Lett Math Phys (2017) 107:1581–1589
DOI 10.1007/s11005-017-0956-9

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The prequantum line bundle on the moduli space of flat $SU(N)$ connections on a Riemann surface and the homotopy of the large N limit

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Received: 21 November 2015 / Revised: 12 December 2016 / Accepted: 7 March 2017 /
Published online: 20 March 2017

Abstract We show that the prequantum line bundle on the moduli space of flat $SU(2)$ connections on a closed Riemann surface of positive genus has degree 1. It then follows from work of Lawton and the second author that the classifying map for this line bundle induces a homotopy equivalence between the stable moduli space of flat $SU(n)$ connections, in the limit as n tends to infinity, and $\mathbb{C}P^\infty$. Applications to the stable moduli space of flat unitary connections are also discussed.

Keywords Prequantum line bundle · Moduli space of flat connections

Mathematics Subject Classification 53C05 · 32L05 · 58J28

Lisa C. Jeffrey was partially supported by a grant from NSERC.
Daniel A. Ramras was partially supported by a grant from the Simons Foundation (#279007).
Jonathan Weitsman was partially supported by NSF Grant DMS-12/11819.

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1 Introduction

Let G be a simply connected compact Lie group, and let Σ be a closed oriented 2-manifold of genus $g > 0$. In [7] Ramadas, Singer and Weitsman construct a line bundle $\mathcal{L}$ over the moduli space of gauge equivalence classes of flat connections $\mathcal{A}_F(\Sigma)/\mathcal{G}$ on a trivial G-bundle on Σ. This bundle possesses a natural connection, whose curvature is a scalar multiple of Goldman's symplectic form.

The purpose of this paper is to compute the degree (that is, the first Chern class) of the line bundle described in [7] in the case $G = SU(2)$. Our main theorem is

Theorem 1.1 *The degree of the line bundle is 1.*

As we will explain, the second integral cohomology group of the moduli space is infinite cyclic, and the theorem implies that the first Chern class of $\mathcal{L}$ is a generator. There is in fact a preferred generator (depending on the orientation of Σ), which agrees with $c_1(\mathcal{L})$.

In view of Question 5.6 of [5], Theorem 1.1 has the following corollary:

Corollary 1.2 *Let Σ be a closed oriented 2-manifold of genus $g > 0$. Let $\mathcal{G}_n$ be the gauge group of the trivial $SU(n)$-bundle on Σ, and let $\mathcal{A}_F^{SU(n)}(\Sigma)$ denote the space of flat connections on this bundle. The classifying maps for the line bundles*

$$\mathcal{L} \to \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n$$

induce a homotopy equivalence $\operatorname{colim}_{n\to\infty} \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n \simeq \mathbb{C}P^\infty$.

It was previously shown in [5] that the stable moduli space

$$\operatorname{colim}_{n\to\infty} \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_n \cong \operatorname{colim}_{n\to\infty} \operatorname{Hom}(\pi_1\Sigma, SU(n))/SU(n)$$

is a $K(\mathbb{Z}, 2)$ space and hence is homotopy equivalent to $\mathbb{C}P^\infty$. This corollary gives a geometric viewpoint on this homotopy equivalence. In Sect. 4, we also obtain a geometric viewpoint on the homotopy equivalence $\operatorname{colim}_{n\to\infty} \operatorname{Hom}(\pi_1\Sigma, U(n))/U(n) \simeq (S^1)^{2g} \times \mathbb{C}P^\infty$ from [8].

Our computation of $c_1(\mathcal{L})$ in genus 1 (Sect. 3) is similar to Kirk–Klassen [4, Theorem 2.1].[1] For related work in the algebraic category, see Drezet–Narasimhan [3].

We remark that it would be interesting to extend this degree calculation to other simply connected compact Lie groups.

