Lett Math Phys (2017) 107:223–265
DOI 10.1007/s11005-016-0894-y

The BRST complex of homological Poisson reduction

Martin Müller-Lennert[1]

Received: 11 April 2015 / Revised: 31 August 2016 / Accepted: 1 September 2016
Published online: 9 November 2016

Abstract BRST complexes are differential graded Poisson algebras. They are associated with a coisotropic ideal J of a Poisson algebra P and provide a description of the Poisson algebra $(P/J)^J$ as their cohomology in degree zero. Using the notion of stable equivalence introduced in Felder and Kazhdan (Contemporary Mathematics 610, Perspectives in representation theory, 2014), we prove that any two BRST complexes associated with the same coisotropic ideal are quasi-isomorphic in the case $P = \mathbb{R}[V]$ where V is a finite-dimensional symplectic vector space and the bracket on P is induced by the symplectic structure on V. As a corollary, the cohomology of the BRST complexes is canonically associated with the coisotropic ideal J in the symplectic case. We do not require any regularity assumptions on the constraints generating the ideal J. We finally quantize the BRST complex rigorously in the presence of infinitely many ghost variables and discuss the uniqueness of the quantization procedure.

Keywords BRST complex · Reduction · Quantization

Mathematics Subject Classification Primary 17B63; Secondary 17B70 · 17B81

1 Introduction

In the quantization of gauge systems, the so-called BRST complex plays a prominent role [12]. In the Hamiltonian formalism, the theory is called BFV theory and goes back to Batalin, Fradkin, Fradkina and Vilkovisky [1,2,10,11].

✉ Martin Müller-Lennert
martin.mueller-lennert@math.ethz.ch

1 ETH Zurich, Zurich, Switzerland

In the Hamiltonian formulation of gauge theory, the presence of gauge freedom yields constraints in the phase space M of the system. The gauge group still acts on the resulting constraint surface $M_0 \subset M$. The physical observables are the functions on the quotient $\tilde{M}$ of the constraint surface M_0 by this action. One wishes to quantize those observables. In the BRST method, one introduces variables of non-zero degree to the Poisson algebra P of functions on the original phase space. One then constructs the so-called BRST differential on the resulting complex and recovers the functions on the subquotient $\tilde{M}$ as the cohomology of that complex in degree zero. One may then attempt to quantize the system by quantizing the BRST complex instead of the algebra of functions on $\tilde{M}$.

The quantization procedure involves the construction of gauge invariant observables from the cohomology of the BRST complex [3,19]. Kostant and Sternberg gave a mathematically rigorous description of the theory [14] in the case where the constraints arise from a Hamiltonian group action on phase space. They make certain assumptions that allow the BRST complex to be constructed as a double complex combining a Koszul resolution of the vanishing ideal J of the constraint surface $M_0 \subset M$ with the Lie algebra cohomology of the gauge group. In more general cases, the Koszul complex does not yield a resolution and one has to use a much bigger Tate resolution.

More recently, Felder and Kazhdan formalized the corresponding construction in the Lagrangian formulation of the theory [8]. They consider general Tate resolutions. The aim of this note is to perform a similar formalization in the Hamiltonian setting. We consider Poisson algebras P as a starting point, which arise in the Hamiltonian viewpoint as the functions on phase space. We define the notion of a *BFV model* for a coisotropic ideal $J \subset P$. In the Hamiltonian theory, J is given as the vanishing ideal of the constraint surface $M_0 \subset M$. We use techniques from [8,17] to prove the existence of the BFV models and show that they model the Poisson algebra $(P/J)^J$ cohomologically. This latter Poisson algebra is a physically interesting one, since, in the case where P are the functions on phase space and J is the vanishing ideal of the constraint surface, it corresponds to the function on the subquotient $\tilde{M}$, which are the physical observables of the system. The statements about the existence of what we call BFV models and their cohomology are known [12]. However, a rigorous treatment of the question of uniqueness is missing. Under certain local regularity assumptions on the constraint functions, which for instance imply that the constraint surface M_0 is smooth, a construction for a uniqueness proof for the BRST cohomology was given in [9]. Stasheff considers the problem from the perspective of homological perturbation theory [17] and gives further special cases under which such uniqueness theorems hold. For instance, he considers the case where a proper subset of the constraints satisfy a regularity condition. Using the notion of *stable equivalence* from [8], we show that, for a symplectic polynomial algebra $P = \mathbb{R}[V]$ with bracket induced from the symplectic structure on a finite-dimensional vector space V, any two BRST complexes for the same coisotropic ideal $J \subset P$ are quasi-isomorphic. Hence, we rigorously prove the uniqueness of the BRST cohomology for such P. In contrast to previous treatments of the problem, the assumption on P does not force the constraint surface to be smooth. Moreover, we do not assume a subset of the constraints to be regular. Our Tate resolutions are allowed to contain infinitely many generators.

Finally, we quantize the BRST complex. Under a cohomological assumption, we construct a quantum BRST charge and discuss its uniqueness. The obstruction to quantize lies in the second degree of the classical BRST cohomology, while the ambiguity lies in the first degree. We do this analysis in a rigorous fashion. To the best of our knowledge, such a rigorous treatment in our setting for general Tate resolutions is new.

