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Virasoro and KdV

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Abstract We investigate the structure of representations of the (positive half of the) Virasoro algebra and situations in which they decompose as a tensor product of Lie algebra representations. As an illustration, we apply these results to the differential operators defined by the Virasoro conjecture and obtain some factorization properties of the solutions as well as a link to the multicomponent KP hierarchy.

Keywords Virasoro constraints · KP hierarchy · KdV hierarchy · Sato Grassmannian · Semisimple quantum cohomology

Mathematics Subject Classification Primary 14N35; Secondary 17B68 · 22E65 · 81T40

1 Introduction

The breakthrough discovery of Witten–Kontsevich [22,39] established an intimate link between mathematical physics and enumerative geometry. From a general perspective, one aims at studying the Gromov–Witten invariants of a smooth projective variety X in terms of suitable integrable hierarchies. From this point of view, the Witten–

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Kontsevich case corresponds to the situation when X is a point and it was shown that the exponential of the generating function of intersection numbers on the moduli space of curves was a common solution of the Virasoro constraints and of the KdV hierarchy. Therefore, following the generalization for the case of the projective space proposed in [9], on the one hand, one wonders if the generating function fulfills a generalization of the Virasoro constraints. On the other hand, one also wants to know if the generating function is given by (the logarithm of) a particular tau-function of an integrable hierarchy. Nowadays, the case of varieties with semisimple quantum cohomology is well understood and the answer to both questions is affirmative ([7,38]; see also [6,13,14,26,31,32]).

It is worth pointing out that, for each X, the explicit Virasoro operators as well as the relevant integrable hierarchy may vary; for instance, the 2-Toda hierarchy appears when dealing with the equivariant GW invariants of $\mathbb{P}^1$ [32]. Nevertheless, one recognizes some common features that arise among these results. Let us mention some of them. In [15], Givental studied a case in which the total descendent potential is a τ-function for the nKdV-hierarchy using $n-1$ copies of the KdV. Thus, the total descendent potential of a semisimple Frobenius manifold was defined in [14] as (a suitable operator acting on) a product of n copies of Witten–Kontsevich τ-functions. Dealing with a case of orbifold quantum cohomology, it has been proved in [17] that the Virasoro constraints decomposed as n copies of (half of) the Virasoro algebra, that their solution was the product of Witten–Kontsevich τ-functions, and that the relevant integrable hierarchy consisted of n commuting copies of the KdV hierarchy. Finally, in [5,18] it was shown that the solution of the Virasoro constraints in the case of Witten–Kontsevich is unique (up to a constant factor) and this uniqueness also holds in other setups (e.g. [25]).

This paper, making use of the representation theoretic properties of the Virasoro algebra, offers new insights into and results about these properties and provides evidences that the above-mentioned properties rely heavily on the structure of the Virasoro algebra and its representations. Our study of explicit expressions for Virasoro representations (see Sect. 2) is general enough to encode many of the known representations within the framework of Virasoro constraints. Further, it allows us to determine whether a representation is the tensor product of Lie algebra representations and if a solution factorizes as a product of solutions of those representations. An explicit realization of these ideas is carried out in Sect. 3 for the case of smooth projective varieties with trivial odd cohomology and vanishing first Chern class. Thus, we think that our approach may help in determining the explicit expression of the Virasoro operators as well as the corresponding integrable hierarchies for other types of varieties X (see Sect. 3.6). Now, let us be more precise and explain the contents of the paper.

We begin by fixing a pair $(A, (\ ,\ ))$ consisting of a finite dimensional vector space and a non-degenerate bilinear form. Associated to these data, we consider a Heisenberg algebra $\mathbb{H}(A)$ and its universal enveloping algebra $\mathcal{U}(\mathbb{H}(A))$. Let us denote by $\mathcal{W}_>$ the positive half of the Virasoro algebra and recall that it contains $\mathfrak{sl}(2)$ canonically. Section 2 is entirely devoted to the study of Lie algebra maps $\mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A))$. To begin with, we show that, under some homogeneity condition, there is a canonical bijection between $\operatorname{Hom}_{\text{Lie-alg}}(\mathcal{W}_>, \mathcal{U}(\mathbb{H}(A)))$ and $\operatorname{Hom}_{\text{Lie-alg}}(\mathfrak{sl}(2), \mathcal{U}(\mathbb{H}(A)))$ (Theorem 2.9). This is highly non-trivial since, in general, the problem of extending a map defined on

$\mathfrak{sl}(2)$ to $\mathcal{W}_>$ involves infinitely many conditions (see [35]). Accordingly, it is natural to expect that many properties of a map $\mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A))$ can be stated in terms of its restriction to $\mathfrak{sl}(2)$. Actually, we prove that such a map decomposes as tensor product of Lie algebra representations if and only if its restriction does. Moreover, this factorization is possible only if A decomposes as the orthogonal sum of two subspaces (Theorem 2.10). We finish this section by showing that the fact that $\mathcal{W}_>$ admits no non-trivial finite dimensional representations has important consequences for the structure of the solutions of the equations $(\rho_1 \otimes 1 + 1 \otimes \rho_2)(L)(\sum_i f_i \otimes g_i) = 0$, where $L \in \mathcal{W}_>$ (see Theorem 2.12), that is, decompositions of the representation and of the solutions depend strongly on the structure of $\mathcal{W}_>$ and of $(A, (\ ,\ ))$.

Although the previous results are interesting on their own, Sect. 3 explores their application to concrete situations; for instance, relations with integrable hierarchies (e.g. multicomponent KdV). The case we have chosen to illustrate this issue is that of the differential operators appearing in the Virasoro conjecture when X has trivial odd cohomology (for instance, whenever X has semisimple even quantum cohomology) and its first Chern class vanishes. Then, Theorem 3.3 shows explicitly how to obtain these operators as the images of the generators $L_k \in \mathcal{W}_>$ by:

$$\hat{\rho} : \mathcal{W}_> \stackrel{\rho}{\longrightarrow} \mathcal{U}(\mathbb{H}(A)) \stackrel{\frown}{\longrightarrow} \operatorname{End}\left(\mathbb{C}[[\{t_{i,\alpha} | 1 \leq \alpha \leq \dim(A), i = 1, 3, \ldots\}]]\right)$$

for $A = H^*(X, \mathbb{C})$ endowed with the Poincaré pairing. Then, our results of Sect. 2 imply that $\hat{\rho}$ decomposes as the tensor product of Lie algebra representations associated with data $(\mathbb{C}, (\ ,\ ))$, i.e. the one-dimensional case. The detailed study of the one-dimensional case carried out in Sect. 3.4 shows that, up to re-scaling the variables, the corresponding operators *always* come from a representation:

$$\sigma : \mathcal{W}_> \longrightarrow \operatorname{Diff}^1(\mathbb{C}((z)))$$

which means that we can profit from [18,33] to build the unique solution in terms of a τ-function of the KdV hierarchy. Putting everything together, we have the main results of this section. First, in the case of $\dim A = 1$:

Theorem (see Theorem 3.14) *Let $\rho \in \operatorname{Hom}_{Lie\text{-}alg}(\mathcal{W}_>, \operatorname{End}(\mathbb{C}[[t_1, t_3, \ldots]]))$ be such that $\rho(L_k)$ is of type k for $k \geq -1$ and that all coefficients of $\rho(L_{-1})$ are non-zero.*

Then, there exists a unique $\tau(t) \in \mathbb{C}[[t_1, t_3, \ldots]]$, with $\tau(0) = 1$, such that:

$$\rho(L_k)(\tau(t)) = 0 \qquad k \geq -1$$

Further, the solution $\tau(t)$ is a τ-function of the scaled KdV hierarchy.

and, for $\dim A = N \geq 2$:

Theorem (see Theorem 3.16) *Let $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A))$ be as in Sect. 3.3.*

There exist $S \in \operatorname{Gl}(A)$ and functions $\tau_\alpha(t_{1,\alpha}, t_{3,\alpha}, \ldots) \in \mathbb{C}[[t_{1,\alpha}, t_{3,\alpha}, \ldots]]$ such that:

$$\hat{\rho}(L_k)\left(S\left(\prod_\alpha \tau_\alpha(t_\alpha)\right)\right) = 0$$

Further, $\tau_\alpha(t_{1,\alpha}, t_{3,\alpha}, \ldots)$ are τ-functions of the scaled KdV hierarchy.

We hope that our methods shed some light on the explicit expressions of the Virasoro operators and of the relevant integrable hierarchies that appear when studying the Virasoro conjecture. We also think that the techniques presented here can be applied to many instances of representations of $\mathcal{W}_>$ which appear in a variety of problems such as recursion relations, Hurwitz numbers, and knot theory. We sketch some ideas in Sect. 3.6 although all of them deserve further research.

2 Lie algebras

Let $\mathcal{W}$ be the Witt algebra; that is, the $\mathbb{C}$-vector space with basis $\{L_k\}_{k\in\mathbb{Z}}$ endowed with the Lie bracket $[L_i, L_j] = (i-j)L_{i+j}$, and let $\mathcal{W}_>$ be the subalgebra generated by $\{L_k\}_{k\geq -1}$. It contains a copy of $\mathfrak{sl}(2)$ via $\mathfrak{sl}(2) = \langle L_{-1}, L_0, L_1\rangle \subset \mathcal{W}_>$. Recall that $\mathcal{W}_>$ is also called the *positive half of the centerless Virasoro algebra*.

In this section, we study certain maps from $\mathfrak{sl}(2)$ and their extensions to $\mathcal{W}_>$. These results will eventually allow us to relate the representation theories of $\mathcal{W}_>$ and $\mathfrak{sl}(2)$. A further consequence is that, to construct the operators $L_0, L_1, L_2, \ldots$ one only has to start with L_{-1} and follow some simple procedures and choices.

It is worth mentioning that a study of the representation theory of $\mathcal{W}_>$ in terms of the representation theory of its subalgebra $\mathfrak{sl}(2) \subset \mathcal{W}_>$ has been carried out in [35] in full generality.

2.1 Preliminaries

Let us be more precise. Let $(A, (\ ,\ ))$ be given, where A is a finite dimensional $\mathbb{C}$-vector space and $(\ ,\ )$ is a non-degenerated bilinear pairing. For a basis $\{a_\alpha | \alpha = 1, \ldots, n\}$ of A, let $\eta = (\eta_{\alpha\beta})$ denote the matrix associated to the given bilinear product; that is, $\eta_{\alpha\beta} := (a_\alpha, a_\beta)$. The inverse will be denoted with superindices; i.e. $\eta^{\alpha\beta} := (\eta^{-1})_{\alpha\beta}$.

Let us consider unknowns $\{p_i, q_i | i \geq 1\}$ and introduce $p_{i,\alpha} := p_i \otimes a_\alpha$ and $q_{i,\alpha} := q_i \otimes a_\alpha$. Let $\mathbb{H}(A)$ be the Heisenberg algebra generated by $\{1, p_{i,\alpha}, q_{i,\alpha} | i \geq 1, \alpha = 1, \ldots, n\}$, whose elements will be called *operators*, endowed with the Lie bracket:

$$\begin{aligned}
[p_{i,\alpha}, q_{j,\beta}] &= \delta_{i,j}\, i\, \eta^{\alpha\beta} \cdot 1 \\
[p_{i,\alpha}, p_{j,\beta}] = [q_{i,\alpha}, q_{j,\beta}] &= 0 \\
[p_{i,\alpha}, 1] = [q_{i,\alpha}, 1] &= 0
\end{aligned} \tag{1}$$

We define their degree by $\deg(q_{i,\alpha}) = i$, $\deg(p_{i,\alpha}) = -i$ and $\deg(1) = 0$.

Although the definition of the Heisenberg algebra depends on the pair $(A, (\ ,\ ))$, it will be simply denoted by $\mathbb{H}$ if no confusion arises.

For $\mathbb{H}$ as above, let us define $\mathcal{U}(\mathbb{H})$ the universal enveloping algebra of $\mathbb{H}$, which is the quotient of the tensor algebra of $\mathbb{H}$ by the two-sided ideal generated by the relations $u \otimes v - v \otimes u - [u, v]$.

Motivated by the explicit forms of the Virasoro operators considered in the literature [5,6,11,13,14,19,21], we introduce the following notion. Lemma 3.7 will help us to understand the meaning of this notion.

Definition 2.1 An operator $T \in \mathcal{U}(\mathbb{H})$ is *of type* $i \geq -1$ if it is a linear combination of $p_{2i+3,\alpha}$ and double products of degree $-2i$; i.e. $p_{j,\alpha}p_{2i-j,\beta}$, $q_{j,\alpha}p_{2i+j,\beta}$ and $q_{j,\alpha}q_{-j-2i,\beta}$. If $i = 0$ we also allow a constant times the central element $1 \in \mathbb{H}(A)$.

The subset consisting of operators of type $i \geq -1$ will be denoted by $\mathcal{U}(\mathbb{H}(A))_i$ (or, simply, $\mathcal{U}(\mathbb{H})_i$).