2 The Chern–Simons line bundle

Let G be a simply connected, compact Lie group, equipped with a chosen faithful representation into $\mathrm{GL}(n, \mathbb{C})$, and let $\mathfrak{g}$ be the Lie algebra of G (viewed as a subalgebra

[1] Kirk and Klassen conclude that $c_1(L) = -1$. The discrepancy can be explained using the footnote regarding signs in Sect. 3 of the present article.

of $\mathfrak{gl}(n, \mathbb{C})$). The space of connections on the trivial G-bundle over Σ will be denoted by $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$, and the gauge group of this bundle will be denoted by $\mathcal{G} = C^\infty(\Sigma, G)$.

The line bundle from [7] is defined using the Chern–Simons cocycle ([7], p. 411) $\Theta : \mathcal{A} \times \mathcal{G} \to \mathbb{C}$ defined by

$$\Theta(A, g) = \exp i(CS(\mathbf{A}^{\mathbf{g}}) - CS(\mathbf{A})).$$

The Chern–Simons functional $CS(\mathbf{A})$ is defined by

$$CS(\mathbf{A}) = \frac{1}{4\pi} \int_N \mathrm{Trace}(\mathbf{A} d\mathbf{A} + \frac{2}{3}\mathbf{A}^3)$$

where N is a 3-manifold with boundary Σ and $g \in \mathcal{G} = C^\infty(\Sigma, G)$. We have chosen extensions $\mathbf{A}$ and $\mathbf{g}$ of A and g (respectively) over the bounding 3-manifold N (the existence of $\mathbf{g}$ relies on simple connectivity of G). It is shown in [7] that the Chern–Simons cocycle $\Theta(A, g)$ is independent of the choice of these extensions. We define a line bundle $\mathcal{L}$ over $\mathcal{A}_{\mathcal{F}}/\mathcal{G}$ as a $\mathcal{G}$-equivariant bundle over the space of flat connections $\mathcal{A}_F$, where $g \in \mathcal{G}$ acts on $\mathcal{A} \times \mathbb{C}$ by

$$g : (A, z) \mapsto (A^g, \Theta(A, g)z).$$

The definition of $\mathcal{L}$ is

$$\mathcal{L} = \mathcal{A}_F \times_{\mathcal{G}} \mathbb{C}.$$

The symplectic form $\hat{\Omega}$ on $\mathcal{A}$ is defined by (see [7], p. 412):

$$\hat{\Omega}(a, b) = \frac{i}{2\pi} \int_\Sigma \mathrm{Trace}(a \wedge b) \tag{1}$$

for $a, b \in \Omega^1(\Sigma, \mathfrak{g})$. Notice that on the affine space $\mathcal{A}$, the symplectic form is a constant quadratic form; it does not depend on choosing a point in $\mathcal{A}$.

3 Degree of the Chern–Simons line bundle in genus 1

Let N be a 3-manifold with boundary Σ.

The symplectic form on $\mathcal{A}$ from (1) descends to a 2-form Ω on $\mathcal{A}_F/\mathcal{G}$ (the space of flat connections), which is symplectic when restricted to the subspace $\mathcal{A}_F^s \subset \mathcal{A}_F$ of irreducible flat connections.

The authors of [7] exhibit a unitary connection $\hat{\omega}$ on the prequantum line bundle over $\mathcal{A}_F$:

$$\hat{\omega}(a) = \frac{i}{4\pi} \int_\Sigma \mathrm{Trace}(A \wedge a) \tag{2}$$

whose curvature is $\hat{\Omega}$. This is done on p. 412 of [7]. The proof uses the fact that the derivative of the Chern–Simons function is

$$dCS_A(v) = \frac{1}{4\pi}\left(\int_N 2\mathrm{Trace}(v \wedge F_A) - \int_\Sigma \mathrm{Trace}(A \wedge v)\right)$$

for $v \in T_A\mathcal{A} = \mathcal{A} = \Omega^1(N, \mathfrak{g})$. This follows from a straightforward calculation using Stokes' theorem. The above expression restricts on $\mathcal{A}_F$ to

$$dCS_A(v) = -\frac{1}{4\pi}\int_\Sigma \mathrm{Trace}(A \wedge v) = i\hat{\omega}(v)$$

(recalling (2)). It is shown on p. 412 of [7] (second paragraph) that $\hat{\omega}$ is the pullback of a connection ω on $\mathcal{A}_F \times_{\mathcal{G}} \mathbb{C}$. This is demonstrated by introducing a vertical vector field Y for the action of $\mathcal{G}$ and showing that

$$i_Y\hat{\omega} = L_Y\hat{\omega} = 0$$

so $\hat{\omega}$ is basic and therefore descends to a 1-form on $\mathcal{A}_F/\mathcal{G}$.

