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A q-boson representation of Zamolodchikov-Faddeev algebra for stochastic R matrix of $U_q(A_n^{(1)})$

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Abstract We construct a q-boson representation of the Zamolodchikov-Faddeev algebra whose structure function is given by the stochastic R matrix of $U_q(A_n^{(1)})$ introduced recently. The representation involves quantum dilogarithm type infinite products in the $n(n-1)/2$-fold tensor product of q-bosons. It leads to a matrix product formula of the stationary probabilities in the $U_q(A_n^{(1)})$-zero range process on a one-dimensional periodic lattice.

Keywords Yang–Baxter equation · Quantum group · Markov process

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1 Introduction

Exploring integrable models in non-equilibrium statistical mechanics is a significant branch of applications of quantum group theory. In such approaches to Markov processes, one typically tries to construct a Markov matrix whose spectral problem can be solved by Bethe ansatz and the Yang–Baxter equation [1]. Whether it is possible or not relies on the fundamental question; can a quantum R matrix [7,11] be made

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stochastic? This was answered affirmatively in [12] for the $U_q(A_n^{(1)})$ quantum R matrix intertwining the symmetric tensor representations of arbitrary degree. The modified one, called stochastic R matrix, possesses nonnegative elements only and fulfills a local version of total probability conservation called the sum-to-unity condition. The associated continuous and discrete time Markov processes are formulated as stochastic dynamics of n species of particles on one-dimensional lattice obeying a zero range type interaction (cf. [8]). We call them $U_q(A_n^{(1)})$-zero range processes. They include several models studied earlier for $n = 1$ [2,4,19–21] and for n general [13–15,22].

In this paper, we study the Zamolodchikov-Faddeev (ZF) algebra with the structure function given by the $U_q(A_n^{(1)})$ stochastic R matrix $\check{S}(\lambda, \mu)$ in [12]. Symbolically, it is a family of quadratic relations of the form

$$X(\mu) \otimes X(\lambda) = \check{S}(\lambda, \mu)\big[X(\lambda) \otimes X(\mu)\big],$$

where $X(\mu) = (X_\alpha(\mu))_{\alpha \in \mathbb{Z}_{\geq 0}^n}$ denotes a collection of the operator $X_\alpha(\mu)$ associated with each local state α of the $U_q(A_n^{(1)})$-zero range process. The parameters λ, μ are reminiscent of the degrees of the symmetric tensors (magnitude of "spins") which can be utilized to describe the inhomogeneity of the system. See (12) for a concrete description.

The ZF algebra originates in integrable quantum field theory in $(1+1)$ dimension and encodes the factorized scattering of particles [9,23]. The structure function therein is the scattering matrix of the theory which should be properly normalized so as to satisfy the unitarity.

In the realm of integrable Markov processes, the situation is parallel. The ZF algebra serves as a local version of the stationary condition in the matrix product construction of the stationary states. Its infinitesimal version, often called "hat relation" or "cancellation mechanism", has been utilized in many works, e.g., [5,6,14,15,18]. As in the factorized scattering theory, it is crucial to adopt the correct normalization of $\check{S}(\lambda, \mu)$ since the ZF algebra is inhomogeneous in the structure function unlike the more commonly argued $RLL = LLR$ type relation. In our setting, the normalization is canonically fixed by the sum-to-unity condition mentioned above. See (7).

The main result of this paper is a q-boson representation of the so defined ZF algebra. More precisely, we construct an algebra homomorphism from it to the tensor product $\mathcal{B}^{\otimes n(n-1)/2}$ where $\mathcal{B}$ is the q-boson algebra defined in (16). We will give either a recursive characterization with respect to the rank n (Theorem 1) or an explicit formula (Theorem 2). The latter contains quantum dilogarithm type infinite products, which is a distinctive feature reflecting the fact that the stochastic R matrix in [12] formally originates in the symmetric tensor representation of $U_q(A_n^{(1)})$ of infinite degree. It provides the first systematic result beyond the simplest choice of the structure function [3,10] where representations of the ZF algebra associated with the stochastic R matrix of the vector representation of $U_q(A_n^{(1)})$ were studied.

The q-boson algebra $\mathcal{B}$ has a natural representation on the Fock space F as in (37). As an application, we present a matrix product formula for the stationary probabilities of the $U_q(A_n^{(1)})$-zero range process, which is expressed as a trace over $F^{\otimes n(n-1)/2}$ (Theorem 5). It is a corollary of the auxiliary condition [16, Eq. (30)] satisfied by the

q-boson representation as well as the ZF algebra relation. These results extend the earlier ones for $n = 2$ [16] to general n.

The layout of the paper is as follows. In Sect. 2, we quote the stochastic R matrix for $U_q(A_n^{(1)})$ from [12]. In Sect. 3, we introduce the ZF algebra and give a q-boson representation. In Sect. 4, the $U_q(A_n^{(1)})$-zero range process associated with the stochastic R matrix [12] is recalled briefly, and a matrix product formula of the stationary probabilities is presented. Section 5 is a summary. Appendix A contains a proof of Theorem 1.

Throughout the paper we assume that q is generic unless otherwise stated and use the notation $\theta(\text{true}) = 1, \theta(\text{false}) = 0$, the q-Pochhammer symbol $(z)_m = (z; q)_m = \prod_{j=1}^{m}(1 - zq^{j-1})$ and the q-binomial $\binom{m}{k}_q = \theta(k \in [0, m])\frac{(q)_m}{(q)_k(q)_{m-k}}$. The symbols $(z)_m$ appearing in this paper always mean $(z; q)_m$. For integer arrays $\alpha = (\alpha_1, \ldots, \alpha_m), \beta = (\beta_1, \ldots, \beta_m)$ of any length m, we write $|\alpha| = \alpha_1 + \cdots + \alpha_m$. The relation $\alpha \le \beta$ or equivalently $\beta \ge \alpha$ is defined by $\beta - \alpha \in \mathbb{Z}_{\ge 0}^m$. We often denote $0^m := (0, \ldots, 0) \in \mathbb{Z}_{\ge 0}^m$ simply by 0 when it is clear from the context.

2 Stochastic R matrix for $U_q(A_n^{(1)})$

Set $W = \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_{\ge 0}^n} \mathbb{C}|\alpha\rangle$. Define the operator $\mathcal{S}(\lambda, \mu) \in \text{End}(W \otimes W)$ depending on the parameters λ and μ by

$$\mathcal{S}(\lambda, \mu)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma,\delta\in\mathbb{Z}_{\ge 0}^n} \mathcal{S}(\lambda, \mu)_{\alpha,\beta}^{\gamma,\delta} |\gamma\rangle \otimes |\delta\rangle, \tag{1}$$

$$\mathcal{S}(\lambda, \mu)_{\alpha,\beta}^{\gamma,\delta} = \theta(\alpha + \beta = \gamma + \delta)\Phi_q(\gamma|\beta; \lambda, \mu), \tag{2}$$

where $\Phi_q(\gamma|\beta; \lambda, \mu)$ with $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{\ge 0}^n, \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}_{\ge 0}^n$ is given by

$$\Phi_q(\gamma|\beta; \lambda, \mu) = q^{\varphi(\beta-\gamma,\gamma)} \left(\frac{\mu}{\lambda}\right)^{|\gamma|} \frac{(\lambda)_{|\gamma|}(\frac{\mu}{\lambda})_{|\beta|-|\gamma|}}{(\mu)_{|\beta|}} \prod_{i=1}^{n} \binom{\beta_i}{\gamma_i}_q,$$

$$\varphi(\beta, \gamma) = \sum_{1 \le i < j \le n} \beta_i \gamma_j. \tag{3}$$

The sum (1) is finite due to the θ factor in (2). The difference property $\mathcal{S}(\lambda, \mu) = \mathcal{S}(c\lambda, c\mu)$ is absent. We call $\mathcal{S}(\lambda, \mu)$ the stochastic R matrix [12]. It originates in the quantum R matrix [7,11] of the symmetric tensor representation of the quantum affine algebra $U_q(A_n^{(1)})$. It satisfies the Yang–Baxter equation, the inversion relation and the sum-to-unity condition [12,16]:

$$\mathcal{S}_{1,2}(v_1, v_2)\mathcal{S}_{1,3}(v_1, v_3)\mathcal{S}_{2,3}(v_2, v_3) = \mathcal{S}_{2,3}(v_2, v_3)\mathcal{S}_{1,3}(v_1, v_3)\mathcal{S}_{1,2}(v_1, v_2), \tag{4}$$

$$\mathcal{S}_{1,2}^T(v_1, v_2)\mathcal{S}_{1,3}^T(v_1, v_3)\mathcal{S}_{2,3}^T(v_2, v_3) = \mathcal{S}_{2,3}^T(v_2, v_3)\mathcal{S}_{1,3}^T(v_1, v_3)\mathcal{S}_{1,2}^T(v_1, v_2), \tag{5}$$

$$\check{\mathcal{S}}(\lambda, \mu)\check{\mathcal{S}}(\mu, \lambda) = \text{id}_{W^{\otimes 2}}, \tag{6}$$

$$\sum_{\gamma, \delta \in \mathbb{Z}_{\geq 0}^n} \mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} = 1 \quad (\forall \alpha, \beta \in \mathbb{Z}_{\geq 0}^n), \tag{7}$$

where the checked R matrix $\check{\mathcal{S}}(\lambda, \mu)$ and the transposed R matrix $\mathcal{S}^T(\lambda, \mu)$ are defined by $\check{\mathcal{S}}(\lambda, \mu)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma, \delta} \mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} |\delta\rangle \otimes |\gamma\rangle$, and $\mathcal{S}^T(\lambda, \mu)(|\alpha\rangle \otimes |\beta\rangle) = \sum_{\gamma, \delta \in \mathbb{Z}_{\geq 0}^n} \mathcal{S}(\lambda, \mu)_{\gamma, \delta}^{\alpha, \beta} |\gamma\rangle \otimes |\delta\rangle$. The matrix elements are depicted as

$$\mathcal{S}(\lambda, \mu)_{\alpha, \beta}^{\gamma, \delta} = \alpha \xrightarrow[\beta]{\delta} \gamma \tag{8}$$

When preferable, we will exhibit the n-dependence of (3) as $\Phi_q^{(n)}(\gamma|\beta; \lambda, \mu)$. The function $\Phi_q^{(1)}(\gamma|\beta; \lambda, \mu)$ appeared in Povolotsky's chipping model [19], which stimulated many subsequent studies. It was also built in the explicit formula of $U_q(A_1^{(1)})$ R matrix and Q-operators by Mangazeev [17].