This section deals with the study of homomorphisms of Lie algebras:

$$\rho : \mathcal{W}_{>} \longrightarrow \mathcal{U}(\mathbb{H}) \qquad \text{s.t. } \rho(L_i) \in \mathcal{U}(\mathbb{H})_i$$

Let us illustrate the previous definition. From now on, according to Einstein convention, summation over repeated indices will be understood. For instance, an operator of type -1 is of the form:

$$b_{-1}^{0,1} p_1 + q_1 a_{-1}^{1,1} q_1^T + q_{i+2} b_{-1}^{i+2,i} p_i \ \in \mathcal{U}(\mathbb{H})_{-1} \tag{2}$$

(the sum runs over the set of positive integers i), p_i is the column vector $(p_{i,1}, \ldots, p_{i,n})^T$, q_i is the row vector $(q_{i,1}, \ldots, q_{i,n})$, $b_{-1}^{0,1}$ is a row vector, $a_{-1}^{1,1}$ and $b_{-1}^{i+2,i}$ are $n \times n$ matrices. For brevity, we set $a := a_{-1}^{1,1}$.

Similarly, an type 0 operator can be expressed as:

$$b_0^{0,3} p_3 + b_0^{0,0} + q_i b_0^{i,i} p_i \ \in \mathcal{U}(\mathbb{H})_0 \tag{3}$$

while an operator of type $i \geq 1$ is of the form:

$$b_i^{0,2i+3} p_{2i+3} + p_j^T c_i^{j,2i-j} p_{2i-j} + q_j b_i^{j,2i+j} p_{2i+j} \ \in \mathcal{U}(\mathbb{H})_i \qquad i \geq 1 \tag{4}$$

for a row vector $b_i^{0,2i+3}$ and $n \times n$-matrices $b_i^{j,2i+j}$ and $c_i^{j,2i-j}$, where $c_i^{j,2i-j} = (c_i^{2i-j,j})^T$ and the sum runs over j positive.

It is convenient to offer an interpretation of these matrices. Recall that q_i is the row vector $(q_{i,1}, \ldots, q_{i,n})$, which can be thought as an $\mathbb{H}$-valued vector of A. A similar argument holds for the column vector p_i. Thus, under a basis change in A, the matrix b in $q_i \cdot b \cdot p_i$ behaves as a bilinear form on A. The same fact applies to all a, b and c matrices. Similarly, column vectors $b_i^{0,2i+3}$ are understood as vectors on A while row vectors are like linear forms.

It is worth noticing how these operators behave w.r.t. the Lie bracket. Indeed, the computations given in the Appendix and the linearity of the bracket show that it is compatible with the type:

$$[\ ,\ ] : \mathcal{U}(\mathbb{H})_i \times \mathcal{U}(\mathbb{H})_j \longrightarrow \mathcal{U}(\mathbb{H})_{i+j} \tag{5}$$

where $i, j, i+j \geq -1$. In particular, it follows that $\oplus_{i\geq -1}\mathcal{U}(\mathbb{H})_i$ is a partial Lie algebra (see [27]). Note that the notion of partial Lie algebras turned out to be essential in the approach of [35].

2.2 Maps from $\mathfrak{sl}(2)$ to Heisenberg

Let $\mathfrak{sl}(2)$ be the Lie algebra of $\mathrm{Sl}(2,\mathbb{C})$. We fix a basis $\{e, f, h\}$ of $\mathfrak{sl}(2)$ satisfying the relations:

$$[e, f] = h\ , \qquad [h, e] = 2e\ , \qquad [h, f] = -2f\ .$$

In particular, the previous choice yields a natural embedding:

$$\iota : \mathfrak{sl}(2) \hookrightarrow \mathcal{W}_{>} \tag{6}$$

by mapping f to L_{-1}, h to $-2L_0$, and e to $-L_1$.

Lemma 2.2 *Let $F \in \mathcal{U}(\mathbb{H})_{-1}$ be as in (2). Assume that $b_{-1}^{i+2,i}$ is invertible for all i. It holds that: $b_{-1}^{i+2,i}$ is invertible for all i. It holds that:*

$$\left\{ \begin{array}{l} H \in \mathcal{U}(\mathbb{H})_0 \ s.t. \\ [H, F] \ = \ -2F \end{array} \right\} \simeq \left\{ \begin{array}{l} (b, B) \in \mathbb{C} \times \mathrm{Mat}_{n\times n}(\mathbb{C}) \ s.t. \\ (B\eta^{-1} + \mathrm{Id})(a + a^T) + (a + a^T)(B\eta^{-1} + \mathrm{Id})^T = 0 \end{array} \right\}$$

Proof Our task consists of computing the bracket $[H, F]$ explicitly. Recall that, for simplicity, we have set $a = a_{-1}^{1,1}$. Since $H \in \mathcal{U}(\mathbb{H})_0$, it must be of the form $H := b_0^{0,3} p_3 + b_0^{0,0} + q_i b_0^{i,i} p_i$ where $b_0^{0,3}$ is a row vector, $b_0^{0,0}$ is an homothety, and $b_0^{i,i}$ are $n \times n$ matrices.

Having in mind the commutation relations of the Appendix, the bracket $[H, F]$ is a linear combination of $p_1, q_{1\alpha}q_{1\beta}$ and $q_{i+2,\alpha}p_{i,\beta}$. Therefore, the expression $[H, F] = -2F$ is equivalent to the following identities:

$$\begin{aligned} \left(3b_0^{0,3}\eta^{-1}b_{-1}^{3,1} - b_{-1}^{0,1}\eta^{-1}b_0^{1,1}\right) p_1 &= -2b_{-1}^{0,1}p_1 \\ q_1 b_0^{1,1}\eta^{-1}\left(a + a^T\right) q_1^T &= -2q_1 a q_1^T \\ q_{i+2}\left((i+2)b_0^{i+2,i+2}\eta^{-1}b_{-1}^{i+2,i} - i b_{-1}^{i+2,i}\eta^{-1}b_0^{i,i}\right) p_i &= -2q_{i+2}b_{-1}^{i+2,i}p_i \qquad \forall i \geq 1 \end{aligned}$$

Observe that $q_1 A q_1^T = q_1 B q_1^T$ if and only if $A + A^T = B + B^T$. Hence, the above system is equivalent to the following equations:

$$3b_0^{0,3}\eta^{-1}b_{-1}^{3,1} - b_{-1}^{0,1}\eta^{-1}b_0^{1,1} = -2b_{-1}^{0,1} \tag{7a}$$

$$b_0^{1,1}\eta^{-1}(a + a^T) + (a + a^T)(b_0^{1,1}\eta^{-1})^T = -2(a + a^T) \tag{7b}$$

$$(i+2)b_0^{i+2,i+2}\eta^{-1}b_{-1}^{i+2,i} - i b_{-1}^{i+2,i}\eta^{-1}b_0^{i,i} = -2b_{-1}^{i+2,i} \qquad \forall i \geq 1 \tag{7c}$$

Note that, since $b_{-1}^{i+2,i}$ and η are invertible, given a pair (b, B) as in the statement, this system has a unique solution for $b_0^{0,0} = b$ and $b_0^{1,1} = B$; namely,

$$\begin{aligned} b_0^{0,3} &= \frac{1}{3} b_{-1}^{0,1} (\eta^{-1} b_0^{1,1} - 2)(\eta^{-1} b_{-1}^{3,1})^{-1} \\ b_0^{i+2,i+2} &= \frac{1}{i+2} b_{-1}^{i+2,i} (i\eta^{-1} b_0^{i,i} - 2)(\eta^{-1} b_{-1}^{i+2,i})^{-1} \qquad \forall i \geq 1 \end{aligned} \tag{8}$$

The converse is straightforward. □

Example 2.3 Set $F = b_{-1}^{0,1} p_1 + q_1 a q_1^T + \frac{i+2}{2} q_{i+2} p_i$ and $b_0^{1,1} = -\frac{1}{2}$, then $H = -2b_{-1}^{0,1} p_3 + b_0^{0,0} + i q_i p_i$. Note that $i q_i p_i$ is the *degree operator*.

Example 2.4 Let us consider the case where the chosen basis in A is orthonormal, i.e. η is the identity matrix, and suppose that:

$$F = b_{-1}^{0,1} p_1 + q_1 a q_1^T + q_{i+2} p_i \in \mathcal{U}(\mathbb{H})_{-1}$$

Then, operators H given by Lemma (2.2) acquire the form:

$$H = \frac{1}{3} b_{-1}^{0,1} (b_0^{1,1} - 2) p_3 + b_0^{0,0} + \frac{1}{i} q_i (b_0^{1,1} - (i-1)) p_i, \in \mathcal{U}(\mathbb{H})_0$$

where $b_0^{0,0} \in \mathbb{C}$ and $b_0^{1,1}$ verifies $(b_0^{1,1} + \mathrm{Id})a + a(b_0^{1,1} + \mathrm{Id})^T = 0$.

Example 2.5 Finally, let $\dim A = 1$, $a, \eta \in \mathbb{C}^*$ and $F = b_{-1}^{0,1} p_1 + q_1 a q_1^T + q_{i+2} \eta p_i$. Then, $b_0^{i,i} = -\eta$ for all i and $H = -b_{-1}^{0,1} p_3 + b_0^{0,0} - q_i \eta p_i$.

Lemma 2.6 *Let H be as in Eq.* (3) *and $\mathcal{U}(\mathbb{H})'_i$ be the subspace:*

$$\mathcal{U}(\mathbb{H})'_i := \{T \in \mathcal{U}(\mathbb{H})_i \text{ s.t. } [H, T] = 2iT\}$$

Then, for $i, j, i + j \geq -1$, it holds that $[\mathcal{U}(\mathbb{H})'_i, \mathcal{U}(\mathbb{H})'_j] \subseteq \mathcal{U}(\mathbb{H})'_{i+j}$.

Proof The claim follows easily from (5) and the Jacobi identity. □

Theorem 2.7 *Let F and H be as in Eqs.* (2) *and* (3)*, respectively.*
There is a surjective map:

$$\left\{ \begin{array}{l} c \in M_{n\times n}(\mathbb{C}) \text{ such that} \\ b_0^{0,0} = \mathrm{Tr}(c\eta^{-1}(a + a^T)(\eta^{-1})^T) \\ \text{and Eq. (10b) below} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \sigma \in \mathrm{Hom}_{Lie\text{-}alg}(\mathfrak{sl}(2), \mathcal{U}(\mathbb{H})) \\ \text{such that } \sigma(f) = F, \\ \sigma(h) = H \text{ and } \sigma(e) \in \mathcal{U}(\mathbb{H})_1 \end{array} \right\}$$

Moreover, c_1 and c_2 have the same image iff $c_1 + c_1^T = c_2 + c_2^T$. Thus, the restriction of the above map to symmetric matrices yields a bijection.

Proof Giving a map σ as in the r.h.s. is equivalent to set an operator $E \in \mathcal{U}(\mathbb{H})_1$, such that $[E, F] = H$ and $[H, F] = -2F$. Consider:

$$E = b_1^{0,5} p_5 + p_1^T c_1^{1,1} p_1 + q_i b_1^{i,i+2} p_{i+2} \in \mathcal{U}(\mathbb{H})_1 \tag{9}$$

where, for simplicity, we will set $c = c_1^{1,1}$. The identity $[H, E] = 2E$, expressed in terms of the coefficients of the operators, is equivalent to the following equations (thanks to the computations of the Appendix):

$$3b_0^{0,3}\eta^{-1}b_1^{3,5} - 5b_1^{0,5}\eta^{-1}b_0^{5,5} = 2b_1^{0,5} \tag{10a}$$

$$-(c^T + c)\eta^{-1}b_0^{1,1} - (\eta^{-1}b_0^{1,1})^T(c^T + c) = 2(c^T + c) \tag{10b}$$

$$rb_0^{r,r}\eta^{-1}b_1^{r,r+2} - (r+2)b_1^{r,r+2}\eta^{-1}b_0^{r+2,r+2} = 2b_1^{r,r+2} \tag{10c}$$

Analogously, expanding the relation $[E, F] = H$ with the help of the Appendix yields the system:

$$\mathrm{Tr}(c\eta^{-1}(a + a^T)(\eta^{-1})^T) = b_0^{0,0} \tag{11a}$$

$$-b_{-1}^{0,1}\eta^{-1}b_1^{1,3} + 5b_1^{0,5}\eta^{-1}b_{-1}^{5,3} = b_0^{0,3} \tag{11b}$$

$$3b_1^{1,3}\eta^{-1}b_{-1}^{3,1} + (a + a^T)(\eta^{-1})^T(c + c^T) = b_0^{1,1} \tag{11c}$$

$$(r+2)b_1^{r,r+2}\eta^{-1}b_{-1}^{r+2,r} - (r-2)b_{-1}^{r,r-2}\eta^{-1}b_1^{r-2,r} = b_0^{r,r} \quad \forall r > 2 \tag{11d}$$

Having in mind the properties of the trace, one observes that these equations only depend on $c + c^T$.

It remains to show that Eqs. (10) and (11) are equivalent to the conditions of the claim; that is, that they can be reduced to (10b) and (11a).

Assuming (10b) and (11a), one gets $b_1^{1,3}$ from (11c); then, $b_1^{0,5}$ is determined by (11b), and $b_1^{r,r+2}$ is obtained from (11d). We claim that (10a) is fulfilled too. Indeed, a long but straightforward computation shows that (10a) is derived from (8), (10b) together with the case $r = 3$ of (11d). Similarly, (10c) follows from (8), (7c) and (11d). □

2.3 Extending to $\mathcal{W}_>$

To extend a map defined on $\mathfrak{sl}(2)$ to one on $\mathcal{W}_>$, one should choose an endomorphism T, define $\rho(L_i)$ by Eqs. (12) and (13) and check infinitely many constraints (see [35]). However, in our situation the following Lemma simplifies that approach drastically; there will exist a unique T satisfying all the requirements.

