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Strong homotopy Lie algebras, homotopy Poisson manifolds and Courant algebroids

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Abstract We study Maurer–Cartan elements on homotopy Poisson manifolds of degree n. They unify many twisted or homotopy structures in Poisson geometry and mathematical physics, such as twisted Poisson manifolds, quasi-Poisson $\mathfrak{g}$-manifolds, and twisted Courant algebroids. Using the fact that the dual of an n-term L_∞-algebra is a homotopy Poisson manifold of degree $n - 1$, we obtain a Courant algebroid from a 2-term L_∞-algebra $\mathfrak{g}$ via the degree 2 symplectic NQ-manifold $T^*[2]\mathfrak{g}^*[1]$. By integrating the Lie quasi-bialgebroid associated to the Courant algebroid, we obtain a Lie-quasi-Poisson groupoid from a 2-term L_∞-algebra, which is proposed to be the geometric structure on the dual of a Lie 2-algebra. These results lead to a construction of a new 2-term L_∞-algebra from a given one, which could produce many interesting examples.

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1 Introduction

The concept of an L_∞-algebra (sometimes called a strongly homotopy (sh) Lie algebra)
was originally introduced in [22,38] as a model for "Lie algebras that satisfy the
Jacobi identity up to all higher homotopies". A Lie 2-algebra is a categorification of
a Lie algebra. It is well-known that the category of Lie 2-algebras is equivalent to
the category of 2-term L_∞-algebras [4]. The structure of a Lie 2-algebra or a 2-term
L_∞-algebra appears in many areas such as string theory, higher symplectic geometry
[5,6], and Courant algebroids [27].

A homotopy Poisson algebra is a graded commutative algebra with an L_∞-structure
whose brackets satisfy the Leibniz rule. It has appeared in Voronov's work [17,40]
under the name "higher Poisson structures" and Cattaneo and Felder's work [10]
under the name "P_∞-structures". See also [8,26] for more recent work involving this
structure. A homotopy Poisson algebra structure of degree n on a graded commutative
algebra $\mathfrak{a}$ means that there is a homotopy Poisson algebra structure on the shifted
space $\mathfrak{a}[n]$. One interesting example of homotopy Poisson manifolds is given by the
dual $\mathfrak{g}^*[n-1]$ of an n-term L_∞-algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{1-n}$, which turns
out to be a homotopy Poisson manifold of degree $n-1$. This generalizes the fact that
there is a Lie-Poisson structure on the dual space of a Lie algebra to the case of n-term
L_∞-algebras.

It is known that there is a one-to-one correspondence between Poisson manifolds
and symplectic NQ-manifolds of degree 1,

$$(M, \pi) \leftrightsquigarrow (T^*[1]M, Q = \{\pi, \cdot\}),$$

where the bracket $\{\cdot, \cdot\}$ is the canonical Poisson bracket on the cotangent bundle of
the manifold M. One can further study the cotangent bundle of a homotopy Poisson
manifold $\mathcal{M}$ of degree n. The shifted cotangent bundle $T^*[n+1]\mathcal{M}$ is a symplectic
NQ-manifold of degree $n+1$. Then we study Maurer–Cartan elements on a degree n
homotopy Poisson manifold $\mathcal{M}$, which are solutions of the Maurer–Cartan equation
associated to the L_∞-structure on the function ring $C^\infty(\mathcal{M})$. Recall that there are
many types of twisted or homotopy structures in Poisson geometry and mathematical
physics. See [2,14,19,20,29,33,34] for lists of such examples. We provide a unified
framework to describe these structures in which they are viewed as solutions of the
Maurer–Cartan equation. For example, given a manifold M, the cotangent bundle
$T^*[1]M$, equipped with the Schouten bracket $l_2 = [\cdot, \cdot]_S$, is a homotopy Poisson
manifold of degree 1. Furthermore, a degree 2 function $\pi \in C^\infty(T^*[1]M)$ is a Maurer–
Cartan element on $T^*[1]M$ if and only if π is a Poisson tensor on M; in the twisted
case, given a closed 3-form H, the cotangent bundle $T^*[1]M$ with $l_2 = [\cdot, \cdot]_S$ and a
ternary bracket $l_3 = H(\cdot, \cdot, \cdot)$ on $C^\infty(T^*[1]M)$ is a new homotopy Poisson manifold

of degree 1, and a degree 2 function π satisfies the Maurer–Cartan equation if and only if π is a twisted Poisson tensor.

A homotopy Poisson manifold $\mathcal{M}$ of degree n is called symplectic if its binary bracket is non-degenerate (See [17] for the $\mathbb{Z}_2$-graded case). If $\mathcal{M}$ is a symplectic N-manifold of degree n, then a function α of degree $n + 1$, satisfying the Maurer–Cartan equation $l_2(\alpha, \alpha) = 0$, induces a differential $Q = l_2(\alpha, \cdot)$ of degree 1 on $\mathcal{M}$. This shows that a differential graded symplectic manifold or symplectic NQ-manifold is a special homotopy symplectic manifold with a Maurer–Cartan element. A general degree n homotopy symplectic manifold $(\mathcal{M}, \{l_i\}_{1 \le i < \infty})$ with $l_1 = 0$ is, therefore, a homotopy version of a symplectic N-manifold. In this case, the Maurer–Cartan equation becomes

$$\frac{1}{2!}l_2(\alpha, \alpha) - \frac{1}{3!}l_3(\alpha, \alpha, \alpha) + \cdots = 0,$$

which can be viewed as a homotopy version of the classical master equation. To relate this to physical applications, we prove that homotopy symplectic manifolds of degree n with zero l_1 are in one-to-one correspondence with twisted symplectic NQ-manifolds with $\Theta|_{\mathcal{M}} = 0$ (see Definition 3.11), to which the authors in [16] associate sigma models with boundary via AKSZ formalism.

The notion of a Lie bialgebroid was introduced by Mackenzie and Xu [25]. To study the double of a Lie bialgebroid,[1] Liu–Weinstein–Xu introduced the notion of a Courant algebroid [23]. Then Roytenberg gave an equivalent definition in [30]. Symplectic NQ-manifolds of degree 2 are in one-to-one correspondence with Courant algebroids [28]. On the other hand, a Courant algebroid gives rise to a 2-term L_∞-algebra [27]. This paper uses these facts to construct a new 2-term L_∞-algebra from a given one as follows. Given a Lie 2-algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, its dual $\mathfrak{g}^*[1]$ is a homotopy Poisson manifold of degree 1, which implies that $T^*[2]\mathfrak{g}^*[1]$ is a symplectic NQ-manifold of degree 2. Thus, we obtain a Courant algebroid $E = \mathfrak{g}^*_{-1} \times (\mathfrak{g}_0 \oplus \mathfrak{g}^*_0)$, where $\mathfrak{g}^*_{-1}$ is the base manifold. Choosing some special sections of E and linear functions on the base manifold $\mathfrak{g}^*_{-1}$, we obtain a 2-term L_∞-algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_{-1}$, where $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}^*_0)$ and $\tilde{\mathfrak{g}}_{-1} = \mathfrak{g}_{-1}$. We thus have

$$
\begin{array}{ccccc}
\text{2-term } L_\infty\text{-} & & \text{homotopy} & & \text{degree 2} \\
\text{algebra } \mathfrak{g} & \longmapsto & \text{Poisson} & \longmapsto & \text{symplectic} \\
& & \text{manifold} & & \text{NQ-manifold} \\
& & \mathfrak{g}^*[1] & & T^*[2]\mathfrak{g}^*[1] \\
& \longmapsto & \text{Courant algebroid} & \longmapsto & \text{2-term } L_\infty\text{-} \\
& & E = \mathfrak{g}^*_{-1} \times (\mathfrak{g}_0 \oplus \mathfrak{g}^*_0) & & \text{algebra } \tilde{\mathfrak{g}}.
\end{array}
$$

In particular, we obtain the 2-term L_∞-algebra associated to the omni-Lie algebra [36,41] from a vector space V, which is viewed as the abelian 2-term L_∞-algebra $(V \xrightarrow{\text{id}} V, l_2 = 0, l_3 = 0)$, and obtain the 2-term L_∞-algebra of string type [36]

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[1] There are different ways to describe the double of a Lie bialgebroid, e.g. Mackenzie gave the description of Drinfeld doubles for Lie bialgebroids using double Lie algebroids in [24]; Roytenberg and Voronov gave the description of Drinfeld doubles for Lie bialgebroids using graded manifolds in [28,39] respectively.

from the 2-term L_∞-algebra $(\mathbb{R} \xrightarrow{0} \mathfrak{k}, l_2 = [\cdot, \cdot]_{\mathfrak{k}}, l_3 = 0)$, where $(\mathfrak{k}, [\cdot, \cdot]_{\mathfrak{k}})$ is a Lie algebra. These interesting examples show that the construction of $\tilde{\mathfrak{g}}$ from $\mathfrak{g}$ has some important properties and applications, which will be studied in the future, since it is unrelated to the key subject of this paper. Nevertheless, we prove that there is a homomorphism from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$ (see Theorem 4.11).

We also observe that the Courant algebroid $E = \mathfrak{g}^*_{-1} \times (\mathfrak{g}_0 \oplus \mathfrak{g}_0^*)$ obtained above is the double of a Lie quasi-bialgebroid (A, δ, ϕ), where $A = \mathfrak{g}^*_{-1} \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}^*_{-1}$ is an action Lie algebroid. By integration, we get a Lie-quasi-Poisson groupoid (Γ, Π, ϕ), where $\Gamma = \mathfrak{g}^*_{-1} \times \mathfrak{g}_0^* \rightrightarrows \mathfrak{g}^*_{-1}$ is an action groupoid. In summary, we have
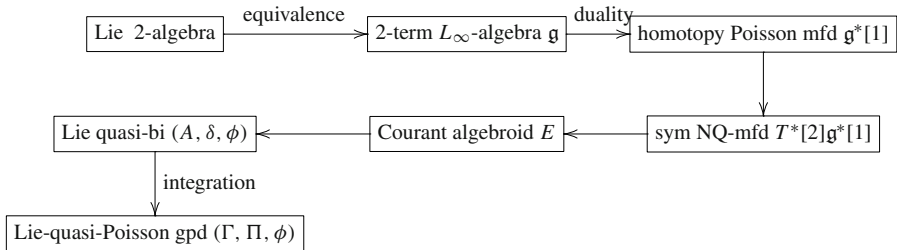
If we view the 2-vector space $\mathfrak{g}^*_{-1} \oplus \mathfrak{g}_0^* \rightrightarrows \mathfrak{g}^*_{-1}$ as the dual of the 2-vector space $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \rightrightarrows \mathfrak{g}_0$, this Lie-quasi-Poisson groupoid can be viewed as the natural geometric structure on the "dual" of a Lie 2-algebra.

Notations: For a graded vector space $V = \sum_{n \in \mathbb{Z}} V_n$, we use $V[l]$ to denote the l-shifted graded vector space, namely $V[l]_n = V_{l+n}$; we use $\mathsf{Sym}(V)$ to denote the symmetric algebra of V. We use $[\cdot, \cdot]_S$ to denote the Schouten bracket of sections of a Lie algebroid.