In the smooth setting, Schätz has dealt with the problem in [16]. See also [5] for the case of a Hamiltonian group action with regular moment map. In [4,13], this regularity assumption is replaced by the weaker assumption that the components of the moment map generate the vanishing ideal J and that the Koszul complex is acyclic. The authors construct a BRST complex and quantize it. In [13], the assumptions are weakened to allow Tate resolutions with *finitetly* many generators.

2 BFV models

We work over $\mathbb{K} = \mathbb{R}$, but any field of characteristic zero will be sufficient. Let P be a unital, Noetherian Poisson algebra. Let $J \subset P$ be a multiplicative ideal satisfying $\{J, J\} \subset J$. Such ideals are called coisotropic. Then the Poisson structure on P induces one on $(P/J)^J$. The purpose of the BRST complex is to model this Poisson algebra cohomologically.

Let $\mathcal{M}$ be a negatively graded real vector space with finite-dimensional homogeneous components $\mathcal{M}^j$. Denote its component-wise dual by $\mathcal{M}^* = \bigoplus_{j>0}(\mathcal{M}^{-j})^*$. Define a Poisson bracket on $\mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$ via the natural pairing between $\mathcal{M}$ and $\mathcal{M}^*$. For details of the construction, we refer to Appendix A.

Form the tensor product $X_0 = P \otimes \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$ of the two Poisson algebras defined above. Let $\mathcal{F}^p X_0$ denote the ideal generated by all elements in X_0 of degree at least p. Using the filtration defined by the $\mathcal{F}^p X_0$, complete the space X_0 to a graded commutative algebra X with homogeneous components

$$X^j = \lim_{\leftarrow p} \frac{X_0^j}{\mathcal{F}^p X_0 \cap X_0^j}.$$

Extend the bracket on X_0 to X, thus turning X into a graded Poisson algebra. Again, we refer to Appendix A for details. Denote the bracket on X by $\{-, -\}$.

Set $I \subset X$ to be the homogeneous ideal with homogeneous components

$$I^j = \lim_{\leftarrow p} \frac{\mathcal{F}^1 X_0 \cap X_0^j}{\mathcal{F}^{p+1} X_0 \cap X_0^j} \subset X^j.$$

The powers of ideals are denoted with exponents in parentheses, e.g. $I^{(k)}$ refers to the k-th power of the ideal I.

An element $R \in X$ of odd degree which solves $\{R, R\} = 0$ defines a differential $d_R = \{R, -\}$ on X by the Jacobi identity. If $R \in X^1$, the differential d_R induces a differential on X/I since it preserves I.

Definition 1 A *BFV model* for P and J is a pair (X, R) where $(X, \{-,-\})$ is a graded Poisson algebra constructed as above and $R \in X^1$ is such that the following conditions hold:

(1) $\{R, R\} = 0$.
(2) $H^j(X/I, d_R) = 0$ for $j \neq 0$.
(3) $H^0(X/I, d_R) = P/J$.

The first equation is called the *classical master equation* and the element R is called a *BRST charge.*

The aim of this note is to prove

Theorem 2 *Let P be a Poisson algebra and $J \subset P$ a coisotropic ideal. BFV models exist and in the case of $P = \mathbb{R}[V]$ with bracket induced from the symplectic structure on a finite-dimensional vector space V, the complexes of any two BFV models for the same ideal J are quasi-isomorphic, whence the cohomology $H(X, d_R)$ is uniquely determined by J up to isomorphism.*

The existence of the BFV models is known [9,12]. The problem of uniqueness has been dealt with under certain regularity assumptions [9,12]. These assumptions imply that the constraint surface is smooth. The novel part is the statement that any two BRST complexes are quasi-isomorphic, which gives the uniqueness of the BRST cohomology as a corollary. We prove this without assuming that the constraint surface is smooth and for Tate resolutions with possibly infinitely many generators. The Noetherian hypothesis ensures that there are only finitely many generators in each degree. This is necessary for our proofs of our convergence results. For completeness, we also include proofs of the already known facts in our framework.

Finally, we quantize the BRST charge rigorously and discuss the uniqueness of the quantization procedure.

3 Existence

3.1 Tate resolutions

To construct BFV models, we first have to construct a suitable commutative graded algebra X. The odd variables are obtained via Tate resolutions.

Let P be a unital, Noetherian Poisson algebra and $J \subset P$ be a coisotropic ideal. Tate constructed resolutions of Noetherian rings by adding certain odd variables to the ring [18]. Consider a Tate resolution $T = P \otimes \mathrm{Sym}(\mathcal{M})$ of P/J given by a negatively graded vector space $\mathcal{M}$ with finite-dimensional homogeneous components together with a differential δ on T of degree 1. Define the dual $\mathcal{M}^*$ degree-wise. Extend δ to $X_0 := P \otimes \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*) = P \otimes \mathrm{Sym}(\mathcal{M}) \otimes \mathrm{Sym}(\mathcal{M}^*)$ by tensoring with the identity. Endow X_0 with the natural extension of the Poisson bracket, define the filtration $\mathcal{F}^p X_0$, and extend the bracket to the completion X as described in Appendix A. We will frequently refer to statements from that section.

3.1.1 The differential δ

Since δ is the identity on $\mathrm{Sym}(\mathcal{M}^*)$, it preserves the filtration on X_0. Hence it extends to the completion X by Remark 80. Call this extension δ. The extension has degree 1 and preserves the filtration on X. The extension is still an odd derivation, whose square is zero. Since δ preserves the filtration, it defines a differential on the associated graded mapping $\mathrm{gr}^p X^n$ into $\mathrm{gr}^p X^{n+1}$.