Lemma 2.8 *Let F, H be as in Eqs. (2) and (3). The map:*

$$\mathrm{ad}(F) : \mathcal{U}(\mathbb{H})'_i \xrightarrow{\sim} \mathcal{U}(\mathbb{H})'_{i-1}$$

is an isomorphism for $i \geq 2$.

Proof First, one has to prove that given an operator:

$$S := b_{i-1}^{0,2i+1} p_{2i+1} + p_j^T c_{i-1}^{j,2i-j-2} p_{2i-j-2} + q_j b_{i-1}^{j,j+2i-2} p_{j+2i-2} \in \mathcal{U}(\mathbb{H})_{i-1}$$

of type $i-1 \geq 1$, there is exactly one operator:

$$T := b_i^{0,2i+3} p_{2i+3} + p_j^T c_i^{j,2i-j} p_{2i-j} + q_j b_i^{j,j+2i} p_{j+2i} \in \mathcal{U}(\mathbb{H})_i$$

of type i satisfying $\mathrm{ad}(F)(T) = S$ where ad denotes the adjoint representation and F is given by Eq. (2).

Now, one proceeds as in the proof of Lemma 2.2 and shows that $\mathrm{ad}(F)(T) = [F, T] = S$ has exactly one solution.

Finally, let us check that if $S \in \mathcal{U}(\mathbb{H})'_{i-1}$ and $\mathrm{ad}(F)(T) = S$, then $T \in \mathcal{U}(\mathbb{H})'_i$. Using the injectivity of $\mathrm{ad}(F)$ and the relation:

$$\begin{aligned} \mathrm{ad}(F)(\mathrm{ad}(H)(T)) &= \mathrm{ad}(H)(\mathrm{ad}(F)(T)) + \mathrm{ad}([F, H])(T) \\ &= \mathrm{ad}(H)(S) + \mathrm{ad}(2F)(T) = 2(i-1)S + 2S = 2iS \end{aligned}$$

one obtains $\mathrm{ad}(H)(T) = 2iT$, as we wanted.

Theorem 2.9 *Let F be as in* (2) *where a is symmetric and $b_{-1}^{i,i-2}$ are invertible. Then, the map ι of* (6) *yields a bijection:*

$$\left\{ \begin{array}{l} \rho \in \mathrm{Hom}_{Lie\text{-}alg}(\mathcal{W}_>, \mathcal{U}(\mathbb{H})) \\ \text{such that } \rho(L_{-1}) = F \qquad\qquad l \\ \text{and } \rho(L_i) \in \mathcal{U}(\mathbb{H})_i \text{ for } i \geq 0 \end{array} \right\} \overset{\iota^*}{\underset{\sim}{\longrightarrow}} \left\{ \begin{array}{l} \sigma \in \mathrm{Hom}_{Lie\text{-}alg}(\mathfrak{sl}(2), \mathcal{U}(\mathbb{H})) \\ \text{such that } \sigma(f) = F\,, \\ \sigma(h) \in \mathcal{U}(\mathbb{H})_0 \text{ and } \sigma(e) \in \mathcal{U}(\mathbb{H})_1 \end{array} \right\}$$

Proof Given ρ, we define $\sigma := \iota^*(\rho) := \rho \circ \iota$ and, therefore, $\sigma(f) = \rho(\iota(f)) = \rho(L_{-1})$, $\sigma(h) = \rho(-2L_0)$ and $\sigma(e) = \rho(-L_1)$.

For the converse, one requires several steps and the previous Lemmas.

Step 1. Let σ be given. There exists a $\mathbb{C}$-linear homomorphism $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H})$ such that $\sigma = \iota^*(\rho)$. First, we set:

$$\rho(L_{-1}) := \sigma(f) = F\,, \quad \rho(L_0) := -\frac{1}{2}\sigma(h)\,, \quad \rho(L_1) := -\sigma(e)$$

The fact that σ is a map of Lie algebras and Lemma 2.2 implies that:

$$\rho(L_0) = -\frac{1}{2}H$$

where $H := \sigma(h)$ is a type 0 operator and has the form given in Eq. (3). Furthermore, it holds that $\rho(L_i) \in \mathcal{U}(\mathbb{H})'_i$ for $i = -1, 1$. Having in mind Lemma 2.8 we obtain that there is a unique $T \in \mathcal{U}(\mathbb{H})'_2$ such that:

$$\mathrm{ad}(\rho(L_{-1}))(T) = \rho(L_1)$$

Then, we define:

$$\rho(L_2) := -3T \in \mathcal{U}(\mathbb{H})'_2 \tag{12}$$

and, recursively,

$$\rho(L_i) := \frac{1}{i-2}[\sigma(e), \rho(L_{i-1})] \qquad \text{for } i > 2\,. \tag{13}$$

Step 2. It holds that $[\rho(L_0), \rho(L_j)] = -j\rho(L_j)$ for $j \geq -1$. This is equivalent to show that $\rho(L_j) \in \mathcal{U}(\mathbb{H})'_j$ for all $j \geq 1$. Bearing in mind that $\rho(L_1) \in \mathcal{U}(\mathbb{H})'_1$ and Lemma 2.6, the conclusion follows.

Step 3. It holds that $[\rho(L_{-1}), \rho(L_j)] = -(1+j)\rho(L_{j-1})$ for $j \geq -1$. The cases $j \leq 1$ follow from the fact that σ is a homomorphism of Lie algebras. The choice of T implies the case $j = 2$. Let us proceed by induction on j. For $j \geq 3$, the definition of $\rho(L_i)$, the Jacobi identity and the induction hypothesis yield:

$$\begin{aligned}
[\rho(L_{-1}), \rho(L_j)] &= \left[\rho(L_{-1}), -\frac{1}{j-2}[\rho(L_1), \rho(L_{j-1})]\right] \\
&= \frac{1}{j-2}\big([\rho(L_1),\ [\rho(L_{j-1}), \rho(L_{-1})]] \\
&\quad +[\rho(L_{j-1}),\ [\rho(L_{-1}), \rho(L_1)]]\big) \\
&= \frac{1}{j-2}\big([\rho(L_1),\ j\rho(L_{j-2})] \\
&\quad +[\rho(L_{j-1}),\ (-2)\rho(L_0)]\big) \\
&= \frac{1}{j-2}\big(j(3-j)\rho(L_{j-1}) - 2(j-1)\rho(L_{j-1})\big) \\
&= (-1-j)\rho(L_{j-1})
\end{aligned}$$

Step 4. The identity:

$$[\rho(L_i), \rho(L_j)] - (i-j)\rho(L_{i+j}) = 0 \tag{14}$$

holds for $i, j \geq 1$. We proceed by induction on $n = i + j$. The case $n = 4$ (i.e. $i, j \geq 1$ and $i + j = 4$) holds by the very definition of $\rho(L_4)$. Now, let us assume that it holds true up to $n - 1 = i + j - 1$ and let us prove the case $n = i + j > 4$. Observe that, by Step 2, the l.h.s of the Eq. (14) lies in $\mathcal{U}(\mathbb{H})'_{i+j}$. By Lemma 2.8, it suffices to show that its image under $\text{ad}(F) = \text{ad}(\rho(L_{-1}))$ vanishes. In fact, the Jacobi identity, the Step 3 and the induction hypothesis show that:

$$\begin{aligned}
&\text{ad}(\rho(L_{-1}))\big([\rho(L_i), \rho(L_j)] - (i-j)\rho(L_{i+j})\big) \\
&\quad = [[\rho(L_{-1}), \rho(L_i)], \rho(L_j)] + [\rho(L_i), [\rho(L_{-1}), \rho(L_j)]] \\
&\qquad -(i-j)[\rho(L_{-1}), \rho(L_{i+j})] \\
&\quad = [-(1+i)\rho(L_{i-1}), \rho(L_j)] + [\rho(L_i), -(1+j)\rho(L_{j-1})]
\end{aligned}$$

$$
\begin{aligned}
&+ (i-j)(1+i+j)\rho(L_{i+j-1}) \\
&= -\left((1+i)(i-j-1) + (1+j)(i-j+1) - (i-j)(1+i+j)\right)\rho(L_{i+j-1}) \\
&= 0
\end{aligned}
$$

Step 5. ρ is a Lie algebra homomorphism. This follows from the properties of σ and Steps 2, 3, 4. □

2.4 Factorization as a product

It is remarkable that if the vector space $(A, (\ ,\ ))$ decomposes as $A_1 \perp A_2$ (i.e. $A = A_1 \oplus A_2$ and $(a_1, a_2) = 0$ for all $a_i \in A_i$), then the very definition of the associated Heisenberg algebra implies that $\mathbb{H}(A_i)$ is a subalgebra of $\mathbb{H}(A)$ and that there is a canonical Lie algebras homomorphism:

$$
\mathcal{U}(\mathbb{H}(A)) \simeq \mathcal{U}(\mathbb{H}(A_1)) \otimes_{\mathbb{C}[K]} \mathcal{U}(\mathbb{H}(A_2))
$$

where we identify $\mathbb{C}[K]$ with the universal enveloping algebra of the center of $\mathbb{H}(A_i)$ for $i = 1, 2$. So, we may wonder under which circumstances a morphism $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A))$ would decompose accordingly. The following Theorem provides an answer in terms of the restriction $\rho|_{\mathfrak{sl}(2)}$. For this goal, recall that matrices a, b and c behave as bilinear forms on A (w.r.t. the action of $\mathrm{Gl}(A)$).

Theorem 2.10 *Let F, H, E be as in (2), (3) and (9). Let $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A))$ be a map of Lie algebras satisfying $\rho(L_{-1}) = F$, $\rho(L_0) = -\frac{1}{2}H$ and $\rho(L_1) = -E$.*

If the vector space A decomposes as $A_1 \perp A_2$ w.r.t. η and this decomposition is compatible with the action of F and with the bilinear forms $b_0^{1,1}$ and $c_1^{1,1}$, then there are Lie algebra maps $\rho_i : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A_i))$ for $i = 1, 2$ such that:

$$
\rho = \rho_1 \otimes 1 + 1 \otimes \rho_2
$$

If this is the case, and $\rho(L_k) \in \mathcal{U}(\mathbb{H}(A))'_k$ for all $k \geq -1$, then $\rho_i(L_k) \in \mathcal{U}(\mathbb{H}(A_i))'_k$ for all $k \geq -1$ and $i = 1, 2$.

Proof Step 1. The case of $\rho(L_{-1})$. The hypothesis says that we can find $\{a_\alpha | \alpha = 1, \ldots, n\}$, a basis of A, and an index m such that, for $1 \leq i < m \leq j \leq n$, the vectors a_i and a_j are orthogonal w.r.t. to the bilinear form defined by a. Equivalently, w.r.t. the splitting $A_1 \oplus A_2$ the matrix of this bilinear form acquires a block decomposition as follows:

$$
a = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}
$$

It is now straightforward that the terms of the operator F (as given in Eq. (2)) can be grouped into two sets, the first one involving $p_{i,\alpha}$ and $q_{i,\alpha}$ for $i \in \mathbb{N}$ and $1 \leq \alpha < m$, and the second one depending only on $p_{i,\alpha}$ and $q_{i,\alpha}$ for $i \in \mathbb{N}$ and $m \leq \alpha \leq n$. Denote these operators as $\bar{L}_{-1,1}$ and $\bar{L}_{-1,2}$, respectively. One checks that:

$$\rho(L_{-1}) = \bar{L}_{-1,1} \otimes 1 + 1 \otimes \bar{L}_{-1,2}$$
$$\bar{L}_{-1,\alpha} \in \mathcal{U}(\mathbb{H}(A_\alpha))_{-1} \qquad \alpha = 1, 2. \tag{15}$$

Step 2. The case of $\rho(L_0)$. Bearing in mind that it is defined as $-\frac{1}{2}H$ and that the coefficients of the latter fulfill the relations (7), one can proceed as in the previous case. More precisely, considering the following block decompositions:

$$\eta = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \qquad a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad c_1^{1,1} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$

and motivated by the computations of the Appendix, one may use the following identity as a defining relation for $\bar{L}_{0,\alpha} \in \mathcal{U}(\mathbb{H}(A_\alpha))_0$:

$$\rho(L_0) - b_0^{0,0} = \left(\bar{L}_{0,1} - 2\,\mathrm{Tr}(c_1\eta_1^{-1}(a_1 + a_1^T)(\eta_1^T)^{-1})\right) + \left(\bar{L}_{0,2} - 2\,\mathrm{Tr}(c_2\eta_2^{-1}(a_2 + a_2^T)(\eta_2^T)^{-1})\right)$$

Step 3. The case of $\rho(L_k)$ for $k \geq 1$. Recall from the proof of Theorem 2.7 that the coefficients $b_1^{r,r+2}$ of $\rho(L_1)$ can be expressed in terms of a, $b_0^{1,1}$ and $c_1^{1,1}$ and that a close look of these expressions shows that $b_1^{r,r+2}$ are compatible w.r.t. to the splitting of A. Thus, we can express $\rho(L_1)$ as the sum of two factors, namely $\bar{L}_{1,\alpha}$ for $\alpha = 1, 2$ which consists of the terms of $\rho(L_1)$ in $p_{i,\alpha}, q_{i,\alpha}$ for $1 \leq \alpha < m$ and for $m \leq \alpha \leq n$, respectively. Now, we proceed as above.

