* [21]), as well as in a striking relation between cryptography and black hole physics, recently discovered [28–33]. It should also be recalled that fascinating exceptional algebraic structures arise in the description of the Attractor Mechanism for black holes in Maxwell–Einstein supergravity theories [34–40], such as the so-called magic exceptional supergravity [41–45].

In this context, the aforementioned, intermediate 190-dimensional Lie algebra $\mathbf{e}_{7\frac{1}{2}}$ is quite novel, and applications to physics are still under investigation, even though recent studies (*cfr.* e.g., [46]) intriguingly seem to connect sextonions to theories *beyond* eleven-dimensional M-theory. It should also be mentioned that $\mathbf{e}_{7\frac{1}{2}}$ can be regarded as a *Freudenthal triple system* over the exceptional Albert algebra $J_3^{\mathbb{O}}$, along with its automorphism algebra $\mathbf{e}_7$ and an extra $\mathbf{e}_7$-singlet generator, acting on the Freudenthal triple system as the multiplication by a scalar; for details on the applications of Freudenthal triple systems to the study of black hole attractors in four dimensions, *cfr.* e.g., [39,47–49], and Refs. therein.

In the present paper, we will apply the formal machinery introduced in [50] and [51], as well as an explicit realization of the sextonions (over the algebraically closed field $\mathbb{C}$), in order to explicitly construct the non-semisimple Lie algebra $\mathbf{e}_{7\frac{1}{2}}$, as well as all algebras occurring in the sextonionic row of the *extended* Freudenthal Magic Square [5,9], in terms of Jordan pairs.

The plan of the paper is as follows.

In Sect. 2, we provide a realization of the sextonions in terms of nilpotents constructed from the traceless octonions, and recall their representation in terms of suitably constrained Zorn matrices.

The intermediate exceptional algebra $\mathbf{e}_{7\frac{1}{2}}$ is then considered in Sect. 3, which focuses on the construction (then developed in Sects. 6, 7) of the sextonionic row and column of the extended Magic Square, by exploiting Jordan pairs for the sextonionic rank-3 Jordan algebra.

The action of $\mathbf{g}_2 = Der(\mathfrak{C})$ on the Zorn matrices is recalled in Sect. 4, and exploited in Sect. 5 to determine the derivations of the sextonions, $Der(\mathbb{S})$.

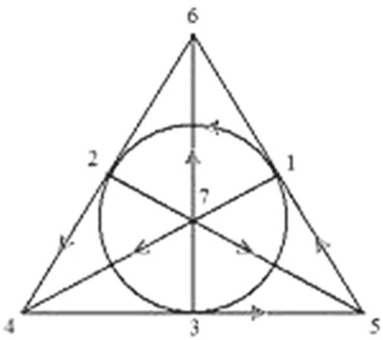
Then, in Sects. 6 and 7, the explicit construction of the intermediate algebras $\mathbf{c}_{3\frac{1}{2}}$ (which analogously holds for $\mathbf{a}_{5\frac{1}{2}}$ and $\mathbf{d}_{6\frac{1}{2}}$) and $\mathbf{e}_{7\frac{1}{2}}$ is presented.

The paper is concluded by App. A, in which we prove that, on the field $\mathbb{R}$, $\mathbb{S}_s$ does not contain the divisional quaternions $\mathbb{H}$.

2 Sextonions and their nilpotent realization

The algebra of sextonions is a six-dimensional subalgebra $\mathbb{S}$ of the octonions. As mentioned above, we denote by $\mathfrak{C}$ the algebra of the octonions over the complex field $\mathbb{C}$, whose multiplication rule goes according to the Fano diagram in Fig. 1.

Fig. 1 Fano diagram for the octonions' products



If $a \in \mathfrak{C}$ we write $a = a_0 + \sum_{j=1}^{7} a_j u_j$ where $a_j \in \mathbb{C}$ for $j = 1, \ldots, 7$ and u_j for $j = 1, \ldots, 7$ denote the octonion imaginary units. We denote by i the imaginary unit in $\mathbb{C}$.

We introduce 2 idempotent elements:

$$\rho^{\pm} = \frac{1}{2}(1 \pm i u_7)$$

and 6 nilpotent elements:

$$\varepsilon_k^{\pm} = \rho^{\pm} u_k \quad , k = 1, 2, 3$$

One can readily check that [50]:

$$\begin{aligned}
&(\rho^{\pm})^2 = \rho^{\pm}, \quad \rho^{\pm}\rho^{\mp} = 0 \\
&\rho^{\pm}\varepsilon_k^{\pm} = \varepsilon_k^{\pm}\rho^{\mp} = \varepsilon_k^{\pm} \\
&\rho^{\mp}\varepsilon_k^{\pm} = \varepsilon_k^{\pm}\rho^{\pm} = 0 \\
&(\varepsilon_k^{\pm})^2 = 0 \\
&\varepsilon_k^{\pm}\varepsilon_{k+1}^{\pm} = -\varepsilon_{k+1}^{\pm}\varepsilon_k^{\pm} = \varepsilon_{k+2}^{\mp} \quad \text{(indices modulo 3)} \\
&\varepsilon_j^{\pm}\varepsilon_k^{\mp} = 0 \quad j \neq k \\
&\varepsilon_k^{\pm}\varepsilon_k^{\mp} = -\rho^{\pm}
\end{aligned} \tag{2.1}$$

We can write $a \in \mathfrak{C}$ as $a = \alpha_0^+ \rho^+ + \alpha_0^- \rho^- + \alpha_k^+ \varepsilon_k^+ + \alpha_k^- \varepsilon_k^-$.