Define $B = P \otimes \mathrm{Sym}(\mathcal{M}^*)$. Then, $X_0 = B \otimes_P T$. Since, by definition, the extension of δ to X leaves elements in $\mathrm{Sym}(\mathcal{M}^*)$ fixed, we have

Remark 3 The natural isomorphism of Lemma 82 identifies the differential δ on the associated graded with $1 \otimes \delta$ on $B \otimes_P T$.

3.1.2 Contracting homotopy

From the Tate resolution, construct a contracting homotopy $s : T \to T$ of degree -1. Then there exists a $\mathbb{K}$-linear split $P/J \to P$ and a map $\overline{\pi} : T \to T$ which is defined as the composition $P \otimes \mathrm{Sym}(\mathcal{M}) \to P \to P/J \to P \to P \otimes \mathrm{Sym}(\mathcal{M})$ such that

$$\delta s + s\delta = 1 - \overline{\pi}. \tag{1}$$

Extend δ, s and $\overline{\pi}$ to X_0 by tensoring with the identity on $\mathrm{Sym}(\mathcal{M}^*)$. From the definition of $\overline{\pi}$, we find

Remark 4 $\overline{\pi} : X_0 \to X_0$ is zero on monomials which contain a factor of negative degree.

The homotopy s does not act on elements in $\mathrm{Sym}(\mathcal{M}^*)$ and hence preserves the filtration. For the same reason, $\bar{\pi}$ preserves the filtration. Both s and $\bar{\pi}$ hence naturally extend to the extension and Eq. 1 is valid in X too. Moreover,

Remark 5 s preserves $I^{(2)}$.

3.2 Constructing the BRST charge

3.2.1 First approximation

Definition 6 Let Q_0 be the differential δ on X/I considered as an element of X.

Hence, the cohomological conditions for Q_0 to be a BRST charge are satisfied. However, Q_0 does not in general satisfy the classical master equation. We are going to prove the existence part of Theorem 2 by adding correction terms to Q_0.

An explicit description of Q_0 is the following. Let e_i be a homogeneous basis of $\mathcal{M}$, e_i^* its dual basis. Set $d_i := \deg e_i = -\deg e_i^* \equiv \deg e_i^* \pmod 2$. Assume that $i \leqslant j$ implies $d_i \geqslant d_j$. Define $Q_0 := \sum_j (-1)^{1+d_j} e_j^* \delta(e_j)$. By Lemma 77, this defines an element of X^1. For each p, let L_p be an integer with $\{j \in \mathbb{N} : -d_j \leqslant p-1\} =$

$\{1, \ldots, L_p\}$ so that $(q_0)_p := \sum_{j=1}^{L_p} (-1)^{1+d_j} e_j^* \delta(e_j)$ defines a representative of the p-th component of Q_0. Of course, the element Q_0 is independent of the choice of basis e_j of $\mathcal{M}$.

Lemma 7 *We have* $\delta = \sum_j (-1)^{1+d_j} \delta(e_j)\{e_j^*, -\}$ *on* X *where the operator on the right hand side is well defined.*

Proof Set $\delta' = \sum_j (-1)^{1+d_j} \delta(e_j)\{e_j^*, -\}$. This defines a map on X. For $x \in X^n$, the elements $\{e_j^*, x\}$ are in $\mathcal{F}^{-d_j+n} X$. Hence, the sum converges by Lemma 77. By linearity, δ' is defined on all of X. We claim that δ' is continuous on each X^n. Let $x^j = (x_p^j + \mathcal{F}^p X_0^n)_p \in X^n$ be a sequence converging to zero. Fix p. Then there exists a K, independent of j, such that a p-th representative of $\delta'(x^j)$ is given by

$$\sum_{k=1}^{K} (-1)^{1+d_k} \delta(e_k)\{e_k^*, x^j_{s_{-d_k,n}(p)}\},$$

since the bracket is in $X_0^{-d_k+n}$. Now, let j_0 be such that for $j \geqslant j_0$ and for all $k \in \{1, \ldots, K\}$ we have $x^j_{s_{-d_k,n}(p)} \in \mathcal{F}^{s_{-d_k,n}(p)} X_0^n$. Then the above representative vanishes modulo $\mathcal{F}^p X_0^{n+1}$ by Corollary 64. Hence, δ' is continuous on X^n. The map δ' descends to a map on X_0, since the sum is then effectively finite because $\{e_j^*, x\}$ becomes zero for j large enough, depending on $x \in X_0$. This restriction agrees with δ, which can be checked on generators since both maps are derivations. Hence, $\delta' = \delta$ on each X^n by continuity. Hence, $\delta = \delta'$. □

Lemma 8 *For* $L_0 := \{Q_0, -\} - \delta$, *we have* $L_0(\mathcal{F}^p X) \subset \mathcal{F}^{p+1} X$.