For the case of $\rho(L_k)$ for $k \geq 2$ one proceeds recursively (using the expressions of the proof of Lemma 2.8).

Step 4. $[\bar{L}_{k,\alpha}, \bar{L}_{l,\beta}] = 0$ for $k, l \geq -1$ and $\alpha \neq \beta$, since these two operators involve disjoint sets of variables.

Step 5. The maps ρ_α. Consider:

$$\rho_\alpha(L_k) := \bar{L}_{k,\alpha} \quad \text{for} \quad k \geq -1 \quad \text{and} \quad \alpha = 1, 2$$

The previous steps show that $\rho = \rho_1 \otimes 1 + 1 \otimes \rho_2$.

It remains to check that ρ_α are morphisms of Lie algebras. For this goal, we will expand both sides of the identity $[\rho(L_k), \rho(L_l)] = (k - l)\rho(L_{k+l})$ using the above facts. The l.h.s. is:

$$[\rho(L_k), \rho(L_l)] = [\bar{L}_{k,1} + \bar{L}_{k,2}, \bar{L}_{l,1} + \bar{L}_{l,2}] = [\bar{L}_{k,1}, \bar{L}_{k,1}] + [\bar{L}_{k,2}, \bar{L}_{l,2}]$$

while the r.h.s. reads:

$$(k - l)\rho(L_{k+l}) = (k - l)(\bar{L}_{k+l,1} + \bar{L}_{k+l,2})$$

Comparing both expressions and having in mind the separation of variables, it follows that:

$$[\bar{L}_{k,\alpha}, \bar{L}_{k,\alpha}] = (k-l)\bar{L}_{k+l,\alpha}$$

and we conclude that ρ_α is a map of Lie algebras $\mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A_\alpha))$.

Step 6. Type of the operators. To show that $\rho(L_k) \in \mathcal{U}(\mathbb{H}(A))'_k$ implies that $\rho_\alpha(L_k) \in \mathcal{U}(\mathbb{H}(A_\alpha))'_k$, it suffices to expand the Lie bracket $[\rho(L_0), \rho(L_k)]$ using $\rho(L_k) = \rho_1(L_k) \otimes 1 + 1 \otimes \rho_2(L_k)$. □

Remark 2.11 It is worth noticing that if a decomposition is compatible with a, it does not need to be compatible with $b_0^{1,1}$. Indeed, for $A = \mathbb{C}^2$, $\eta = a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the general form of $b_0^{1,1}$ is given by $\begin{pmatrix} -1 & \lambda \\ \lambda & -1 \end{pmatrix}$.

For later use, the following general result will be required.

Theorem 2.12 *Let* $\rho_i : \mathcal{W}_> \to \operatorname{End} V_i$, $i = 1, 2$, *be two representations of the Lie algebra* $\mathcal{W}_>$. *And let us consider the product representation:*

$$\rho = \rho_1 \otimes 1 + 1 \otimes \rho_2 : \mathcal{W}_> \longrightarrow \operatorname{End}(V_1 \otimes V_2)$$

Let $\sum_{i=1}^r f_{1,i} \otimes f_{2,i} \in V_1 \otimes V_2$. *Assume that* $f_{i,1}, \ldots, f_{i,r}$ *are linearly independent* (*for* $i = 1, 2$). *It then holds that:*

$$\rho(L_k)\left(\sum_i f_{1,i} \otimes f_{2,i}\right) = 0 \quad \forall k \geq -1$$

if and only if:

$$\rho_i(L_k)(f_{i,j}) = 0 \;\; \textit{for all } i, j \;\; \textit{and} \;\; k \geq -1.$$

Proof The converse is obvious.

The direct implication is more subtle. The hypothesis and the decomposition of ρ yield:

$$\begin{aligned} 0 = \rho(L_k)\left(\sum_i f_{1,i} \otimes f_{2,i}\right) &= (\rho_1 \otimes 1 + 1 \otimes \rho_2)(L_k)\left(\sum_i f_{1,i} \otimes f_{2,i}\right) \\ &= \sum_i \rho_1(L_k)(f_{1,i}) \otimes f_{2,i} + \sum_i f_{1,i} \otimes \rho_2(L_k)(f_{2,i}) \end{aligned}$$

Let E be the vector space generated by $\{f_{1,1}, \ldots, f_{1,r}\} \subset V_1$. Suppose that there exists l such that $\rho_1(L_k)(f_{1,l})$ does not belong to E. Then, let $\chi : V_1 \to \mathbb{C}$ be a linear form such that $\chi(f_{1,i}) = 0$ for all i and $\chi(\rho_1(L_k)(f_{1,l})) \neq 0$. Applying χ to the above equation, one obtains:

$$0 = \sum_i \chi\left(\rho_1(L_k)(f_{1,i})\right) f_{2,i} \in V_2$$

which contradicts the fact that $f_{2,1}, \dots, f_{2,r}$ are linearly independent. Therefore, it follows that $\rho_1(f_{1,l})$ belongs to E for all l or, equivalently,

$$\begin{aligned} \rho_{1,E} : \mathcal{W}_{>} &\longrightarrow \mathrm{End}(E) \\ L_k &\longmapsto \rho_{1,E}(L_k) := \rho_1(L_k)|_E \end{aligned}$$

is a Lie algebra homomorphism. Recall that, being $\mathcal{W}_{>}$ simple, the non-trivial representations of $\mathcal{W}_{>}$ are faithful. Since E is finite dimensional, $\rho_{1,E}$ must be trivial; that is, $\rho_{1,E}(L_k) = 0$ for all k. In particular,

$$0 = \rho_{1,E}(L_k)(f_{1,j}) = \rho_1(L_k)(f_{1,j}) \qquad \forall j$$

The identities $\rho_2(L_k)(f_{2,j}) = 0$ are proven similarly. □

3 An application

As an application of the previous sections, we offer here an example that illustrates how our results can be used for studying the representation of $\mathcal{W}_{>}$ appearing in the study of the Virasoro conjecture. Regarding the Virasoro conjecture, our main references are the works of Dubrovin–Zhang, Eguchi–Hori–Xiong, Getzler, Givental and Liu–Tian [6,9,13,14,26].

In our example, will consider $(A, (,))$ to be the cohomology ring of a smooth projective variety X, with $c_1(X) = 0$ and trivial odd cohomology groups, endowed with the Poincaré pairing. Recall that the hypothesis on the first Chern class is equivalent to the vanishing of the operator R in [6,13]; however, it does not seem difficult to extend the results of Sect. 2 to include this case. On the other hand, the hypothesis on the odd cohomology groups is fulfilled if X has generically semisimple even quantum cohomology [16]. It seems to be very hard to weaken this assumption.

Finally, let us point out that bilinear form $(,)$ is not necessary symmetric in Sects. 3.1–3.2 while it will be assumed to be symmetric from Sect. 3.3 on.

3.1 Preliminaries

Let A be a n-dimensional vector space over $\mathbb{C}$ endowed with a bilinear form $(,)$. Let $\{a_1, \dots, a_n\}$ be a basis and η be the matrix associated to the pairing, $\eta_{\alpha\beta} := (a_\alpha, a_\beta)$. Let us consider the subspace $\mathbb{C}[[t_1, t_3, t_5, \dots]]$ of the boson Fock space $\mathbb{C}[[t_1, t_2, \dots]]$ and the subalgebra of $\mathbb{C}[[t_1, \dots]]\hat{\otimes}_{\mathbb{C}} S^\bullet A$ generated by $t_{i,\alpha} := t_i \otimes a_\alpha$ with i odd:

$$V_{\mathrm{odd}}(A) := \mathbb{C}[[\{t_{i,\alpha} | 1 \le \alpha \le n, i \text{ odd}\}]] \subseteq \mathbb{C}[[t_1, \dots]]\hat{\otimes}_{\mathbb{C}} S^\bullet A \tag{16}$$

If no confusion arises, we will simply write V_{odd}.

Now we study a distinguished representation of $\mathcal{W}_{>}$ in V_{odd}; eventually, we will see that it is the representation coming from the action of the Heisenberg algebra via

Givental's quantization [14]. More precisely, we will combine the chain of inclusions of Lie algebras:

$$\mathfrak{sl}(2) \hookrightarrow \mathcal{W}_{>} \hookrightarrow \mathcal{U}(\mathbb{H})$$

which has been studied in the previous section, with a map:

$$\widehat{\ } : \mathcal{U}(\mathbb{H}(A)) \longrightarrow \mathrm{End}_{\mathbb{C}}(V_{\mathrm{odd}}(A))$$
$$P \longmapsto \hat{P}$$

whose obstruction to be compatible with the Lie brackets is governed by a cocycle. This map is defined following the results of Dubrovin–Zhang, Givental and Kazarian [6,14,20]; namely, we set:

$$\hat{1} = 1\,, \qquad \hat{p}_{i,\alpha} = \eta^{\alpha\beta}\frac{\partial}{\partial t_{i,\beta}}\,, \qquad \hat{q}_{i,\alpha} = i t_{i,\alpha} \tag{17}$$

(recall that i is a positive odd integer number).

Remark 3.1 Givental has developed a beautiful formalism for this construction in terms of quantization of quadratic Hamiltonians [14]. An alternative approach, originated in the Japanese school and strongly linked to the Sato Grassmannian, can be found in [19]. The forthcoming section (Sect. 3.4) is deeply inspired by the latter.

3.2 The representation

Bearing in mind the results of Sect. 2.2, we know that the operator:

$$F := b^{0,1}_{-1} p_1 + q_1 \eta q_1^T + q_{i+2}\eta p_i$$

together with the data:

- $b^{0,1}_{-1}$ arbitrary,
- $b^{1,1}_{0}$ such that (7b) holds, and
- $c^{1,1}_{1} := \frac{1}{16}\eta^T b^{1,1}_0 \eta^{-1}(b^{1,1}_0\eta^{-1} + 2)$,

determine a map $\sigma : \mathfrak{sl}(2) \to \mathcal{U}(\mathbb{H})$. Indeed, Eqs. (7), (11) and (10) allow us to obtain the explicit expressions for H and F:

$$H = \frac{1}{3} b^{0,1}_{-1}(\eta^{-1} b^{1,1}_0 - 2) p_3 + \frac{1}{i} q_i (b^{1,1}_0 - (i-1)\eta) p_i$$
$$+ \frac{1}{16}\,\mathrm{Tr}(b^{1,1}_0 \eta^{-1}(b^{1,1}_0\eta^{-1} + 2)(1 + \eta^{-1}\eta^T))$$

$$\begin{aligned} E = & \frac{1}{5!!} b_{-1}^{0,1} \left(2\eta^{-1} b_0^{1,1} - 2 - (\eta^{-1} + (\eta^{-1})^T)(c_1^{1,1} + (c_1^{1,1})^T) \right) p_5 \\ & + \frac{1}{16} p_1^T \eta^T b_0^{1,1} \eta^{-1} (b_0^{1,1} \eta^{-1} + 2) p_1 \\ & - \frac{1}{4i(i+2)} q_i (b_0^{1,1} \eta^{-1} - (i-1))(b_0^{1,1} \eta^{-1} - (i+1)) \eta p_{i+2} \end{aligned} \tag{18}$$

Now, by Theorem 2.9, the map σ extends uniquely to an homomorphism $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H})$. And one can now compute the induced action on V_{odd}. Let us write down the first operators:

$$\begin{aligned} \hat{L}_{-1} := & (\rho(L_{-1}))\hat{} = \hat{F} = b_{-1}^{0,1} \eta^{-1} \frac{\partial}{\partial t_1} + t_1 \eta t_1^T + (i+2) t_{i+2} \frac{\partial}{\partial t_i} \\ \hat{L}_0 := & (\rho(L_0))\hat{} = -\frac{1}{2} \hat{H} = -\frac{1}{6} b_{-1}^{0,1} (\eta^{-1} b_0^{1,1} - 2) \eta^{-1} \frac{\partial}{\partial t_3} \\ & - \frac{1}{2} t_i (b_0^{1,1} \eta^{-1} - (i-1)) \frac{\partial}{\partial t_i} - \frac{1}{32} \operatorname{Tr}(b_0^{1,1} \eta^{-1} (b_0^{1,1} \eta^{-1} + 2)(1 + \eta^{-1} \eta^T)) \\ \hat{L}_1 := & (\rho(L_1))\hat{} = -\hat{E} \\ & - \frac{1}{5!!} b_{-1}^{0,1} \left(2\eta^{-1} b_0^{1,1} - 2 - (\eta^{-1} + (\eta^{-1})^T)(c_1^{1,1} + (c_1^{1,1})^T) \right) \eta^{-1} \frac{\partial}{\partial t_5} \\ & - \frac{1}{16} (\frac{\partial}{\partial t_1})^T \eta^T b_0^{1,1} \eta^{-1} (b_0^{1,1} \eta^{-1} + 2) \frac{\partial}{\partial t_1} \\ & + \frac{1}{4(i+2)} t_i (b_0^{1,1} \eta^{-1} - (i-1))(b_0^{1,1} \eta^{-1} - (i+1)) \frac{\partial}{\partial t_{i+2}} \end{aligned}$$

where, as usual, we write t_i for the row vector $(t_{i,1}, \dots, t_{i,n})$ and $\frac{\partial}{\partial t_i}$ for the column vector $(\frac{\partial}{\partial t_{i,1}}, \dots, \frac{\partial}{\partial t_{i,n}})^T$.