For n general it is zero unless $\gamma \leq \beta$, and satisfies the sum rule [12]:

$$\sum_{\gamma \in \mathbb{Z}_{\geq 0}^n} \Phi_q^{(n)}(\gamma|\beta; \lambda, \mu) = 1 \quad (\forall \beta \in \mathbb{Z}_{\geq 0}^n), \tag{9}$$

which may be viewed as a corollary of (7).

For an array of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ with any length m, we use the notation

$$\overline{\alpha} = (\alpha_2, \ldots, \alpha_m), \quad \underline{\alpha} = (\alpha_1, \ldots, \alpha_{m-1}). \tag{10}$$

Then it is straightforward to check

$$\begin{aligned} \Phi_q^{(n)}(\gamma|\alpha; \lambda, \mu) &= \Phi_q^{(1)}(\gamma_1|\alpha_1; \lambda, \mu) \Phi_q^{(n-1)}(\overline{\gamma}|\overline{\alpha}; q^{\gamma_1}\lambda, q^{\alpha_1}\mu) \\ &= \Phi_q^{(n-1)}(\underline{\gamma}|\underline{\alpha}; \lambda, \mu) \Phi_q^{(1)}(\gamma_n|\alpha_n; q^{|\underline{\gamma}|}\lambda, q^{|\underline{\alpha}|}\mu). \end{aligned} \tag{11}$$

3 q-Boson representation of Zamolodchikov-Faddeev algebra

3.1 Zamolodchikov-Faddeev algebra

With an array $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ and a parameter μ, we associate an operator $X_\alpha(\mu)$. By the Zamolodchikov-Faddeev algebra for the stochastic R matrix $\mathcal{S}(\lambda, \mu)$ we mean the following family of quadratic relations:

$$X_\alpha(\mu) X_\beta(\lambda) = \sum_{\gamma, \delta \in \mathbb{Z}_{\geq 0}^n} \mathcal{S}(\lambda, \mu)_{\gamma, \delta}^{\beta, \alpha} X_\gamma(\lambda) X_\delta(\mu). \tag{12}$$

It is associative due to the Yang–Baxter equation (5). From (2) it reads more explicitly as

$$X_\alpha(\mu)X_\beta(\lambda) = \sum_{\gamma \leq \alpha} \Phi_q(\beta|\alpha + \beta - \gamma; \lambda, \mu)X_\gamma(\lambda)X_{\alpha+\beta-\gamma}(\mu), \qquad (13)$$

where the omitted condition $\gamma \in \mathbb{Z}_{\geq 0}^n$ should always be taken for granted. We find it convenient to work also with another normalization $Z_\alpha(\mu)$ specified by

$$X_\alpha(\mu) = g_\alpha(\mu)Z_\alpha(\mu), \qquad g_\alpha(\mu) = \frac{\mu^{-|\alpha|}(\mu)_{|\alpha|}}{\prod_{i=1}^n (q)_{\alpha_i}}. \qquad (14)$$

The ZF algebra for the latter takes the form

$$Z_\alpha(\mu)Z_\beta(\lambda) = \sum_{\gamma \leq \alpha} q^{\varphi(\alpha-\gamma,\beta-\gamma)}\Phi_q(\gamma|\alpha; \lambda, \mu)Z_\gamma(\lambda)Z_{\alpha+\beta-\gamma}(\mu) \qquad (15)$$

due to the identity

$$\frac{g_\gamma(\lambda)g_{\alpha+\beta-\gamma}(\mu)}{g_\alpha(\mu)g_\beta(\lambda)}\Phi_q(\beta|\alpha + \beta - \gamma; \lambda, \mu) = q^{\varphi(\alpha-\gamma,\beta-\gamma)}\Phi_q(\gamma|\alpha; \lambda, \mu).$$

3.2 q-Boson representation

Let $\mathcal{B}$ be the algebra generated by 1, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{k}$ obeying the relations

$$\mathbf{kb} = q\mathbf{bk}, \qquad \mathbf{kc} = q^{-1}\mathbf{kc}, \qquad \mathbf{bc} = 1 - \mathbf{k}, \qquad \mathbf{cb} = 1 - q\mathbf{k}. \qquad (16)$$

We call it the q-boson algebra. It has a basis $\{\mathbf{b}^i\mathbf{c}^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$. See (37) for a representation on the Fock space.

The ZF algebra (15) admits a "trivial" representation $Z_\alpha(\zeta) = K_\alpha$ in terms of an operator K_α satisfying $K_0 = 1$ and $K_\alpha K_\beta = q^{\varphi(\alpha,\beta)}K_{\alpha+\beta}$ as shown in [16, Prop.7]. See (3) for the definition of $\varphi(\alpha, \beta)$. Such a K_α is easily constructed, for instance as[1]

$$K_{\alpha_1,\dots,\alpha_n} = \mathbf{k}^{\alpha_1^+}\mathbf{c}^{\alpha_1} \otimes \cdots \otimes \mathbf{k}^{\alpha_{n-1}^+}\mathbf{c}^{\alpha_{n-1}} \in \mathcal{B}^{\otimes n-1}, \qquad \alpha_i^+ := \alpha_{i+1} + \cdots + \alpha_n. \qquad (17)$$

However, this is not the representation we are after because it does not contain a creation operator $\mathbf{b}$ and leads to vanishing trace in the forthcoming matrix product formula (38). Our construction of $Z_\alpha(\zeta)$ below may be viewed as a series expansion starting from the trivial representation in terms of creation operators.

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[1] We write K_α with $\alpha = (\alpha_1, \dots, \alpha_n)$ as $K_{\alpha_1,\dots,\alpha_n}$ rather than $K_{(\alpha_1,\dots,\alpha_n)}$ for simplicity. A similar convention will also be used for $g_\alpha(\zeta)$, $X_\alpha(\zeta)$ and $Z_\alpha(\zeta)$.

For $\alpha_i \in \mathbb{Z}_{\geq 0}$, we define the operator $Z_{\alpha_1,\dots,\alpha_n}(\zeta) \in \mathcal{B}^{\otimes n(n-1)/2}$ by the $n = 1$ case and the recursion relation with respect to n by

$$Z_{\alpha_1}(\zeta) = 1, \tag{18}$$

$$Z_{\alpha_1,\dots,\alpha_n}(\zeta) = \sum_{l=(l_1,\dots,l_{n-1})\in\mathbb{Z}_{\geq 0}^{n-1}} X_l(\zeta) \otimes \mathbf{b}^{l_1}\mathbf{k}^{\alpha_1^+}\mathbf{c}^{\alpha_1} \otimes \cdots \otimes \mathbf{b}^{l_{n-1}}\mathbf{k}^{\alpha_{n-1}^+}\mathbf{c}^{\alpha_{n-1}}, \tag{19}$$

where $X_l(\zeta) = g_l(\zeta)Z_l(\zeta)$ as in (14) and α_i^+ is defined by (17). Now we present the first half of the main result of the paper.

Theorem 1 *The $Z_\alpha(\zeta)$ defined by* (18)–(19) *satisfies the ZF algebra* (15) *for general n.*

We present a proof in Appendix.

3.3 Explicit formula

One can take the infinite sum in (19) and write down an explicit formula of $Z_{\alpha_1,\dots,\alpha_n}(\zeta)$ in terms of a product of $(\zeta^{-1}Q)_\infty^{\pm 1}$ with various monomials $Q \in \mathcal{B}^{\otimes n(n-1)/2}$. Note that (19) tells the simple dependence on α as

$$Z_{\alpha_1,\dots,\alpha_n}(\zeta) = Z_{0^n}(\zeta)\big(1^{\otimes\frac{1}{2}(n-1)(n-2)} \otimes K_{\alpha_1,\dots,\alpha_n}\big). \tag{20}$$

Here and in what follows, $K_{\alpha_1,\dots,\alpha_n}$ is to be understood as the one in (17). Thus, our task is reduced to the calculation of the special case of the sum (19):

$$Z_{0^n}(\zeta) = \sum_{l_1,\dots,l_{n-1}\in\mathbb{Z}_{\geq 0}} g_{l_1,\dots,l_{n-1}}(\zeta)Z_{l_1,\dots,l_{n-1}}(\zeta) \otimes \mathbf{b}^{l_1} \otimes \cdots \otimes \mathbf{b}^{l_{n-1}}. \tag{21}$$