Proof Fix $x \in \mathcal{F}^p X^n$. Then, by Lemma 74,

$$\{Q_0, x\} = \lim_{m\to\infty} \left(\sum_{j=1}^{m} (-1)^{1+d_j} \delta(e_j)\{e_j^*, x\} + \sum_{j=1}^{m} (-1)^{1+d_j} e_j^* \{\delta(e_j), x\} \right).$$

The first part converges to $\delta(x)$ by Lemma 7. The second part converges by Lemma 77 and hence equals L_0. Fix j. By Lemma 75, it suffices to prove that $e_j^*\{\delta(e_j), x\} \in \mathcal{F}^{p+1} X$. By the derivation property it suffices to consider $x = e_l^*$ for some l. The term $\delta(e_j)$ is a sum of monomials whose factors have degrees in $\{d_j + 1, \ldots, 0\}$. Hence, all elementary factors e_r in $\delta(e_j)$ that could possibly kill e_l^* have degree d_l and get compensated by a factor e_j^* with $\deg(e_j^*) > \deg(e_l^*)$. □

Moreover, we have

Lemma 9 $\{Q_0, Q_0\} \in X^2 \cap I^{(2)} \subset \mathcal{F}^2 X \cap I^{(2)}$.

Proof We compute $\deg\{Q_0, Q_0\} = 2\deg Q_0 = 2$ and hence $\{Q_0, Q_0\} \in \mathcal{F}^2 X$. For the last statement, we need to calculate. By Lemma 74,

$$\begin{aligned}\{Q_0, Q_0\} &= \lim_{m\to\infty} (-1)^{d_j+d_k} \sum_{j,k=1}^{m} \{\delta(e_j)e_j^*, \delta(e_k)e_k^*\} \\ &= \lim_{m\to\infty} \sum_{j,k=1}^{m} \Big(2(-1)^{1+d_k}\Big((-1)^{1+d_j}\delta(e_j)\{e_j^*, \delta(e_k)\}\Big)e_k^* \\ &\quad + (-1)^{d_j+d_k} e_j^*\{\delta(e_j), \delta(e_k)\}e_k^*\Big).\end{aligned}$$

By Lemma 7, the first term is a sum in k with summands that contain factors $\delta(\delta(e_k)) = 0$ and hence the first term vanishes. By Lemma 75, $\{Q_0, Q_0\} = \sum_{j,k}(-1)^{d_j+d_k} e_j^*\{\delta(e_j), \delta(e_k)\}e_k^* \in I^{(2)}$. □

Corollary 10 $\delta\{Q_0, Q_0\} \in X^3 \subset \mathcal{F}^3 X$.

3.2.2 Recursive construction

We now inductively construct out of Q_0 a sequence of elements $R_n \in X^1$ by setting

$$R_n = \sum_{j=0}^{n} Q_j, \qquad Q_0 \text{ as defined above}, \qquad Q_{n+1} = -\frac{1}{2}s\{R_n, R_n\}.$$

The elements R_n have degree 1, since Q_0 has and s is of degree -1. The idea for the construction is taken from [17]. Also, the proof of the following theorem is adapted from that paper.

Theorem 11 *For all n,* $\{R_n, R_n\} \in \mathcal{F}^{n+2} X \cap I^{(2)}$ *and* $\delta\{R_n, R_n\} \in \mathcal{F}^{n+3} X$.

Proof The base step was done in Lemma 9 and Corollary 10. We assume the statement is true for $0 \leqslant j \leqslant n$ and consider

$$\{R_{n+1}, R_{n+1}\} = \{R_n, R_n\} + 2\{R_n, Q_{n+1}\} + \{Q_{n+1}, Q_{n+1}\}.$$

By construction and assumption, $Q_{n+1} = -\frac{1}{2}s\{R_n, R_n\} \in \mathcal{F}^{n+2} X^1$. Hence, by Corollary 64,

$$\{R_{n+1}, R_{n+1}\} \equiv \{R_n, R_n\} + 2\{R_n, Q_{n+1}\} \pmod{\mathcal{F}^{n+3} X}.$$

Expand $\{R_n, Q_{n+1}\} = \sum_{j=1}^{n}\{Q_j, Q_{n+1}\} + \{Q_0, Q_{n+1}\}$. We have, for $j \in \{1, \dots, n+1\}$ by inductive hypothesis, that $Q_j = -\frac{1}{2}s\{R_{j-1}, R_{j-1}\} \in \mathcal{F}^{j+1} X^1 \cap I^{(2)}$. Hence, by Lemma 65,

$$\{R_{n+1}, R_{n+1}\} \equiv \{R_n, R_n\} + 2\{Q_0, Q_{n+1}\} \pmod{\mathcal{F}^{n+3} X}.$$

We split $\{Q_0, Q_{n+1}\} = \delta Q_{n+1} + L_0 Q_{n+1}$ and, by Lemma 8,

$$\{R_{n+1}, R_{n+1}\} \equiv \{R_n, R_n\} + 2\delta Q_{n+1} \pmod{\mathcal{F}^{n+3} X}.$$

Commuting δ and s,

$$2\delta Q_{n+1} = -\delta s\{R_n, R_n\} = s\delta\{R_n, R_n\} - \{R_n, R_n\} + \overline{\pi}\{R_n, R_n\}.$$

Since $\{R_n, R_n\} \in \mathcal{F}^{n+2} X^2$, we have that $\overline{\pi}\{R_n, R_n\} = 0$ for $n > 0$ by Remark 4. For $n = 0$, we obtain $\overline{\pi}\{R_0, R_0\} = 0$ from $\{Q_0, Q_0\} = \sum_{j,k} \pm e_j^*\{\delta(e_j), \delta(e_k)\}e_k^*$ and the fact that $\overline{\pi}$ is zero on $\{J, J\} \subset J$. Hence,

$$\{R_{n+1}, R_{n+1}\} \equiv s\delta\{R_n, R_n\} \pmod{\mathcal{F}^{n+3}, X}.$$

which vanishes modulo $\mathcal{F}^{n+3} X$ by the assumption on $\delta\{R_n, R_n\}$.