2 Preliminaries

Lie algebras can be categorified to Lie 2-algebras. For a good introduction on this subject see [4,21]. Vector spaces can be categorified to 2-vector spaces. Let Vect be the category of vector spaces. A 2-vector space is a category in the category Vect. A 2-vector space C is a category with a vector space of objects C_0 and a vector space of morphisms C_1, such that all the structure maps are linear. Let $s, t : C_1 \longrightarrow C_0$ be the source and target maps respectively.

The 2-category of 2-vector spaces is equivalent to the 2-category of 2-term complexes of vector spaces. Roughly speaking, given a 2-vector space C, $\ker(s) \xrightarrow{t} C_0$ is a 2-term complex. Conversely, any 2-term complex of vector spaces $V_1 \xrightarrow{d} V_0$ gives rise to a 2-vector space of which the set of objects is V_0, the set of morphisms is $V_0 \oplus V_1$, the source map s is given by $s(v + m) = v$, and the target map t is given by $t(v + m) = v + dm$, where $v \in V_0$, $m \in V_1$.

A Lie 2-algebra is a 2-vector space C equipped with a skew-symmetric bilinear functor, such that the Jacobi identity is controlled by a natural isomorphism, which satisfies a coherence law of its own. The category of Lie 2-algebras is equivalent to the category of 2-term L_∞-algebras (see [4]).

Definition 2.1 An L_∞-algebra is a graded vector space $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ equipped with a system $\{l_k | 1 \le k < \infty\}$ of linear maps $l_k : \wedge^k \mathfrak{g} \longrightarrow \mathfrak{g}$ with degree $\deg(l_k) = 2 - k$, where the exterior powers are interpreted in the graded sense and the following relation with Koszul sign "Ksgn" is satisfied for all $n \ge 0$:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_\sigma \text{sgn}(\sigma) \text{Ksgn}(\sigma) l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0. \quad (1)$$

Here the summation is taken over all $(i, n-i)$-unshuffles with $i \ge 1$.

We denote an L_∞-algebra by $(\oplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \{l_i\}_{i \ge 1})$, or simply by $\mathfrak{g}$ if there is no confusion. In particular, we denote a 2-term L_∞-algebra by $(\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2, l_3)$.

Definition 2.2 Let $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2, l_3)$ and $\mathfrak{g}' = (\mathfrak{g}'_{-1} \xrightarrow{l'_1} \mathfrak{g}'_0, l'_2, l'_3)$ be two 2-term L_∞-algebras. An L_∞-algebra homomorphism F from $\mathfrak{g}$ to $\mathfrak{g}'$ consists of: linear maps $F_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0$, $F_1 : \mathfrak{g}_{-1} \to \mathfrak{g}'_{-1}$ and $F_2 : \mathfrak{g}_0 \wedge \mathfrak{g}_0 \to \mathfrak{g}'_{-1}$, such that the following equalities hold for all $x, y, z \in \mathfrak{g}_0, m \in \mathfrak{g}_{-1}$,

(i) $F_0 \circ l_1 = l'_1 \circ F_1$,
(ii) $F_0 l_2(x, y) - l'_2(F_0(x), F_0(y)) = l'_1 F_2(x, y)$,
(iii) $F_1 l_2(x, m) - l'_2(F_0(x), F_1(m)) = F_2(x, l_1(m))$,
(iv) $F_2(l_2(x, y), z) + c.p. + F_1(l_3(x, y, z)) = l'_2(F_0(x), F_2(y, z)) + c.p. + l'_3(F_0(x), F_0(y), F_0(z))$.

Given a graded vector space $V = \sum_{n \in \mathbb{Z}} V_n$, it is well-known that there is a graded version of Kosmann-Schwarzbach's big bracket [18], which we denote by $\{\cdot, \cdot\}$, on $\text{Sym}(V^*[l]) \otimes \text{Sym}(V[k]) \cong \text{Sym}(V^*[l] \oplus V[k]) \cong \text{Sym}(T^*[l+k]V[k])$ by extending the usual pairing between V^* and V via a graded Leibniz rule

$$\{u, v \wedge w\} = \{u, v\} \wedge w + (-1)^{(|u|+l+k)|v|} v \wedge \{u, w\}, \quad (2)$$

$$\{u, v\} = -(-1)^{(|u|+k+l)(|v|+k+l)} \{v, u\}, \quad (3)$$

where $u \in \text{Sym}(V^*[l] \oplus V[k])_{|u|}$ and $v \in \text{Sym}(V^*[l] \oplus V[k])_{|v|}$. The big bracket is in fact the canonical graded Poisson bracket on $T^*[-l-k]V^*[-k]$. Thus, we have a graded Jacobi identity:

$$\{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{(|u|+k+l)(|v|+k+l)} \{v, \{u, w\}\}. \quad (4)$$

We view $l_i : \wedge^i \mathfrak{g} \longrightarrow \mathfrak{g}$ as elements in $\text{Sym}^i(\mathfrak{g}^*[-1]) \otimes \mathfrak{g}[k]$. Then we have

Lemma 2.3 *A series of degree $2 - k$ elements $l_i \in (\text{Sym}^i(\mathfrak{g}^*[-1]) \otimes \mathfrak{g}[k])$ with $i = 1, 2, \dots$ on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \cdots$ gives an L_∞-algebra structure if and only if $\{\sum_{i=1}^\infty l_i, \sum_{i=1}^\infty l_i\} = 0$.*

Definition 2.4 A Courant algebroid is a vector bundle $E \longrightarrow M$, together with a fiber metric $(\cdot, \cdot)_+$ (so we can identify E with E^*), a bundle map $\rho : E \longrightarrow TM$ (called

the anchor), a bilinear bracket operation (Dorfman bracket) $\lceil \cdot, \cdot \rceil$ on $\Gamma(E)$, such that for all $e_1, e_2, e_3 \in \Gamma(E)$, we have

$$\lceil e, e \rceil = \frac{1}{2} D \left(e, e \right)_+ , \tag{5}$$

$$\rho(e_1) \left(e_2, e_3 \right)_+ = (\lceil e_1, e_2 \rceil, e_3)_+ + (e_2, \lceil e_1, e_3 \rceil)_+ , \tag{6}$$

$$\lceil e_1, \lceil e_2, e_3 \rceil \rceil = \lceil \lceil e_1, e_2 \rceil, e_3 \rceil + \lceil e_2, \lceil e_1, e_3 \rceil \rceil, \tag{7}$$

where the operator $D : C^\infty(M) \longrightarrow \Gamma(E)$ is the map defined by

$$(e, Df)_+ = \rho(e) f.$$

One can also use the skew-symmetric Courant bracket $[\![\cdot, \cdot]\!]$: $[\![e_1, e_2]\!]$ = $\frac{1}{2}(\lceil e_1, e_2 \rceil - \lceil e_2, e_1 \rceil)$. But the equality (7) does not hold anymore. We have

$$[\![[\![e_1, e_2]\!], e_3]\!] + [\![[\![e_2, e_3]\!], e_1]\!] + [\![[\![e_3, e_1]\!], e_2]\!] = DT(e_1, e_2, e_3),$$

where $T(e_1, e_2, e_3) \in C^\infty(M)$ is defined by

$$T(e_1, e_2, e_3) = \frac{1}{6} \left([\![e_1, e_2]\!], e_3 \right)_+ + c.p..$$

Theorem 2.5 ([27]) *A Courant algebroid structure on a vector bundle E gives rise to a 2-term L_∞-algebra structure on $C^\infty(M) \oplus \Gamma(E)$, where l_i are given by*

$$\begin{cases} l_1(f) = Df, & \forall f \in C^\infty(M), \\ l_2(e_1, e_2) = [\![e_1, e_2]\!] & \forall e_1, e_2 \in \Gamma(E), \\ l_2(e_1, f) = \frac{1}{2} (e_1, Df)_+ & \forall e_1 \in \Gamma(E), f \in C^\infty(M), \\ l_3(e_1, e_2, e_3) = -T(e_1, e_2, e_3) & \forall e_1, e_2, e_3 \in \Gamma(E). \end{cases} \tag{8}$$

A Courant algebroid can be described by a symplectic NQ-manifold of degree 2 [28]. Explicitly, let $(E, (\cdot, \cdot)_+)$ be a pseudo-Euclidean vector bundle over a manifold M. Then, $E[1]$ is a Poisson N-manifold of degree -2. Let $\mathcal{M}$ be its minimal symplectic realization. An NQ-structure on $\mathcal{M}$ is determined by a cubic Hamiltonian function Θ on $\mathcal{M}$ satisfying $\{\Theta, \Theta\} = 0$, where $\{\cdot, \cdot\}$ is the even Poisson bracket on $\mathcal{M}$ of degree -2. Such a Θ corresponds to a Courant algebroid structure on E. In fact, we define the anchor ρ and the Dorfman bracket $\lceil \cdot, \cdot \rceil$ as the derived bracket by

$$\rho(e) f = \{\{e, \Theta\}, f\}, \quad \lceil e_1, e_2 \rceil = \{\{e_1, \Theta\}, e_2\}, \quad \forall f \in C^\infty(M), \ \forall e, e_1, e_2 \in \Gamma(E).$$

Here $C^\infty(M)$ and $\Gamma(E)$ are the algebras of degree 0 and degree 1 polynomial functions on $\mathcal{M}$, respectively. We refer to [39] for a general construction of doubles for graded QS-manifolds and QP-manifolds.

For a vector bundle A, consider the graded manifold $T^*[2]A[1]$. It is canonically equipped with a Poisson bracket of degree -2, and is actually the minimal symplectic realization of $A \oplus A^*$. This Poisson bracket, called the big bracket in [18], is denoted

here by $\{\cdot, \cdot\}$. Let (x^i, ξ^a) be local coordinates on $A[1]$, we denote by $(x^i, \xi^a, \theta_a, p_i)$ the local coordinates on $T^*[2]A[1]$. About their degrees, we have

$$\text{degree}(x^i, \xi^a, \theta_a, p_i) = (0, 1, 1, 2).$$

The big bracket satisfies

$$\{x^i, p_j\} = \delta^i_j = -\{p_j, x^i\}, \quad \{\xi^a, \theta_b\} = \delta^i_j = \{\theta_b, \xi^a\}.$$

A Lie algebroid structure on A is equivalent to a degree 3 function $\mu = \rho^i_b p_i \xi^b + \frac{1}{2}\mu^a_{bc}\xi^b\xi^c\theta_a$ such that $\{\mu, \mu\} = 0$.