Let us illustrate it for $n = 2$ and 3. For $n = 2$, one has

$$Z_{0,0}(\zeta) = \sum_{l_1\geq 0} \frac{(\zeta)_{l_1}\zeta^{-l_1}}{(q)_{l_1}}\mathbf{b}^{l_1} = \frac{(\mathbf{b})_\infty}{(\zeta^{-1}\mathbf{b})_\infty},$$

$$Z_{\alpha_1,\alpha_2}(\zeta) = Z_{0,0}(\zeta)K_{\alpha_1,\alpha_2} = \frac{(\mathbf{b})_\infty}{(\zeta^{-1}\mathbf{b})_\infty}\mathbf{k}^{\alpha_2}\mathbf{c}^{\alpha_1} \in \mathcal{B}$$

by means of the formula

$$\frac{(zw)_\infty}{(z)_\infty} = \sum_{j\geq 0} \frac{(w)_j}{(q)_j}z^j.$$

This result agrees with [16, Eq. (39)]. For $n = 3$, the sum (21) is calculated using $(\zeta)_{l_1+l_2} = (\zeta)_{l_2}(q^{l_2}\zeta)_{l_1}$ as

$$
\begin{aligned}
Z_{0,0,0}(\zeta) &= \sum_{l_1,l_2} \frac{\zeta^{-l_1-l_2}(\zeta)_{l_1+l_2}}{(q)_{l_1}(q)_{l_2}} \frac{(\mathbf{b})_\infty}{(\zeta^{-1}\mathbf{b})_\infty} \mathbf{k}^{l_2}\mathbf{c}^{l_1} \otimes \mathbf{b}^{l_1} \otimes \mathbf{b}^{l_2} \\
&= \frac{(\mathbf{b} \otimes 1 \otimes 1)_\infty}{(\zeta^{-1}\mathbf{b} \otimes 1 \otimes 1)_\infty} \sum_{l_2} \frac{\zeta^{-l_2}(\zeta)_{l_2}(\mathbf{k} \otimes 1 \otimes \mathbf{b})^{l_2}}{(q)_{l_2}} \sum_{l_1} \frac{\zeta^{-l_1}(q^{l_2}\zeta)_{l_1}(\mathbf{c} \otimes \mathbf{b} \otimes 1)^{l_1}}{(q)_{l_1}} \\
&= \frac{(\mathbf{b} \otimes 1 \otimes 1)_\infty}{(\zeta^{-1}\mathbf{b} \otimes 1 \otimes 1)_\infty} \sum_{l_2} \frac{\zeta^{-l_2}(\zeta)_{l_2}(\mathbf{k} \otimes 1 \otimes \mathbf{b})^{l_2}}{(q)_{l_2}} \frac{(q^{l_2}\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty}{(\zeta^{-1}\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty} \\
&= \frac{(\mathbf{b} \otimes 1 \otimes 1)_\infty}{(\zeta^{-1}\mathbf{b} \otimes 1 \otimes 1)_\infty}(\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty \sum_{l_2} \frac{\zeta^{-l_2}(\zeta)_{l_2}(\mathbf{k} \otimes 1 \otimes \mathbf{b})^{l_2}}{(q)_{l_2}} \frac{1}{(\zeta^{-1}\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty} \\
&= \frac{(\mathbf{b} \otimes 1 \otimes 1)_\infty}{(\zeta^{-1}\mathbf{b} \otimes 1 \otimes 1)_\infty}(\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty \frac{(\mathbf{k} \otimes 1 \otimes \mathbf{b})_\infty}{(\zeta^{-1}\mathbf{k} \otimes 1 \otimes \mathbf{b})_\infty} \frac{1}{(\zeta^{-1}\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty},
\end{aligned}
$$
$$
Z_{\alpha_1,\alpha_2,\alpha_3}(\zeta) = Z_{0,0,0}(\zeta)(1 \otimes K_{\alpha_1,\alpha_2,\alpha_3}) = Z_{0,0,0}(\zeta)(1 \otimes \mathbf{k}^{\alpha_2+\alpha_3}\mathbf{c}^{\alpha_1} \otimes \mathbf{k}^{\alpha_3}\mathbf{c}^{\alpha_2}). \tag{22}
$$

These results on $Z_{0,\dots,0}(\zeta)$ are neatly presented as

$$
\begin{aligned}
Z_{0,0}(\zeta) &= V_1(1)V_1(\zeta)^{-1}, \quad V_1(\zeta) = (\zeta^{-1}\mathbf{b})_\infty, \\
Z_{0,0,0}(\zeta) &= \big(Z_{0,0}(\zeta) \otimes 1 \otimes 1\big)V_2(1)V_2(\zeta)^{-1}, \\
V_2(\zeta) &= (\zeta^{-1}\mathbf{c} \otimes \mathbf{b} \otimes 1)_\infty(\zeta^{-1}\mathbf{k} \otimes 1 \otimes \mathbf{b})_\infty.
\end{aligned} \tag{23}
$$

Let us proceed to general $n \ (\geq 2)$ case. Substitution of (20)$|_{n \to n-1}$ into the RHS of (21) gives

$$
\begin{aligned}
Z_{0^n}(\zeta) &= \big(Z_{0^{n-1}}(\zeta) \otimes 1^{\otimes n-1}\big)Y_n(\zeta), \\
Y_n(\zeta) &= \sum_{l_1,\dots,l_{n-1}\in\mathbb{Z}_{\geq 0}} g_{l_1,\dots,l_{n-1}}(\zeta)\, 1^{\otimes \frac{1}{2}(n-2)(n-3)} \otimes K_{l_1,\dots,l_{n-1}} \otimes \mathbf{b}^{l_1} \otimes \cdots \otimes \mathbf{b}^{l_{n-1}}.
\end{aligned}
$$
$$\tag{24}$$

To systematize the calculation, we introduce copies $\mathcal{B}_{i,j} = \langle 1, \mathbf{b}_{i,j}, \mathbf{c}_{i,j}, \mathbf{k}_{i,j}\rangle$ of the q-boson algebras and the generators for $1 \leq i \leq j \leq n-1$ obeying (16) within each $\mathcal{B}_{i,j}$ and $[\mathcal{B}_{i,j}, \mathcal{B}_{i',j'}] = 0$ if $(i,j) \neq (i',j')$. We take them so that $Z_{\alpha_1,\dots,\alpha_n}(\zeta) \in \bigotimes_{1 \leq i \leq j \leq n-1} \mathcal{B}_{i,j}$ and (24) reads

$$
Y_n(\zeta) = \sum_{l_1,\dots,l_{n-1}\in\mathbb{Z}_{\geq 0}} g_{l_1,\dots,l_{n-1}}(\zeta)\,\big(\mathbf{k}_{1,n-2}^{l_1^+}\mathbf{c}_{1,n-2}^{l_1} \cdots \mathbf{k}_{n-2,n-2}^{l_{n-2}^+}\mathbf{c}_{n-2,n-2}^{l_{n-2}}\big)\big(\mathbf{b}_{1,n-1}^{l_1} \cdots \mathbf{b}_{n-1,n-1}^{l_{n-1}}\big),
$$

where $l_j^+ = l_{j+1} + \cdots + l_{n-1}$. This corresponds to labeling the components in the tensor product $\mathcal{B}^{\otimes n(n-1)/2}$ as

$$
(1,1), (1,2), (2,2), (1,3), (2,3), (3,3), \dots\dots, (1, n-1), (2, n-1), \dots, (n-1, n-1). \tag{25}
$$

As exemplified in the above formula of $Y_n(\zeta)$, using the q-bosons $\mathcal{B}_{i,j}$ with indices allow us to avoid the cumbersome factor $1^{\otimes N}$ as in (24).

One can rearrange the summand in $Y_n(\zeta)$ by reordering the commuting generators only as

$$Y_n(\zeta) = \sum_{l=(l_1,\ldots,l_{n-1})\in\mathbb{Z}_{\geq 0}^{n-1}} \frac{\zeta^{-|l|}(\zeta)_{|l|}}{\prod_{1\leq i\leq n-1}(q)_{l_i}} A_{n-1,n-1}^{l_{n-1}} A_{n-2,n-1}^{l_{n-2}} \cdots A_{1,n-1}^{l_1},$$

$$A_{j,n-1} = \mathbf{k}_{1,n-2}\mathbf{k}_{2,n-2}\cdots\mathbf{k}_{j-1,n-2}\mathbf{c}_{j,n-2}\mathbf{b}_{j,n-1} \quad (\mathbf{c}_{n-1,n-2} = 1). \tag{26}$$

In particular, $A_{1,n-1} = \mathbf{c}_{1,n-2}\mathbf{b}_{1,n-1}$ and $A_{n-1,n-1} = \mathbf{k}_{1,n-2}\cdots\mathbf{k}_{n-2,n-2}\mathbf{b}_{n-1,n-1}$. By utilizing the decomposition $(\zeta)_{|l|} = (\zeta)_{l_1+\cdots+l_{n-2}}(q^{l_1+\cdots+l_{n-2}}\zeta)_{l_{n-1}}$, the sum over l_{n-1} is taken, leading to

$$Y_n(\zeta) = \sum_{l_1,\ldots,l_{n-2}\in\mathbb{Z}_{\geq 0}} \frac{\zeta^{-l_1-\cdots-l_{n-2}}(\zeta)_{l_1+\cdots+l_{n-2}}}{\prod_{1\leq i\leq n-2}(q)_{l_i}} \frac{(q^{l_1+\cdots+l_{n-2}}A_{n-1,n-1})_\infty}{(\zeta^{-1}A_{n-1,n-1})_\infty} A_{n-2,n-1}^{l_{n-2}} \cdots A_{1,n-1}^{l_1}$$

$$= \frac{1}{(\zeta^{-1}A_{n-1,n-1})_\infty} \sum_{l_1,\ldots,l_{n-2}\in\mathbb{Z}_{\geq 0}} \frac{\zeta^{-l_1-\cdots-l_{n-2}}(\zeta)_{l_1+\cdots+l_{n-2}}}{\prod_{1\leq i\leq n-2}(q)_{l_i}} A_{n-2,n-1}^{l_{n-2}} \cdots A_{1,n-1}^{l_1}(A_{n-1,n-1})_\infty,$$

where the second step is due to $A_{n-1,n-1}A_{j,n-1} = q^{-1}A_{j,n-1}A_{n-1,n-1}$ $(1 \leq j \leq n-2)$. Now the sum over l_{n-2} can be taken in the same manner. Repeating this process we arrive at

$$Y_n(\zeta) = V_{n-1}(\zeta)^{-1}V_{n-1}(1) = V_{n-1}(1)V_{n-1}(\zeta)^{-1}, \tag{27}$$

$$V_{n-1}(\zeta) = (\zeta^{-1}A_{1,n-1})_\infty(\zeta^{-1}A_{2,n-1})_\infty\cdots(\zeta^{-1}A_{n-1,n-1})_\infty, \tag{28}$$

where the rightmost expression in (27) follows from a similar calculation taking the sum (26) in the order $l_1, l_2, \ldots, l_{n-1}$ applying the decomposition $(\zeta)_{|l|} = (\zeta)_{l_2+\cdots+l_{n-1}}(q^{l_2+\cdots+l_{n-1}}\zeta)_{l_1}$ first.