Next, by the graded Jacobi identity, we have $0 = \{R_{n+1}, \{R_{n+1}, R_{n+1}\}\}$. From Lemmas 8 and 65, we find that $L_{n+1} := \{R_{n+1}, -\} - \delta = L_0 + \sum_{j=1}^{n+1}\{Q_j, -\}$ increases filtration degree. Hence, $L_{n+1}\{R_{n+1}, R_{n+1}\} \in \mathcal{F}^{n+4} X$ and thus $\delta\{R_{n+1}, R_{n+1}\} \in \mathcal{F}^{n+4} X$.

Finally, we prove that $\{R_{n+1}, R_{n+1}\} = \{R_n, R_n\} + 2\{R_n, Q_{n+1}\} + \{Q_{n+1}, Q_{n+1}\} \in I^{(2)}$. By hypothesis, $\{R_n, R_n\} \in I^{(2)}$. Next, $\{Q_{n+1}, Q_{n+1}\} \in \{I^{(2)}, I^{(2)}\} \subset I^{(2)}$ by Lemma 76. Now, by the same lemma, for $j \in \{1, \ldots, n\}$, $\{Q_j, Q_{n+1}\} \in \{I^{(2)}, I^{(2)}\} \subset I^{(2)}$ and $\{Q_0, Q_{n+1}\} \in \{I, I^{(2)}\} \subset I^{(2)}$ which concludes the proof. □

From $Q_{n+1} = -\frac{1}{2}s\{R_n, R_n\} \in \mathcal{F}^{n+2} X^1$, it follows that the $R_n = \sum_{j=0}^n Q_j$ converge to an element $R \in X^1$ by Lemma 77. From Lemma 74, we obtain $\{R_n, R_n\} \longrightarrow \{R, R\}$ as $n \longrightarrow \infty$. We obtain

Corollary 12 $\{R, R\} = 0$.

Proof We have $\{R_{n+l}, R_{n+l}\} \in \mathcal{F}^{n+2} X^2$ for all $l \geqslant 0$. Hence, $\{R, R\} \in \mathcal{F}^{n+2} X^2$ for all n by Lemma 75. Hence, $\{R, R\} = 0$. □

We also remark that R as defined above satisfies $R \equiv Q_0 \pmod{I^{(2)}}$, since for $j > 0$ we have $Q_j \in I^{(2)}$. We are left to consider the cohomology of $d_R = \{R, -\}$ on X/I.

Lemma 13 *The action of d_R preserves the filtration and hence defines a differential on gr X, which is identified with $1 \otimes \delta$ under the natural isomorphism of Lemma 82.*

Proof $R \in X^1$ and Lemma 63 imply that d_R preserves the filtration and hence descends to the associated graded. We have $\{Q_0, -\} = L_0 + \delta$. Since L_0 increases filtration degree by Lemma 8, we have that $\{Q_0, -\}$ and δ induce the same maps on gr X. Moreover, $K := R - Q_0 \in I^{(2)} \cap X^1$ by the remark above. Hence by Lemma 65, $\{K, -\}$ increases the filtration degree and thus $d_R = \{R, -\}$ and $\{Q_0, -\}$ induce the same maps on the associated graded. □

Corollary 14 *$H^j(X/I, d_R) \cong P/J$ if $j = 0$ and zero otherwise.*

Proof $H^j(X/I, d_R) = H^j(\mathrm{gr}^0 X, d_R) \cong H^j(B^0 \otimes_P T, 1 \otimes \delta) \cong H^j(T, \delta)$. □

Given a unital, Noetherian Poisson algebra P with a coisotropic ideal J, we thus have constructed a BFV model for (P, J).

4 Properties

In this section, we describe the general properties of the BFV models. We postpone the discussion of their cohomology to Sect. 6.

Let (X, R) be a BFV model of (P, J) with X being the completion of $P \otimes \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$. Since R is of degree one, the differential d_R preserves the filtration and hence descends to $\mathrm{gr}\, X$. Let $\pi : X \to X/I = T = P \otimes \mathrm{Sym}(\mathcal{M})$ be the canonical projection. Let $j : T \to X_0 \to X$ be the inclusion given by $t \mapsto 1 \otimes t \in \mathrm{Sym}(\mathcal{M}^*) \otimes T = X_0$. Define $\delta = \pi \circ d_R \circ j : T \to T$.

Lemma 15 *The map $\delta : T \to T$ is a derivation and a differential of degree 1.*

Proof The derivation property follows immediately. Let $a \in T$. We have $(j \circ \pi - \mathrm{id}_X)(d_R(j(a))) \in I$ and hence $d_R(j(\pi(d_R(j(a)))) = d_R((j \circ \pi - \mathrm{id}_X)(d_R(j(a)))) \in I$ is in the kernel of π. The statement about the degree is obvious. □

Lemma 16 *Under the identification of Lemma 82, the differential d_R induced on $\mathrm{gr}\, X$ corresponds to the differential $1 \otimes \delta$ on $B \otimes_P T$.*

Proof Let $x \in \mathrm{gr}^p X$ and pick a representative $a \otimes b \in B^p \otimes_P T$. (It suffices to consider the case where this is a pure tensor.) Then, $d_R(ab) = d_R(a)b + (-1)^p a d_R(b)$. The first summand is in $\mathcal{F}^{p+1} X$ and the second is equivalent to $1 \otimes \delta(a \otimes b)$ modulo $\mathcal{F}^{p+1} X$. □

Let Q_0 be the differential δ on X/I as an element of X.