A Lie quasi-bialgebroid structure on A is given by a degree 3 function $\mu + \gamma + \phi$, which can be locally written as

$$\mu = \rho^i_b p_i \xi^b + \frac{1}{2}\mu^a_{bc}\xi^b\xi^c\theta_a, \quad \gamma = \varrho^{ib} p_i \theta_b + \frac{1}{2}\gamma^{bc}_a \xi^a \theta_b \theta_c, \quad \phi = \frac{1}{6}\phi^{abc}\theta_a\theta_b\theta_c,$$

and they satisfy

$$\{\mu + \gamma + \phi, \mu + \gamma + \phi\} = 0.$$

More precisely, we have

$$\{\mu, \mu\} = 0, \quad \{\mu, \gamma\} = 0, \quad \frac{1}{2}\{\gamma, \gamma\} + \{\mu, \phi\} = 0, \quad \{\gamma, \phi\} = 0. \tag{9}$$

See [19] for more details. On $A \oplus A^*$, there is a natural Courant algebroid structure, in which the degree 3 function Θ is exactly $\mu + \gamma + \phi$. Note that there are two canonical fiber metric $(\cdot, \cdot)_\pm$ on $A \oplus A^*$:

$$(x + \xi, y + \eta)_\pm = \langle \xi, y \rangle \pm \langle x, \eta \rangle, \tag{10}$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing without any degree involved in.

3 Maurer–Cartan elements on homotopy Poisson manifolds

First, we recall the notions of homotopy Poisson algebras and homotopy Poisson manifolds. They are also called P_∞-manifolds in [10,32] and higher Poisson manifolds in [8,17,40]. Here, we follow the convention in [26].

Definition 3.1 A homotopy Poisson algebra of degree n is a graded commutative algebra $\mathfrak{a}$ over a field of characteristic zero with an L_∞-algebra structure $\{l_m\}_{m\geq 1}$ on $\mathfrak{a}[n]$, such that the map

$$x \longrightarrow l_m(x_1, \ldots, x_{m-1}, x), \quad x_1, \ldots, x_{m-1}, \quad x \in \mathfrak{a}$$

is a derivation of $\mathfrak{a}$ of degree $\kappa := 2 - m - n(m-1) + \sum_{i=1}^{m-1} |x_i|$, i.e., for all x, $y \in \mathfrak{a}$,

$$l_m(x_1, \ldots, x_{m-1}, xy) = l_m(x_1, \ldots, x_{m-1}, x)y + (-1)^{\kappa|x|}x l_m(x_1, \ldots, x_{m-1}, y).$$

Here, $|x|$ denotes the degree of $x \in \mathfrak{a}$.

A homotopy Poisson algebra of degree n is of finite type if there exists a q such that $l_m = 0$ for all $m > q$.

A homotopy Poisson manifold of degree n is a graded manifold $\mathcal{M}$ whose algebra of functions $C^\infty(\mathcal{M})$ is equipped with a degree n homotopy Poisson algebra structure of finite type.

Throughout this paper, we use $(\mathcal{M}, \{l_i\}_{1 \leq i < \infty})$ to denote a homotopy Poisson manifold. Obviously, a usual Poisson manifold is a homotopy Poisson manifold of degree 0.

Remark 3.2 In this remark, we compare the related structures in the literature.

(i) In [10], the authors introduced the notion of a P_∞-algebra, which is a graded commutative algebra $\mathfrak{a}$ with an L_∞-algebra structure such that the Leibniz rule is satisfied. So a P_∞-algebra is a homotopy Poisson algebra of degree 0.

(ii) The notion of a higher Poisson structure was introduced in [40], and further studied in [8,17], where the authors used the superized version of an L_∞-algebra, i.e., $\mathbb{Z}_2$-graded.

(iii) A graded Poisson algebra of degree k is a graded commutative algebra $\mathfrak{a}$ with a degree $-k$ Lie bracket, such that the bracket is a biderivation of the product, namely

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{|y|(|x|+k)} y \cdot [x, z].$$

See [9] for more details. Thus, a graded Poisson algebra of degree k is a homotopy Poisson algebra of degree k. In particular, the associated L_∞-algebra has only one non-zero map l_2.

In the following, we list a few interesting examples of different kinds.

Example 3.3 Let A be a Lie algebroid. Consider its dual vector bundle, $A^*[1]$, which is an N-manifold of degree 1. Its algebra of polynomial functions is

$$\cdots \oplus \Gamma(A) \oplus C^\infty(M),$$

where $\Gamma(A)$ is of degree 1, and $C^\infty(M)$ is of degree 0. The Poisson bracket is in fact the Schouten bracket $[\cdot, \cdot]_S$ on $\Gamma(\wedge^\bullet A)$. It is straightforward to see that $A^*[1]$ is a homotopy Poisson manifold of degree 1.

Example 3.4 For an arbitrary manifold M, the shifted cotangent bundle $T^*[1]M$ is a symplectic N-manifold of degree 1 (see [11,28]). Its algebra of polynomial functions is

$$\cdots \oplus \mathfrak{X}(M) \oplus C^\infty(M),$$

where $\mathfrak{X}(M)$ is of degree 1, and $C^\infty(M)$ is of degree 0. The Poisson bracket is exactly the Schouten bracket $[\cdot, \cdot]_S$ on $\Gamma(\wedge^\bullet TM)$. It is straightforward to see that $(T^*[1]M, l_2 = [\cdot, \cdot]_S)$ is a homotopy Poisson manifold of degree 1. Similarly, any symplectic N-manifold of degree n is a homotopy Poisson manifold of degree n. The L_∞-algebra structure here only contains one nonzero l_2.

Example 3.5 Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2, l_3)$, its graded dual space $\mathfrak{g}^*[1] = \mathfrak{g}_0^*[1] \oplus \mathfrak{g}_{-1}^*[1]$ is an N-manifold of degree 1 with the base manifold $\mathfrak{g}_{-1}^*$. Its algebra of polynomial functions is

$$\cdots \oplus \left(C^\infty(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0\right) \oplus C^\infty(\mathfrak{g}_{-1}^*).$$

There is a degree 1 homotopy Poisson algebra structure on it obtained by extending the original 2-term L_∞-algebra structure using the Leibniz rule. Thus, the dual of a 2-term L_∞-algebra is a homotopy Poisson manifold of degree 1. This generalizes the fact that the dual of a Lie algebra is a linear Poisson manifold.

Similarly, the dual of an n-term L_∞-algebra is a homotopy Poisson manifold of degree $n - 1$.

Example 3.6 Given a splitting Lie n-algebroid[2] $\mathcal{A} = A_0 \oplus \cdots \oplus A_{1-n}$, its dual $\mathcal{A}^*[n] = A_0^*[n] \oplus \cdots \oplus A_{1-n}^*[n]$ is an N-manifold of degree n. Its algebra of polynomial functions is

$$\cdots \oplus \left(\Gamma(A_0) \oplus \Gamma(\wedge^n A_{1-n}) \oplus \cdots\right) \oplus \cdots \oplus \Gamma(A_{1-n}) \oplus C^\infty(M).$$

There is a degree n homotopy Poisson algebra structure on it. Thus, the dual of a splitting Lie n-algebroid is a homotopy Poisson manifold of degree n.

It is known that (M, π) is a Poisson manifold if and only if $(T^*[1]M, Q)$ is a symplectic NQ-manifold of degree 1, where the homological vector field Q is given by $Q = \{\pi, \cdot\}$. More generally, the cotangent bundle of a homotopy Poisson manifold of degree n gives rise to a symplectic NQ-manifold of degree $n + 1$. Consider the homotopy Poisson manifold given in Example 3.5, we have

Proposition 3.7 *Given an n-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{-n+1}, \{l_i\}_{1 \leq i \leq n+1})$, the cotangent bundle $T^*[n]\mathfrak{g}^*[n - 1]$ is a symplectic NQ-manifold of degree n, where the degree 1 homological vector field Q is given by*

$$Q = \left\{\sum l_i, \cdot\right\}, \tag{11}$$

in which $\{\cdot, \cdot\}$ is the canonical Poisson structure, and $\sum l_i \in \mathsf{Sym}(\mathfrak{g}^*[-1]) \otimes \mathfrak{g}[1-n]$ is viewed as a polynomial function of degree $n + 1$ on $T^*[n]\mathfrak{g}^*[n - 1]$.

---

[2] By definition, there is an L_∞-algebra structure on the complex of section spaces together with some compatibility conditions. See [7,35] for more details.

Similar to [13], a Maurer–Cartan element on a degree n homotopy Poisson manifold $\mathcal{M}$ is given by a function α on $\mathcal{M}$ satisfying the Maurer–Cartan equation

$$\sum_i \frac{(-1)^i}{i!} l_i(\alpha, \dots, \alpha) = 0. \tag{12}$$

Example 3.8 (quasi-Poisson $\mathfrak{g}$-manifolds) Let M be a manifold and $(\mathfrak{k}, [\cdot, \cdot]_{\mathfrak{k}}, K)$ a quadratic Lie algebra. Let $\{e_a\}$ be an orthogonal basis of $\mathfrak{k}$ with respect to the metric K, i.e. $K(e_a, e_b) = \delta_{ab}$, and $\{e^a\}$ its dual basis. Then K induces an isomorphism, which we denote by $K^\sharp$, from $\mathfrak{k}^*$ to $\mathfrak{k}$ via $K(K^\sharp(\xi), u) = \langle \xi, u \rangle$. More precisely, we have $K^\sharp(e^a) = e_a$. We define $R \in \wedge^3 \mathfrak{k}^*$ by

$$R(u, v, w) = K([u, v]_{\mathfrak{k}}, w), \quad \forall u, v, w \in \mathfrak{k}.$$

Then $\mathcal{M} := T^*[1]M \times \mathfrak{k}[1]$ is a homotopy Poisson manifold of degree 1. In fact, we can define an L_∞-algebra structure on $C^\infty(\mathcal{M})$ generated by

$$l_2(X, f) = Xf, \ \ l_2(X, Y) = [X, Y]_S, \ \ l_1(\xi) = \delta(\xi), \ \ l_3(\xi, \eta, \gamma) = K^\sharp(R)(\xi, \eta, \gamma),$$

where $f \in C^\infty(M), X, Y \in \mathfrak{X}(M), \xi, \eta, \gamma \in \mathfrak{k}^*$, and $\delta : \wedge^\bullet \mathfrak{k}^* \longrightarrow \wedge^{\bullet+1} \mathfrak{k}^*$ is the coboundary operator associated to the Lie algebra $\mathfrak{k}$.

A degree 2 function $\alpha = \pi + \rho$, where $\pi \in \wedge^2 \mathfrak{X}(M)$ and $\rho \in \mathfrak{k}^* \otimes \mathfrak{X}(M)$, is a Maurer–Cartan element, i.e. $-l_1(\alpha) + \frac{1}{2} l_2(\alpha, \alpha) - \frac{1}{3!} l_3(\alpha, \alpha, \alpha) = 0$, if and only if the following three conditions hold:

$$l_1(\rho) = \frac{1}{2}[\rho, \rho]_S, \quad [\pi, \rho]_S = 0, \quad \frac{1}{2}[\pi, \pi]_S = \frac{1}{6} K^\sharp(R)(\rho, \rho, \rho).$$

These conditions are equivalent to that $\rho : \mathfrak{k} \longrightarrow \mathfrak{X}(M)$ is a Lie algebra morphism, π is $\mathfrak{k}$-invariant and $\frac{1}{2}[\pi, \pi]_S = \wedge^3 \rho(K^\sharp(R))$, respectively. Therefore, a quasi-Poisson $\mathfrak{g}$-manifold [1,2] gives rise to a Maurer–Cartan element on $\mathcal{M}$.