The explicit formulas derived in this way supplement the recursive characterization in Theorem 1. They constitute the latter half of the main result of the paper. We summarize them in

Theorem 2 *The ZF algebra* (15) *has the following representation in* $\bigotimes_{1\leq i\leq j\leq n-1} \mathcal{B}_{i,j}$:

$$Z_{\alpha_1,\ldots,\alpha_n}(\zeta) = Z_{0^n}(\zeta)\,\mathbf{k}_{1,n-1}^{\alpha_1^+}\mathbf{c}_{1,n-1}^{\alpha_1}\cdots\mathbf{k}_{n-1,n-1}^{\alpha_{n-1}^+}\mathbf{c}_{n-1,n-1}^{\alpha_{n-1}} \quad (\alpha_i^+ = \alpha_{i+1} + \cdots + \alpha_n),$$

$$Z_{0^n}(\zeta) = Y_2(\zeta)Y_3(\zeta)\cdots Y_n(\zeta),$$

$$Y_j(\zeta) = V_{j-1}(1)V_{j-1}(\zeta)^{-1} = V_{j-1}(\zeta)^{-1}V_{j-1}(1),$$

$$V_j(\zeta) = (\zeta^{-1}A_{1,j})_\infty(\zeta^{-1}A_{2,j})_\infty\cdots(\zeta^{-1}A_{j,j})_\infty,$$

$$A_{i,j} = \mathbf{k}_{1,j-1}\mathbf{k}_{2,j-1}\cdots\mathbf{k}_{i-1,j-1}\mathbf{c}_{i,j-1}\mathbf{b}_{i,j} \quad (\mathbf{c}_{j,j-1} = 1).$$

The cases $n = 2, 3$ reproduce (23) under the identification $\mathbf{x}_{1,1} = \mathbf{x} \otimes 1 \otimes 1$, $\mathbf{x}_{1,2} = 1 \otimes \mathbf{x} \otimes 1$, $\mathbf{x}_{2,2} = 1 \otimes 1 \otimes \mathbf{x}$ in accordance with (25).

Remark 3 An interesting corollary of the ZF algebra (12) and $\mathcal{S}(\lambda, \mu)_{\gamma, \delta}^{0,0} = \theta(\gamma = \delta = 0)$ is the commutativity:

$$[X_0(\mu), X_0(\lambda)] = 0, \qquad [Z_0(\mu), Z_0(\lambda)] = 0.$$

In addition to it, we have

$$[V_m(\mu), V_m(\lambda)] = 0 \qquad (1 \le m \le n - 1).$$

To see this, note that the transformation $(\mathbf{b}_{i,j}, \mathbf{c}_{i,j}, \mathbf{k}_{i,j}) \to (\eta_{i,j}\mathbf{b}_{i,j}, \eta_{i,j}^{-1}\mathbf{c}_{i,j}, \mathbf{k}_{i,j})$ is an automorphism of the q-boson algebra $\bigotimes_{1 \le i \le j \le m} \mathcal{B}_{i,j}$ for any m and $\eta_{i,j} \ne 0$. Choosing $\eta_{i,j} = \eta^{\theta(j=m)}$ leads to $A_{i,m} \to \eta A_{i,m}$ hence $V_m(\zeta) \to V_m(\zeta/\eta)$. Therefore the above commutativity follows by applying this automorphism to the equality $V_m(1)V_m(\zeta)^{-1} = V_m(\zeta)^{-1}V_m(1)$ in Theorem 2.

Before closing the section, let us explain the relation to the work [15] where an inhomogeneous generalization of an n-species totally asymmetric zero range process was introduced and a matrix product formula of the stationary states was obtained. Let $X_{\alpha_1,...,\alpha_n}^{(n)}$ be the homogeneous case $w_1 = \cdots = w_n = 1$ of the matrix product operator defined from the initial condition $X_{\alpha_1}^{(1)} = 1$ recursively by [15, Eq. (3.4)], i.e.,

$$X_{\alpha_1,...,\alpha_n}^{(n)} = \sum_{l_1,...,l_{n-1} \in \mathbb{Z}_{\ge 0}} X_{l_1,...,l_{n-1}}^{(n-1)} \otimes \mathbf{b}^{l_1}\mathbf{k}^{\alpha_1^+}\mathbf{c}^{\alpha_1} \otimes \cdots \otimes \mathbf{b}^{l_{n-1}}\mathbf{k}^{\alpha_{n-1}^+}\mathbf{c}^{\alpha_{n-1}} \qquad (29)$$

for $n \ge 2$, where α_i^+ is given by (17). The operators $\mathbf{b}, \mathbf{c}, \mathbf{k}$ here are regarded as representations (37) of q-boson generators at $q = 0$, which are given by $\mathbf{a}^+, \mathbf{a}^-$ and $\mathbf{k}$ in [14, Eq. (2.3)], respectively. Let us consider an automorphism of the q-boson algebra given by the replacement

$$\mathbf{b}_{i,j} \to \zeta^{j-i+1}\mathbf{b}_{i,j}, \quad \mathbf{c}_{i,j} \to \zeta^{i-j-1}\mathbf{c}_{i,j} \quad (1 \le i \le j \le n - 1). \qquad (30)$$

We claim that (29) is reproduced from the corresponding representation of $Z_{\alpha_1,...,\alpha_n}(\zeta)$ in this paper by

$$X_{\alpha_1,...,\alpha_n}^{(n)} = \lim_{\zeta, q \to 0} \zeta^{(n-1)\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1}} Z_{\alpha_1,...,\alpha_n}(\zeta)|_{(30)}. \qquad (31)$$

To see (31), note that it holds as $1 = 1$ for $n = 1$. Moreover, the recursion (19) is equivalently presented as

$$\zeta^{(n-1)\alpha_1 + \cdots + 2\alpha_{n-2} + \alpha_{n-1}} Z_{\alpha_1,...,\alpha_n}(\zeta)|_{(30)} = \sum_{l_1,...,l_{n-1} \in \mathbb{Z}_{\ge 0}} \frac{(\zeta)_{l_1 + \cdots + l_{n-1}}}{\prod_{1 \le i \le n-1}(q)_{l_i}}$$
$$\times \zeta^{(n-2)l_1 + \cdots + 2l_{n-3} + l_{n-2}} Z_{l_1,...,l_{n-1}}(\zeta)|_{(30)_{n \to n-1}}$$
$$\otimes \mathbf{b}^{l_1}\mathbf{k}^{\alpha_1^+}\mathbf{c}^{\alpha_1} \otimes \cdots \otimes \mathbf{b}^{l_{n-1}}\mathbf{k}^{\alpha_{n-1}^+}\mathbf{c}^{\alpha_{n-1}}.$$

The point here is that $\zeta^{-l_1-\cdots-l_{n-1}}$ that was contained in the coefficient in (19) via (14) has been absorbed away into q-bosons. Now the limits $q, \zeta \to 0$ can be smoothly taken reducing the above relation to (29).

4 Application to $U_q(A_n^{(1)})$-zero range process

4.1 $U_q(A_n^{(1)})$-zero range process

Let us briefly recall the discrete time inhomogeneous $U_q(A_n^{(1)})$-zero range process. Among a few versions of the models introduced in [12], it corresponds to the discrete time inhomogeneous one described in Sect. 3.3 therein. As we will remark after Theorem 5, it covers the continuous time version mentioned in (36).

Let L be a positive integer. Introduce the operator

$$T(\lambda|\mu_1, \ldots, \mu_L) = \text{Tr}_W \left(\mathcal{S}_{0,L}(\lambda, \mu_L) \cdots \mathcal{S}_{0,1}(\lambda, \mu_1) \right) \in \text{End}(W^{\otimes L}). \qquad (32)$$

In the terminology of the quantum inverse scattering method, it is the row transfer matrix of the $U_q(A_n^{(1)})$ vertex model of length L with periodic boundary condition whose quantum space is $W^{\otimes L}$ with inhomogeneity parameters $\mu_1, \ldots, \mu_L$ and the auxiliary space W carrying a parameter λ. If these spaces are labeled as $W_1 \otimes \cdots \otimes W_L$ and W_0, the stochastic R matrix $\mathcal{S}_{0,i}(\lambda, \mu_i)$ acts as $\mathcal{S}(\lambda, \mu_i)$ on $W_0 \otimes W_i$ and as the identity elsewhere. Owing to (4) and (6), the matrix (32) forms a commuting family (cf. [1]):

$$[T(\lambda|\mu_1, \ldots, \mu_L), T(\lambda'|\mu_1, \ldots, \mu_L)] = 0. \qquad (33)$$

We write the vector $|\alpha_1\rangle \otimes \cdots \otimes |\alpha_L\rangle \in W^{\otimes L}$ representing a state of the system as $|\alpha_1, \ldots, \alpha_L\rangle$ and the action of $T = T(\lambda|\mu_1, \ldots, \mu_L)$ as

$$T|\beta_1, \ldots, \beta_L\rangle = \sum_{\alpha_1, \ldots, \alpha_L \in \mathbb{Z}_{\geq 0}^n} T_{\beta_1, \ldots, \beta_L}^{\alpha_1, \ldots, \alpha_L} |\alpha_1, \ldots, \alpha_L\rangle \in W^{\otimes L}.$$

Then the matrix element is depicted by the concatenation of (8) as

$$T_{\beta_1,\ldots,\beta_L}^{\alpha_1,\ldots,\alpha_L} = \sum_{\gamma_1,\ldots,\gamma_L \in \mathbb{Z}_{\geq 0}^n} \gamma_L \xrightarrow{\;\alpha_1\;} \gamma_1 \xrightarrow{\;\alpha_2\;} \gamma_2 \cdots \gamma_{L-1} \xrightarrow{\;\alpha_L\;} \gamma_L,$$
$$\qquad\qquad\qquad\qquad\qquad\quad \beta_1 \qquad\quad \beta_2 \qquad\qquad\qquad \beta_L \qquad (34)$$

where the summand means $\prod_{i=1}^{L} \mathcal{S}(\lambda, \mu_i)_{\gamma_{i-1}, \beta_i}^{\gamma_i, \alpha_i}$ with $\gamma_0 = \gamma_L$. By the construction it satisfies the weight conservation, i.e., $T_{\beta_1,\ldots,\beta_L}^{\alpha_1,\ldots,\alpha_L} = 0$ unless $\alpha_1 + \cdots + \alpha_L = \beta_1 + \cdots + \beta_L \in \mathbb{Z}_{\geq 0}^n$.