The subalgebra $\mathbb{S} \in \mathfrak{C}$ generated by $\rho^{\pm}, \varepsilon_1^{\pm}, \varepsilon_2^{+}, \varepsilon_3^{-}$ (namely $a \in \mathbb{S}$ *iff* $\alpha_2^- = \alpha_3^+ = 0$) provides an explicit realization of the sextonions. The existence of the non-divisional sextonionic elements can be easily understood. Indeed, in order to construct divisional sextonions, one would need to combine a nilpotent with its complex conjugate; but, as given by the above construction, this is not possible for ε_2^+ nor for ε_3^-.

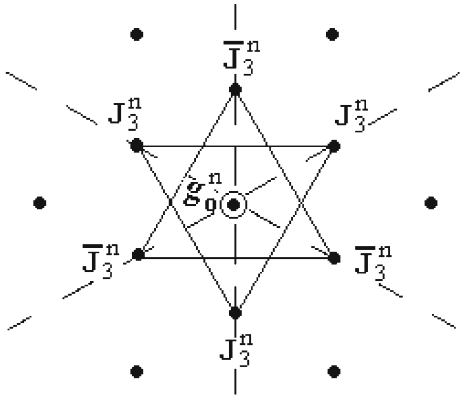


Fig. 2 A unifying view of the roots of exceptional Lie algebras through *Jordan pairs* [50]. For $\mathbf{n} = 8$, the root diagram of $\mathbf{e}_8$ is obtained

Octonions can be represented by Zorn matrices [52]. After the treatment of Sect. 3 of [51], we can represent the sextonions as a Zorn matrix, as long as A^+ and A^- are $\mathbb{C}^3$-vectors of the type

$$A^+ = (a^+, c^+, 0) \quad \text{and} \quad A^- = (a^-, 0, c^-)$$

Notice that A^+ and A^- lie on orthogonal $\mathbb{C}^3$-planes sharing the line along the first component.

3 $\mathbf{e}_{7\frac{1}{2}}$

In recent papers [50,51], a unifying view of all exceptional Lie algebras in terms of $\mathbf{a_2}$ subalgebras and *Jordan Pairs* has been presented, and a *Zorn matrix-like* representation of these algebras has been introduced.

The root diagram related to this view is shown in Fig. 2, where the roots of the exceptional Lie algebras are projected on a complex $\mathbf{su(3)} = \mathbf{a}_2$ plane, recognizable by the dots forming the external hexagon, and it exhibits the *Jordan pair* content of each exceptional Lie algebra. There are three Jordan pairs $(\mathbf{J_3^n}, \overline{\mathbf{J}}\mathbf{_3^n})$, each of which lies on an axis symmetrically with respect to the center of the diagram. Each pair doubles a simple Jordan algebra of rank 3, $\mathbf{J_3^n}$, with involution—the conjugate representation $\overline{\mathbf{J}}\mathbf{_3^n}$, which is the algebra of 3×3 Hermitian matrices over $\mathbb{A}$, where $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathfrak{C}$ for $\mathbf{n} = \dim_{\mathbb{R}} \mathbb{A} = 1, 2, 4, 8$, respectively, stands for real, complex, quaternion, octonion algebras, the four normed division algebras according to Hurwitz's Theorem; see, e.g., [53]. Exceptional Lie algebras $\mathbf{f_4}, \mathbf{e_6}, \mathbf{e_7}, \mathbf{e_8}$ are obtained for $\mathbf{n} = 1, 2, 4, 8$, respectively. $\mathbf{g_2}$ (corresponding to $\mathbf{n} = -2/3$) can be also represented in the same way, with the Jordan algebra reduced to a single element. For further detail, *cfr.* [50].

We expand that view in this paper to include $\mathbf{e}_{7\frac{1}{2}}$ [9], a Lie subalgebra of $\mathbf{e}_8$ of dimension 190. If we consider the $\mathbf{e}_8$ root diagram (obtained in Fig. 2 for $\mathbf{n} = 8$), then the sub-diagram of $\mathbf{e}_{7\frac{1}{2}}$ is shown in Fig. 3, (for $\mathbf{n} = 8$, as well).

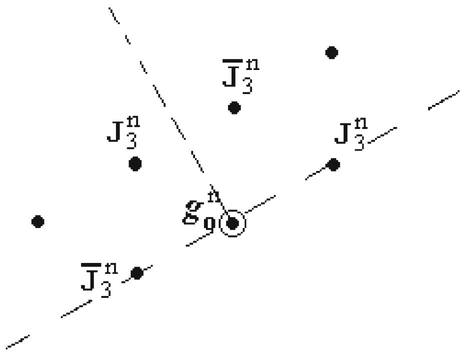


Fig. 3 Root diagram of $\mathbf{e}_{7\frac{1}{2}}$ (for $\mathbf{n} = 8$)

Table 1 Third and fourth row of the magic square

n	1	2	4	8
$\mathfrak{g}_{III}$	$\mathbf{c}_3$	$\mathbf{a}_5$	$\mathbf{d}_6$	$\mathbf{e}_7$
$\mathfrak{g}_{IV}$	$\mathbf{f}_4$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_8$

In general, one can do the same for all algebras in the fourth and third row of the Magic square [54,55], that we denote by $\mathfrak{g}_{IV}$ and $\mathfrak{g}_{III}$, respectively (see Table 1). In this way, the algebras in the intermediate (fourth) row of the extended Magic Square [5,9] are explicitly constructed in terms of Jordan pairs.