3.3 The operators of the Virasoro conjecture: a baby model

Now, we are ready to show how the operators appearing in the Virasoro conjecture agree with our approach for the case of manifolds with trivial odd cohomology and whose first Chern class vanishes.

From now on, we suppose we are given X, whose first Chern class is zero, and with trivial odd cohomology. Under this hypothesis, the Poincaré pairing defines on $A := H^\bullet(X, \mathbb{C})$ a symmetric non-degenerated bilinear form:

$$(a, b) = \int_X a \cup b \quad \text{for } a, b \in A,$$

Let $r := \dim(X)$ and fix a basis $\{a_\alpha | \alpha = 1, \dots, n\}$ of A, with $a_1 = 1 \in H^0(X, \mathbb{C})$, such that it is homogeneous w.r.t. the Hodge decomposition; that is, $a_\alpha \in H^{p_\alpha, q_\alpha}(X)$ for certain p_α, q_α. Let $\bar{\eta}$ the matrix associated to the Poincaré pairing w.r.t. the chosen basis and let us define $\mu_\alpha := p_\alpha - \frac{r}{2}$ and μ the matrix with $\mu_1, \dots, \mu_r$ along its diagonal and 0 elsewhere. Observe that the compatibility of the Poincaré pairing w.r.t. the Hodge decomposition yields:

$$\bar{\eta}_{\alpha\beta} \neq 0 \implies \mu_\alpha + \mu_\beta = 0 \tag{19}$$

The operators appearing in the Virasoro conjecture when the first Chern class vanishes [13, Equation (1.2)] are as follows:

$$\begin{aligned} \bar{L}_{-1} &:= -\frac{\partial}{\partial \bar{t}_{0,1}} + \frac{1}{2\hbar}\bar{t}_0\bar{\eta}\bar{t}_0^T + \bar{t}_{i+1}\frac{\partial}{\partial \bar{t}_i} \\ \bar{L}_0 &:= -\frac{3-r}{2}\frac{\partial}{\partial \bar{t}_{1,1}} + \left(\mu_\alpha + i + \frac{1}{2}\right)\bar{t}_{i,\alpha}\frac{\partial}{\partial \bar{t}_{i,\alpha}} + \frac{1}{48}(3-r)\int_X c_r(X) \end{aligned} \tag{20}$$

and, for $k \geq 1$, as:

$$\begin{aligned} \bar{L}_k :=& -\frac{\Gamma(k+\frac{5-r}{2})}{\Gamma(\frac{3-r}{2})}\frac{\partial}{\partial \bar{t}_{k+1,1}} + \frac{\Gamma(\mu_\alpha + i + k + \frac{3}{2})}{\Gamma(\mu_\alpha + i + \frac{1}{2})}\bar{t}_{i,\alpha}\frac{\partial}{\partial \bar{t}_{k+i,\alpha}} \\ &+ \frac{\hbar}{2}(-1)^i \frac{\Gamma(\mu_\alpha + i + k + \frac{3}{2})}{\Gamma(\mu_\alpha + i + \frac{1}{2})}\bar{\eta}^{\alpha\beta}\frac{\partial}{\partial \bar{t}_{-1-i,\alpha}}\frac{\partial}{\partial \bar{t}_{k+i,\beta}} \end{aligned} \tag{21}$$

where $c_r(X)$ is the rth Chern class and we have used variables $\bar{t}_{i,\alpha}$ with $\alpha = 1, \ldots, n$ and $i = 0, 1, 2, \ldots$.

Similarly to the case of $\mathcal{U}(\mathbb{H})$, we say that a second-order differential operator in $\{\bar{t}_{i,\alpha}\}$ is of type i if it is a linear combination of $\frac{\partial}{\partial \bar{t}_{i+1,\alpha}}$ and the following terms $\frac{\partial}{\partial \bar{t}_{j-1,\alpha}}\frac{\partial}{\partial \bar{t}_{i-j,\beta}}$, $\bar{t}_{j,\alpha}\frac{\partial}{\partial \bar{t}_{j+i,\beta}}$ and $\bar{t}_{j-1,\alpha}\bar{t}_{-i-j,\beta}$ and, if $i = 0$, a constant term. Observe that $\bar{L}_k$ is of type k. Now, we offer a simple proof of a folk statement.

Proposition 3.2 *The operators $\{\bar{L}_k | k \geq 2\}$ are uniquely determined by $\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ and the condition that $\bar{L}_k$ is of type k for all $k \geq -1$.*

Proof Under the change of variables $\bar{t}_i := \sqrt{2\hbar}(2i+1)!!t_{2i+1}$, it is clear that a second-order differential operator in $\bar{t}_i$s is of type k if and only if is equal to $\hat{T}$ for $T \in \mathcal{U}(\mathbb{H})_k$. Now, it is easy to check that the hypothesis of Theorem 2.9 hold; namely, $\hat{F} = \bar{L}_{-1}$ and $\bar{L}_k$ are of type k for $k = 0, 1$. The conclusion follows.

Theorem 3.3 *It holds:*

$$\bar{L}_i = \hat{L}_i \qquad i = -1, 0, 1, \ldots$$

for the choice $\bar{t}_i := \sqrt{2\hbar}(2i+1)!!t_{2i+1}$, $\eta = \bar{\eta}$, $b_{-1}^{0,1} = (0, \ldots, 0, \frac{-1}{\sqrt{2\hbar}})$ and:

$$b_0^{1,1} := -(2\mu+1)\eta = -\begin{pmatrix} 0 & & 2\mu_1+1 \\ & \iddots & \\ 2\mu_n+1 & & 0 \end{pmatrix}$$

Proof Theorem 2.9 implies that it suffices to show that $\bar{L}_i = \hat{L}_i$ for $i = -1, 0, 1$. Indeed, this fact follows from the explicit substitution of t_i, η, etc. as in the statement in the operators $\hat{L}_i$. The only identity which is not obvious is the one corresponding to the constant term of $\hat{L}_0$. Bearing in mind the definitions and the fact that η is symmetric, this term is:

$$\begin{aligned} -\frac{1}{32}\operatorname{Tr}(b_0^{1,1}\eta^{-1}(b_0^{1,1}\eta^{-1}+2)(1+\eta^{-1}\eta^T)) \\ = -\frac{1}{16}\sum_{\alpha=1}^{n}(2p_\alpha - r + 1)(2p_\alpha - r - 1) \\ = \frac{1}{4}\sum_{p,q} h^{p,q}(\frac{r+1}{2} - p)(p - \frac{r-1}{2}) \end{aligned}$$

where $h^{p,q} = \dim H^p(X, \Omega^q)$.

Now, observe that the Libgober–Wood identity [24, Proposition 2.3] can be stated as:

$$\begin{aligned} \sum_{p,q}(-1)^{p+q}h^{p,q}\left(\frac{r+1}{2} - p\right)\left(p - \frac{r-1}{2}\right) \\ = \frac{1}{6}\int_X\left(\frac{3-r}{2}c_r(X) - c_1(X)c_{r-1}(X)\right) \end{aligned}$$

Recalling that we are assuming that X has trivial odd cohomology, the constant term equals:

$$\frac{1}{48}\int_X((3-r)c_r(X) - 2c_1(X)c_{r-1}(X))$$

which agrees with the free term of $\bar{L}_0$ (see (20)) since $c_1(X) = 0$. □

Remark 3.4 It is worth noticing that up to rescaling the variables and a Dilaton shift, these operators coincide with those of [5, Equation (3.5)] and [14, §3] (for $b_1 = 0$) and with those of [4, Equation (7.33)] and [39, Equation (2.59)] (for $b_1 = -\frac{1}{3}\sqrt{\frac{\eta}{2\hbar}}$).

Now, we will go one step further in the study of the above representation. Recall that in Sect. 2.1 it was stated that matrices a, $b_i^{j-2i,j}$ and $c_i^{j,2i-j}$ behave as bilinear forms under the action of $\mathrm{Gl}(A)$. *A fundamental observation is that all results and equations above are invariant under the action of the general linear group* (acting as base changes on the given basis $\{a_1, \ldots, a_n\}$). Let us briefly discuss this statement. For instance, let $S \in \mathrm{Gl}(A)$, then the row vector $q_i = (q_{i,1}, \ldots, q_{i,n})$ is transformed to $q_i S^T$, accordingly the column vector p_i goes to Sp_i. The action of S sends the bilinear form η to $(S^{-1})^T\eta S^{-1}$ and analogously with a, etc. Note that, since η and $b_0^{1,1}$ behave as bilinear forms, $\eta^{-1}b_0^{1,1}$ defines an endomorphism of A. Finally, the Heisenberg algebra is also affected.

Definition 3.5 Let $\mathbb{H}^\eta$ be the Heisenberg algebra defined in (1). Given a map of Lie algebras $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}^\eta)$ and $S \in \mathrm{Gl}(A)$, we denote by ρ^S the map of Lie algebras:

$$\mathcal{W}_> \xrightarrow{\rho} \mathcal{U}(\mathbb{H}^\eta) \xrightarrow{\sim} \mathcal{U}(\mathbb{H}^{(S^{-1})^T \eta S^{-1}})$$

where the last map sends q_i to $q_i S^T$ and p_i to Sp_i.

With the hypothesis and choices of above, we have the following,

Theorem 3.6 *Let $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}^\eta)$ be as above; i.e. $\hat{\rho}$ defines the Virasoro constraints,* (20) *and* (21)*, of a smooth projective variety with trivial odd cohomology and vanishing first Chern class.*

Then there exists $S \in \mathrm{Gl}(A)$ such that ρ^S decomposes as the product of n representations of dimension 1; that is, there exists $\rho_i : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(\mathbb{C}))$ such that:

$$\rho^S = \rho_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \rho_n \tag{22}$$

Proof Let us consider a basis which is orthonormal for η. Let $S \in \mathrm{Gl}(A)$ be the matrix associated to this change of basis. Due to the choices of a, $b_{-1}^{i+2,i}$, $b_0^{1,1}$ and $c_1^{1,1}$, it is trivial that S also brings them into diagonal form; or, equivalently, there is a common orthogonal basis for all these bilinear pairings. Applying Theorem 2.10, one concludes. □

In this situation, for each $\alpha = 1, \ldots, n$, one obtains a one-dimensional representation ρ_α or, what is tantamount, our study essentially reduces to the case of Example 2.5. That is, $\dim A = 1$, $a = \eta \in \mathbb{C}^*$ and, thus, $b_0^{1,1} = -\eta$. Setting $b_0 := b_0^{0,0}$, one has that (18) gives:

$$\begin{aligned} F &= b_{-1} p_1 + q_1 \eta q_1 + q_{i+2} \eta p_i \\ H &= -b_{-1} p_3 - q_i \eta p_i - \frac{1}{8} \\ E &= -\frac{1}{4} b_{-1} p_5 - \frac{1}{4} p_1 \eta p_1 - \frac{1}{4} q_i \eta p_{i+2} \end{aligned}$$

where b_{-1} and η are computed from the n-dimensional setup (20).

These three operators determine ρ completely and, according to the map (17) and Theorem 3.3, one has:

$$\begin{aligned} \bar{L}_{-1} &:= b_{-1}\sqrt{2\hbar}\eta^{-1} \frac{\partial}{\partial \bar{t}_0} + \frac{1}{2\hbar}\eta \bar{t}_0^2 + \bar{t}_{i+1} \frac{\partial}{\partial \bar{t}_i} \\ \bar{L}_0 &:= \frac{3}{2} b_{-1}\sqrt{2\hbar}\eta^{-1} \frac{\partial}{\partial \bar{t}_1} + \left(i + \frac{1}{2}\right) \bar{t}_i \frac{\partial}{\partial \bar{t}_i} + \frac{1}{16} \\ \bar{L}_1 &:= \frac{5!!}{4} b_{-1}\sqrt{2\hbar}\eta^{-1} \frac{\partial}{\partial \bar{t}_2} + \frac{\hbar}{2}\eta^{-1} \frac{\partial}{\partial \bar{t}_0} \frac{\partial}{\partial \bar{t}_0} + \left(i + \frac{1}{2}\right)\left(i + \frac{3}{2}\right) \bar{t}_i \frac{\partial}{\partial \bar{t}_{i+1}} \end{aligned} \tag{23}$$

3.4 On the solutions for the one-dimensional situation

Once the representation has been decomposed in terms of one-dimensional parts, we wonder if one could deduce some properties of the solutions of the Virasoro constraints. Our approach follows closely our previous work [33] which is inspired in [18]. Briefly, the idea is to show that each of our representations ρ_i come from an action of $\mathcal{W}_>$ on the Sato Grassmannian and that they admit exactly one solution τ_i, which are τ-functions for the KP hierarchy and, then, conclude that the product $\tau_1 \cdot \ldots \cdot \tau_n$ is a solution for ρ^S.