For the rest of the section, we restrict to $G = SU(2)$. Let x, y be the flat coordinates on the genus 1 surface (see the proof of Lemma 3.1 for more details). Inside the space $\mathcal{A}$ we can consider the space $\mathcal{W}$ of all connections of the form $a\,\mathrm{d}x + b\,\mathrm{d}y$ where $a, b \in \mathrm{Lie}(T)$ and $T = \left\{\mathrm{diag}(\lambda, \lambda^{-1}) : \lambda \in S^1\right\}$ is the diagonal maximal torus of $SU(2)$. Note that $\mathrm{Lie}(T) = \{xX : x \in \mathbb{R}\}$, where $X = \mathrm{diag}(i, -i) \in su(2)$.

Now $\mathcal{W}$ is a subspace of $\mathcal{A}$ so the bundle $\mathcal{L}$ restricts to $\mathcal{W}$ as a bundle with connection. This bundle is invariant under that part of the gauge group that preserves $\mathcal{W}$. This consists of $(\mathbb{Z} \times \mathbb{Z}) \ltimes \mathbb{Z}_2$. Here $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ is identified with the gauge transformation $(e^{ix}, e^{iy}) \mapsto e^{imx}e^{iny}$, and $\mathbb{Z}_2 = \{\pm 1\}$ is the Weyl group of $SU(2)$.

Taking the quotient by $\mathbb{Z} \times \mathbb{Z}$ we get a bundle $\mathcal{L}'$ on $T \times T = \mathcal{W}/(\mathbb{Z} \times \mathbb{Z})$ with a connection ω'. We will show, via a direct computation (Lemma 3.1), that the curvature Ω' of this connection has integral equal to $-4\pi i$, and using Chern–Weil theory we will be able to conclude that $\mathcal{L}'$ has degree 2, while $\mathcal{L}$ has degree 1.

The computation goes as follows.

Lemma 3.1 *We have*

$$\int_{T\times T} \Omega' = -4\pi i.$$

Proof As above, let $X = \mathrm{diag}(i, -i) \in su(2)$. Then $\mathrm{Trace}(X^2) = -2$.

Let Γ be a fundamental domain for the action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathrm{Lie}(T) \oplus \mathrm{Lie}(T)$. Parameterize Γ by $(x, y) \in [0, 2\pi] \times [0, 2\pi]$. Under the exponential map

$$\exp : \mathrm{Lie}(T) \to T,$$

the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ on Γ are identified with the constant vector field X on T, so with the above formula (1) for $\hat{\Omega}$ we have that

$$\begin{aligned}\Omega'_{(x,y)} &:= \Omega'_{(x,y)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \hat{\Omega}_{(x,y)}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{i}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mathrm{Trace}(X^2)\mathrm{d}x\mathrm{d}y \\ &= -4\pi i.\end{aligned}$$

The value of $\Omega'_{(x,y)}$ is independent of x and y, so the integral of Ω' over $T \times T$ (integrating using an area form of total area 1) is also $-4\pi i$. □

Proof of Theorem 1.1 in genus 1. By Lemma 3.1, the cohomology class $[\Omega'] \in H^2(T \times T; \mathbb{R})$ associated with Ω' is $-4\pi i\alpha$, where

$$\alpha = \frac{1}{(2\pi)^2}[\mathrm{d}x \wedge \mathrm{d}y] \in H^2(T \times T; \mathbb{R})$$

denotes the fundamental class. By Chern–Weil theory,[2] we have

$$[\Omega'] = -2\pi i\, c_1(\mathcal{L}'),$$

so $c_1(\mathcal{L}') = \frac{2}{(2\pi)^2}[\mathrm{d}x \wedge \mathrm{d}y] = 2\alpha$, and $\mathcal{L}'$ has degree two.