We get a subalgebra of $\mathfrak{g}_{IV}$, that we denote here by $\mathfrak{g}_{III\frac{1}{2}}$, given by $\mathfrak{g}_{III}$ plus a $(6n+8)$-dimensional irreducible representation of $\mathfrak{g}_{III}$ plus a $\mathfrak{g}_{III}$-singlet, as shown in Fig. 4.[1]

In particular, the irreps. of $\mathfrak{g}_{III}$ are symplectic (*i.e.,*, they admit a skew-symmetric invariant form), and they have complex dimension $6n + 8 = 14, 20, 32, 56$ for $n = 1, 2, 4, 8$, respectively; the algebras $\mathfrak{g}_{III\frac{1}{2}}$ are their corresponding Heisenberg algebras (denoted by $\mathbf{H}$) through such an invariant tensor [9], $\mathbf{c}_{3\frac{1}{2}} = \mathbf{c}_3 \bullet \mathbf{H}_{14}$, $\mathbf{a}_{5\frac{1}{2}} = \mathbf{a}_5 \bullet \mathbf{H}_{20}$, $\mathbf{d}_{6\frac{1}{2}} = \mathbf{d}_6 \bullet \mathbf{H}_{32}$, $\mathbf{e}_{7\frac{1}{2}} = \mathbf{e}_7 \bullet \mathbf{H}_{56}$, of complex dimension 36, 56, 99, 190.

Let us here present a brief account of the *Jordan pairs* for sextonions $\mathbb{S}$ by means of suitable embeddings. We start with the maximal, non-symmetric embedding:

$$\mathbf{e}_7 \supset \mathbf{a}_2 \oplus \mathbf{a}_5 \tag{3.1}$$

$$\mathbf{133} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{35}) + (\mathbf{3}, \mathbf{15}) + \left(\overline{\mathbf{3}}, \overline{\mathbf{15}}\right) \tag{3.2}$$

$$\mathbf{56} = (\mathbf{3}, \mathbf{6}) + \left(\overline{\mathbf{3}}, \overline{\mathbf{6}}\right) + (\mathbf{1}, \mathbf{20}), \tag{3.3}$$

implying that:

[1] There are some variations on the definition of *intermediate* algebra [5,9], based on the grading induced by an highest root. Our realization of $Der(\mathbb{S})$ and $\mathbf{e}_{7\frac{1}{2}}$ corresponds to the algebra denoted by $\mathfrak{g}''$ in the Introduction of [9].

Table 2 Sixth column of the magic square

n	6
$\mathfrak{g}_{I}$	$\mathbf{c}_{3\frac{1}{2}}$
$\mathfrak{g}_{II}$	$\mathbf{a}_{5\frac{1}{2}}$
$\mathfrak{g}_{III}$	$\mathbf{d}_{6\frac{1}{2}}$
$\mathfrak{g}_{III\frac{1}{2}}$	$\mathbf{d}_{6\frac{1}{2}\frac{1}{2}}$
$\mathfrak{g}_{IV}$	$\mathbf{e}_{7\frac{1}{2}}$

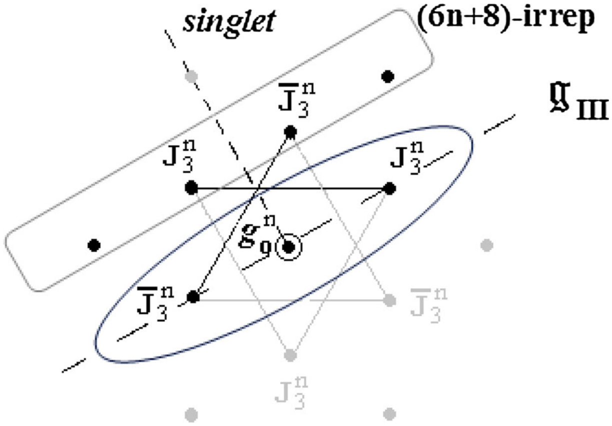


Fig. 4 Diagram of $\mathfrak{g}_{III\frac{1}{2}}$

$$\mathbf{e}_7 \ltimes \mathbf{56} \supset [\mathbf{a}_2 \oplus (\mathbf{a}_5 \ltimes \mathbf{20})] \ltimes (\mathbf{3}, \mathbf{15} + \mathbf{6}) + \left(\overline{\mathbf{3}}, \overline{\mathbf{15}} + \overline{\mathbf{6}}\right). \tag{3.4}$$

Thus, the *Jordan pairs* for the sextonionic Jordan algebra of rank 3, $\mathbf{J}_3^{\mathbf{n}=6}$, are given by $(\mathbf{3}, \mathbf{15} + \mathbf{6}) + \left(\overline{\mathbf{3}}, \overline{\mathbf{15}} + \overline{\mathbf{6}}\right)$ in (3.4).

In order to reconstruct the extended Magic Square [5,9], one needs also to add the extra column shown in Table 2 [9], where a further algebra $\mathbf{d}_{6\frac{1}{2}\frac{1}{2}} = \mathbf{d}_{6\bullet}\mathbf{H}_{32\bullet}\mathbf{H}_{44}$ is introduced.

This column corresponds to the Jordan algebra that we denote by $\mathbf{J}_\mathbf{3}^\mathbf{6}$ of 3×3 Hermitian matrices over the sextonions. The *new* element $\mathbf{d}_{6\bullet}\mathbf{H}_{32\bullet}\mathbf{H}_{44}$ can be easily seen in the diagram of Fig. 4 for $n = 6$: $\mathbf{g}_\mathbf{0}^\mathbf{6} = \mathbf{a}_{5\frac{1}{2}}$ is the reduced structure algebra of $\mathbf{J}_\mathbf{3}^\mathbf{6}$, $\mathfrak{g}_{III} = \mathbf{d}_{6\frac{1}{2}}$ the super-structure algebra of $\mathbf{J}_\mathbf{3}^\mathbf{6}$ and finally $\mathbf{d}_{6\frac{1}{2}\frac{1}{2}} = \mathbf{d}_{6\frac{1}{2}\bullet}\mathbf{H}_{44} = \mathbf{d}_{6\bullet}\mathbf{H}_{32\bullet}\mathbf{H}_{44}$. Notice that the 44-dimensional representation of $\mathbf{d}_{6\frac{1}{2}}$ is made of $\mathbf{J}_\mathbf{3}^\mathbf{6} \oplus \overline{\mathbf{J}_\mathbf{3}^\mathbf{6}} \oplus 2$. Finally, the algebra $\mathbf{e}_{7\frac{1}{2}}$ at the end of the column is viewed as in the diagram of Fig. 2 for $n = 6$, with $\mathbf{g}_\mathbf{0}^\mathbf{6} = \mathbf{a}_{5\frac{1}{2}}$ and the subalgebra $\mathbf{e_7}$ represented by the same diagram for $n = 4$.