Let us begin recalling that the Sato Grassmannian is the set of subspaces $U \subset \mathbb{C}((z))$ such that the kernel and cokernel of $\pi_U : U \to \mathbb{C}((z))/\mathbb{C}[[z]]$ are finite dimensional [36,37]. Actually, it is an infinite dimensional scheme [1] and carries a distinguished line bundle, the determinant line bundle $\mathbb{D}$. Each integer n corresponds to a connected component, Gr^n; namely, those subspaces U such that $\dim\ker\pi_U - \dim\operatorname{coker}\pi_U = n$. Sato-Sato's achievement was to show that there was a bijection between the set of those U s.t. π_U is an isomorphism and the set of functions $\tau(t) \in \mathbb{C}[t_1, t_2, \ldots]]$ with $\tau(0) = 1$ and fulfilling the KP hierarchy (thus, each U has a τ-function; see [1,36,37] for details). The same holds for the Sato grassmannian of $\mathbb{C}((z))^{\oplus n}$ and the n-multicomponent KP hierarchy.

The fact that the space of global section of $\mathbb{D}^*$ is isomorphic to the semi-infinite wedge product or Fermion Fock space:

$$H^0(\mathrm{Gr}^n, \mathbb{D}^*) \simeq \wedge^{\frac{\infty}{2}}\mathbb{C}((z)) = \left\langle \left\{ \begin{array}{l} z^{i_1} \wedge z^{i_2} \wedge \cdots \text{ s.t. } i_1 < i_2 < \cdots \\ \text{and } i_k = k + n \ \forall k \gg 0 \end{array} \right\} \right\rangle$$

have allowed its extensive use in CFTs (in particular, by the Japanese school, see [19] and references therein). Recall that the boson–fermion correspondence is the isomorphism (we restrict us to Gr^0; that is, the charge 0 sector):

$$\wedge^{\frac{\infty}{2}}\mathbb{C}((z)) \simeq \mathbb{C}[[t_1, t_2, \ldots]]$$

that maps $z^{i_1} \wedge z^{i_2} \wedge \ldots$ to the Schur polynomial associated with the partition $1 - i_1 \geq 2 - i_2 \geq \ldots$. Similarly, the space of global sections of $\mathbb{D}^*$ over the Sato grassmannian of $\mathbb{C}((z))^{\oplus n}$ is isomorphic to $\mathbb{C}[[\{t_{i,\alpha} | \alpha = 1, \ldots, n, i = 1, 2, \ldots\}]]$.

Given a subgroup of the restricted linear group of $\mathbb{C}((z))$ (see [37]), one has an induced action on $\mathrm{Gr}^n(\mathbb{C}((z)))$. Moreover, if the action preserves the determinant bundle, it will yield a projective action on the space of global sections. In fact, an analogous statement holds for the case of Lie algebras. Let us illustrate this issue with the case of the Lie algebra $\mathrm{Diff}^1(\mathbb{C}((z)))$ of first-order differential operators on $\mathbb{C}((z))$. An operator $D \in \mathrm{Diff}^1(\mathbb{C}((z)))$ acts on sections as follows. If the matrix (d_{ij}) corresponding to D w.r.t. the basis $\{z^i\}$ has no non-trivial diagonal elements, then:

$$D(z^{i_1} \wedge z^{i_2} \wedge \cdots) := D(z^{i_1}) \wedge z^{i_2} \wedge \cdots + z^{i_1} \wedge D(z^{i_2}) \wedge \cdots + \cdots$$

If the matrix (d_{ij}) is diagonal, then:

$$D(z^{i_1}\wedge z^{i_2}\wedge\cdots) := \sum_{j=1}^{\infty}(d_{i_j i_j}-d_{jj})z^{i_1}\wedge z^{i_2}\wedge\cdots$$

Having in mind the boson–fermion correspondence, the above construction gives rise to a linear map:

$$\begin{aligned}\mathrm{Diff}^1(\mathbb{C}((z))) &\xrightarrow{\beta} \mathrm{End}(\mathbb{C}[[t_1,t_2,\ldots]])\\ D &\longmapsto \beta(D)\end{aligned}\tag{24}$$

which defines a projective representation. Note, nevertheless, that if we are given a map of Lie algebras $\sigma:\mathcal{W}_>\to\mathrm{Diff}^1(\mathbb{C}((z)))$, then, $\beta\circ\sigma$ can be canonically promoted to a linear representation since $\mathcal{W}_>$ has no non-trivial central extensions. Indeed, for this goal, if suffices to add a constant to $\beta\circ\sigma(L_0)$.

The following results will show that the operators of Sect. 3.3 arise from the previous setup.

Lemma 3.7 *Let $D\in\mathrm{Diff}^1(\mathbb{C}((z)))$. Then, $\beta(D)$ is of type i if and only if D is a linear combination of* 1, $z^{-(2i+3)}$ *and* $z^{-2i}(z\partial_z+\frac{1-2i}{2})$.

Proof Recall that $\mathrm{Diff}^1(\mathbb{C}((z)))$ is generated as $\mathbb{C}$-vector space by 1, z^m for $m\in\mathbb{Z}$ acting as an homothety and $z^m(z\partial_z+\frac{m+1}{2})$ for $m\in\mathbb{Z}$. Let us recall from [20, Table 1] the description of the operators induced by them via the boson–fermion correspondence:

$$\beta(z^m) = \begin{cases} mt_m & \text{for } m>0\\ 0 & \text{for } m=0\\ \frac{\partial}{\partial t_{-m}} & \text{for } m<0\end{cases}$$

and, for $m>0$,

$$\beta\left(z^m(z\partial_z+\frac{1+m}{2})\right) = \frac{1}{2}\sum_{j=1}^{m-1} j(m-j)t_j t_{m-j} + \sum_{j=1}^{\infty}(j+m)t_{m+j}\frac{\partial}{\partial t_j}$$

Analogously, the action of $z^{-m}(z\partial_z+\frac{1-m}{2})$ on $\mathbb{C}((z))$ corresponds to the action of:

$$\beta\left(z^{-m}(z\partial_z+\frac{1-m}{2})\right) = \sum_{j=1}^{\infty} jt_j\frac{\partial}{\partial t_{m+j}} + \frac{1}{2}\sum_{j=1}^{m-1}\frac{\partial}{\partial t_j}\frac{\partial}{\partial t_{m-j}}$$

Finally, recall that the case $m = 0$ is regularized as follows:

$$\beta\left(z^{-m}(z\partial_z + \frac{1-m}{2})\right) := \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{-j}}$$

Checking the degrees, the conclusion follows. □

Lemma 3.8 *Let* $\sigma \in \mathrm{Hom}_{Lie\text{-}alg}(\mathcal{W}_>, \mathrm{Diff}^1(\mathbb{C}((z))))$. *Recall that* $V_{\mathrm{odd}} = \mathbb{C}[[t_1, t_3, \ldots]]$.

$\beta(\sigma(L_i))|_{V_{\mathrm{odd}}}$ *takes values in* V_{odd} *and it is of type* i *for all* i*, if and only if there exist* $s, t \in \mathbb{C}$ *such that:*

$$\sigma(L_i) = t^i\left(\frac{1}{2}z^{-2i}(z\partial_z + \frac{1-2i}{2}) + sz^{-2i-3}\right) \qquad \forall i \geq -1$$

Proof The "if" part follows from Lemma 3.7 and the fact that σ as in the statement defines a map of Lie algebras. Let us now deal with the "only if" part.

We know from [33, §2] (see also [34]) that there is a 1–1 correspondence:

$$\left\{\begin{array}{l}\sigma \in \mathrm{Hom}_{\mathrm{Lie\text{-}alg}}(\mathcal{W}_>, \mathrm{Diff}^1(\mathbb{C}((z)))) \\ \text{such that } \sigma \neq 0\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{triples } (h(z), c, b(z)) \text{ such that} \\ h'(z) \in \mathbb{C}((z))^*,\ c \in \mathbb{C},\ b(z) \in \mathbb{C}((z))\end{array}\right\}$$

which is explicitly given by:

$$\sigma(L_i) = \frac{-h(z)^{i+1}}{h'(z)}\partial_z - (i+1)c \cdot h(z)^i + \frac{h(z)^{i+1}}{h'(z)}b(z) \tag{25}$$

On the other hand, due to Lemma 3.7, the fact that $\sigma(L_i)$ is of type i implies that there exist $r_i, s_i, t_i \in \mathbb{C}$ satisfying:

$$\sigma(L_i) = r_i \cdot 1 + s_i \cdot z^{-(2i+3)} + t_i \cdot z^{-2i}\left(z\partial_z + \frac{1-2i}{2}\right) \tag{26}$$

Comparing the coefficients of ∂_z in the previous identities, it follows that $h(z) = \frac{t_i}{t_{i-1}}z^{-2}$. Hence, the quotients $\frac{t_i}{t_{i-1}}$ are all equal to a constant, say t. Hence, $t_i = t^i t_0$ and $h(z) = tz^{-2}$. Further, the case $i = 0$ yields $t_0 = \frac{1}{2}$.

Plugging this in Eqs. (25) and (26), one gets:

$$-(i+1)c(tz^{-2})^i - \frac{(tz^{-2})^{i+1}}{2tz^{-3}}b(z) = r_i + s_i z^{-(2i+3)} + \frac{1}{2}t^i z^{-2i}\left(\frac{1-2i}{2}\right)$$

and, thus:

$$b(z) = -2(i+1)cz^{-1} - 2t^{-i}r_i z^{2i-1} - 2s_i t^{-i} z^{-4} - \frac{1}{2}z^{-1}(1-2i)$$

Observe that the l.h.s. does not depend on i, one gets many conditions. First, for $i \neq 0$ the term z^{2i-1} is an odd power of z different from z^{-1} that cannot be canceled with any other term; consequently, $r_i = 0$ for $i \neq 0$. Further, since the coefficient of z^{-4} in $b(z)$ has to be independent of i, it follows that $t^{-i}s_i$ is a constant independent of i, and, thus, equal to s_0. Finally, the coefficient of z^{-1} in $b(z)$ is:

$$-2(i+1)c - 2r_0\delta_{i,0} - \frac{1}{2}(1-2i)$$

Since it has to be independent of i, it follows that $c = \frac{1}{2}$, $r_0 = 0$ and, thus:

$$b(z) = -\frac{3}{2}z^{-1} - 2s_0 z^{-4}$$

Substituting $h(z), c, b(z)$ into expression (25) and setting $s = s_0$, one obtains the result. □

Let us recall that the rescaling of the variables yields an action on the boson Fock space. More precisely, $\lambda = \{\lambda_i\} \in \prod_{i \text{ odd}} \mathbb{C}^*$ maps t_i to $\lambda_i t_i$. Accordingly, it acts on $\mathrm{Hom}_{\text{Lie-alg}}(\mathcal{W}_>, \mathrm{End}(V_{\text{odd}}))$ and sends ρ to $\rho^\lambda := \lambda \circ \rho \circ \lambda^{-1}$.

Definition 3.9 The λ-scaled KP hierarchy is the hierarchy obtained by replacing t_i by $\lambda_i t_i$ in the KP hierarchy (for given $\lambda = (\lambda_i) \in \prod_{i\in\mathbb{N}} \mathbb{C}^*$). A function $\tau_1(t) \in \mathbb{C}[[t_1, t_2, \ldots]]$ is called τ-function of the λ-scaled KP hierarchy if $\tau(\lambda^{-1}t) := \tau(\lambda_1^{-1}t_1, \lambda_2^{-1}t_2, \ldots)$ is a τ-function of the KP hierarchy. For brevity, we simply say scaled KP. We do similarly for KdV, multicomponent KP.

Note that the λ-scaled KP hierarchy for $\lambda = (\mu^i)$ for $\mu \in \mathbb{C}^*$ coincides with the KP hierarchy. However, this does not happen in general.

The following Lemma is the key point to go from Virasoro to KdV.