The generator of $W = \mathbb{Z}/2\mathbb{Z}$ acts on $T \times T$ by complex conjugation on each factor, inducing a quotient map $f\colon\ T \times T \to (T \times T)/W$, and we have a homeomorphism $(T \times T)/W \cong S^2$. Our bundle $\mathcal{L}'$ is $\mathbb{Z}_2$-equivariant, and it descends to the bundle $\mathcal{L}$ on $(T \times T)/W$ (note here that if $(z, w) \in T \times T$ is fixed by W, then the action of W on the fiber of $\mathcal{L}'$ over (z, w) is *trivial*: this action is defined in terms of the cocycle $\Theta(A, g)$, which is zero whenever g fixes A). So $f^*(\mathcal{L}) = \mathcal{L}'$, and hence, $\deg(\mathcal{L}) = \frac{1}{\deg(f)}\deg(\mathcal{L}')$. An elementary calculation (e.g., using a $\mathbb{Z}_2$-equivariant CW complex structure on $T \times T$) shows that $\deg(f) = 2$, completing the proof. □

The key point is that we have computed the degree on $T \times T$, which has a canonical smooth manifold structure, so we can use Chern–Weil theory. The proof of Theorem 1.1 in higher genus is given in Sect. 5.

4 The conjecture of Lawton and Ramras on the Chern–Simons line bundle

Let Σ be a closed oriented 2-manifold of genus $g > 0$. Let $\mathcal{G}_{SU(n)} = \mathcal{G}_{SU(n)}(\Sigma)$ and $\mathcal{G}_{U(n)} = \mathcal{G}_{U(n)}(\Sigma)$ denote the gauge groups of the trivial $SU(n)$ and $U(n)$-bundles on Σ (respectively), and let $\mathcal{A}_F^{SU(n)}(\Sigma)$ and $\mathcal{A}_F^{U(n)}(\Sigma)$ denote the spaces of flat $SU(n)$- and $U(n)$-connections on these bundles. Define

$$\mathcal{M}_U(\Sigma) = \operatorname{colim}_n \mathcal{A}_F^{U(n)}(\Sigma)/\mathcal{G}_{U(n)} \text{ and } \mathcal{M}_{SU}(\Sigma) = \operatorname{colim}_n \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_{SU(n)}.$$

We refer to these as the *stable moduli spaces* of flat unitary (or special unitary) connections over Σ. Let $\mathcal{L}_n \to \mathcal{A}_F^{SU(n)}(\Sigma)/\mathcal{G}_{SU(n)}$ denote the prequantum line bundle. As n varies, these bundles are compatible with the inclusions $SU(n) \hookrightarrow SU(n+1)$ and hence induce a line bundle $\mathcal{L}_\infty \to \mathcal{M}_{SU}(\Sigma)$, which we call the stable prequantum line bundle.

[2] In [6, Appendix C], the Chern–Weil formula for characteristic classes is stated without signs, because they use a version of Fubini's theorem with signs [6, p. 304]. Since we have integrated using the usual version of Fubini's theorem, we need a sign in our formula for $c_1(\mathcal{L}')$.

The homotopy types of the stable moduli spaces were determined in [5,8]:

$$\mathcal{M}_U(\Sigma) \simeq \mathbb{C}P^\infty \times (S^1)^{2g}, \quad \mathcal{M}_{SU}(\Sigma) \simeq \mathbb{C}P^\infty. \tag{3}$$

These results are computational and rely on the uniqueness of Eilenberg–MacLane spaces; that is, no explicit homotopy equivalences between these spaces have been constructed. Here we offer bundle-theoretic descriptions of these homotopy equivalences.