A homotopy Poisson manifold $(\mathcal{M}, \{l_i\}_{1 \le i < \infty})$ is called symplectic if the binary bracket l_2 is non-degenerate. We refer the reader to [17] for a thorough discussion of homotopy symplectic structures in the setting of $\mathbb{Z}_2$-graded manifolds.

We turn to study Maurer–Cartan elements on a homotopy symplectic manifold of degree n. First, we present some examples with mathematical and physical interests.

Example 3.9 (Twisted Poisson structures) The shifted cotangent bundle $T^*[1]M$ of a manifold M is canonically a symplectic N-manifold of degree 1. Slightly different from Example 3.4, with a choice of a closed 3-form H, we introduce a nontrivial l_3 on the algebra of functions of $T^*[1]M$ by $l_3(X, Y, Z) = H(X, Y, Z)$, for all $X, Y, Z \in \mathfrak{X}(M)$. The compatibility of l_2 and l_3 is due to the fact that H is closed. Thus, $(T^*[1]M, l_2 = [\cdot, \cdot]_S, l_3 = H)$ is a homotopy symplectic manifold of degree 1. The idea of adding a closed 3-form to obtain a new L_∞-algebra was first introduced in [31].

A degree 2 function π is a Maurer–Cartan element of $T^*[1]M$ if and only if

$$\frac{1}{2}l_2(\pi, \pi) - \frac{1}{3!}l_3(\pi, \pi, \pi) = 0,$$

which is equivalent to $\frac{1}{2}[\pi, \pi] = \wedge^3 \pi^\sharp H$, that is, π is a twisted Poisson structure ([29,34]) on M.

Example 3.10 (Twisted Courant algebroids) Let $E \longrightarrow M$ be a vector bundle with a fiber metric $(\cdot, \cdot)_+$, and H a closed 4-form on M. Let $\mathcal{M}$ be its minimal symplectic realization. See [28] for details. Then $\mathcal{A}^2$ is the section space of a vector bundle $\mathbb{A}$, which fits in the following exact sequence:

$$0 \longrightarrow \wedge^2 E^* \longrightarrow \mathbb{A} \xrightarrow{a} TM \longrightarrow 0.$$

Similar to the treatment in Example 3.9, we can define a new degree 2 homotopy Poisson algebra structure on the algebra of functions of $\mathcal{M}$ by adding

$$l_4(\chi_1, \chi_2, \chi_3, \chi_4) = H\big(a(\chi_1), a(\chi_2), a(\chi_3), a(\chi_4)\big), \quad \forall \chi_i \in \mathcal{A}^2.$$

Choose a local coordinate (x^i, p_i, ξ^a). A degree 3 function $\alpha = \rho^i_a p_i \xi^a - \frac{1}{3!} f_{abc} \xi^a \xi^b \xi^c$ is a Maurer–Cartan element if and only if $\frac{1}{2}l_2(\alpha, \alpha) + \frac{1}{24}l_4(\alpha, \alpha, \alpha, \alpha) = 0$. Define an anchor $\rho : E \to TM$ and a derived bracket $\lceil \cdot, \cdot \rceil$ on $\Gamma(E)$ by

$$\rho(e)f = l_2(l_2(e, \alpha), f), \quad \lceil e_1, e_2 \rceil = l_2(l_2(e_1, \alpha), e_2).$$

According to [28, Theorem 4.5], we have

$$l_2(\alpha, \alpha) = (\rho^*(dx^i), \rho^*(dx^j)) p_i p_j + \xi^a \xi^b dx^j \big([\rho(e_a), \rho(e_b)]_S - \rho(\lceil e_a, e_b \rceil)\big) p_j$$
$$+ \frac{1}{12} \left(\lceil \lceil e_a, e_b \rceil, e_c \rceil + \lceil e_b, \lceil e_a, e_c \rceil \rceil - \lceil e_a, \lceil e_b, e_c \rceil \rceil, e_d\right)_+ \xi^a \xi^b \xi^c \xi^d.$$

On the other hand, a straightforward calculation gives that

$$l_4(\alpha, \alpha, \alpha, \alpha) = \rho^* H.$$

Thus, the condition for that α is a Maurer–Cartan element is equivalent to that $(E, K, \rho, \lceil \cdot, \cdot \rceil, H)$ is a twisted Courant algebroid, which arises from the study of three dimensional sigma models with Wess–Zumino term [14]. See [42] for its close relation with coisotropic Cartan geometry.

We have seen that the study of a general homotopy symplectic manifold of degree n and its Maurer–Cartan elements is inspired by many interesting geometric structures. Next, we will see that these structures naturally appear in the study of topological field theory. To be precise, it is known that there exists a systematic method to construct a topological sigma model from a symplectic NQ-manifold, which is called the

Alexandrov–Kontsevich–Schwartz–Zaboronsky (AKSZ) formalism [3]. We will show that this construction can be extended to the case of a degree n homotopy symplectic manifold $(\mathcal{M}, \{l_i\}_{2 \le i < \infty})$ with a degree $n + 1$ Maurer–Cartan element α. To do this, we need the notion of a twisted symplectic NQ-manifold, which was introduced in [16] to describe the structure on the target supermanifold of the AKSZ sigma model with boundary.

Definition 3.11 [16] Let $(\mathcal{M}, \{\cdot, \cdot\}_s)$ be a symplectic N-manifold of degree n and α a degree $n + 1$ function on it. Then $(\mathcal{M}, \{\cdot, \cdot\}_s, \alpha)$ is called a twisted symplectic NQ-manifold if there is a degree $n + 1$ symplectic NQ-manifold $(T^*[n + 1]\mathcal{M}, \{\cdot, \cdot\}, \Theta)$ such that

(a) $\{\cdot, \cdot\}_s = \{\cdot, \{\cdot, \Theta\}\}|_{\mathcal{M}}$;
(b) it satisfies the canonical transformation equation $e^{-\alpha}\Theta|_{\mathcal{M}} = 0$, where

$$e^{-\alpha}\Theta = \Theta - \{\alpha, \Theta\} + \frac{1}{2}\{\alpha, \{\alpha, \Theta\}\} - \cdots.$$

The following theorem states that a homotopy symplectic manifold with a Maurer–Cartan element naturally gives rise to a twisted symplectic NQ-manifold.

Theorem 3.12 *Degree n homotopy symplectic manifolds $(\mathcal{M}, \{l_i\}_{2 \le i < \infty})$ with a degree $n + 1$ Maurer–Cartan element α are in one-to-one correspondence with twisted symplectic NQ-manifolds $(\mathcal{M}, \{\cdot, \cdot\}_s, \alpha)$ with $\Theta|_{\mathcal{M}} = 0$.*

Proof Let $(\mathcal{M}, \{l_i\}_{2 \le i < \infty})$ be a homotopy symplectic manifold of degree n and α a degree $n + 1$ Maurer–Cartan element. Then $(T^*[n + 1]\mathcal{M}, \{\cdot, \cdot\}, \Theta = \sum l_i)$ is a symplectic NQ-manifold of degree $n + 1$, and the following equality holds:

$$l_k(a_1, \ldots, a_k) = \left\{a_k, \ldots, \left\{a_2, \left\{a_1, \sum l_i\right\}\right\} \cdots \right\}|_{\mathcal{M}}. \tag{13}$$

Define $\{\cdot, \cdot\}_s = l_2$, which is a Poisson bracket determined by a nondegenerate closed 2-form. Now we extend α to a function on $T^*[n + 1]\mathcal{M}$, which is constant along the fiber. By (13), we have $\Theta|_{\mathcal{M}} = \{\alpha, \Theta\}|_{\mathcal{M}} = 0$ and

$$e^{-\alpha}\Theta|_{\mathcal{M}} = \frac{1}{2}\{\alpha, \{\alpha, \Theta\}\}|_{\mathcal{M}} - \frac{1}{6}\{\alpha, \{\alpha, \{\alpha, \Theta\}\}\}|_{\mathcal{M}} + \cdots$$

$$= \frac{1}{2}l_2(\alpha, \alpha) - \frac{1}{3!}l_3(\alpha, \alpha, \alpha) + \cdots.$$

Thus, the condition for that α satisfies the Maurer–Cartan equation on $(\mathcal{M}, \{l_i\}_{2 \le i < \infty})$ is equivalent to that the canonical transformation equation $e^{-\alpha}\Theta|_{\mathcal{M}} = 0$ holds on $T^*[n + 1]\mathcal{M}$.   □

One immediate consequence of the main construction in [16] is that, associated to a degree n homotopy symplectic manifold $(\mathcal{M}, \{l_i\}_{2 \le i < \infty})$ with a degree $n + 1$ Maurer–Cartan element α, there is an AKSZ sigma model with boundary. This can be viewed as a generalization of AKSZ formalism to the setting of homotopy symplectic manifolds. In particular, if $(\mathcal{M}, \{l_i\}_{2 \le i < \infty})$ is a symplectic NQ-manifold, i.e. $l_i = 0$ for $i \ge 3$, it becomes the AKSZ construction.

4 2-term L_∞-algebras and Courant algebroids

In this section, we obtain a Courant algebroid from a 2-term L_∞-algebra. Consequently, we give a general construction of a 2-term L_∞-algebra from a 2-term L_∞-algebra, which could produce many interesting examples including the 2-term L_∞-algebra associated to an omni-Lie algebra and the 2-term L_∞-algebra of string type.

Let $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$ be a 2-term L_∞-algebra, where

$$l_1 \in \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0, \quad l_2^0 \in \wedge^2 \mathfrak{g}_0^* \otimes \mathfrak{g}_0, \quad l_2^1 \in \mathfrak{g}_0^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}, \quad l_3 \in \wedge^3 \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}.$$

By Proposition 3.7, the cotangent bundle $T^*[2]\mathfrak{g}^*[1]$ is a symplectic NQ-manifold of degree 2. Now the canonical Poisson structure $\{\cdot, \cdot\}$ is given by (2)–(4), in which $k = l = -1$. We give the relation between l_i and $\{\cdot, \cdot\}$ by the following lemma.