This completes the explicit construction of the relevant rows and columns (pertaining to the sextonions) of the extended Magic Square[2] [5,9].

4 $\mathbf{g_2}$ action on Zorn matrices

In our previous paper [51], we have introduced the following adjoint representation ϱ of the Lie algebra $\mathbf{g_2}$:

$$\begin{bmatrix} a & A^+ \\ A^- & 0 \end{bmatrix} \tag{4.1}$$

where $a \in \mathbf{a_2}$, $A^+, \ A^- \in \mathbb{C}^3$, viewed as column and row vector, respectively. The commutator of two such matrices reads [51]:

$$\begin{array}{l} \left[\begin{bmatrix} a & A^+ \\ A^- & 0 \end{bmatrix}, \begin{bmatrix} b & B^+ \\ B^- & 0 \end{bmatrix}\right] \\ = \begin{bmatrix} [a,b] + A^+ \circ B^- - B^+ \circ A^- & aB^+ - bA^+ + 2A^- \wedge B^- \\ A^- b - B^- a + 2A^+ \wedge B^+ & 0 \end{bmatrix} \end{array} \tag{4.2}$$

where

$$A^+ \circ B^- = t(A^+B^-)I - t(I)A^+B^- \tag{4.3}$$

(with standard matrix products of row and column vectors and with I denoting the 3×3 identity matrix); $A \wedge B$ is the standard vector product of A and B, and $t(a)$ denotes the trace of a.

The $\mathbf{g_2}$ generators are [50]:

$$\begin{array}{l} \varrho(d_k^{\pm}) = E_{k\pm1\ k\pm2} \pmod 3, \quad k = 1,2,3 \\ \varrho(\sqrt{2}H_1) = E_{11} - E_{22} \qquad \varrho(\sqrt{6}H_2) = E_{11} + E_{22} - 2E_{33} \\ \varrho(g_k^+) = E_{k4} := \mathbf{e}_k^+ \quad \varrho(g_k^-) = E_{4k} := \mathbf{e}_k^- \quad , \ k = 1,2,3 \end{array} \tag{4.4}$$

where E_{ij} denotes the matrix with all zero elements except a 1 in the $\{ij\}$ position: $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$ and $\mathbf{e}_k^+$ are the standard basis vectors of $\mathbb{C}^3$ ($\mathbf{e}_k^-$ are their transpose). The correspondence with the roots of $\mathbf{g_2}$ is shown in Fig. 5.

5 Derivations of $\mathbb{S}$

We now use the representation ϱ to get a representation of the Lie algebra of $Der(\mathbb{S})$, which indeed is a non-reductive subalgebra of $\mathbf{g}_2 = Der(\mathfrak{C})$.

It was shown in [5] that the map from the subalgebra of derivations of $\mathfrak{C}$ preserving $\mathbb{S}$, that we here denote by $Der_{\mathfrak{C}}(\mathbb{S})$, to $Der(\mathbb{S})$ is surjective with one-dimensional kernel; the corresponding statement at the level of automorphism group was made in [9].

[2] It is once again worth stressing that in the present investigation, as well as in the previous papers [50,51], we only consider complex forms of the Lie algebras.

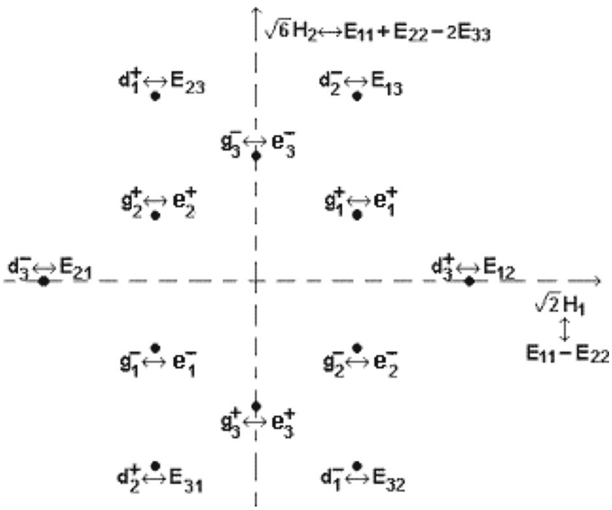


Fig. 5 Diagram of $\mathbf{g_2}$ with corresponding generators and *matrix-like* elements

Within our formalism, this result is achieved by restricting $\varrho(\mathbf{g_2})$ to the matrices that preserve $\mathbb{S}$. One easily gets:

$$\begin{bmatrix} a & S^+ \\ S^- & 0 \end{bmatrix} : a = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad S^+ = \begin{pmatrix} s_1^+ \\ s_2^+ \\ 0 \end{pmatrix}, \quad S^- = (s_1^-, 0, s_3^-) \quad (5.1)$$