Remark 4.1 The proof that $\mathcal{M}_U(\Sigma)$ and $\mathcal{M}_{SU}(\Sigma)$ have the homotopy types stated above relies on an independent result showing that these spaces have the homotopy types of CW complexes. In the unitary case, this is proven in [8, Lemma 5.7], and the same argument works in the special unitary case.

Theorem 4.2 *The classifying map*

$$\mathcal{M}_{SU}(\Sigma) \longrightarrow \mathbb{C}P^\infty$$

for the stable prequantum line bundle $\mathcal{L}_\infty$ is a homotopy equivalence.

Proof Writing $\Sigma = \Sigma' \# T$, where T is a torus, the quotient map induces a homeomorphism $\Sigma \to \Sigma/\Sigma' \cong T$. Together with the inclusions $SU(2) \hookrightarrow SU(n)$, this induces a map

$$\mathcal{A}_F^{SU(2)}(T)/\mathcal{G}_{SU(2)}(T) \longrightarrow \mathcal{M}_{SU}(\Sigma),$$

which induces an isomorphism on $H^2(-;\mathbb{Z})$ by [5, Theorem 5.3]. We have shown in the previous section that the classifying map for the bundle

$$\mathcal{L}_2 \to \mathcal{A}_F^{SU(2)}(T)/\mathcal{G}_{SU(2)}(T)$$

induces an isomorphism on $H^2(-;\mathbb{Z})$. Since the classifying map for $\mathcal{L}_\infty$ restricts to a classifying map for $\mathcal{L}_2$, we find that the classifying map for $\mathcal{L}_\infty$ must also induce an isomorphism on $H^2(-;\mathbb{Z})$. But up to homotopy, this is a self-map of $\mathbb{C}P^\infty$, and any self-map of $\mathbb{C}P^\infty$ that induces an isomorphism on $H^2(-;\mathbb{Z})$ is a homotopy equivalence. □

We now turn to the unitary case. Fix generators α_i, β_i ($i = 1, \ldots, g$) for $\pi_1(\Sigma)$ (with $\prod_i [\alpha_i, \beta_i] = 1$). The determinant map

$$\det\colon \mathcal{M}_U(\Sigma) \to (S^1)^{2g}$$

is defined by

$$[A] \mapsto (\det(\rho_A(\alpha_1)), \det(\rho_A(\beta_1)), \ldots, \det(\rho_A(\alpha_g)), \det(\rho_A(\beta_g))),$$

where $A \in \mathcal{A}_F^{U(n)}(\Sigma)$ and $\rho_A\colon \pi_1(\Sigma) \to U(n)$ is its holonomy representation.

Corollary 4.3 *There is a line bundle* $\mathcal{L}^U \to \mathcal{M}_U(\Sigma)$ *that restricts to*

$$\mathcal{L}_\infty \to \mathcal{M}_{SU}(\Sigma),$$

and if α *is a classifying map for* $\mathcal{L}^U$*, then the map*

$$\mathcal{M}_U(\Sigma) \xrightarrow{(\det,\alpha)} (S^1)^{2g} \times \mathbb{C}P^\infty$$

is a homotopy equivalence.

Proof Let $f\colon \mathcal{M}_{SU} \xrightarrow{\simeq} \mathbb{C}P^\infty$ be a classifying map for $\mathcal{L}_\infty$, and choose a homotopy equivalence $\phi\colon \mathcal{M}_U(\Sigma) \xrightarrow{\simeq} \mathbb{C}P^\infty \times (S^1)^{2g}$, as in (3). Let $\mathbb{C}P^\infty \times (S^1)^{2g} \xrightarrow{p_1} \mathbb{C}P^\infty$ be the projection, and let $i\colon \mathcal{M}_{SU}(\Sigma) \hookrightarrow \mathcal{M}_U(\Sigma)$ be the inclusion. By [5, Theorem 5.3], i induces an isomorphism on π_2, so the composite

$$i_1 := p_1 \circ \phi \circ i\colon \mathcal{M}_{SU}(\Sigma) \longrightarrow \mathbb{C}P^\infty$$