Lemma 3.10 *The map β of* (24) *induces a bijection between:*

- *the set of $\sigma \in \mathrm{Hom}_{Lie\text{-}alg}(\mathcal{W}_>, \mathrm{Diff}^1(\mathbb{C}((z))))$ such that there exists $s \in \mathbb{C}$ satisfying:*

$$\sigma(L_i) = \frac{1}{2}z^{-2i}\left(z\partial_z + \frac{1-2i}{2}\right) + sz^{-2i-3}$$

- *the set of scale equivalence classes of $\rho \in \mathrm{Hom}_{Lie\text{-}alg}(\mathcal{W}_>, \mathrm{End}(V_{\text{odd}}))$ whose coefficients of quadratic terms in $\rho(L_{-1})$ do not vanish and such that $\rho(L_i)$ is of type i for $i \geq -1$.*

Proof First, we prove the statement with no reference to $r(z)$ on the first item and with no mention to a linear function on the second item. Under these circumstances, given σ as in the statement, Lemma 3.7 shows that $(\beta \circ \sigma)(L_i)|_{V_{\text{odd}}}$ takes values in V_{odd} and it is of type i for all i. An explicit computation yields:

$$\begin{aligned}(\beta\circ\sigma)(L_{-1}) &= s\frac{\partial}{\partial t_1}+\frac{1}{4}t_1^2+\frac{1}{2}\sum_{j=1}^{\infty} jt_{j+2}\frac{\partial}{\partial t_j}\\ (\beta\circ\sigma)(L_0) &= s\frac{\partial}{\partial t_3}+\frac{1}{2}\sum_{j=1}^{\infty} jt_j\frac{\partial}{\partial t_j}\\ (\beta\circ\sigma)(L_i) &= s\frac{\partial}{\partial t_{2i+3}}+\frac{1}{4}\sum_{j=1}^{2i-1}\frac{\partial}{\partial t_j}\frac{\partial}{\partial t_{2i-j}}+\frac{1}{2}\sum_{j=1}^{\infty} jt_j\frac{\partial}{\partial t_{2i-j}}\end{aligned}$$

(where j, as usual, is odd) and thus:

$$[(\beta\circ\sigma)(L_{-1}),(\beta\circ\sigma)(L_1)]\ -\beta([\sigma(L_{-1}),\sigma(L_1)]) = -\frac{1}{8}$$

which implies that we have a map of Lie algebras defined by:

$$\rho(L_i) := (\beta\circ\sigma)(L_i)+\frac{1}{16}\delta_{i,0} \tag{27}$$

Conversely, let us start with ρ as in the second set of the statement. The assumptions yield the following expression:

$$\rho(L_{-1}) = b_{-1}^{0,1}\frac{\partial}{\partial t_1}+at_1^2+b_{-1}^{i+2,i}t_{i+2}\frac{\partial}{\partial t_i}$$

with $a, b_{-1}^{i+2,i}\neq 0$. Considering the action of $\prod_{i \text{ odd}}\mathbb{C}^*$ by conjugation, one finds $\lambda=\{\lambda_i\in\mathbb{C}^*|i \text{ odd}\}$ and $s\in\mathbb{C}$ such that:

$$\rho^\lambda(L_{-1}) = \lambda\circ\rho(L_{-1})\circ\lambda^{-1} = s\frac{\partial}{\partial t_1}+\frac{1}{4}t_1^2+\left(\frac{i+2}{2}\right)t_{i+2}\frac{\partial}{\partial t_i}$$

Lemma 3.8 and the previous discussion show that ρ^λ is the representation associated to the map $\sigma:\mathcal{W}_>\to\mathrm{Diff}^1(\mathbb{C}((z)))$ defined by:

$$\sigma(L_i)=\frac{1}{2}z^{-2i}\left(z\partial_z+\frac{1-2i}{2}\right)+sz^{-2i-3}\qquad\forall i\geq -1$$

□

Remark 3.11 The statement can be generalized. On the one hand, we may consider the conjugation of σ by an operator of the type $\exp(r(z))$ while, on the other hand, we replace ρ by its conjugate by $\exp(\beta(r(z)))$. For instance, for $r(z)\in\mathbb{C}[[z^2]]$, one has that $\beta(r(z))$ is a linear function on $t_1, t_3,\ldots$. Thus, the first representation is:

$$\sigma(L_i) = \frac{1}{2} z^{-2i} \left(z(-r(z) + \partial_z) + \frac{1-2i}{2} \right) + s z^{-2i-3}$$

while ρ is as in the statement up to a linear function on t_is.

Remark 3.12 It is worth noticing that the Virasoro operators studied by Witten [39] correspond to the case $s = -\frac{1}{2}, r(z) = 0$. Kac–Schwarz [18], using the fact that these operators come from a representation in $\mathrm{Diff}^1(\mathbb{C}((z)))$, proved that there is a point in the Sato Grassmannian whose τ-function is a solution of these equations and, hence, is a solution of KdV hierarchy too. A study of common solutions of Virasoro-like constraints and KdV has been carried out in [33].

Lemma 3.13 *Let ρ be as in Lemma 3.10 and let $\tau(t) \in V_{\mathrm{odd}} = \mathbb{C}[[t_1, t_3, \ldots]]$. Then, the Virasoro constraints:*

$$\rho(L_k)(\tau(t)) = 0 \qquad k \geq -1$$

with the initial condition $\tau(0) = 1$ admits no solution for $s = 0$ and at most one solution for $s \neq 0$.

Proof Since $\tau(0) = 1$, let us consider the problem in terms of a formal function $F(t) \in V_{\mathrm{odd}} = \mathbb{C}[[t_1, t_3, \ldots]]$ with $F(0) = 0$ and $\tau(t) = \exp(F(t))$. The function $F(t)$ has a series expansion:

$$F(t) = \sum_{\mathbf{n}} f_{\mathbf{n}} \mathbf{t}^{\mathbf{n}} \tag{28}$$

where $\mathbf{n} := \{n_1, n_3, \ldots\}$ is a sequence of non-negative integers such that $n_i = 0$ for all $i \gg 0$, $f_{\mathbf{n}} \in \mathbb{C}$ and $\mathbf{t}^{\mathbf{n}} := \prod_{i \geq 1} t_i^{n_i}$. Further, the topology of $V_{\mathrm{odd}} = \mathbb{C}[[t_1, t_3, \ldots]]$ comes from the definition $\deg(t_i) = i$. In particular, the degree of $\mathbf{t}^{\mathbf{n}}$ is given by $|\mathbf{n}| := \sum_{i \geq 0} i n_i$.

For the sake of brevity, let us denote by $f_{n_1 n_3 \ldots n_k} = f_{\mathbf{n}}$ for $\mathbf{n} = \{n_1, n_3, \ldots\}$ with $n_k \neq 0$ and $n_i = 0$ for all $i > k$ and we set $f_0 = F(0) = 0$. As a brief summary, let us write down the monomials and their coefficients up to degree 5:

degree	0	1	2	3	4	5
monomials	1	t_1	t_1^2	t_1^3, t_3	$t_1^4, t_1 t_3$	$t_1^5, t_1^2 t_3, t_5$
coefficient	f_0	f_1	f_2	f_3, f_{01}	f_4, f_{11}	f_5, f_{21}, f_{001}

After rescaling t_is and conjugation by an exponential, if needed, we may assume that ρ is given by (27). The hypothesis $\rho(L_k)(\tau(t)) = 0$ is equivalent to the vanishing of the corresponding homogeneous parts of degree i for $i = 0, 1, 2, \ldots$. An explicit computation for low values of k and i yields:

k	i	part of degree i in $\rho(L_k)(\tau(t))$
-1	0	sf_1
-1	1	$2sf_2t_1$
-1	2	$3sf_3t_1^2 + \frac{1}{2}t_1^2$
-1	3	$s\left(4f_4t_1^3 + f_{13}t_3\right) + t_3f_1$
0	0	$sf_{01} + \frac{1}{16}$
0	1	$sf_{11}t_1 + t_1f_1$
0	2	$sf_{21}t_1^2 + 2f_2t_1^2$
1	0	$sf_{001} + \frac{1}{2}f_2 + \frac{1}{4}f_1^2$
1	1	$sf_{101} + \frac{3}{2}f_3t_1 + f_{11}t_1$

Thus, it is clear that if a solution F does exist, then $s \neq 0$. In this case, the vanishing of the above polynomials implies that $f_1 = 0$, $f_2 = 0$, $f_3 = -\frac{1}{3!s}$, $f_{01} = -\frac{1}{16s}$, $f_{11} = 0$, $f_4 = 0$, $f_{11} = 0$, etc. . Writing down the general expression for the homogeneous part of degree i of $\rho(L_k)(\tau(t))$, one observes that it allows us to determine $f_{\mathbf{n}}$ with $|n| = i$ and $n_k \neq 0$ in terms of $f_{\mathbf{n}}$ with $|n| \leq i - 2$. Thus, if a solution F exists, the coefficients $f_{\mathbf{n}}$ can be recursively determined. □

Theorem 3.14 *Let $\rho \in \mathrm{Hom}_{Lie\text{-}alg}(\mathcal{W}_>, \mathrm{End}(\mathbb{C}[[t_1, t_3, \ldots]]))$ be such that $\rho(L_k)$ is of type k for $k \geq -1$ and that all coefficients of $\rho(L_{-1})$ are non-zero.*

Then, there exists a unique $\tau(t) \in \mathbb{C}[[t_1, t_3, \ldots]]$, with $\tau(0) = 1$, such that:

$$\rho(L_k)(\tau(t)) = 0 \qquad k \geq -1$$

Further, the solution $\tau(t)$ is a τ-function of the scaled KdV hierarchy.

Proof Lemma 3.10 implies that there is λ and $\sigma : \mathcal{W}_> \to \mathrm{Diff}^1(\mathbb{C}((z)))$ such that $\rho^\lambda = \beta_*(\sigma)$. Recalling Theorem 3.12 of [33], one knows that there is a function $\tau_0(t)$ which satisfies that $\rho^\lambda(L_n)(\tau_0(t)) = 0$ and that it is a τ-function of the KdV hierarchy. Then, $\tau(t) := \tau_0(\lambda t)$ fulfills the requirements. Since Lemma 3.13 implies the uniqueness of the solution, the conclusion follows. □

Remark 3.15 Let us make two comments on the solutions. First, an instance of the notion of scaled KdV appears already in Kontsevich's Theorem when it is claimed that the exponential of the generating function in variables $T_{2i+1} := t_i/(2i+1)!!$ is a τ-function for the KdV hierarchy [22, Theorem 1.2]. On the other hand, although the dilaton shift $\bar{t}_i \mapsto \bar{t}_i - \delta_{i,0}$ transforms the operators $\rho(L_k)$, it should be noted that it does not induce an automorphism of the algebra $\mathbb{C}[[\bar{t}_0, \bar{t}_1, \ldots]]$.

3.5 On the solutions for the n-dimensional situation

Let us now focus in the n-dimensional situation. That is, we aim at studying the interplay between Virasoro representations and multicomponent KP hierarchy. Special attention will be paid at their common solutions.

Recall that $V_{\mathrm{odd}}(A)$ is the subalgebra of $\mathbb{C}[[t_1, t_3, \ldots]]\widehat{\otimes}_{\mathbb{C}} S^\bullet A$ generated by $t_i \otimes a$. Then, $S \in \mathrm{Gl}(A)$ acts on it by the automorphism of algebras $t_i \otimes a \mapsto t_i \otimes S(a)$.

Theorem 3.16 *Let* $\rho : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(A))$ *be as in Sect.* 3.3.

There exist $S \in \mathrm{Gl}(A)$ *and functions* $\tau_\alpha(t_{1,\alpha}, t_{3,\alpha}, \dots) \in \mathbb{C}[[t_{1,\alpha}, t_{3,\alpha}, \dots]]$ *such that:*

$$\hat{\rho}(L_k)\left(S\left(\prod_\alpha \tau_\alpha(t_\alpha)\right)\right) = 0 \tag{29}$$

Further, $\tau_\alpha(t_{1,\alpha}, t_{3,\alpha}, \dots)$ *are* τ*-functions of the scaled KdV hierarchy.*

Proof Theorem 3.6 shows that there is $S \in \mathrm{Gl}(A)$ such that ρ^S decomposes as the tensor product of n one-dimensional Lie algebra representations of $\mathcal{W}_>$. More precisely, if $\{a_\alpha\}$ is the chosen basis for A, then $\{S(a_\alpha)\}$ is a orthogonal basis for η. Consequently, there are $\rho_\alpha : \mathcal{W}_> \to \mathcal{U}(\mathbb{H}(\langle S(a_\alpha)\rangle))$ such that (22) holds.

Now, apply the results of Sect. 3.4 on the one-dimensional case. Indeed, since η is non-degenerated and $\{S(a_\alpha)\}$ is a orthogonal basis, from Theorem 3.14 one obtains functions $\tau_\alpha(t_\alpha)$, such that $\tau_\alpha(0) = 1$, $\rho_\alpha(L_k)(\tau_\alpha) = 0$ for all α, k and they are τ-functions for the scaled KdV hierarchy.

Observe that (29) holds if and only if $\hat{\rho}^S(L_k)(\prod_\alpha \tau_\alpha(t_\alpha))$ vanishes. Applying the converse of Theorem 2.12 one concludes. □

Remark 3.17 The previous Theorem means that, assuming the uniqueness of the solution [6, Theorem 3.10.20], the solution of the Virasoro constraints has to be of the above form; that is, an operator acting on a product of Witten–Kontsevich τ-functions. Thus, it agrees with the results of Givental [14] for the total descendent potential. It would be interesting to relate both expressions explicitly (see also [12,15,23]). Alternatively, one could combine Teleman's classification of semisimple cohomological field theories [38] with Givental's results to deduce that this is the right expression for the solution. Nevertheless, our result can be applied on other frameworks, as it will be mentioned in Sect. 3.6.

Corollary 3.18 *Let* ρ *be as in the Theorem* 3.16.

If S, τ_α *satisfy* (29)*, then* $\rho^S = \rho_1 + \dots + \rho_n$ *and* $\hat{\rho}_\alpha(L_k)(\tau_\alpha) = 0$.