Lemma 4.1 *For all* $x, y \in \mathfrak{g}_0$ *and* $m \in \mathfrak{g}_{-1}$, *we have*

$$\begin{aligned}
l_1(m) &= \{m, l_1\} = -\{l_1, m\}, \\
l_2^0(x, y) &= \{y, \{x, l_2^0\}\} = -\{\{x, l_2^0\}, y\}, \\
l_2^1(x, m) &= \{m, \{x, l_2^1\}\} = -\{\{x, l_2^1\}, m\}, \\
l_3(x, y, \cdot) &= \{y, \{x, l_3\}\} = -\{\{x, l_3\}, y\}.
\end{aligned}$$

On the other hand, symplectic NQ-manifolds of degree 2 are in one-to-one correspondence with Courant algebroids [28]. Thus, from $T^*[2]\mathfrak{g}^*[1]$, we obtain a Courant algebroid E:

$$E = \mathfrak{g}_{-1}^* \times (\mathfrak{g}_0^* \oplus \mathfrak{g}_0) \longrightarrow \mathfrak{g}_{-1}^*, \tag{14}$$

in which the anchor and the Dorfman bracket are defined by the derived bracket using the degree 3 function $l = -\sum l_i$. Note that E is the direct sum of two vector bundles A and A^*, where $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ and $A^* = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0 \longrightarrow \mathfrak{g}_{-1}^*$. The fiber metric is given by the canonical pairing between A and A^*.

It is necessary to give the precise structures on the Courant algebroid E.

Proposition 4.2 *Consider the Courant algebroid E given above. For constant sections* $x, y \in \mathfrak{g}_0, \xi, \eta \in \mathfrak{g}_0^*$ *and a linear function* $m \in \mathfrak{g}_{-1}$, *we have*

(i) *the anchor of x is a linear vector field, more precisely,* $\rho(x)(m) = l_2^1(x, m)$;
(ii) *the anchor of ξ is a constant vector field, more precisely,* $\rho(\xi) = -l_1^*(\xi)$, *where* l_1^* *is defined by* $\langle l_1^*(\xi), m \rangle = -\langle \xi, l_1(m) \rangle$;
(iii) *the image of a linear function under the operator D is not a constant section, we have*
$$Dm = l_1(m) - l_2^1(m, \cdot) \in \mathfrak{g}_0 + \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}; \tag{15}$$
(iv) *the constant sections in $\mathfrak{g}_0$ under the Dorfman bracket are not closed, but we have*

$$\lceil x, y \rceil = l_2^0(x, y) + l_3(x, y, \cdot) \in \mathfrak{g}_0 + \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1};$$

(v) the Dorfman bracket of two constant sections in $\mathfrak{g}_0^*$ is zero, i.e. $\lceil \xi, \eta \rceil = 0$;

(vi) the Dorfman bracket of a constant section in $\mathfrak{g}_0^*$ and a constant section in $\mathfrak{g}_0$ is a constant section in $\mathfrak{g}_0^*$,

$$\lceil x, \xi \rceil = -\lceil \xi, x \rceil = l_2^0(x, \cdot)^* \xi \in \mathfrak{g}_0^*,$$

where $l_2^0(x, \cdot) \in \mathfrak{gl}(\mathfrak{g}_0)$ can be viewed as the adjoint map, and $l_2^0(x, \cdot)^*$ is its dual map.

Proof By the relation $\rho(x)(m) = \{\{x, -l\}, m\}$ and Lemma 4.1, (i) is obvious. (ii) follows from

$$\begin{aligned} \rho(\xi)(m) &= \{\{\xi, -l\}, m\} = \{\{\xi, -l_1\}, m\} = -\{\xi, \{l_1, m\}\} \\ &= \{\xi, \{m, l_1\}\} = \{\xi, l_1(m)\} = -\langle l_1^*(\xi), m \rangle. \end{aligned}$$

(iii) follows from the relation $Dm = \{-l, m\}$ and Lemma 4.1. (iv)-(vi) also follow from the relation $\lceil a, b \rceil = \{\{a, -l\}, b\}$ for all sections a, b and Lemma 4.1. □

We have seen that we can obtain linear sections through the Dorfman bracket of constant sections. Thus, it is not enough if we only consider constant sections.

Corollary 4.3 *Let $x \in \mathfrak{g}_0$ be a constant section, and $\phi = \xi \otimes m$, $\psi = \eta \otimes n$: $\mathfrak{g}_{-1}^* \longrightarrow \mathfrak{g}_0^*$ be linear sections, where $\xi, \eta \in \mathfrak{g}_0^*$ and $m, n \in \mathfrak{g}_{-1}$. Then we have*

(i) $\lceil x, \xi \otimes m \rceil = \xi \otimes l_2^1(x, m) + (l_2^0(x, \cdot)^* \xi) \otimes m$;

(ii) $\lceil \xi \otimes m, x \rceil = -\xi \otimes l_2^1(x, m) - (l_2^0(x, \cdot)^* \xi) \otimes m + \xi(x)(l_1(m) - l_2^1(m, \cdot))$;

(iii) $\lceil \xi \otimes m, \eta \otimes n \rceil = \langle l_1^* \eta, m \rangle \xi \otimes n - \langle l_1^* \xi, n \rangle \eta \otimes m$, *equivalently, we have*

$$\lceil \phi, \psi \rceil = \phi \circ l_1^* \circ \psi - \psi \circ l_1^* \circ \phi. \tag{16}$$

Proof Using the property of the Dorfman bracket, it is straightforward to obtain (i) and (ii). We only give the proof of (iii). We have

$$\begin{aligned} \lceil \xi \otimes m, \eta \otimes n \rceil &= \eta \otimes \rho(\xi \otimes m)(n) + \lceil \xi \otimes m, \eta \rceil \otimes n \\ &= \rho(\xi)(n)\eta \otimes m - \lceil \eta, \xi \otimes m \rceil \otimes n \\ &= -l_1^* \xi(n)\eta \otimes m + l_1^* \eta(m)\xi \otimes n. \end{aligned}$$

Let it act on an arbitrary $\alpha \in \mathfrak{g}_{-1}^*$. We obtain

$$\lceil \xi \otimes m, \eta \otimes n \rceil(\alpha) = \alpha(n)l_1^* \eta(m)\xi - \alpha(m)l_1^* \xi(n)\eta = (\phi \circ l_1^* \circ \psi - \psi \circ l_1^* \circ \phi)(\alpha).$$

This finishes the proof. □

Remark 4.4 Note that the bracket (16) is naturally skew-symmetric and satisfies the Jacobi identity. Thus, it is a Lie bracket. It has already appeared in other places, e.g. see [37, Proposition 3.1] for more details.

Given a Courant algebroid $E \longrightarrow M$, using the skew-symmetric Courant bracket, we get a 2-term L_∞-algebra structure on $C^\infty(M) \oplus \Gamma(E)$. Now consider the Courant algebroid (14) obtained from a 2-term L_∞-algebra. Since it is linear, we pick linear functions on $\mathfrak{g}^*_{-1}$ as the degree -1 part and $\mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}^*_0)$ as the degree 0 part. Before we give the precise structure of the corresponding 2-term L_∞-algebra, we give some properties of the skew-symmetric Courant bracket $[\![\cdot, \cdot]\!]$.

Corollary 4.5 *With the above notations, for any $x, y, z \in \mathfrak{g}_0$, and $\xi \otimes m, \eta \otimes n \in \mathfrak{g}^*_0 \otimes \mathfrak{g}_{-1}$, we have*

$$
\begin{cases}
[\![x, y]\!] = x \circ y = l_2^0(x, y) + l_3(x, y, \cdot); \\
[\![x, \xi \otimes m]\!] = \xi \otimes l_2^1(x, m) + (l_2^0(x, \cdot)^*\xi) \otimes m - \frac{1}{2}\xi(x)\big(l_1(m) - l_2^1(m, \cdot)\big); \\
[\![\xi \otimes m, \eta \otimes n]\!] = l_1^*\eta(m)\xi \otimes n - l_1^*\xi(n)\eta \otimes m.
\end{cases}
\tag{17}
$$

Now we can give the main theorem in this section.

Theorem 4.6 *Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$, we can obtain a new 2-term L_∞-algebra $\tilde{\mathfrak{g}} = (\mathfrak{g}_{-1} \xrightarrow{\tilde{l}_1} \mathfrak{g}_0 \oplus (\mathfrak{g}_{-1} \otimes \mathfrak{g}^*_0), \tilde{l}_2 = \tilde{l}_2^0 + \tilde{l}_2^1, \tilde{l}_3)$ from the corresponding Courant algebroid (14), in which $\tilde{l}_1 = D$ (given by (15)), $\tilde{l}_2^0$ is given by (17), $\tilde{l}_2^1$ and $\tilde{l}_3$ are given by*

$$
\tilde{l}_2^1(x + \xi \otimes m, n) = \frac{1}{2}l_2^1(x, n) + \frac{1}{2}\langle\xi, l_1(n)\rangle m,
\tag{18}
$$

and

$$
\begin{aligned}
\tilde{l}_3(x_1 &+ \xi_1 \otimes m_1, x_2 + \xi_2 \otimes m_2, x_3 + \xi_3 \otimes m_3) \\
&= -\frac{1}{2}l_3(x_1, x_2, x_3) - \frac{1}{2}\big(\langle l_2^0(x_1, x_2), \xi_3\rangle m_3 + c.p.\big) \\
&\quad -\frac{1}{4}\big(\langle\xi_1, x_2\rangle\langle\xi_3, l_1 m_1\rangle m_3 - \langle\xi_2, x_1\rangle\langle\xi_3, l_1 m_2\rangle m_3 + c.p.\big) \\
&\quad -\frac{1}{4}\big(\langle\xi_2, x_3\rangle l_2^1(x_1, m_2) - \langle\xi_3, x_2\rangle l_2^1(x_1, m_3) + c.p.\big).
\end{aligned}
\tag{19}
$$

Proof It follows from Theorem 2.5, Proposition 4.2 and Corollary 4.5. $\qquad\square$

Now we give some examples to illustrate that the construction given in the above theorem is full of interest.

Example 4.7 (omni-Lie algebra) Let V be a vector space. Consider the abelian 2-term L_∞-algebra $(V \xrightarrow{\text{id}} V, l_2 = 0, l_3 = 0)$. By Theorem 4.6, we get a new 2-term L_∞-algebra $(V \xrightarrow{\text{i}} V \oplus \mathfrak{gl}(V), \tilde{l}_2, \tilde{l}_3)$, in which i is the natural inclusion, and

$$
\begin{cases}
\tilde{l}_2^1(u + A, m) = \frac{1}{2}Am, \\
\tilde{l}_2^0(u + A, v + B) = \frac{1}{2}(Av - Bu) + [A, B], \\
\tilde{l}_3(u + A, v + B, w + C) = -\frac{1}{4}([A, B]w + [B, C]u + [C, A]v),
\end{cases}
$$

for all $u, v, w \in V_0 = V$, $m \in V_{-1} = V$ and $A, B, C \in \mathfrak{gl}(V)$. This 2-term L_∞-algebra is the one associated to the omni-Lie algebra $V \oplus \mathfrak{gl}(V)$. See [36,41] for more details.