induces an isomorphism on π_2 and hence is a homotopy equivalence. Define

$$\alpha := f \circ i_1^{-1} \circ p_1 \circ \phi\colon \mathcal{M}_U(\Sigma) \to \mathbb{C}P^\infty,$$

where i_1^{-1} is a homotopy inverse to i_1. Then we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{SU}(\Sigma) & \xhookrightarrow{\quad i \quad} & \mathcal{M}_U(\Sigma) \\ & {\scriptstyle f}\searrow^{\simeq} \quad \swarrow{\scriptstyle \alpha} & \\ & \mathbb{C}P^\infty & \end{array}$$

and we define $\mathcal{L}^U$ to be the pullback, under α, of the universal bundle over $\mathbb{C}P^\infty$. It remains to show that $(\det, \alpha)$ is a homotopy equivalence. Since det is split by the inclusion of $(S^1)^{2g} \cong \text{Hom}(\pi_1(\Sigma), U(1))/U(1)$ into $\mathcal{M}_U(\Sigma)$, we see that on fundamental groups, $\det_*$ is a surjection between free abelian groups of rank $2g$, hence an isomorphism. Since α_* is an isomorphism on π_2, the result follows from the Whitehead theorem (and Remark 4.1) □

5 The degree of the line bundle in higher genus

Let $\mathcal{L}_g$ denote the prequantum line bundle on the moduli space $\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}$ of flat connections on a trivial $SU(2)$-bundle over the genus g surface Σ^g. We now show that $\mathcal{L}_g$ has degree 1 for every genus g surface ($g > 0$), not just $g = 1$ (thereby completing the proof of Theorem 1.1). This statement is meaningful, since we have:

Lemma 5.1 *For any* $g \geq 1$*, we have* $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z}) \cong \mathbb{Z}$.

Proof In [5], it was proven that $\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}$ is simply connected (and a second proof of this fact was given in [1]). Now, triviality of $H_1(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$ implies that $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$ is torsion free (by the universal coefficient theorem), and a simple direct analysis of the Poincaré polynomial of $\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}$, as determined by Cappell–Lee–Miller [2], shows that $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$ has rank 1. □

A map $f\colon\ \Sigma^g \to \Sigma^h$ induces a map

$$f^{\#}\colon\ \mathcal{A}_F^{SU(2)}(\Sigma^h)/\mathcal{G} \to \mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G},$$

and as noted in [7, Remark 3, p. 412], if f has degree 1 then $(f^{\#})^*(\mathcal{L}_g) = \mathcal{L}_h$. This implies that

$$(f^{\#})^*(c_1(\mathcal{L}_g)) = c_1(\mathcal{L}_h),$$

and taking $h = 1$ we find that $(f^{\#})^*(c_1(\mathcal{L}_g)) = c_1(\mathcal{L}_1) = 1$ (by the result in Sect. 3). Now Lemma 5.1 implies that $c_1(\mathcal{L}_g)$ is a generator of $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$. This completes the proof of Theorem 1.1. □

Remark 5.2 The results in [5] in fact show that the map

$$H^2(\mathcal{M}_{SU}(\Sigma^g); \mathbb{Z}) \longrightarrow H^2(\mathcal{M}_{SU}(\Sigma^1); \mathbb{Z})$$

is determined by

$$f^*\colon\ H^2(\Sigma^1) \to H^2(\Sigma^g).$$

Thus, a choice of generator in $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^1)/\mathcal{G}; \mathbb{Z})$ and a choice of orientations on Σ^1 and Σ^g give a choice of generator in $H^2(\mathcal{A}_F^{SU(2)}(\Sigma^g)/\mathcal{G}; \mathbb{Z})$, and the above discussion shows that this generator coincides with $c_1(\mathcal{L}_g)$.

Acknowledgements The second author thanks Simon Donaldson for suggesting that the results of [8] should be connected to Goldman's symplectic form. Additionally, we thank Jacques Hurtubise for asking about the higher genus case considered in Sect. 5 and the anonymous referees for helping to improve the exposition.

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