The matrix S *is unique up to an orthogonal matrix.*

Proof If S and τ_α are such that (29) vanishes, then the following expression also vanishes:

$$0 = \exp\left(-\sum_\alpha \tilde{\tau}_\alpha(t_\alpha)\right) S^{-1}\hat{\rho}(L_k)\left(S\left(\prod_\alpha \tau_\alpha(t_\alpha)\right)\right) = \frac{\hat{\rho}^S(L_k)(\exp(\sum_\alpha \tilde{\tau}_\alpha(t_\alpha)))}{\exp(\sum_\alpha \tilde{\tau}_\alpha(t_\alpha))}$$

Recall that an operator $\rho^S(L_k)$ of type (4) is the same as $\rho(L_k)$ where the matrix a has been replaced by $(S^{-1})^T a S^{-1}$ (and, accordingly, b_k, c_k, etc.). Expanding the case $k = -1$ of the last identity, one obtains that $(S^{-1})^T \eta S^{-1}$ is diagonal. Then, Theorem 2.10, implies that ρ^S decomposes as a sum $\rho_1 + \dots + \rho_n$ and Theorem 2.12 implies that $\hat{\rho}_\alpha(L_k)(\tau_\alpha(t_\alpha)) = 0$.

It is straightforward that S is unique up to an orthogonal matrix. □

Corollary 3.19 *Let* ρ *be as in the Theorem* 3.16. *If either* S *is diagonal or* τ_α *are* τ*-functions of the same scaled hierarchy, then the solution is a* τ*-function for the scaled multicomponent KP hierarchy.*

Proof In particular, the product $S(\prod_\alpha \tau_\alpha(t_\alpha))$ is uniquely determined by ρ. Each function $\tau_\alpha(t_\alpha)$ satisfies the scaled KdV and, thus, there are $\lambda_\alpha := (\lambda_{i,\alpha}) \in \prod_{i \text{ odd}} \mathbb{C}^*$ such that $\tau_\alpha(\lambda_\alpha^{-1} t_\alpha)$ defines a point $U_\alpha \in \mathrm{Gr}(\mathbb{C}((z)))$. If ρ is expressed w.r.t. a basis $\{a_1, \dots, a_n\}$, then S determines a second basis $\{S(a_1), \dots, S(a_n)\}$ or, equivalently, an isomorphism $\mathbb{C} \oplus \ldots \oplus \mathbb{C} \simeq A$. This isomorphism induces:

$$\mathrm{Gr}(\mathbb{C}((z))) \times \cdots \times \mathrm{Gr}(\mathbb{C}((z))) \hookrightarrow \mathrm{Gr}(\mathbb{C}((z)) \oplus \cdots \oplus \mathbb{C}((z))) \simeq \mathrm{Gr}(A \otimes \mathbb{C}((z)))$$

Since the τ-function of the image of $(U_1, \dots, U_n)$, which is $U_1 \oplus \ldots \oplus U_n$, is given by $\prod_\alpha \tau_\alpha(\lambda_\alpha^{-1} t_\alpha)$ it follows that $S \prod_\alpha \tau_\alpha(t_\alpha)$ is a τ-function of the scaled multicomponent KP in the two cases of the statement. □

Remark 3.20 Recalling Remark 3.11, we observe that Theorems 3.14 and 3.16 could be weaken and stated for representations satisfying the hypothesis up to a linear function on t_i's.

3.6 Final comments

Let us finish with some brief comments. From a general perspective, we hope that our methods shed some light on the explicit expressions of the Virasoro operators and of the relevant integrable hierarchies that appear in the Virasoro conjecture. Furthermore, they can also be applied to many instances of representations of $\mathcal{W}_>$ such as recursion relations, Hurwitz numbers, and knot theory.

As an illustration, let us point out the results of [2,21] on Hurwitz numbers. In both cases, the authors study the generating functions of the number of coverings of $\mathbb{P}_1 \setminus \{0, 1, \infty\}$ with some properties. It is shown that these functions satisfy Virasoro constraints, KP hierarchy and topological recursion (of the Eynard–Orantin type [11]). It is remarkable that the Virasoro constraints are explicitly expressed as differential operators of the form considered in Sect. 2 for the case $A = \mathbb{C}$. Thus, the results of Sect. 3.4 can be directly applied to conclude that Virasoro constraints imply the scaled KP hierarchy.

Our results could also be of interest within the context of Eynard–Orantin topological recursion [11]. Indeed, we learn from [30] that Mirzakhani's recursion formula for the Weil–Petersson volumes [29] is indeed a Virasoro constraint imposed on a generating function of these volumes and that this function satisfies the KdV hierarchy. It is worth pointing out some recent results on the relation of topological recursion and Virasoro constraints [10,28]. On the one hand, it has been shown in [10] that these Virasoro constraints are actually equivalent to Eynard–Orantin topological recursion for some spectral curve. On the other hand, one knows from [28] that the correlation functions of a semisimple cohomological field theory satisfy the Eynard–Orantin topological recursion and that these recursion formulas are equivalent to n copies of the Virasoro constraints for the ancestor potential. Therefore, two problems can be faced with our techniques. First, we think that Theorem 3.16 should imply some bilinear relations of Hirota type for the solution of the Eynard–Orantin topological recursion. Second, due to the uniqueness of the solution and the fact that the solution satisfies the KP hierarchy, there must be a relation of the Eynard–Orantin spectral curve and the Krichever construction.

Similarly, it would be interesting to interpret the recent papers [3,8] from our perspective.

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Appendix

This appendix only collects the explicit computations of some Lie brackets used in Sect. 2. From a formal point of view, we are dealing with generators of $\mathcal{U}(\mathbb{H})$, $1, q_{i,\alpha}, p_{i,\alpha}$, with $i = 1, 2, \ldots$ and $\alpha = 1, \ldots, n$, that satisfy the following relations:

$$[p_{i,\alpha}, q_{j,\beta}] = \delta_{i,j} i\eta^{\alpha\beta} \cdot 1$$
$$[p_{i,\alpha}, p_{j,\beta}] = [q_{i,\alpha}, q_{j,\beta}] = [p_{i,\alpha}, 1] = [q_{i,\alpha}, 1] = 0$$

and, because of the associativity of composition, we will also use:

$$[a, bc] = [a, b]c + b[a, c]$$

We will use the Einstein convention; that is, repeated subindices of the variables p, qs imply the summation is to be done. Recall that $b_i^{0,2i+3}$ and $q_i := (q_{i,1}, \ldots, q_{i,n})$ denote row vectors, $p_i := (p_{i,1}, \ldots, p_{i,n})^T$ are column vectors (the superscript T denotes the transpose), and $a, b_i^{j,2i+j}, c_i^{j,2i-j}$ are $n \times n$ square matrices.

Let us compute some Lie brackets. For instance,

$$\begin{aligned}
&[b_i^{0,2i+3} p_{2i+3}, b_j^{0,2j+3} p_{2j+3}] = [(b_i^{0,2i+3})_\alpha (p_{2i+3})_\alpha\,,\ (b_j^{0,2j+3})_\beta (p_{2j+3})_\beta] \\
&\quad = (b_i^{0,2i+3})_\alpha [(p_{2i+3})_\alpha\,,\ (p_{2j+3})_\beta](b_j^{0,2j+3})_\beta = 0
\end{aligned}$$

where subindices α, β denote the corresponding entries of the vectors. Analogously, we have the following identities:

$$\begin{aligned}
&[q_1 a q_1^T, b_i^{0,2i+3} p_{2i+3}] = 0 \qquad \forall i \geq 0 \\
&[q_r b_i^{r,r+2i} p_{r+2i}, b_j^{0,2j+3} p_{2j+3}] \\
&\quad = -[b_j^{0,2j+3} p_{2j+3}, (q_r)_\alpha](b_i^{r,r+2i} p_{r+2i})_\alpha \\
&\qquad - (q_r b_i^{r,r+2i})_\alpha [b_j^{0,2j+3} p_{2j+3}, (p_{r+2i})_\alpha] \\
&\quad = -(b_j^{0,2j+3})_\beta [(p_{2j+3})_\beta, (q_r)_\alpha](b_i^{r,r+2i})_{\alpha\gamma}(p_{r+2i})_\gamma \\
&\quad = -(2j+3)(b_j^{0,2j+3})_\beta \eta^{\beta\alpha}(b_i^{2j+3,2j+3+2i})_{\alpha\gamma}(p_{2j+3+2i})_\gamma \\
&\quad = -(2j+3) b_j^{0,2j+3} \eta^{-1} b_i^{2j+3,2j+3+2i} p_{2j+3+2i}
\end{aligned}$$

$$[p_r c_i^{r,2i-r} p_{2i-r}, b_j^{0,2j+3} p_{2j+3}] = 0$$

$$[q_1 a q_1^T, q_r b_j^{r,r+2j} p_{r+2j}] = 0 \qquad \forall j \geq 1$$

$$\begin{aligned}
&[q_1 a q_1^T, q_r b_0^{r,r} p_r] \\
&\quad = (q_1 b_0^{1,1})_\alpha [q_1 a q_1^T, (p_1)_\alpha] \\
&\quad = -(q_1 b_0^{1,1})_\alpha \left([(p_1)_\alpha, (q_1)_\beta](a q_1^T)_\beta + (q_1 a)_\beta [(p_1)_\alpha, (q_1)_\beta] \right) \\
&\quad = -q_1 b_0^{1,1} \eta^{-1} \left(a + a^T \right) q_1^T
\end{aligned}$$

$$\begin{aligned}
&[q_1 a q_1^T, p_r^T c_j^{r,2j-r} p_{2j-r}] \\
&\quad = [q_1 a q_1^T, (p_r)_\alpha](c_j^{r,2j-r} p_{2j-r})_\alpha + (p_r^T c_j^{r,2j-r})_\alpha [q_1 a q_1^T, (p_{2j-r})_\alpha] \\
&\quad = -\left([(p_1)_\alpha, (q_1)_\beta](a q_1^T)_\beta + (q_1 a)_\beta [(p_1)_\alpha, (q_1)_\beta] \right) (c_j^{1,2j-1} p_{2j-1})_\alpha \\
&\qquad - (p_{2j-1}^T c_j^{2j-1,1})_\alpha \left([(p_1)_\alpha, (q_1)_\beta](a q_1^T)_\beta + (q_1 a)_\beta [(p_1)_\alpha, (q_1)_\beta] \right) \\
&\quad = -q_1 \left(a + (a)^T \right) (\eta^{-1})^T c_j^{1,2j-1} p_{2j-1} - p_{2j-1}^T c_j^{2j-1,1} \eta^{-1} \left(a + (a)^T \right) q_1^T \\
&\quad = -q_1 \left(a + (a)^T \right) (\eta^{-1})^T \left(c_j^{1,2j-1} + (c_j^{2j-1,1})^T \right) p_{2j-1} \\
&\qquad - \delta_{j1} \operatorname{Tr} \left(c_1^{1,1} \eta^{-1} (a + a^T)(\eta^{-1})^T \right)
\end{aligned}$$

$$\begin{aligned}
&[q_r b_i^{r,r+2i} p_{r+2i}, p_s^T c_j^{s,2j-s} p_{2j-s}] \\
&\quad = [q_r b_i^{r,r+2i} p_{r+2i}, (p_s)_\alpha](c_j^{s,2j-s} p_{2j-s})_\alpha \\
&\qquad + (p_s^T c_j^{s,2j-s})_\alpha [q_r b_i^{r,r+2i} p_{r+2i}, (p_{2j-s})_\alpha] \\
&\quad = -[(p_s)_\alpha, (q_r)_\beta](b_i^{r,r+2i} p_{r+2i})_\beta (c_j^{s,2j-s} p_{2j-s})_\alpha \\
&\qquad - (p_s^T c_j^{s,2j-s})_\alpha [(p_{2j-s})_\alpha, (q_r)_\beta](b_i^{r,r+2i} p_{r+2i})_\beta \\
&\quad = -r p_{2j-r}^T (c_j^{r,2j-r})^T \eta^{-1} b_i^{r,r+2i} p_{r+2i} - r p_{2j-r}^T c_j^{2j-r,r} \eta^{-1} b_i^{r,r+2i} p_{r+2i} \\
&\quad = -r p_{2j-r}^T \left((c_j^{r,2j-r})^T + c_j^{2j-r,r} \right) \eta^{-1} b_i^{r,r+2i} p_{r+2i}
\end{aligned}$$

$$\begin{aligned}
&[q_r b_i^{r,r+2i} p_{r+2i}, q_s b_j^{s,s+2j} p_{s+2j}] \\
&\quad = [q_r b_i^{r,r+2i} p_{r+2i}, (q_s)_\alpha](b_j^{s,s+2j} p_{s+2j})_\alpha \\
&\qquad + (q_s b_j^{s,s+2j})_\alpha [q_r b_i^{r,r+2i} p_{r+2i}, (p_{s+2j})_\alpha] \\
&\quad = -(q_r b_i^{r,r+2i})_\beta [(q_s)_\alpha, (p_{r+2i})_\beta](b_j^{s,s+2j} p_{s+2j})_\alpha \\
&\qquad - (q_s b_j^{s,s+2j})_\alpha [(p_{s+2j})_\alpha, (q_r)_\beta](b_i^{r,r+2i} p_{r+2i})_\beta \\
&\quad = (r + 2i) q_r b_i^{r,r+2i} \eta^{-1} b_j^{r+2i,r+2i+2j} p_{r+2i+2j} - r q_{r-2j} b_j^{r-2j,r} \eta^{-1} b_i^{r,r+2i} p_{r+2i}
\end{aligned}$$

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