Example 4.8 (2-term L_∞-algebra of string type) Let $(\mathfrak{k}, [\cdot, \cdot]_\mathfrak{k})$ be a Lie algebra. Consider the 2-term L_∞-algebra $(\mathbb{R} \xrightarrow{0} \mathfrak{k}, l_2 = [\cdot, \cdot]_\mathfrak{k}, l_3 = 0)$. By Theorem 4.6, we get a new 2-term L_∞-algebra $(\mathbb{R} \xrightarrow{0} \mathfrak{k} \oplus \mathfrak{k}^*, \tilde{l}_2, \tilde{l}_3)$, where $\tilde{l}_2$ and $\tilde{l}_3$ are given by

$$\begin{cases} \tilde{l}_2^1(u + \xi, r) = 0, \\ \tilde{l}_2^0(u + \xi, v + \eta) = [u, v]_\mathfrak{k} + \mathrm{ad}_u^*\eta - \mathrm{ad}_v^*\xi, \\ \tilde{l}_3(u + \xi, v + \eta, w + \zeta) = -\frac{1}{2}(\langle[u, v]_\mathfrak{k}, \zeta\rangle + \langle[v, w]_\mathfrak{k}, \xi\rangle + \langle[w, u]_\mathfrak{k}, \eta\rangle), \end{cases}$$

for all $u, v, w \in \mathfrak{k}, \xi, \eta, \zeta \in \mathfrak{k}^*$ and $r \in \mathbb{R}$. This is exactly the 2-term L_∞-algebra of string type studied in [36].

Example 4.9 Let $(\mathfrak{k}, [\cdot, \cdot]_\mathfrak{k}, K)$ be a quadratic Lie algebra, and $(\mathbb{R} \xrightarrow{0} \mathfrak{k}, l_2, l_3)$ the corresponding string Lie 2-algebra. More precisely,

$$l_2^0(u, v) = [u, v]_\mathfrak{k}, \ l_2^1(u, r) = 0, \ l_3(u, v, w) = K([u, v]_\mathfrak{k}, w), \ \forall u, v, w \in \mathfrak{k}, \ r \in \mathbb{R}.$$

Then we get a new 2-term L_∞-algebra $(\mathbb{R} \xrightarrow{0} \mathfrak{k} \oplus \mathfrak{k}^*, \tilde{l}_2, \tilde{l}_3)$, in which

$$\begin{cases} \tilde{l}_2^1(u + \xi, r) = 0, \\ \tilde{l}_2^0(u + \xi, v + \eta) = [u, v]_\mathfrak{k} + K([u, v]_\mathfrak{k}, \cdot) + \mathrm{ad}_u^*\eta - \mathrm{ad}_v^*\xi, \\ \tilde{l}_3(u + \xi, v + \eta, w + \zeta) = -\frac{1}{2}K([u, v]_\mathfrak{k}, w) - \frac{1}{2}(\langle[u, v]_\mathfrak{k}, \zeta\rangle + \langle[v, w]_\mathfrak{k}, \xi\rangle + \langle[w, u]_\mathfrak{k}, \eta\rangle), \end{cases}$$

for all $u, v, w \in \mathfrak{k}, r \in \mathbb{R}$ and $\xi, \eta, \zeta \in \mathfrak{k}^*$.

Example 4.10 Consider the 2-term L_∞-algebra $(\mathfrak{k} \xrightarrow{\mathrm{id}} \mathfrak{k}, l_2 = [\cdot, \cdot]_\mathfrak{k}, l_3 = 0)$, where $(\mathfrak{k}, [\cdot, \cdot]_\mathfrak{k})$ is a Lie algebra. By Theorem 4.6, we obtain a new 2-term L_∞-algebra $(\mathfrak{k} \xrightarrow{\tilde{l}_1} \mathfrak{k} \oplus \mathfrak{gl}(\mathfrak{k}), \tilde{l}_2, \tilde{l}_3)$, where

$$\begin{cases} \tilde{l}_1(m) = m - \mathrm{ad}_m, \\ \tilde{l}_2^1(u + A, m) = \frac{1}{2}[u, m]_\mathfrak{k} + \frac{1}{2}Am, \\ \tilde{l}_2^0(u + A, v + B) = [u, v]_\mathfrak{k} + \frac{1}{2}(Av - Bu) + [\mathrm{ad}_u, B] + [A, \mathrm{ad}_v] + \frac{1}{2}(\mathrm{ad}_{Bu} - \mathrm{ad}_{Av}) + [A, B], \\ \tilde{l}_3(u + A, v + B, w + C) = -\frac{1}{2}C[u, v]_\mathfrak{k} - \frac{1}{4}[A, B]w - \frac{1}{4}([u, Bw]_\mathfrak{k} + [Bu, w]_\mathfrak{k}) + c.p., \end{cases}$$

for all $u, v, w \in \mathfrak{k}, m \in \mathfrak{k}$ and $A, B, C \in \mathfrak{gl}(\mathfrak{k})$. This 2-term L_∞-algebra can be viewed as a deformation of the one associated to the omni-Lie algebra. It can also be viewed as a linearization of the Courant algebroid $TM \oplus T_\pi^*M$, where T_π^*M denotes the Lie algebroid associated to the Poisson manifold (M, π). We will study its full properties in a separate paper.

Now let us consider the relation between the 2-term L_∞-algebra given in the above theorem and the original 2-term L_∞-algebra.

Theorem 4.11 *With the above notations, there is an L_∞-algebra homomorphism $F = (F_0 = \mathrm{pr}_1, F_1 = \mathrm{id}, F_2)$ from $\tilde{\mathfrak{g}}$ to $\mathfrak{g}$, in which pr_1 is the projection to the first component, and $F_2 : \wedge^2(\mathfrak{g}_0 \oplus (\mathfrak{g}_0^* \otimes \mathfrak{g}_{-1})) \longrightarrow \mathfrak{g}_{-1}$ is given by*

$$F_2(x + \xi \otimes m, y + \eta \otimes n) = \frac{1}{2}\big(\langle \xi, y\rangle m - \langle x, \eta\rangle n\big) = \frac{1}{2}(x + \xi \otimes m, y + \eta \otimes n)_- ,$$
(20)

for all $x, y \in \mathfrak{g}_0$, $\xi \otimes m, \eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$.

Proof We verify the conditions of an L_∞-algebra homomorphism step by step. It is obvious that

$$F_0 \circ D = l_1 \circ F_1.$$
(21)

Then, by Corollary 4.5, we obtain

$$F_0 \tilde{l}_2^0(x + \xi \otimes m, y + \eta \otimes n) = l_2^0(x, y) + \frac{1}{2}\big(\langle \xi, y\rangle l_1 m - \langle x, \eta\rangle l_1 n\big).$$

On the other hand, it is obvious that

$$l_2^0(F_0(x + \xi \otimes m), F_0(y + \eta \otimes n)) = l_2^0(x, y).$$

Therefore, we have

$$
\begin{aligned}
F_0 \tilde{l}_2^0(x &+ \xi \otimes m, y + \eta \otimes n) - l_2^0(F_0(x + \xi \otimes m), F_0(y + \eta \otimes n)) \\
&= \frac{1}{2}\big(\langle \xi, y\rangle l_1 m - \langle x, \eta\rangle l_1 n\big) \\
&= l_1 F_2(x + \xi \otimes m, y + \eta \otimes n).
\end{aligned}
$$
(22)

By (18), we obtain

$$F_1 \tilde{l}_2^1(x + \xi \otimes m, n) = \frac{1}{2}l_2^1(x, n) + \frac{1}{2}\langle \xi, l_1(n)\rangle m,$$

which implies that

$$F_1 \tilde{l}_2^1(x + \xi \otimes m, n) - l_2^1(F_0(x + \xi \otimes m), F_1(n)) = -\frac{1}{2}l_2^1(x, n) + \frac{1}{2}\langle \xi, l_1(n)\rangle m.$$

On the other hand, we have

$$F_2(x + \xi \otimes m, Dn) = F_2(x + \xi \otimes m, l_1(n) - l_2^1(n, \cdot)) = -\frac{1}{2}l_2^1(x, n) + \frac{1}{2}\langle \xi, l_1(n)\rangle m.$$

Thus, we have

$$F_1 \tilde{l}_2^1(x + \xi \otimes m, n) - l_2^1(F_0(x + \xi \otimes m), F_1(n)) = F_2(x + \xi \otimes m, Dn).$$
(23)

It is left to prove the compatibility condition, that is, for all $a, b, c \in \mathfrak{g}_0 \oplus \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$,

$$l_2^1(F_0 a, F_2(b, c)) + c.p. + l_3(F_0 a, F_0 b, F_0 c) = F_2(\tilde{l}_2^0(a, b), c) + c.p. + F_1 \tilde{l}_3(a, b, c). \tag{24}$$

First, for all $a, b, c \in \mathfrak{g}_{-1} \otimes \mathfrak{g}_0^*$, it is obvious that both sides of the equation are 0. While for all $a, b, c \in \mathfrak{g}_0$ (we use the notation x, y, z instead), it is obvious that the left hand side is equal to $l_3(x, y, z)$. It is not hard to see that

$$F_2(\tilde{l}_2^0(x, y), z) + c.p. = F_2(l_3(x, y, \cdot), z) + c.p. = \frac{1}{2} l_3(x, y, z) + c.p.$$

$$= \frac{3}{2} l_3(x, y, z),$$

$$F_1 \tilde{l}_3(x, y, z) = -\frac{1}{2} l_3(x, y, z), \quad \text{by Theorem 4.6}$$

which implies that (24) holds for all $x, y, z \in \mathfrak{g}_0$. For x, y in $\mathfrak{g}_0$ and $\xi \otimes m$ in $\mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, we have

$$l_2^1(F_0 x, F_2(y, \xi \otimes m)) + c.p. + l_3(F_0 x, F_0 y, F_0(\xi \otimes m))$$

$$= l_2^1\left(x, -\frac{1}{2} \langle \xi, y \rangle m\right) + l_2^1\left(y, \frac{1}{2} \langle \xi, x \rangle m\right)$$

$$= -\frac{1}{2} \langle y, \xi \rangle l_2^1(x, m) + \frac{1}{2} \langle x, \xi \rangle l_2^1(y, m),$$

and

$$F_2(\tilde{l}_2^0(x, y), \xi \otimes m) + c.p. + F_1 \tilde{l}_3(x, y, \xi \otimes m)$$

$$= \left(-\frac{1}{2} - \frac{1}{6}\right) \langle [\![x, y]\!], \xi \rangle m + \left(\frac{1}{2} - \frac{1}{6}\right) \langle [\![y, \xi \otimes m]\!], x \rangle + \left(\frac{1}{2} - \frac{1}{6}\right) \langle [\![\xi \otimes m, x]\!], y \rangle$$

$$= -\frac{2}{3} \langle l_2^0(x, y), \xi \rangle m + \frac{1}{3} \left\{ \langle \xi, x \rangle l_2^1(y, m) - \langle \xi, l_2^0(y, x) \rangle m + \frac{1}{2} \langle \xi, y \rangle l_2^1(m, x) \right\}$$

$$+ \frac{1}{3} \left\{ -\langle \xi, y \rangle l_2^1(x, m) + \langle \xi, l_2^0(x, y) \rangle m - \frac{1}{2} \langle \xi, x \rangle l_2^1(m, y) \right\}$$