Remark 17 The complex $(X/I, d_R) = (T, \delta)$ is a Tate resolution of P/J. Hence, the results from Appendix A and Sects. 3.1 and 3.2.1 apply.

Lemma 18 *We have $R \equiv Q_0 \pmod{I^{(2)}}$. Moreover, $\{R, -\} \equiv \{Q_0, -\} \pmod{I}$.*

Proof We have $\{R, -\} \equiv \{Q_0, -\} \pmod{I}$ by construction. Expand $R - Q_0 = \sum_{j \geqslant 0} h_j$ with $h_j \in B^j \otimes_P T^{1-j}$. Such a decomposition exists by Lemma 78. Decompose $h_j = \alpha_j + \beta_j$ with $\alpha_j \in B^j \otimes_P T^{1-j} \cap I_0^{(2)}$ and $\beta_j \in B^j \otimes_P T^{1-j} \setminus I_0^{(2)}$. Let $\{e_k^{(l)}\}_k$ be a basis of $\mathcal{M}^{-l}$ with dual basis $\{e_k^{(l)*}\}_k$. By the Leibnitz rule, $\sum_j \{\beta_j, e_k^{(l)}\} = \{R - Q_0, e_k^{(l)}\} - \sum_j \{\alpha_j, e_k^{(l)}\} \in I$. Expand each $\beta_j = \sum_s a_{j,s} e_s^{(j)*}$ with $a_{j,s} \in T$. We obtain $a_{l,k} = (-1)^{1+l} \sum_j \{\beta_j, e_k^{(l)}\} \in I$; hence all $a_{l,k}$ vanish. □

5 Uniqueness

Fix a unital, Noetherian Poisson algebra P and a coisotropic ideal J. In a first step, we prove that two BFV models for (P, J) related to the same Tate resolutions have isomorphic cohomologies. This is a known fact [9,12] and is presented in Sects. 5.1–5.2. The key tool will be the notion of gauge equivalences. In a second step, we prove that BFV models for $(P = \mathbb{R}[V], J)$, V a finite-dimensional symplectic vector space, on different spaces X have isomorphic cohomologies too. We present this result in Sects. 5.3–5.5. Here, the key tool will be the notion of stable equivalence, introduced in the corresponding Lagrangian setting in [8]. The novel part is that we do not require regularity assumptions, which would imply that the constraint surface is smooth.

5.1 Gauge equivalences

We adapt the language of [8] and call the elements in $\mathfrak{g} = X^0 \cap I^{(2)}$ generators of gauge equivalences. Different BRST charges for the same Tate resolution will be related by these equivalences.

Lemma 19 *The set of generators of gauge equivalences $\mathfrak{g}$ is a closed subset which forms a Lie algebra acting nilpotently on $X/\mathcal{F}^p X$ via the adjoint representation. The Lie algebra $ad(\mathfrak{g})$ exponentiates to a group G acting on X by Poisson automorphisms.*

Proof By Lemma 75, the set is closed. By Lemma 76 and the fact that $\{X^0, X^0\} \subset X^0$, this is a Lie algebra. By Corollary 64, $\mathfrak{g}$ acts on $X/\mathcal{F}^p X$. By Lemma 76, this action is nilpotent. Hence, $\operatorname{ad}\mathfrak{g}$ exponentiates to a group acting on X by vector space automorphisms. Since $\operatorname{ad}\mathfrak{g}$ consists of derivations both for the product and the bracket, those automorphisms are Poisson. □

The elements of G are called gauge equivalences.

Lemma 20 *For $x \in X^1$ and a gauge equivalence g, we have $gx \equiv x \pmod{I^{(2)}}$.*

Proof Let $c \in X^0 \cap I^{(2)}$ be a generator. Then, $gx - x = \sum_{j>0} \frac{1}{j!} \operatorname{ad}_c^j x \in I^{(2)}$ by Lemmas 75 and 76. □

5.2 Uniqueness for fixed Tate resolution

In this section, we prove that two solutions R, R' of the classical master equation in the same space X which induce the same map on X/I are related by a gauge equivalence. Since by Lemma 19, gauge equivalences are Poisson automorphisms, this implies that they have isomorphic cohomologies. We use known techniques, which are adapted from [8].

Remark 21 If R solves $\{R, R\} = 0$ and $g \in G$ is a gauge equivalence, then also $\{gR, gR\} = 0$.