We also realize very easily that the generator corresponding to d_1^+, namely the element E_{23} in $\varrho(\mathbf{g_2})$, acts trivially on $\mathbb{S}$, hence it can be set to 0. The commutator (4.2) must be modified accordingly, by setting the $\{23\}$ element of a equivalent to zero, that is by replacing the standard matrix product of two matrices

$$a = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} \quad (5.2)$$

with the new product

$$a \,\mathbf{.}\, b = ab - E_{22}\, ab\, E_{33} \quad (5.3)$$

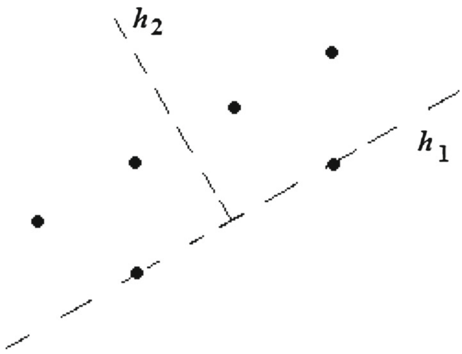
and the product $S^+ \circ S^-$ with

$$S^+ \underline{\circ}\, S^- = t(S^+S^-)I - t(I)(S^+S^- - E_{22}S^+S^-E_{33}) \quad (5.4)$$

We thus have $Der(\mathbb{S}) = \mathbf{a_1} \oplus \mathbb{C} \oplus V_4$, where V_4 is a 4-dimensional[3] (spin-3/2) irreducible representation of $\mathbf{a_1}$ (as confirmed by the entry in the first column, fourth

[3] This representation also characterizes $\mathbf{a}_1$ as the smallest Lie group "of type E_7" [56], and it pertains to the so-called T^3 model of $N = 2$, $D = 4$ supergravity.

Fig. 6 Root diagram of $Der(\mathbb{S})$



row in the extended Magic Square; *cfr.* e.g., [9]). The corresponding root diagram is shown in Fig. 6, where we have also included the axes corresponding the linear span of the Cartan generators, represented by the matrices:

$$\begin{bmatrix} h_{1,2} & 0 \\ 0 & 0 \end{bmatrix} : h_1 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (5.5)$$

Proposition 5.1 *The algebra spanned by the generators corresponding to the roots in Fig. 6 is a Lie algebra.*

Proof By looking at the diagram in Fig. 5, these generators are $d_2^-, d_3^-, g_2^+, g_3^-$ spanning a subspace L_1 of $\mathbf{g_2}$, plus the generators $g_1^{\pm}, h_1, h_2$ spanning the Lie subalgebra $L_0 := a_1 \oplus \mathbb{C}$. We have $[L_0, L_0] \subset L_0$, $[L_0, L_1] \subset L_1$, $[L_1, L_1] \subset L_2 \sim 0$, where L_2 is the span of d_1^+. The notation is that of the grading with respect to h_2.

We consider the $\mathbf{g_2}$ commutation relations among these generators and identify $d_1^+ \sim 0$. We only need to prove that the Jacobi identity is consistent with this identification. Let $X, Y, Z \in L_0 \oplus L_1$, then consistency must be checked in only two cases (up to cyclic permutation):

Case 1: $[X, Y] \propto d_1^+$;

Case 2: $[[X, Y], Z] \propto d_1^+$.

Case 1: Consistency requires $[[Y, Z], X] + [[Z, X], Y] \sim 0$. This is true if $[d_1^+, Z] = 0$, since it is true in $\mathbf{g_2}$. On the other hand, if $[d_1^+, Z] \neq 0$ then $Z \propto h_2$ and $[Z, X] = \lambda X$, $[Z, Y] = \lambda Y$,since X, Y must be in L_1 by hypothesis. Therefore, $[[Y, Z], X] + [[Z, X], Y] = 2\lambda[X, Y] \propto d_1^+ \sim 0$.

Case 2: Both $[X, Y]$ and Z must be in L_1. In particular either X or Y must be in L_1. Suppose $X \in L_1$. Then, $[Y, Z] \in L_1$ hence we have both $[X, Z] \sim 0$ and $[[Y, Z], X] \sim 0$. Similarly if $Y \in L_1$.

This concludes the proof □

6 n = 1: Matrix representation of $\mathbf{c}_{3\frac{1}{2}}$

We denote by a dot the Jordan product $x \cdot y = \frac{1}{2}(xy + yx)$ and by $t()$ the ordinary trace of 3×3 matrices. We also set $t(x, y) := t(x \cdot y)$. For $\mathbf{J}_3^1$ and $\mathbf{J}_3^2$, obviously $t(x, y) = t(xy)$.

We use in this section the representation ϱ of $\mathbf{f_4}$ in the form of a matrix introduced in [51], restricted to the subalgebra $\mathbf{c}_{3\frac{1}{2}}$:

$$\varrho(\mathfrak{f}) = \begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{s}^+ \\ \mathbf{s}^- & -I \otimes a_1^T \end{pmatrix} \tag{6.1}$$

where

$$a = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \quad t(a) = 0, \quad \mathbf{s}^+ = \begin{pmatrix} s_1^+ \\ s_2^+ \\ 0 \end{pmatrix}, \quad \mathbf{s}^- = (s_1^-, 0, s_3^-) \tag{6.2}$$

and $a_1 \in \mathbf{a_2}$, a_1^T is the transpose of a_1, I is the 3×3 identity matrix, $s_i^\pm \in \mathbf{J}_3^1$, $i = 1, 2, 3$.

The commutator is set to be:

$$\begin{aligned} &\left[\begin{pmatrix} a \otimes I + I \otimes a_1 & \mathbf{s}^+ \\ \mathbf{s}^- & -I \otimes a_1^T \end{pmatrix}, \begin{pmatrix} b \otimes I + I \otimes b_1 & \mathbf{r}^+ \\ \mathbf{r}^- & -I \otimes b_1^T \end{pmatrix}\right] \\ &\quad := \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{aligned} \tag{6.3}$$

where denoting by $[a \centerdot b]$ the commutator with respect to the product (5.3)

$$[a \centerdot b] = a \centerdot b - b \centerdot a = [a, b] - E_{22}[a, b]E_{33}, \tag{6.4}$$

it holds that:

$$\begin{aligned} C_{11} &= [a \centerdot b] \otimes I + I \otimes [a_1, b_1] + \mathbf{s}^+ \diamond \mathbf{r}^- - \mathbf{r}^+ \diamond \mathbf{s}^- \\ C_{12} &= (a \otimes I)\mathbf{r}^+ - (b \otimes I)\mathbf{s}^+ + (I \otimes a_1)\mathbf{r}^+ + \mathbf{r}^+(I \otimes a_1^T) \\ &\quad - (I \otimes b_1)\mathbf{s}^+ - \mathbf{s}^+(I \otimes b_1^T) + \mathbf{s}^- \times \mathbf{r}^- \\ C_{21} &= -\mathbf{r}^-(a \otimes I) + \mathbf{s}^-(b \otimes I) - (I \otimes a_1^T)\mathbf{r}^- - \mathbf{r}^-(I \otimes a_1) \\ &\quad + (I \otimes b_1^T)\mathbf{s}^- + \mathbf{s}^-(I \otimes b_1) + \mathbf{s}^+ \times \mathbf{r}^+ \\ C_{22} &= I \otimes [a_1^T, b_1^T] + \mathbf{s}^- \bullet \mathbf{r}^+ - \mathbf{r}^- \bullet \mathbf{s}^+ \end{aligned} \tag{6.5}$$

with the following definitions (summing over repeated indices):

$$
\begin{aligned}
\mathbf{s}^+ \diamond \mathbf{r}^- \quad &:= \left(\tfrac{1}{3}t(s_1^+, r_1^-)I - (1-(E_{23})_{ij})t(s_i^+, r_j^-)E_{ij}\right) \otimes I \\
&\quad + I \otimes \left(\tfrac{1}{3}t(s_1^+, r_1^-)I - s_1^+ r_1^-\right) \\
\mathbf{s}^- \bullet \mathbf{r}^+ \quad &:= I \otimes (\tfrac{1}{3}t(s_1^-, r_1^+)I - s_1^- r_1^+) \\
(\mathbf{s}^\pm \times \mathbf{r}^\pm)_i &:= \epsilon_{ijk}[s_j^\pm r_k^\pm + r_k^\pm s_j^\pm - s_j^\pm t(r_k^\pm) - r_k^\pm t(s_j^\pm) \\
&\quad -(t(s_j^\pm, r_k^\pm) - t(s_j^\pm)t(r_k^\pm))I] \\
&:= \epsilon_{ijk}(s_j^\pm \# r_k^\pm)
\end{aligned}
\tag{6.6}
$$

Notice that:

1. $s \in \mathbf{J_3^1}$ is a symmetric complex matrix;
2. writing $\mathbf{s}^+ \diamond \mathbf{r}^- := c \otimes I + I \otimes c_1$ we have that both c and c_1 are traceless hence c ia a matrix like a in (5.1), $c_1 \in \mathbf{a_2}$ and $\mathbf{r}^- \bullet \mathbf{s}^+ = I \otimes c_1^T$
3. terms like $(I \otimes a_1)\mathbf{r}^+ + \mathbf{r}^+(I \otimes a_1^T)$ are in $\mathbb{C}^3 \otimes \mathbf{J_3^1}$, namely they are matrix valued vectors with symmetric matrix elements;
4. the *sharp* product # of $\mathbf{J_3^1}$ matrices appearing in $\mathbf{s}^\pm \times \mathbf{r}^\pm$ is a fundamental product in the theory of Jordan Algebras [53]. It is the linearization of $x^\# := x^2 - t(x)x - \frac{1}{2}(t(x^2) - t(x)^2)I$, in terms of which we may write the fundamental cubic identity for $\mathbf{J_3^n}$, $n = 1, 2, 4, 8$:

$$
x^\# \cdot x = \frac{1}{3}t(x^\#, x)I \quad \text{or} \quad x^3 - t(x)x^2 + t(x^\#)x - \frac{1}{3}t(x^\#, x)I = 0 \tag{6.7}
$$

where $x^3 = x^2 \cdot x$ (notice that for $\mathbf{J_3^8}$, because of non-associativity, $x^2 x \neq x x^2$ in general).

The validity of the Jacobi identity for the algebra of matrices (6.1) with Lie product given by (6.3–6.6) derives from the Jacobi identity for $\rho(\mathbf{f_4})$ proven in [51] together with Proposition 5.1, applied to $\mathbf{c_{3\frac{1}{2}}}$ by trivially extending the three grading argument. The validity of the Jacobi identity, together with the fact that the representation ϱ fulfills the root diagram of $\mathbf{c_{3\frac{1}{2}}}$ (as can be easily seen) proves that ϱ is indeed a representation of $\mathbf{c_{3\frac{1}{2}}}$.

Before passing to $\mathbf{e_{7\frac{1}{2}}}$, let us point out that the cases of $\mathbf{a_{5\frac{1}{2}}}$ ($\mathbf{n} = 2$) and $\mathbf{d_{6\frac{1}{2}}}$ ($\mathbf{n} = 4$) can be worked out in the same fashion as for $\mathbf{c_{3\frac{1}{2}}}$, starting from the representations of $\mathbf{e_6}$ and $\mathbf{e_7}$ introduced in [51].

7 n = 8: Matrix representation of $\mathbf{e_{7\frac{1}{2}}}$

We recall a few concepts and notations from [51]. We use the notation $L_x z := x \cdot z$ and, for $\mathbf{x} \in \mathbb{C}^3 \otimes \mathbf{J_3^8}$ with components (x_1, x_2, x_3), $L_\mathbf{x} \in \mathbb{C}^3 \otimes L_{\mathbf{J_3^8}}$ denotes the corresponding operator valued vector with components $(L_{x_1}, L_{x_2}, L_{x_3})$. We can write an element a_1 of $\mathbf{e_6}$ as $a_1 = L_x + \sum[L_{x_i}, L_{y_i}]$ where $x, x_i, y_i \in \mathbf{J_3^8}$ and $t(x) = 0$. The adjoint is defined by $a_1^\dagger := L_x - [L_{x_1}, L_{x_2}]$. Notice that the operators $F := [L_{x_i}, L_{y_i}]$

span the $\mathbf{f_4}$ subalgebra of $\mathbf{e_6}$, the derivation algebra of $\mathbf{J_3^8}$. (Recall that the Lie algebra of the structure group of $\mathbf{J_3^8}$ is $\mathbf{e_6} \oplus \mathbb{C}$.)