Let t be a time variable and consider the evolution equation

$$|P(t+1)\rangle = T(\lambda|\mu_1, \ldots, \mu_L)|P(t)\rangle \in W^{\otimes L}. \tag{35}$$

Although this is an equation in an infinite-dimensional vector space, it splits into finite-dimensional subspaces which we call sectors due to the weight conservation property mentioned in the above. For an array $m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$ and the set $S(m) = \{(\sigma_1, \ldots, \sigma_L) \in (\mathbb{Z}_{\geq 0}^n)^L \mid \sigma_1 + \cdots + \sigma_L = m\}$, the corresponding sector which will also be referred to as m, is given by $\bigoplus_{(\sigma_1, \ldots, \sigma_L) \in S(m)} \mathbb{C}|\sigma_1, \ldots, \sigma_L\rangle$. We interpret a vector $|\sigma_1, \ldots, \sigma_L\rangle \in W^{\otimes L}$ with $\sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,n}) \in \mathbb{Z}_{\geq 0}^n$ as a state in which the i th site from the left is populated with $\sigma_{i,a}$ particles of the a th species. Thus, $m = (m_1, \ldots, m_n)$ means that there are m_a particles of species a in total in the corresponding sector.

To interpret (35) as the master equation of a discrete time Markov process, the matrix $T = T(\lambda|\mu_1, \ldots, \mu_L)$ should fulfill the conditions (i) non-negativity; all the elements (34) belong to $\mathbb{R}_{\geq 0}$ and (ii) sum-to-unity property; $\sum_{\alpha_1, \ldots, \alpha_L \in \mathbb{Z}_{\geq 0}^n} T_{\beta_1, \ldots, \beta_L}^{\alpha_1, \ldots, \alpha_L} = 1$ for any $(\beta_1, \ldots, \beta_L) \in (\mathbb{Z}_{\geq 0}^n)^L$.

The property (i) holds if $\Phi_q(\gamma|\beta; \lambda, \mu_i) \geq 0$ for all $i \in \mathbb{Z}_L$. This is achieved by taking $0 < \mu_i^\epsilon < \lambda^\epsilon < 1, 0 < q^\epsilon < 1$ in the either alternative $\epsilon = \pm 1$. The property (ii) means the total probability conservation and can be shown using (9) as in [12, Sec.3.2].

We call $T(\lambda|\mu_1, \ldots, \mu_L)$ Markov transfer matrix assuming $0 < \mu_i < \lambda < 1, 0 < q < 1$. The equation (35) represents a stochastic dynamics of n-species of particles hopping to the right periodically via an extra lane (horizontal arrows in (34)) which particles get on or get off when they leave or arrive at a site. The rate of these local processes is specified by (2), (3) and (8). For $n = 1$ and the homogeneous choice $\mu_1 = \cdots = \mu_L$, it reduces to the model introduced in [19].

From the homogeneous case $\mu_1 = \cdots = \mu_L = \mu$ of the Markov transfer matrix $T(\lambda|\mu_1, \ldots, \mu_L)$ (32), one can deduce the continuous time $U_q(A_n^{(1)})$-zero range process by a derivative with respect to λ at appropriate points [12, Sec.3.4]. The resulting Markov matrix H in the master equation $\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle$ consists of pairwise interaction terms as $H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}$ where $h_{i,i+1}$ acts on the $(i, i+1)$ th sites as h and as the identity elsewhere. The local Markov matrix h is the $\epsilon = 1$ case of [12, Rem.9], which reads as

$$
\begin{aligned}
h(|\alpha\rangle \otimes |\beta\rangle) = {} & a \sum_{0 < \gamma \leq \alpha} \frac{q^{\varphi(\alpha-\gamma,\gamma)}\mu^{|\gamma|-1}(q)_{|\gamma|-1}}{(\mu q^{|\alpha|-|\gamma|}; q)_{|\gamma|}} \prod_{i=1}^n \binom{\alpha_i}{\gamma_i}_q |\alpha - \gamma\rangle \otimes |\beta + \gamma\rangle \\
& + b \sum_{0 < \gamma \leq \beta} \frac{q^{\varphi(\gamma,\beta-\gamma)}(q)_{|\gamma|-1}}{(\mu q^{|\beta|-|\gamma|}; q)_{|\gamma|}} \prod_{i=1}^n \binom{\beta_i}{\gamma_i}_q |\alpha + \gamma\rangle \otimes |\beta - \gamma\rangle \\
& - \left(\sum_{i=0}^{|\alpha|-1} \frac{aq^i}{1 - \mu q^i} + \sum_{i=0}^{|\beta|-1} \frac{b}{1 - \mu q^i} \right) |\alpha\rangle \otimes |\beta\rangle,
\end{aligned} \tag{36}
$$

where the constraint $\gamma > 0$ for $\gamma \in \mathbb{Z}_{\geq 0}^n$ is equivalent to $|\gamma| \geq 1$. The parameters a, b are arbitrary as long as $a, b \in \mathbb{R}_{\geq 0}$ since the contributions proportional to them are commuting. See [12, Eq. (60)].

4.2 Stationary states

By definition a stationary state of the discrete time $U_q(A_n^{(1)})$-zero range process (35) is a vector $|\overline{P}\rangle \in W^{\otimes L}$ such that

$$|\overline{P}\rangle = T(\lambda|\mu_1, \ldots, \mu_L)|\overline{P}\rangle.$$

The stationary state is unique in each sector m, which we denote by $|\overline{P}(m)\rangle$. Apart from m, it depends on q and the inhomogeneity parameters $\mu_1, \ldots, \mu_L$ but not on λ thanks to the commutativity (33). Sectors $m = (m_1, \ldots, m_n)$ such that $\forall m_a \geq 1$ are called basic. Non-basic sectors are equivalent to a basic sector of some $n' < n$ models with a suitable relabeling of the species. Henceforth, we concentrate on the basic sectors. The coefficient appearing in the expansion

$$|\overline{P}(m)\rangle = \sum_{(\sigma_1, \ldots, \sigma_L) \in S(m)} \mathbb{P}(\sigma_1, \ldots, \sigma_L)|\sigma_1, \ldots, \sigma_L\rangle$$

is the stationary probability if it is properly normalized as $\sum_{(\sigma_1, \ldots, \sigma_L) \in S(m)} \mathbb{P}(\sigma_1, \ldots, \sigma_L) = 1$. In this paper, unnormalized ones will also be referred to as stationary probabilities by abuse of terminology.

If the dependence on the inhomogeneity parameters are exhibited as $\mathbb{P}(\sigma_1, \ldots, \sigma_L; \mu_1, \ldots, \mu_L)$, we have the cyclic symmetry $\mathbb{P}(\sigma_1, \ldots, \sigma_L; \mu_1, \ldots, \mu_L) = \mathbb{P}(\sigma_L, \sigma_1, \ldots, \sigma_{L-1}; \mu_L, \mu_1, \ldots, \mu_{L-1})$ by the construction. Examples of stationary states for $U_q(A_2^{(1)})$-zero range process have been given in [12,16].

Example 4 Consider $U_q(A_3^{(1)})$-zero range process in the minimum sector $m = (1, 1, 1)$ and system size $L = 2$, which is an eight-dimensional space. For the homogeneous case $\mu_1 = \mu_2 = \mu$, the stationary state is given up to normalization by

$$
\begin{aligned}
|\overline{P}(1, 1, 1)\rangle = {} & 2(1 - \mu q^2)\big(3 + q - \mu(1 + 3q)\big)|\emptyset, 123\rangle \\
& + 2(1 - \mu)\big(1 + q + 2q^2 - \mu(2q + q^2 + q^3)\big)|3, 12\rangle \\
& + (1 - \mu)(1 + 5q + q^2 + q^3 - \mu(1 + q + 5q^2 + q^3))|2, 13\rangle \\
& + (1 + q^2)(1 - \mu)\big(3 + q - \mu(1 + 3q)\big)|23, 1\rangle + \text{cyclic},
\end{aligned}
$$

where "cyclic" means further four terms obtained by the change $|\sigma_1, \sigma_2\rangle \to |\sigma_2, \sigma_1\rangle$. We have employed the multiset notation $|3, 12\rangle$ to mean $|(0, 0, 1), (1, 1, 0)\rangle$ etc. In the inhomogeneous case, we have

$$\mathbb{P}(23,1)/\mathbb{P}(\emptyset,123) = \frac{\mu_2^2(1-\mu_1)(1-\mu_1 q)(\mu_2 - \mu_1\mu_2 + \mu_1 q^2 - \mu_1\mu_2 q^2)}{\mu_1^2(1-\mu_2 q)(1-\mu_2 q^2)(\mu_1+\mu_2 - 2\mu_1\mu_2)},$$

$$\mathbb{P}(1,23)/\mathbb{P}(\emptyset,123) = \frac{\mu_2(1-\mu_1)(\mu_1 - \mu_1\mu_2 + \mu_2 q^2 - \mu_1\mu_2 q^2)}{\mu_1(1-\mu_2 q^2)(\mu_1+\mu_2 - 2\mu_1\mu_2)}$$

for example. The other ratios contain bulky factors. We expect that there is a normalization such that all the stationary probabilities belong to $\mathbb{Z}_{\geq 0}[q, -\mu_1, \ldots, -\mu_n]$.