We now discriminate elements in the associated graded $\mathrm{gr}^p X^n$ according to how many positive factors they contain at least by defining $A^n_{p,q} := \{v \in \mathrm{gr}^p X^n : v$ has representative in $I^{(q)}\}$. From the proof of Lemma 82, we see that $A^n_{p,q}$ can be identified with $(B^p \cap I_0^{(q)}) \otimes_P T$, where $I_0 = \mathcal{F}^1 X_0$. We now use Remark 3 to see that $A^\bullet_{p,q}$ is a subcomplex and bound its cohomology:

Lemma 22 *Fix p and q. We have $H^j(A^\bullet_{p,q}, \delta) = 0$ for $j < p$.*

Proof From Remark 3, we have $H^j(A^\bullet_{p,q}, \delta) \cong H^j((B^p \cap I_0^{(q)}) \otimes_P T, 1 \otimes \delta)$. Now, we may factor this space into $(B^p \cap I_0^{(q)}) \otimes_P H^{j-p}(T, \delta)$, since $B^p \cap I_0^{(q)}$ is a free P-module. For $j < p$, the second factor vanishes, since T is a resolution of P/J. □

Lemma 23 *Fix $p \geqslant 2$. Let $R, R' \in X^1$ be two solutions of the classical master equation which induce the same maps on X/I. Then, for $2 \leqslant q \leqslant p$, we have that $R \equiv R' \pmod{I^{(q)} \cap \mathcal{F}^p X^1 + \mathcal{F}^{p+1} X^1}$ implies the existence of a gauge equivalence g with generator $c \in \mathcal{F}^p X^0 \cap I^{(2)}$ such that $gR \equiv R' \pmod{I^{(q+1)} \cap \mathcal{F}^p X^1 + \mathcal{F}^{p+1} X^1}$. Moreover, the element $gR \in X^1$ sill satisfies the classical master equation and induces the same map on X/I as R and R'.*

Proof Let δ be the common differential on X/I and Q_0 be the map δ as an element of X. Hence, $R \equiv Q_0 \equiv R' \pmod{I^{(2)}}$ by Lemma 18. Define $v := R - R' \in I^{(q)} \cap \mathcal{F}^p X^1 + \mathcal{F}^{p+1} X^1 \subset \mathcal{F}^p X^1$. We have

$$\begin{aligned} 0 = \{R + R', R - R'\} &= 2\{Q_0, v\} + \{R - Q_0, v\} + \{R' - Q_0, v\} \\ &\equiv 2\{Q_0, v\} \pmod{\mathcal{F}^{p+1} X^2} \end{aligned}$$

by Lemma 65. By Lemma 16, the maps d_R and $d_{R'}$ also induce the same map on of gr X which we denote by δ too. Since $\delta v = \{Q_0, v\} - L_0 v \equiv \{Q_0, v\} \pmod{\mathcal{F}^{p+1} X^2}$ by Lemma 8, the above implies $\delta v \equiv 0 \pmod{\mathcal{F}^{p+1} X^2}$. Hence, v defines a cocycle $\bar{v}$ in $\mathrm{gr}^p X^1$. We have $p > 1$. By Lemma 22, there exists $\bar{c} \in \mathrm{gr}^p X^0$ with $\delta \bar{c} = \bar{v}$ and a corresponding representative $c \in \mathcal{F}^p X^0 \cap I^{(q)}$, so that $\delta c \equiv v \pmod{\mathcal{F}^{p+1} X^1}$. This c will be the generator of the gauge equivalence we seek. Set $g := \exp \mathrm{ad}_c$. We have

$$\begin{aligned} gR - R' = v + \sum_{j=1}^{\infty} \frac{1}{j!} \mathrm{ad}_c^j R &\equiv \delta(c) - d_R(c) + \sum_{j=2}^{\infty} \frac{1}{j!} \mathrm{ad}_c^j R \\ &\equiv \sum_{j=2}^{\infty} \frac{1}{j!} \mathrm{ad}_c^j R \pmod{\mathcal{F}^{p+1} X^1}. \end{aligned}$$

From Lemma 63, we know that this sum is in $\mathcal{F}^p X^1$. We are left to show that the sum is in $I^{(q+1)}$. By Lemma 76, we have $\mathrm{ad}_c R \in I^{(q)}$. By Lemma 76, we obtain $\mathrm{ad}_c^j R \in I^{(q+1)}$ for all $j \geqslant 2$ since $q \geqslant 2$.

By Remark 21, gR still satisfies the classical master equation and $gR \equiv Q_0 \pmod{I^{(2)}}$ by Lemma 20, whence all maps R, R', gR induce the same map on X/I. □

Theorem 24 *Let R, $R' \in X^1$ be solutions of the classical master equation with differentials inducing the same maps on X/I. Then there exists a gauge equivalence $g \in G$ with $R' = gR$.*

Proof First, we inductively construct a sequence of gauge equivalences $g_2, g_3, \ldots$ such that for all $p \geqslant 2$ we have $g_p \cdots g_2 R \equiv R' \pmod{\mathcal{F}^{p+1} X^1 \cap I^{(2)}}$. By Lemma 20, it suffices to ensure that $g_p \cdots g_2 R \equiv R' \pmod{\mathcal{F}^{p+1} X^1}$.

For $p = 2$, note that $R - R' \in I^{(2)} \subset (I^{(2)} \cap \mathcal{F}^2 X^1) + \mathcal{F}^3 X^1$ by Lemma 18. Now, apply Lemma 23 with $q = p$ to obtain g_2.