We remark that $(a_1, -a_1^\dagger)$ is a derivation in the Jordan Pair $(\mathbf{J_3^8}, \overline{\mathbf{J}}_\mathbf{3}^\mathbf{8})$, and it is useful to recall that the relationship between the structure group of a Jordan algebra J and the automorphism group of a Jordan Pair $V = (J, J)$ goes as follows [57]: if $g \in Str(J)$ then $(g, U_{g(I)}^{-1} g) \in Aut(V)$. In our case, for $g = 1 + \epsilon(L_x + F)$, at first order in ϵ (namely, in the tangent space of the corresponding group manifold) we get $U_{g(I)}^{-1} g = 1 + \epsilon(-L_x + F) + O(\epsilon^2)$.

Next, we introduce a product $\star$ such that $L_x \star L_y := L_{x \cdot y} + [L_x, L_y]$, $F \star L_x := 2FL_x$ and $L_x \star F := 2L_x F$ for each component x of $\mathbf{x} \in \mathbb{C}^3 \otimes \mathbf{J_3^8}$ and y of $\mathbf{y} \in \mathbb{C}^3 \otimes \mathbf{J_3^8}$. If we denote by [;] the commutator with respect to the $\star$ product, we also require that $[F_1; F_2] := 2[F_1, F_2]$. We have that, $L_x \star L_y + L_y \star L_x = 2L_{x \cdot y}$ and $[F; L_x] := F \star L_x - L_x \star F = 2[F, L_x] = 2L_{F(x)}$, where the last equality holds because F is a derivation in $\mathbf{J_3^8}$.

Therefore, for $\mathfrak{f} \in \mathbf{e}_{7\frac{1}{2}}$, we write:

$$\varrho(\mathfrak{f}) = \begin{pmatrix} a \otimes Id + I \otimes a_1 & L_{\mathbf{s}^+} \\ L_{\mathbf{s}^-} & -I \otimes a_1^\dagger \end{pmatrix} \tag{7.1}$$

where a, $\mathbf{s}^\pm$ are the same as in (5.1), $a_1 \in \mathbf{e_6}$, I is the 3×3 identity matrix, $Id := L_I$ is the identity operator in $L_{\mathbf{J_3^8}}$: $L_I L_x = L_x$. Notice that Id is the identity also with respect to the $\star$ product.

By extending the $\star$ product in an obvious way to the matrix elements (7.1), one achieves that $(I \otimes a_1) \star L_{\mathbf{r}^+} + L_{\mathbf{r}^+} \star (I \otimes a_1^\dagger) = 2L_{(I \otimes a_1)\mathbf{r}^+}$ and $(I \otimes a_1^\dagger) \star L_{\mathbf{r}^-} + L_{\mathbf{r}^-} \star (I \otimes a_1) = 2L_{(I \otimes a_1^\dagger)\mathbf{r}^-}$.

After some algebra, the commutator of two matrices like (7.1) can be computed to read:

$$\begin{aligned} &\left[\begin{pmatrix} a \otimes Id + I \otimes a_1 & L_{\mathbf{s}^+} \\ L_{\mathbf{s}^-} & -I \otimes a_1^\dagger \end{pmatrix}, \begin{pmatrix} b \otimes Id + I \otimes b_1 & L_{\mathbf{r}^+} \\ L_{\mathbf{r}^-} & -I \otimes b_1^\dagger \end{pmatrix} \right] \\ &:= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \end{aligned} \tag{7.2}$$

where:

$$\begin{aligned} C_{11} &= [a \centerdot b] \otimes Id + 2I \otimes [a_1, b_1] + L_{\mathbf{s}^+} \diamond L_{\mathbf{r}^-} - L_{\mathbf{r}^+} \diamond L_{\mathbf{s}^-} \\ C_{12} &= (a \otimes Id) L_{\mathbf{r}^+} - (b \otimes Id) L_{\mathbf{s}^+} + 2L_{(I \otimes a_1)\mathbf{r}^+} \\ &\quad - 2L_{(I \otimes b_1)\mathbf{s}^+} + L_{\mathbf{s}^-} \times L_{\mathbf{r}^-} \\ C_{21} &= -L_{\mathbf{r}^-}(a \otimes Id) + L_{\mathbf{s}^-}(b \otimes Id) - 2L_{(I \otimes a_1^\dagger)\mathbf{r}^-} \\ &\quad + 2L_{(I \otimes b_1^\dagger)\mathbf{s}^-} + L_{\mathbf{s}^+} \times L_{\mathbf{r}^+} \\ C_{22} &= 2I \otimes [a_1^\dagger, b_1^\dagger] + L_{\mathbf{s}^-} \bullet L_{\mathbf{r}^+} - L_{\mathbf{r}^-} \bullet L_{\mathbf{s}^+}. \end{aligned} \tag{7.3}$$