$$= -\frac{1}{2} \langle y, \xi \rangle l_2^1(x, m) + \frac{1}{2} \langle x, \xi \rangle l_2^1(y, m),$$

which implies that (24) holds for two elements in $\mathfrak{g}_0$ and one element in $\mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$. At last, for $x \in \mathfrak{g}_0$ and $\xi \otimes m, \eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, it is obvious that the left hand side of (24) is equal to 0. Furthermore, we have

$$F_2(\tilde{l}_2^0(x, \xi \otimes m), \eta \otimes n) + c.p. + F_1 \tilde{l}_3(x, \xi \otimes m, \eta \otimes n)$$

$$= \left(-\frac{1}{2} - \frac{1}{6}\right) \langle [\![x, \xi \otimes m]\!], \eta \rangle n + \left(\frac{1}{2} - \frac{1}{6}\right) \langle [\![\xi \otimes m, \eta \otimes n]\!], x \rangle$$

$$+\left(-\frac{1}{2}-\frac{1}{6}\right)\langle[\![\eta\otimes n,x]\!],\xi\rangle m$$

$$=-\frac{2}{3}\times\left(-\frac{1}{2}\right)\langle\xi,x\rangle\langle l_1m,\eta\rangle n+\frac{1}{3}\left\{-\langle l_1m,\eta\rangle\langle\xi,x\rangle n+\langle l_1n,\xi\rangle\langle\eta,x\rangle m\right\}$$

$$-\frac{2}{3}\times\frac{1}{2}\langle l_1n,\xi\rangle\langle\eta,x\rangle m$$

$$=0.$$

Thus, (24) holds for all $a, b, c \in \mathfrak{g}_0 \oplus \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$. By (21)–(24), we deduce that F is an L_∞-algebra homomorphism. □

5 Lie 2-algebras and quasi-Poisson groupoids

In this section, we show that the Courant algebroid associated to a 2-term L_∞-algebra is the double of a Lie quasi-bialgebroid. Then, by integration, we obtain a quasi-Poisson groupoid. Since a 2-term L_∞-algebra is equivalent to a Lie 2-algebra, this quasi-Poisson groupoid can serve as the geometric structure on the dual of a Lie 2-algebra.

Let $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$ be a 2-term L_∞-algebra. The vector bundle $E \longrightarrow M$ given by (14) can be decomposed as $A \oplus A^*$, where $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ is a trivial vector bundle. By Lemma 2.3 ($k = -1$), the 2-term L_∞-algebra structure $l = l_1 + (l_2^1 + l_2^0) + l_3$, where $l_1 \in \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$, $l_2^0 \in \wedge^2\mathfrak{g}_0^* \otimes \mathfrak{g}_0$, $l_2^1 \in \mathfrak{g}_0^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$, $l_3 \in \wedge^3\mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, satisfies $\{l, l\} = 0$. By the degree reason, $\{l, l\} = 0$ implies that

$$\{l_1, l_1\} = 0 \quad \{l_1, l_2\} = 0, \quad \frac{1}{2}\{l_2, l_2\} + \{l_1, l_3\} = 0, \quad \{l_2, l_3\} = 0. \tag{25}$$

Comparing this with (9), we can see that Eq. (25) implies Eq. (9) for $\mu = -l_1$, $\gamma = -l_2 = -l_2^0 - l_2^1$, $\phi = -l_3$. Therefore, we have

Theorem 5.1 *Let* $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$ *be a 2-term L_∞-algebra. Then $(A, -l)$ is a Lie quasi-bialgebroid, where $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ and $l = l_1 + (l_2^1 + l_2^0) + l_3$, and the Courant algebroid E is the double of the Lie quasi-bialgebroid $(A, -l)$.*

Now we give the precise structure on A and A^* using the general method of differential geometry. By Theorem 5.1 and Proposition 4.2, we have

Corollary 5.2 *The Lie algebroid structure on A, determined by $-l_1$, is given by*

(i) *for any constant section $\xi \in \mathfrak{g}_0^*$, the anchor ρ_A is given by $\rho_A(\xi) = -l_1^*(\xi)$;*
(ii) *for any constant sections $\xi, \eta \in \mathfrak{g}_0^*$, we have $[\xi, \eta]_A = 0$;*
(iii) *for any constant section $\xi \in \mathfrak{g}_0^*$ and linear section $\eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, we have*

$$[\xi, \eta \otimes n]_A = \langle \xi, l_1(n) \rangle \eta;$$

(iv) for any linear sections $\xi \otimes m, \eta \otimes n \in \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1}$, we have

$$[\xi \otimes m, \eta \otimes n]_A = \langle l_1^* \eta, m \rangle \xi \otimes n - \langle l_1^* \xi, n \rangle \eta \otimes m.$$

Thus, A is an action Lie algebroid of the abelian Lie algebra $\mathfrak{g}_0^*$ acting on $\mathfrak{g}_{-1}^*$ via $-l_1^*$, which sends an element $\xi \in \mathfrak{g}_0^*$ to a constant vector field $-l_1^*(\xi) \in \mathfrak{g}_{-1}^*$. The corresponding Chevalley-Eilenberg coboundary operator $d_A : \Gamma(\wedge^\bullet A^*) \longrightarrow \Gamma(\wedge^{\bullet+1} A^*)$ is determined by

$$d_A m = \{-l_1, m\} = l_1(m).$$

Corollary 5.3 *For all constant section $x \in \mathfrak{g}_0$ of A^*, $-l_2^1$ gives rise to the anchor map ρ_{A^*} of A^* via*

$$\rho_{A^*}(x) = l_2^1(x, \cdot),$$

which is a linear vector field. For all constant sections $x, y \in \mathfrak{g}_0$, $-l_2^0$ gives rise to the bracket operation on A^*:

$$[x, y]_{A^*} = l_2^0(x, y).$$

The Jacobi identity of $[\cdot, \cdot]_{A^*}$ is controlled by $\phi = -l_3 \in \wedge^3 \mathfrak{g}_0^* \otimes \mathfrak{g}_{-1} \subset \Gamma(\wedge^3 A)$. More precisely, we have

$$[[x, y]_{A^*}, z]_{A^*} + c.p. = d_A \phi(x, y, z) + \phi(d_A x, y, z) - \phi(x, d_A y, z) + \phi(x, y, d_A z).$$

Now we can restate Theorem 5.1 using usual language of differential geometry. This is necessary since we will consider the integration of Lie quasi-bialgebroids at the end of this section.

Corollary 5.4 *Given a 2-term L_∞-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0, l_2 = l_2^0 + l_2^1, l_3)$, we obtain a Lie quasi-bialgebroid (A, δ, ϕ), where the Lie algebroid $A = \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \longrightarrow \mathfrak{g}_{-1}^*$ is given by Corollary 5.2, $\delta : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k+1} A)$ is the generalized Chevalley-Eilenberg operator determined by the anchor ρ_{A^*} and the bracket $[\cdot, \cdot]_{A^*}$ given in Corollary 5.3, and $\phi = -l_3$.*

In the last section, we have seen that, given a 2-term L_∞-algebra, there is a new 2-term L_∞-algebra associated to the Courant algebroid E (Theorem 4.6), and there is an L_∞-algebra homomorphism from the new 2-term L_∞-algebra to the original one (Theorem 4.11). In this section, we see that the Courant algebroid is exactly the double of the Lie quasi-bialgebroid (A, δ, ϕ). In fact, we can obtain a more general result than Theorem 4.11. Namely, given any Lie quasi-bialgebroid (A, δ, ϕ), we can obtain a 2-term L_∞-algebra $C^\infty(M) \oplus \ker(d_A|_{\Gamma(A^*)})$, where d_A is the coboundary operator associated to Lie algebroid A and $d_A|_{\Gamma(A^*)} : \Gamma(A^*) \longrightarrow \Gamma(\wedge^2 A^*)$ is the restriction of d_A on $\Gamma(A^*)$. Furthermore, we construct an L_∞-algebra homomorphism from the 2-term L_∞-algebra associated the Courant algebroid $A \oplus A^*$ to $C^\infty(M) \oplus \ker(d_A|_{\Gamma(A^*)})$.

Theorem 5.5 *Let (A, δ, ϕ) be a Lie quasi-bialgebroid. Then there is a 2-term L_∞-algebra structure on $C^\infty(M) \oplus \ker(d_A|_{\Gamma(A^*)})$, where $C^\infty(M)$ is of degree -1, $\ker(d_A|_{\Gamma(A^*)})$ is of degree 0, and $\{l_i\}$ are given by*

$$
\begin{cases}
\quad l_1(f) = d_A f, & \forall\, f \in C^\infty(M), \\
\quad l_2(e_1, e_2) = [e_1, e_2]_{A^*} & \forall\, e_1, e_2 \in \ker(d_A|_{\Gamma(A^*)}), \\
\quad l_2(e_1, f) = \rho_{A^*}(e_1)(f) & \forall\, e_1 \in \ker(d_A|_{\Gamma(A^*)}),\, f \in C^\infty(M), \\
\quad l_3(e_1, e_2, e_3) = -\phi(e_1, e_2, e_3) & \forall\, e_1, e_2, e_3 \in \ker(d_A|_{\Gamma(A^*)}).
\end{cases}
\tag{26}
$$

Furthermore, there is an L_∞-algebra homomorphism $F = (F_0 = \mathrm{pr}_2, F_1 = \mathrm{id}, F_2)$ from the 2-term L_∞-algebra associated to the Courant algebroid $A \oplus A^*$, $C^\infty(M) \oplus \big(\Gamma(A) \oplus \ker(d_A|_{\Gamma(A^*)})\big)$, to $C^\infty(M) \oplus \ker(d_A|_{\Gamma(A^*)})$, where pr_2 is the projection to the second component, and F_2 is given by

$$
F_2(e_1, e_2) = \frac{1}{2}\, (e_1, e_2)_-, \quad \forall e_1, e_2 \in \Gamma(A) \oplus \ker(d_A|_{\Gamma(A^*)}).
\tag{27}
$$

Proof Since A is a Lie algebroid, we have $d_A^2 = 0$, i.e. $\mathrm{Im}(d_A) \subset \ker(d_A)$. Thus, l_1 is well-defined. By the fact that (A, δ, ϕ) is a Lie quasi-bialgebroid, it is straightforward to see that $\{l_i\}$ is a 2-term L_∞-algebra structure. Furthermore, it is not hard to see that the Courant bracket on $\Gamma(A) \oplus \ker(d_A|_{\Gamma(A^*)})$ is closed, which implies that the 2-term L_∞-algebra $C^\infty(M) \oplus (\Gamma(A) \oplus \Gamma(A^*))$ associated to the Courant algebroid $A \oplus A^*$ can be reduced to $C^\infty(M) \oplus (\Gamma(A) \oplus \ker(d_A|_{\Gamma(A^*)}))$. We can also prove that F is indeed a homomorphism similar as the proof of Theorem 4.11.   □

Next, we return to the Lie quasi-bialgebroid (A, δ, ϕ) associated to a 2-term L_∞-algebra. Clearly, the Lie algebroid A can be integrated to an action groupoid Γ : $\mathfrak{g}_{-1}^* \times \mathfrak{g}_0^* \rightrightarrows \mathfrak{g}_{-1}^*$ with the abelian group structure on $\mathfrak{g}_0^*$, where the source, target and inclusion maps are given by

$$
s(\alpha, \xi) = \alpha, \ t(\alpha, \xi) = \alpha + l_1^* \xi, \ i(\alpha) = (\alpha, 0),
$$

for all $(\alpha, \xi) \in \mathfrak{g}_{-1}^* \times \mathfrak{g}_0^*$. By Theorem 4.9 in [15], we know that Γ has a quasi-Poisson structure, such that its corresponding Lie quasi-bialgebroid is exactly (A, δ, ϕ). Recall that a quasi-Poisson groupoid is a triple (Γ, Π, ϕ), where Γ is a Lie groupoid whose Lie algebroid is A, $\Pi \in \wedge^2 \mathfrak{X}(\Gamma)$, and $\phi \in \Gamma(\wedge^3 A)$, satisfying $\frac{1}{2}[\Pi, \Pi]_S = \overleftarrow{\phi} - \overrightarrow{\phi}$, $[\Pi, \overleftarrow{\phi}] = 0$. Generally, it is a rather difficult work to get the bivector field from the data of an ordinary Lie quasi-bialgebroid. Nevertheless, we can elaborate the quasi-Poisson structure in our specific case, since all the structures are determined by the information of the 2-term L_∞-algebra.