4.3 Matrix product construction

Let $F = \bigoplus_{m\geq 0} \mathbb{C}(q)|m\rangle$ be the Fock space and $F^* = \bigoplus_{m\geq 0} \mathbb{C}(q)\langle m|$ be its dual on which the q-boson operators $\mathbf{b}, \mathbf{c}, \mathbf{k}$ act as

$$
\begin{aligned}
\mathbf{b}|m\rangle &= |m+1\rangle, & \mathbf{c}|m\rangle &= (1-q^m)|m-1\rangle, & \mathbf{k}|m\rangle &= q^m|m\rangle, \\
\langle m|\mathbf{c} &= \langle m+1|, & \langle m|\mathbf{b} &= \langle m-1|(1-q^m), & \langle m|\mathbf{k} &= \langle m|q^m,
\end{aligned}
\tag{37}
$$

where $|-1\rangle = \langle -1| = 0$. They satisfy the defining relations (16). We specify the bilinear pairing of F^* and F as $\langle m|m'\rangle = \theta(m=m')(q)_m$. Then $\langle m|(X|m'\rangle) = (\langle m|X)|m'\rangle$ holds and the trace is given by $\mathrm{Tr}(X) = \sum_{m\geq 0} \frac{\langle m|X|m\rangle}{(q)_m}$. As a vector space, the q-boson algebra $\mathcal{B}$ has the direct sum decomposition $\mathcal{B} = \mathbb{C}(q)1 \oplus \mathcal{B}_{\mathrm{fin}}$, where $\mathcal{B}_{\mathrm{fin}} = \bigoplus_{r\geq 1}(\mathcal{B}_+^r \oplus \mathcal{B}_-^r \oplus \mathcal{B}_0^r)$ with $\mathcal{B}_+^r = \bigoplus_{s\geq 0} \mathbb{C}(q)\mathbf{k}^s\mathbf{b}^r$, $\mathcal{B}_-^r = \bigoplus_{s\geq 0} \mathbb{C}(q)\mathbf{k}^s\mathbf{c}^r$ and $\mathcal{B}_0^r = \mathbb{C}(q)\mathbf{k}^r$. The trace $\mathrm{Tr}(X)$ is convergent if $X \in \mathcal{B}_{\mathrm{fin}}$. It vanishes unless $X \in \bigoplus_{r\geq 1} \mathcal{B}_0^r$ when it is evaluated by $\mathrm{Tr}(\mathbf{k}^r) = (1-q^r)^{-1}$.

In what follows, we regard $X_{\alpha_1,\ldots,\alpha_n}(\zeta) \in \bigotimes_{1\leq i\leq j\leq n-1} \mathcal{B}_{i,j}$ constructed in Sect. 3 as a linear operator on $F^{\otimes n(n-1)/2} = \bigotimes_{1\leq i\leq j\leq n-1} F_{i,j}$, where $F_{i,j}$ is a copy of F on which q-boson operators from $\mathcal{B}_{i,j}$ acts as (37). Now we state the main corollary of Theorem 1.

Theorem 5 *Stationary probabilities of the discrete time $U_q(A_n^{(1)})$-zero range process in Sect. 4.1 in basic sectors are expressed in the matrix product form*

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(X_{\sigma_1}(\mu_1) \cdots X_{\sigma_L}(\mu_L)), \tag{38}$$

where the trace Tr *is taken over* $F^{\otimes n(n-1)/2}$.

Proof From the expression (20), it immediately follows that

$$Z_\beta(\mu)Z_{0^n}(\lambda)^{-1}Z_\gamma(\lambda) = q^{\varphi(\beta,\gamma)}Z_{\beta+\gamma}(\mu) \quad (\beta,\gamma \in \mathbb{Z}_{\geq 0}^n),$$

which agrees with [16, Eq. (34)] called the auxiliary condition. In [16, Prop.6] it was proved that the ZF algebra (15) and the above relation imply the matrix product formula provided that the trace is convergent and not identically zero. The trace is convergent since (19) implies via an inductive argument with respect to n that nonzero contributions to it contains at least one $\mathbf{k}$ in every component in $\bigotimes_{1\leq i\leq j\leq n-1} \mathcal{B}_{i,j}$. The trace is neither zero. In fact (31) shows that $\mathrm{Tr}(Z_{\sigma_1}(\mu_1) \cdots Z_{\sigma_L}(\mu_L))$ is still nonzero even at $q=0$, $\forall \mu_i = 0$ coinciding with the homogeneous case of [15]. □

The stationary probabilities of the continuous time model (36) is obtained just by specializing (38) to $\mu_1 = \cdots = \mu_L = \mu$. Under this homogeneous choice, one can slightly simplify the matrix product formula (38) by the replacements (cf. [16, Eq. (42)] for $n = 2$):

$$X_{\alpha_1,\ldots,\alpha_n}(\mu) \rightarrow \frac{(\mu)_{|\alpha|}}{\prod_{i=1}^{n}(q)_{\alpha_i}} \left(Z_{0^n}(\mu)|_{A_{i,j}\rightarrow\mu A_{i,j}}\right) \mathbf{k}_{1,n-1}^{\alpha_1^+} \mathbf{c}_{1,n-1}^{\alpha_1} \cdots \mathbf{k}_{n-1,n-1}^{\alpha_{n-1}^+} \mathbf{c}_{n-1,n-1}^{\alpha_{n-1}}$$

with α_i^+ given by (17). This is a consequence of the automorphism of q-bosons (30) and removal of a common overall factor in the matrix product (38) within a sector for the homogeneous choice. After these changes the formula (38) with $\mu_1 = \cdots = \mu_L = \mu$ becomes regular at $\mu = 0$.

Example 6 Set $\sigma_i = (\sigma_{i,1}, \sigma_{i,2}, \sigma_{i,3}) \in \mathbb{Z}_{\geq 0}^3$. Up to an overall normalization, Example 4 is reproduced by the $L = 2$ case of

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \left(\prod_{i=1}^{L} \frac{\mu_i^{-|\sigma_i|}(\mu_i)_{|\sigma_i|}}{(q)_{\sigma_{i,1}}(q)_{\sigma_{i,2}}(q)_{\sigma_{i,3}}}\right) \mathrm{Tr}_{F^{\otimes 3}}\left(Z_{\sigma_1}(\mu_1)\cdots Z_{\sigma_L}(\mu_L)\right),$$

$$Z_{\alpha_1,\alpha_2,\alpha_3}(\mu) = \frac{(\mathbf{b}_1)_\infty}{(\mu^{-1}\mathbf{b}_1)_\infty}(\mathbf{c}_1\mathbf{b}_2)_\infty \frac{(\mathbf{k}_1\mathbf{b}_3)_\infty}{(\mu^{-1}\mathbf{k}_1\mathbf{b}_3)_\infty} \frac{1}{(\mu^{-1}\mathbf{c}_1\mathbf{b}_2)_\infty} \mathbf{k}_2^{\alpha_2+\alpha_3}\mathbf{c}_2^{\alpha_1}\mathbf{k}_3^{\alpha_3}\mathbf{c}_3^{\alpha_2}.$$

See (22). We have $\mathbf{x}_1 = \mathbf{x}_{1,1}, \mathbf{x}_2 = \mathbf{x}_{1,2}, \mathbf{x}_3 = \mathbf{x}_{2,2}$ for $\mathbf{x} = \mathbf{b}, \mathbf{c}$ and $\mathbf{k}$ in the notation in Theorem 2.

For the homogeneous case $\mu_1 = \cdots = \mu_L = \mu$, this may be replaced, up to normalization, by a slightly simplified version

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \left(\prod_{i=1}^{L} \frac{(\mu)_{|\sigma_i|}}{(q)_{\sigma_{i,1}}(q)_{\sigma_{i,2}}(q)_{\sigma_{i,3}}}\right) \mathrm{Tr}_{F^{\otimes 3}}\left(Z_{\sigma_1}(\mu)\cdots Z_{\sigma_L}(\mu)\right),$$

$$Z_{\alpha_1,\alpha_2,\alpha_3}(\mu) = \frac{(\mu\mathbf{b}_1)_\infty}{(\mathbf{b}_1)_\infty}(\mu\mathbf{c}_1\mathbf{b}_2)_\infty \frac{(\mu\mathbf{k}_1\mathbf{b}_3)_\infty}{(\mathbf{k}_1\mathbf{b}_3)_\infty} \frac{1}{(\mathbf{c}_1\mathbf{b}_2)_\infty} \mathbf{k}_2^{\alpha_2+\alpha_3}\mathbf{c}_2^{\alpha_1}\mathbf{k}_3^{\alpha_3}\mathbf{c}_3^{\alpha_2},$$

which is suitable for studying the $\mu = 0$ case.

5 Summary

We have studied the Zamolodchikov-Faddeev algebra (12), (15) whose structure function is the $U_q(A_n^{(1)})$ stochastic R matrix (1)–(3) introduced in [12]. A q-boson representation has been constructed either by a recursion relation with respect to the rank n (Theorem 1) or by giving the explicit formula (Theorem 2). It yields a matrix product formula for the stationary probabilities in the $U_q(A_n^{(1)})$-zero range process (Theorem 5). They extend the earlier results for $n = 2$ [16] to general n, although the method of the proof of the ZF algebra relation is different. At $q = 0$, the q-boson representation of the matrix product operators in this paper coincides with the homogeneous case $w_1 = \cdots\cdots = w_n$ of [15] as shown in (31). At $q = 0$, there is another

set of matrix product operators originating in the combinatorial R in crystal theory [13] and the tetrahedron equation [14]. They agree with the $q = 0$ case of the present paper for $n = 2$ upon adjustment of conventions. Their relation for $n \geq 3$ still requires a further investigation.

Appendix A: Proof of Theorem 1

We will use the same symbol $\varphi(\beta, \gamma)$ (3) to mean $(3)|_{n \to n+1}$. Moreover, $\sum_{1 \leq i < j \leq n+1} \beta_i \gamma_j$ with $\beta \in \mathbb{Z}_{\geq 0}^n, \gamma \in \mathbb{Z}_{\geq 0}^{n+1}$ will also be denoted by $\varphi(\beta, \gamma)$.