The products in (7.3) are defined as follows:

$$
\begin{aligned}
L_{\mathbf{s}^+} \diamond L_{\mathbf{r}^-} &:= \Big(\tfrac{1}{3}t(s_1^+, r_1^-)I - (1-(E_{23})_{ij})t(s_i^+, r_j^-)E_{ij}\Big) \otimes Id + \\
&\quad I \otimes \Big(\tfrac{1}{3}t(s_1^+, r_1^-)Id - L_{s_1^+ \cdot r_1^-} - [L_{s_1^+}, L_{r_1^-}]\Big) \\
L_{\mathbf{s}^-} \bullet L_{\mathbf{r}^+} &:= I \otimes (\tfrac{1}{3}t(s_1^-, r_1^+)Id - L_{s_1^- \cdot r_1^+} - [L_{s_1^-}, L_{r_1^+}]) \\
L_{\mathbf{s}^\pm} \times L_{\mathbf{r}^\pm} &:= L_{\mathbf{s}^\pm \times \mathbf{r}^\pm} = L_{\epsilon_{ijk}(s_j^\pm \# r_k^\pm)}
\end{aligned}
\tag{7.4}
$$

From the properties of the triple product of Jordan algebras, it holds that $L_{s_1^+ \cdot r_1^-} + [L_{s_1^+}, L_{r_1^-}] = \frac{1}{2}V_{s_1^+, r_1^-} \in \mathbf{e_6} \oplus \mathbb{C}$ [51]. Moreover, one can readily check that $[a_1^\dagger, b_1^\dagger] = -[a_1, b_1]^\dagger$ and $L_{\mathbf{r}^-} \bullet L_{\mathbf{s}^+} = I \otimes (\frac{1}{3}t(s_1^+, r_1^-)Id - L_{s_1^+ \cdot r_1^-} - [L_{s_1^+}, L_{r_1^-}])^\dagger$; this result implies that we are actually considering an algebra.

The validity of the Jacobi identity for the algebra of matrices (7.1) with Lie product given by (7.2–7.4) derives from the Jacobi identity[4] for $\rho(\mathbf{e_8})$ (proven in [51]), together with Proposition 5.1, applied to $\mathbf{e}_{\mathbf{7\frac{1}{2}}}$ by trivially extending the three grading argument. That the Lie algebra so represented is $\mathbf{e}_{\mathbf{7\frac{1}{2}}}$ is made obvious by a comparison with the root diagram in Fig. 3.

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A Real forms

We use the notations of [51]. From the treatment in [51], a real form of octonions is obtained by taking $\alpha_0^\pm, \alpha_k^\pm \in \mathbf{R}$. The quaternionic subalgebra generated by $\rho^\pm, \varepsilon_1^\pm$ is obviously a split form with nilpotent $\varepsilon_1^\pm$.

Another real form is obtained by taking complex coefficients with complex conjugation denoted by '$*$' subject to the conditions:

$$
\alpha_0^- = (\alpha_0^+)^*, \quad \alpha_1^- = -(\alpha_1^+)^*, \quad \alpha_3^- = (\alpha_2^+)^* \tag{A.1}
$$

Its quaternionic subalgebra, generated by $1, u_7, iu_1, iu_4$, is also split with nilpotent $u_7 + iu_k$, $k = 1, 4$. It is equivalent to the one obtained with all real coefficients, which is generated by $1, u_1, iu_4, iu_7$ upon cyclic permutation of the indices 7, 4, 1.

Let us now restrict the Zorn matrix product [51] to the sextonions and introduce the vectors

$$
E_1 = (1,0,0) \quad E_2 = (0,1,0) \quad E_3 = (0,0,1)
$$

4 We would like to recall that the proof of the Jacobi identity given in [51] strongly relies on identities deriving from the Jordan Pair axioms [57].

We get:

$$\begin{bmatrix} \alpha_0^+ & A^+ \\ A^- & \alpha_0^- \end{bmatrix} \begin{bmatrix} \beta_0^+ & B^+ \\ B^- & \beta_0^- \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0^+\beta_0^+ - \alpha_1^+\beta_1^- & (\alpha_0^+\beta_1^+ + \beta_0^-\alpha_1^+)E_1^+ \\ (\alpha_0^-\beta_1^- + \beta_0^+\alpha_1^-)E_1^- & \alpha_0^-\beta_0^- - \alpha_1^- \cdot \beta_1^+ \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & (\alpha_0^+\beta_2^+ + \beta_0^-\alpha_2^+ + \alpha_3^-\beta_1^- - \alpha_1^-\beta_3^-)E_2^+ \\ (\alpha_0^-\beta_3^- + \beta_0^+\alpha_3^- + \alpha_1^+\beta_2^+ - \alpha_2^+\beta_1^+)E_3^- & 0 \end{bmatrix} \tag{A.2}$$

The algebra generated by $\rho^\pm, \varepsilon_1^\pm$ is the quaternionic subalgebra. Its divisible real form is obtained by setting: $\alpha_0^- = (\alpha_0^+)^*$ and $\alpha_1^- = (\alpha_1^+)^*$ and it is a real linear span of $1, u_7, u_4, u_1$.

We now show that it is impossible to have e sextonion real algebra that has divisible quaternions as a subalgebra. To this aim, we suppose $\alpha_0^- = (\alpha_0^+)^*$ and $\alpha_1^- = (\alpha_1^+)^*$ and take in (A.2) $\alpha_0^+ = \beta_0^+ = \beta_1^+ = 0$ - which implies $\alpha_0^- = \beta_0^- = \beta_1^- = 0$. The product (A.2) shows that the coefficients of ε_2^+ and ε_3^- must be complex, hence each coefficient, say α_2^+ contains 2 real parameters a and b, and, in order to have a six-dimensional real algebra, α_3^- viewed in $\mathbf{R}_2$ must be a linear transformation T of (a, b), linearity being enforced by the linearity of the algebra.

We loosely write $\alpha_2^+ = T\alpha_3^-$. It is easy to show that $T^2 = Id$, namely T is an involution. By playing with the coefficients in (A.2), we can easily obtain $\alpha_2^+ = -T^2\alpha_2^+ = -\alpha_2^+$ and similarly for α_3^-, a contradiction unless $\alpha_2^+ = \alpha_3^- = 0$.

This ends our proof.

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