Theorem 5.6 *The quasi-Poisson groupoid corresponding to (A, δ, ϕ) is $(\Gamma, \Pi, \phi = -l_3)$, where Π is characterized by*

$$
\Pi(\mathrm{d}x, \mathrm{d}y) = -l_2^0(x, y), \quad \Pi(\mathrm{d}x, \mathrm{d}m) = -l_2^1(x, m), \quad \Pi(\mathrm{d}m, \mathrm{d}n) = -l_2^1(l_1(m), n),
$$

where d *is the usual de Rham differential, and* $x, y \in \mathfrak{g}_0, m, n \in \mathfrak{g}_{-1}$ *are linear functions on* $\mathfrak{g}_{-1}^* \times \mathfrak{g}_0^*$.

We need to make some preparations to prove the theorem. Let us take a general action groupoid into account.

Let $M \lhd G$ be an action groupoid and $M \lhd \mathfrak{g}$ the corresponding Lie algebroid, where $\mathfrak{g}$ is the Lie algebra of Lie group G, and the anchor is given by the infinitesimal action of $\mathfrak{g}$ on M, i.e. a Lie algebra homomorphism $\widehat{\cdot} : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$,

$$\widehat{X}(x) = \tfrac{\mathrm{d}}{\mathrm{d}t} x \cdot \exp(-tX), \qquad \forall X \in \mathfrak{g}, \ x \in M.$$

We assume that the derivatives are all taken at $t = 0$. We use $l_a (resp. \ r_a)$ to denote the left(right)-translation on the Lie group G and $l_{(x,a)}(resp. \ r_{(x,a)})$ the left(right)-translation on the groupoid $M \lhd G$, respectively. Let $(l_a)_*, (r_a)_*$ and $(l_{(x,a)})_*, (r_{(x,a)})_*$ be the corresponding tangent maps.

For the Lie algebroid A of a Lie groupoid Γ, when we say the left-translation $\overleftarrow{\Lambda}$ (right-translation $\overrightarrow{\Lambda}$) of $\Lambda \in \Gamma(A)$, we are identifying $\Gamma(A)$ with left-invariant (right-invariant) vector fields on Γ.

Lemma 5.7 *Let* $M \lhd G$ *be an action Lie groupoid and* $M \lhd \mathfrak{g}$ *the corresponding Lie algebroid. For all* $\phi \in \Gamma(M \lhd \mathfrak{g}), (x, a) \in M \times G$, *we have*

$$\overleftarrow{\phi}_{(x,a)} = (0_x, (l_a)_{*e}\phi_{x \cdot a}), \qquad \overrightarrow{\phi}_{(x,a)} = ((\widehat{\phi_x})_x, (r_a)_{*e}\phi_x).$$

Proof For this action groupoid, the s-fiber over x is $s^{-1}(x) = \{(x, a); \forall a \in G\} = \{x\} \times G$, and the t-fiber over x is $t^{-1}(x) = \{(x \cdot a^{-1}, a); \forall a \in G\}$. The tangent map of $l_{(x,a)} : s^{-1}(x \cdot a) \longrightarrow s^{-1}(x)$ is $(l_{(x,a)})_{*(x \cdot a, e)} : T_{(x \cdot a, e)}s^{-1}(x \cdot a) \longrightarrow T_{(x,a)}s^{-1}(x)$. Let $\phi : M \longrightarrow \mathfrak{g}$ be a section of $M \lhd \mathfrak{g}$ and $\gamma(t)$ a smooth curve on G satisfying $\gamma(0) = e, \gamma'(0) = \phi(x \cdot a)$. Then we have

$$\overleftarrow{\phi}_{(x,a)} = (l_{(x,a)})_* \phi_{x \cdot a} = \frac{\mathrm{d}}{\mathrm{d}t}(x, a)(x \cdot a, \gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(x, a\gamma(t)) = (0_x, (l_a)_{*e}\phi_{x \cdot a}).$$

Similarly, the tangent map of $r_{(x,a)} : t^{-1}(x) \longrightarrow t^{-1}(x \cdot a)$ is $(r_{(x,a)})_{*(x,e)} : T_{(x,e)}t^{-1}(x) \longrightarrow T_{(x,a)}t^{-1}(x \cdot a)$. Let $\gamma(t)$ be a smooth curve on G satisfying $\gamma(0) = e, \gamma'(0) = \phi(x)$. When ϕ is considered as a right-invariant vector field, we have

$$\phi_x = \frac{\mathrm{d}}{\mathrm{d}t}(x, \gamma(-t))^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}(x \cdot \gamma(-t), \gamma(-t)^{-1}).$$

Thus, we get

$$\overrightarrow{\phi}_{(x,a)} = (r_{(x,a)})_* \phi_x = \frac{\mathrm{d}}{\mathrm{d}t}(x \cdot \gamma(-t), \gamma(-t)^{-1})(x, a)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(x \cdot \gamma(-t), \gamma(t)a) = ((\widehat{\phi_x})_x, (r_a)_{*e}\phi_x).$$

$\square$

Proof of Theorem 5.6 According to Theorems 2.34 and 4.9 in [15], we only need to prove $\delta_\Pi = \delta$, that is, for any function $f \in C^\infty(\mathfrak{g}_{-1}^*)$ and any section $e \in \Gamma(A)$, we have $\overleftarrow{\delta f} = [t^* f, \Pi]_S$, $\overleftarrow{\delta e} = [\overleftarrow{e}, \Pi]_S$. We only give the proof for a linear function $m \in \mathfrak{g}_{-1}$ and a constant section $\xi \in \mathfrak{g}_0^*$.

By calculation, we have $t^*(m) = m - l_1(m)$. Considering the linear action of the abelian group $\mathfrak{g}_0^*$ on $\mathfrak{g}_{-1}^*$ and by Lemma 5.7, we have,

$$\overleftarrow{\delta m} = \delta m - l_1(\delta m) = -l_2^1(m, \cdot) + l_1(l_2^1(m, \cdot)),$$
$$\overleftarrow{\delta \xi} = \delta \xi - l_1(\delta \xi) = \delta \xi,$$
$$\overleftarrow{\xi} = \xi.$$

Thus, for a linear function $n + x$ on $\mathfrak{g}_{-1}^* \times \mathfrak{g}_0^*$, we have

$$\overleftarrow{\delta m}(n + x) = -l_2^1(m, x) + l_1(l_2^1(m, x)),$$

while

$$[t^* m, \Pi]_S(n + x) = \Pi(\mathrm{d}m, \mathrm{d}n) + \Pi(\mathrm{d}m, \mathrm{d}x) - \Pi(\mathrm{d}l_1(m), \mathrm{d}n) - \Pi(\mathrm{d}l_1(m), \mathrm{d}x)$$
$$= -l_2^1(m, x) + l_2^0(l_1(m), x).$$

Then, we have $\overleftarrow{\delta m} = [t^* m, \Pi]_S$. To prove $\delta \xi = [\xi, \Pi]_S$ as bivector fields, we just need to verify that it holds on two 1-forms on $\mathfrak{g}_0^*$, since it vanishes for all the other cases. By straightforward calculation, we have $\delta \xi(\mathrm{d}x, \mathrm{d}y) = -\langle \xi, [x, y]_{A^*} \rangle = -\langle \xi, l_2^0(x, y) \rangle$, and

$$[\xi, \Pi]_S(\mathrm{d}x, \mathrm{d}y) = L_\xi \Pi(\mathrm{d}x, \mathrm{d}y)$$
$$= -\langle \xi, l_2^0(x, y) \rangle - \Pi(L_\xi \mathrm{d}x, \mathrm{d}y) - \Pi(\mathrm{d}x, L_\xi \mathrm{d}y)$$
$$= -\langle \xi, l_2^0(x, y) \rangle.$$

Therefore, we complete the proof. □

There is a one-to-one correspondence between 2-term chain complex $V_1 \xrightarrow{d} V_0$ and 2-vector space $\mathbb{V} : V_1 \oplus V_0 \rightrightarrows V_0$. Actually, the 2-vector space $\mathbb{V}$ is an action Lie groupoid $V_0 \triangleleft V_1$, where V_1 is seen as an abelian group and the action is given by $v_0 \cdot v_1 := v_0 + dv_1$, for all $v_0 \in V_0, v_1 \in V_1$.

Definition 5.8 Let $\mathbb{V}$ be a 2-vector space. If $(\mathbb{V}, \Pi, \phi)$ is a quasi-Poisson groupoid such that the bivector field Π and the trisection ϕ are both linear, we call $(\mathbb{V}, \Pi, \phi)$ a **Lie-quasi-Poisson groupoid**.

Consider the 2-vector space $\mathfrak{g}_{-1}^* \oplus \mathfrak{g}_0^* \rightrightarrows \mathfrak{g}_{-1}^*$ given by the 2-term chain complex $\mathfrak{g}_0^* \xrightarrow{l_1^*} \mathfrak{g}_{-1}^*$, which is the dual of the complex $\mathfrak{g}_{-1} \xrightarrow{l_1} \mathfrak{g}_0$. Consequently, by Theorem 5.6, we generalize the fact that there is a Lie-Poisson structure on the dual space of a Lie algebra to the case of Lie 2-algebras.

Theorem 5.9 *The dual of a Lie 2-algebra is a Lie-quasi-Poisson groupoid.*

We obtain our result through the integration of the Lie quasi-bialgebroid associated to a Lie 2-algebra. One can also obtain a quasi-Poisson Lie 2-group from a Lie 2-algebra via the integration of Lie 2-bialgebras. See [12] for more details.

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