We are going to prove Theorem 1 by induction on n. The relation $(15)|_{n=1}$ is valid since it is equivalent to $(9)|_{n=1}$. (The case $n = 2$ has been shown in [16] by a method different from here.) Thus, our task in the sequel is to show $(15)|_{n \to n+1}$ assuming $(15)|_{n=n}$.

Lemma 7 *Under the assumption* (15) $|_{n=n}$*, the relation* (15) $|_{n \to n+1}$ *follows from the equality*

$$\sum_{m \leq s} q^{\varphi(m,\alpha)} \Phi_q^{(n)}(m|s; \lambda, \mu) \bigotimes_{i=1}^{n} \mathbf{b}^{s_i - m_i} \mathbf{c}^{\alpha_i} \mathbf{b}^{m_i}$$

$$= \sum_{\gamma \leq \alpha} q^{\varphi(\gamma,\alpha) + \varphi(s,\gamma) - \varphi(\alpha,\gamma)} \Phi_q^{(n+1)}(\gamma|\alpha, \lambda, \mu) \bigotimes_{i=1}^{n} \mathbf{c}^{\gamma_i} \mathbf{b}^{s_i} \mathbf{c}^{\alpha_i - \gamma_i} \quad (\forall s \in \mathbb{Z}_{\geq 0}^n, \forall \alpha \in \mathbb{Z}_{\geq 0}^{n+1}).$$

$$(39)$$

Proof Substitute $(19)|_{n \to n+1}$ into $(15)|_{n \to n+1}$. Applying (13) on the LHS, we get

$$\text{LHS} = \sum_{m, l \in \mathbb{Z}_{\geq 0}^n} \sum_{t \leq m} \Phi_q^{(n)}(l|m + l - t; \lambda, \mu) X_t(\lambda) X_{m+l-t}(\mu)$$

$$\otimes \bigotimes_{i=1}^{n} (m_i, \alpha_i^+, \alpha_i)(l_i, \beta_i^+, \beta_i),$$

$$\text{RHS} = \sum_{\gamma \leq \alpha} q^{\varphi(\alpha-\gamma, \beta-\gamma)} \Phi_q^{(n+1)}(\gamma|\alpha, \lambda, \mu) \sum_{t, s \in \mathbb{Z}_{\geq 0}^n} X_t(\lambda) X_s(\mu) \otimes \bigotimes_{i=1}^{n} (t_i, \gamma_i^+, \gamma_i)$$

$$\times (s_i, \alpha_i^+ + \beta_i^+ - \gamma_i^+, \alpha_i + \beta_i - \gamma_i),$$

where the temporal notation $(i, j, k) := \mathbf{b}^i \mathbf{k}^j \mathbf{c}^k$ is used. The symbol α_i^+ is defined by $(17)|_{n \to n+1}$ and β_i^+, γ_i^+ are similar. Thus, to prove LHS = RHS, it is sufficient, though not a priori necessary, to show that the coefficients of $X_t(\lambda) X_s(\mu)$ are equal for each choice of $t, s \in \mathbb{Z}_{\geq 0}^n$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}^{n+1}$. Explicitly, it reads

$$\sum_{t \leq m \leq t+s} \Phi_q^{(n)}(t+s-m|s; \lambda, \mu) \bigotimes_{i=1}^{n} (m_i, \alpha_i^+, \alpha_i)(t_i + s_i - m_i, \beta_i^+, \beta_i)$$

$$= \sum_{\gamma \leq \alpha} q^{\varphi(\alpha-\gamma, \beta-\gamma)} \Phi_q^{(n+1)}(\gamma|\alpha, \lambda, \mu) \bigotimes_{i=1}^{n} (t_i, \gamma_i^+, \gamma_i)$$

$$\times (s_i, \alpha_i^+ + \beta_i^+ - \gamma_i^+, \alpha_i + \beta_i - \gamma_i).$$

One can pull out the common factor $\bigotimes_{i=1}^{n} \mathbf{k}^{\alpha_i^+ + \beta_i^+}$ from this equality to the left. The result reads

$$\sum_{t \leq m \leq t+s} q^{\varphi(t+s-m, \alpha)} \Phi_q^{(n)}(t+s-m|s; \lambda, \mu) \bigotimes_{i=1}^{n} (m_i, 0, \alpha_i)(t_i + s_i - m_i, 0, \beta_i)$$

$$= \sum_{\gamma \leq \alpha} q^{\varphi(\gamma, \alpha) + \varphi(s, \gamma) - \varphi(\alpha, \gamma)} \Phi_q^{(n+1)}(\gamma|\alpha, \lambda, \mu) \bigotimes_{i=1}^{n} (t_i, 0, \gamma_i)(s_i, 0, \alpha_i + \beta_i - \gamma_i).$$

Note that this further contains a common rightmost factor $\bigotimes_{i=1}^{n} \mathbf{c}^{\beta_i}$ and a common leftmost factor $\bigotimes_{i=1}^{n} \mathbf{b}^{t_i}$. Removing them leads to (39). □

Lemma 8 *The following identities hold:*

$$\mathbf{c}^m \mathbf{b}^s = \sum_{j=0}^{s} q^{j(m-s+j)} (q^m; q^{-1})_{s-j} \binom{s}{j}_q \mathbf{b}^j \mathbf{c}^{m-s+j} \qquad (m, s \in \mathbb{Z}_{\geq 0}), \qquad (40)$$

$$z^s = \sum_{r=0}^{s} (-1)^r q^{r(r-1)/2} \binom{s}{r}_q (z; q^{-1})_r \qquad (s \in \mathbb{Z}_{\geq 0}). \qquad (41)$$

Proof These relations can easily be checked by means of the q-binomial theorem. □

Lemma 9 *The equality* (39) *is equivalent to*

$$\sum_{m \leq s} q^{\varphi(m, \alpha) + \sum_{i=1}^{n} \alpha_i m_i} \Phi_q^{(n)}(m|s; \lambda, \mu)$$

$$= \sum_{\gamma \leq \alpha} q^{\varphi(\gamma, \alpha) + \varphi(s, \gamma) - \varphi(\alpha, \gamma) + \sum_{i=1}^{n} s_i \gamma_i} \Phi_q^{(n+1)}(\gamma|\alpha; \lambda, \mu) \qquad (42)$$

for any $s \in \mathbb{Z}_{\geq 0}^n$ *and* $\alpha \in \mathbb{Z}_{\geq 0}^{n+1}$.

Proof Comparing the coefficient of the basis vector $\bigotimes_{i=1}^{n} \mathbf{b}^{s_i - p_i} \mathbf{c}^{\alpha_i - p_i}$ of $\mathcal{B}^{\otimes n}$ on the both sides of (39) by means of (40), it is translated into the equality of the coefficients

$$\sum_{m \leq s} q^{\varphi(m, \alpha)} \Phi_q^{(n)}(m|s; \lambda, \mu) \prod_{i=1}^{n} q^{(m_i - p_i)(\alpha_i - p_i)} \binom{\alpha_i}{p_i}_q \binom{m_i}{p_i}_q$$

$$= \sum_{\gamma \leq \alpha} q^{\varphi(\gamma, \alpha) + \varphi(s^\vee - \alpha, \gamma)} \Phi_q^{(n+1)}(\gamma|\alpha; \lambda, \mu) \prod_{i=1}^{n} q^{(\gamma_i - p_i)(s_i - p_i)} \binom{s_i}{p_i}_q \binom{\gamma_i}{p_i}_q$$

$$(43)$$

for any arrays of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, $s = (s_1, \ldots, s_n)$, $p = (p_1, \ldots, p_n)$ such that $p_i \le \min(s_i, \alpha_i)$ for all $1 \le i \le n$. On the RHS of (43), we have introduced the notation

$$s^\vee = (s_1, \ldots, s_n, 0) \tag{44}$$

for later convenience. Of course $\varphi(s^\vee, \gamma) = \varphi(s, \gamma)$ by the definition. By substituting (3) into (43) and removing a common overall factor from the both sides, it becomes

$$\sum_{p \le m \le s} q^{\varphi(m, \alpha - m^\vee) + \varphi(s, m)} \nu^{|m|} \frac{(\lambda)_{|m|} (\nu)_{|s|-|m|}}{(\mu)_{|s|}} \prod_{i=1}^n q^{(m_i - p_i)(\alpha_i - p_i)} \binom{s_i - p_i}{m_i - p_i}_q$$
$$= \sum_{p^\vee \le \gamma \le \alpha} q^{\varphi(\gamma, \alpha - \gamma) + \varphi(s, \gamma)} \nu^{|\gamma|} \frac{(\lambda)_{|\gamma|} (\nu)_{|\alpha|-|\gamma|}}{(\mu)_{|\alpha|}} \binom{\alpha_{n+1}}{\gamma_{n+1}}_q \prod_{i=1}^n q^{(s_i - p_i)(\gamma_i - p_i)} \binom{\alpha_i - p_i}{\gamma_i - p_i}_q,$$

where $\nu = \mu/\lambda$ and $m^\vee$, $p^\vee$ are defined similar to (44). By the replacement

$$s \to s + p, \quad m \to m + p, \quad \alpha \to \alpha + p^\vee, \quad \gamma \to \gamma + p^\vee,$$
$$\lambda \to q^{-|p|}\lambda, \quad \mu \to q^{-|p|}\mu,$$

the above relation is cast into

$$\sum_{m \le s} q^{\varphi(m, \alpha) + \varphi(s-m, m) + \sum_{i=1}^n \alpha_i m_i} \nu^{|m|} \frac{(\lambda)_{|m|} (\nu)_{|s|-|m|}}{(\mu)_{|s|}} \prod_{i=1}^n \binom{s_i}{m_i}_q$$
$$= \sum_{\gamma \le \alpha} q^{\varphi(\gamma, \alpha) + \varphi(s^\vee - \gamma, \gamma) + \sum_{i=1}^n s_i \gamma_i} \nu^{|\gamma|} \frac{(\lambda)_{|\gamma|} (\nu)_{|\alpha|-|\gamma|}}{(\mu)_{|\alpha|}} \prod_{i=1}^{n+1} \binom{\alpha_i}{\gamma_i}_q, \tag{45}$$

which turns out to be free from $p = (p_1, \ldots, p_n)$. This coincides with (42). □

So far we have shown that Theorem 1 is a corollary of (42). Let us proceed to a proof of the latter.

Lemma 10 *The equality* (42) *holds for* $n \in \mathbb{Z}_{\ge 0}$.

Proof Again we invoke the induction on n. At $n = 0$, (42) reads $1 = \sum_{\gamma_1 \le \alpha_1} \Phi_q^{(1)}(\gamma_1 | \alpha_1; \lambda, \mu)$, which is indeed valid thanks to (9). Assume (42)$|_{n \to n-1}$. Applying (11) to the LHS of (42)$|_{n=n}$, we get

$$\text{LHS} = \sum_{m_1 \le s_1} q^{m_1 |\alpha|} \Phi_q^{(1)}(m_1 | s_1; \lambda, \mu) \sum_{\overline{m} \le \overline{s}} q^{\varphi(\overline{m}, \overline{\alpha}) + \sum_{i=2}^n m_i \alpha_i} \Phi_q^{(n-1)}(\overline{m} | \overline{s}; q^{m_1}\lambda, q^{s_1}\mu),$$

where $\overline{m}, \overline{s} \in \mathbb{Z}_{\geq 0}^{n-1}$ and $\overline{\alpha} \in \mathbb{Z}_{\geq 0}^n$ are defined by (10). Rewriting the sum over $\overline{m}$ by the induction assumption $(42)|_{n \to n-1}$ yields ($\nu = \mu/\lambda$ as before)

$$
\begin{aligned}
\text{LHS} &= \sum_{m_1 \leq s_1} q^{m_1|\alpha|} \Phi_q^{(1)}(m_1|s_1; \lambda, \mu) \\
&\quad \times \sum_{\overline{\gamma} \leq \overline{\alpha}} q^{\varphi(\overline{\gamma}, \overline{\alpha}) + \varphi(\overline{s}, \overline{\gamma}) - \varphi(\overline{\alpha}, \overline{\gamma}) + \sum_{i=2}^n s_i \gamma_i} \Phi_q^{(n)}(\overline{\gamma}|\overline{\alpha}; q^{m_1}\lambda, q^{s_1}\mu) \\
&= \sum_{m_1 \leq s_1} q^{m_1(|\alpha| - |\overline{\gamma}|)} \nu^{m_1} \binom{s_1}{m_1}_q \\
&\quad \times \sum_{\overline{\gamma} \leq \overline{\alpha}} q^{\xi(s, \overline{\alpha}, \overline{\gamma})} \nu^{|\overline{\gamma}|} \frac{(\lambda)_{m_1 + |\overline{\gamma}|} (\nu)_{m_1 - s_1 + |\overline{\alpha}| - |\overline{\gamma}|}}{(\mu)_{s_1 + |\overline{\alpha}|}} \prod_{i=2}^{n+1} \binom{\alpha_i}{\gamma_i}_q,
\end{aligned}
\tag{46}
$$

where $\xi(s, \overline{\alpha}, \overline{\gamma}) = \varphi(\overline{\gamma}, \overline{\alpha}) + \varphi(\overline{s}, \overline{\gamma}) + \sum_{i=2}^n s_i \gamma_i - \varphi(\overline{\gamma}, \overline{\gamma}) + s_1 |\overline{\gamma}|$. On the other hand, the RHS of (42) has been written out in the RHS of (45), which is expressed using the above $\xi(s, \overline{\alpha}, \overline{\gamma})$ as

$$
\text{RHS} = \sum_{\gamma_1 \leq \alpha_1, \overline{\gamma} \leq \overline{\alpha}} q^{\gamma_1(|\overline{\alpha}| - |\overline{\gamma}| + s_1) + \xi(s, \overline{\alpha}, \overline{\gamma})} \nu^{|\gamma|} \frac{(\lambda)_{|\gamma|} (\nu)_{|\alpha| - |\gamma|}}{(\mu)_{|\alpha|}} \prod_{i=1}^{n+1} \binom{\alpha_i}{\gamma_i}_q.
\tag{47}
$$

Denote the summand in (46) by $\text{LHS}(m_1, \gamma_2, \dots, \gamma_{n+1})$ and the one in (47) by $\text{RHS}(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$. We claim $\sum_{m_1 \leq s_1} \text{LHS}(m_1, \gamma_2, \dots, \gamma_{n+1}) = \sum_{\gamma_1 \leq \alpha_1} \text{RHS}(\gamma_1, \gamma_2, \dots, \gamma_{n+1})$ holds for each fixed $\overline{\gamma} = (\gamma_2, \dots, \gamma_{n+1})$. In fact, the two sides possess a common overall factor $q^{\xi(s, \overline{\alpha}, \overline{\gamma})} \nu^{|\overline{\gamma}|} \frac{(\lambda)_{|\overline{\gamma}|} (\nu)_{|\overline{\alpha}| - |\overline{\gamma}|}}{(\mu)_{|\overline{\alpha}|}} \prod_{i=2}^{n+1} \binom{\alpha_i}{\gamma_i}_q$. By removing it, the claim becomes

$$
\begin{aligned}
&\sum_{m_1 \leq s_1} q^{m_1(|\alpha| - |\overline{\gamma}|)} \nu^{m_1} \frac{(q^{|\overline{\gamma}|}\lambda)_{m_1} (q^{|\overline{\alpha}| - |\overline{\gamma}|}\nu)_{m_1 - s_1}}{(q^{|\overline{\alpha}|}\mu)_{s_1}} \binom{s_1}{m_1}_q \\
&= \sum_{\gamma_1 \leq \alpha_1} q^{\gamma_1(|\overline{\alpha}| - |\overline{\gamma}| + s_1)} \nu^{\gamma_1} \frac{(q^{|\overline{\gamma}|}\lambda)_{\gamma_1} (q^{|\overline{\alpha}| - |\overline{\gamma}|}\nu)_{\alpha_1 - \gamma_1}}{(q^{|\overline{\alpha}|}\mu)_{\alpha_1}} \binom{\alpha_1}{\gamma_1}_q.
\end{aligned}
$$

This is simply stated as $f(\alpha_1, s_1; q^{|\overline{\gamma}|}\lambda, q^{|\overline{\alpha}|}\mu) = f(s_1, \alpha_1; q^{|\overline{\gamma}|}\lambda, q^{|\overline{\alpha}|}\mu)$ in terms of the function defined for $s, t \in \mathbb{Z}_{\geq 0}$ and $\nu = \mu/\lambda$ by

$$
f(s, t; \lambda, \mu) = \sum_{i=0}^t q^{si} \nu^i \frac{(\lambda)_i (\nu)_{t-i}}{(\mu)_t} \binom{t}{i}_q.
\tag{48}
$$

This is verified in Lemma 11.   □

Lemma 11 *The function (48) enjoys the symmetry $f(s, t; \lambda, \mu) = f(t, s; \lambda, \mu)$ for $s, t \in \mathbb{Z}_{\geq 0}$.*

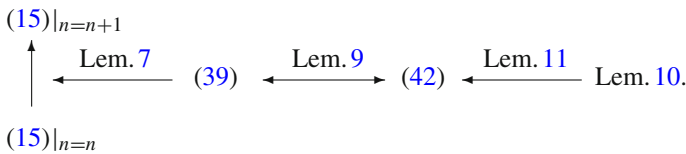*Proof* By applying (41) to the factor q^{si} in (48), $f(s, t; \lambda, \mu)$ is rewritten as follows:

$$f(s, t; \lambda, \mu) = \sum_{i=0}^{t} \nu^i \frac{(\lambda)_i (\nu)_{t-i}}{(\mu)_t} \binom{t}{i}_q \sum_{r=0}^{s} (-1)^r q^{r(r-1)/2} \binom{s}{r}_q (q^i; q^{-1})_r$$

$$= \sum_{r=0}^{\min(s,t)} \sum_{i=r}^{t} \nu^i \frac{(\lambda)_r (q^r \lambda)_{i-r} (\nu)_{t-i}}{(\mu)_r (q^r \mu)_{t-r}} (q^t; q^{-1})_r \binom{t-r}{i-r}_q (-1)^r q^{r(r-1)/2} \binom{s}{r}_q.$$

Replacing i by $i + r$, we have

$$f(s, t; \lambda, \mu) = \sum_{r=0}^{\min(s,t)} \nu^r (-1)^r q^{r(r-1)/2} (q)_r \binom{s}{r}_q \binom{t}{r}_q \frac{(\lambda)_r}{(\mu)_r} h(r, t; \lambda, \mu),$$

$$h(r, t; \lambda, \mu) = \sum_{i=0}^{t-r} \nu^i \frac{(q^r \lambda)_i (\nu)_{t-r-i}}{(q^r \mu)_{t-r}} \binom{t-r}{i}_q. \qquad (49)$$

From $\sum_{i=0}^{t-r} \Phi_q^{(1)}(i | t - r; q^r \lambda, q^r \mu) = 1$ (9), we find $h(r, t; \lambda, \mu) = 1$. Then the expression (49) tells that $f(s, t; \lambda, \mu) = f(t, s; \lambda, \mu)$. $\qquad \square$

Proof of Theorem 1 As a summary of the arguments so far, the induction step from $(15)|_{n=n}$ to $(15)|_{n=n+1}$ has been established by the following scheme:

$$(15)|_{n=n+1}$$
$$\uparrow \quad \xleftarrow{\text{Lem. } 7} \quad (39) \quad \xleftarrow{\text{Lem. } 9} \quad (42) \quad \xleftarrow{\text{Lem. } 11} \quad \text{Lem. } 10.$$
$$(15)|_{n=n}$$

Since $(15)|_{n=1}$ is valid as explained in the beginning of the appendix, the ZF relation (15) holds for any n. This completes the proof of Theorem 1. $\qquad \square$

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