Next, assume the $g_2, \ldots, g_p$ have been constructed to fulfill

$$R'' := g_p \cdots g_2 R \equiv R' \pmod{\mathcal{F}^{p+1} X^1 \cap I^{(2)}}.$$

By Remark 21, R'' solves the classical master equation. Moreover, $R'' \equiv Q_0 \pmod{I^{(2)}}$ by Lemmas 18 and 20. Hence, the pair (R'', R') satisfies the requirements of Lemma 23 with $q = 2$. We obtain a gauge equivalence $g_{p+1,2}$ with generator $c_{p+1,2} \in \mathcal{F}^{p+1} X^0 \cap I^{(2)}$ and

$$g_{p+1,2} R'' \equiv R' \pmod{I^{(3)} \cap \mathcal{F}^{p+1} X^1 + \mathcal{F}^{p+2} X^1}.$$

If we continue to apply the lemma for $q = 3, \ldots, p+1$, we obtain gauge equivalences $g_{p+1,3}, \ldots, g_{p+1,p+1}$ with generators $c_{p+1,3}, \ldots, c_{p+1,p+1} \in \mathcal{F}^{p+1} X^0 \cap I^{(2)}$ such that

$$g_{p+1,p+1} \cdots g_{p+1,2} R'' \equiv R' \pmod{\mathcal{F}^{p+2} X^1 \cap I^{(2)}}.$$

Set $g_{p+1} := g_{p+1,p+1} \cdots g_{p+1,2}$. The construction of the sequence is complete.

We claim that $\lim_{m \to \infty} g_m g_{m-1} \cdots g_2$ converges pointwise to a gauge equivalence g. Since all generators $c_{m,j}$ are in $\mathcal{F}^m X^0 \cap I^{(2)}$ and this set is closed under the bracket, the Campbell–Baker–Hausdorff formula implies that the generator c_m of g_m is also in $\mathcal{F}^m X^0 \cap I^{(2)}$. Now, denote the generator of $g_m \cdots g_2$ by γ_m. We have $\gamma_m \in I^{(2)}$ by the CBH formula. Moreover, the CBH formula implies that the generator γ_{m+1} of $g_{m+1} g_m \cdots g_2$ satisfies

$$\gamma_{m+1} = c_{m+1} + \gamma_m + \text{higher terms},$$

where "higher terms" are those involving commutators of c_{m+1} and γ_m where each contains at least one instance of $c_{m+1} \in \mathcal{F}^{m+1} X^0$. Since $\gamma_m \in X^0$, all these terms are in $\mathcal{F}^{m+1} X^0$. Hence,

$$\gamma_{m+1} \equiv \gamma_m \pmod{\mathcal{F}^{m+1} X^0}.$$

Hence, there exists $\gamma \in X^0$ with $\gamma_m \to \gamma$ as $m \to \infty$. We set $g := \exp \operatorname{ad}_\gamma$. By Lemma 75, this element defines a gauge equivalence. We claim that $\exp \operatorname{ad}_{\gamma_m} = g_m \cdots g_2 \to g$ pointwise. Let $x \in X^n$. Then,

$$\exp \operatorname{ad}_{\gamma_m} x - \exp \operatorname{ad}_\gamma x = \{\gamma_m - \gamma, x\} + \frac{1}{2}\{\gamma_m, \{\gamma_m, x\}\} - \frac{1}{2}\{\gamma, \{\gamma, x\}\} + \cdots .$$

Modulo a fixed $\mathcal{F}^k X$, this sum is finite and the number of terms does not depend on m, since all γ_m are at least in $I^{(2)}$. Since $\gamma_m \to \gamma$ and the bracket are continuous in fixed degree by Lemma 74, we obtain the claim.

Finally, $\exp \mathrm{ad}_{\gamma_{m+l}} R - R' \in \mathcal{F}^m X^1$ implies $gR - R' \in \mathcal{F}^m X^1$ for all m which shows that $gR = R'$. □

5.3 Trivial BFV models

The key construction in the proof of uniqueness for different spaces X in Theorem 2 is the notion of stable equivalence. The idea of adding variables that do not change the cohomology was already present in [12]. It was first explicitly formalized in [8] in a similar situation in the Lagrangian setting. Roughly speaking, one proves that different BRST complexes for the same pair (P, J) are quasi-isomorphic by adding more variables of non-zero degree. This is formalized by taking products with so-called trivial BFV models.

Let $P = \mathbb{R}$ with zero bracket and $J = 0$. Then P is a unital, Noetherian Poisson algebra and J is a coisotropic ideal. Let $\mathcal{N}$ be a negatively graded vector space and $\mathcal{N}[1]$ the same space with degree shifted by -1. Define the differential δ on $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}[1]$ by $\delta(a \oplus b) = b \oplus 0$. Set $T = P \otimes \mathrm{Sym}(\mathcal{M})$ and extend δ to an odd, P-linear derivation on T.

Lemma 25 *The complex (T, δ) has trivial cohomology and hence defines a Tate resolution of $P/J = \mathbb{R}$.*

Proof On $\mathcal{M}$, there is a map $s(a \oplus b) = 0 \oplus a$ with $s\delta + \delta s = \mathrm{id}_{\mathcal{M}}$. Extend s to an odd, P-linear derivation on T. Then, $s\delta + \delta s$ is an even derivation on T which is the identity on $\mathcal{M}$ and hence

$$s\delta + \delta s = k\,\mathrm{id} \quad \text{on } P \otimes \mathrm{Sym}^k(\mathcal{M}).$$

Since both s and δ preserve the k-degree, we have

$$\begin{aligned} H^j(T, \delta) &= \oplus_k H^j(P \otimes \mathrm{Sym}^k(\mathcal{M}), \delta) = H^j(P \otimes \mathrm{Sym}^0(\mathcal{M}), \delta) \\ &= \begin{cases} P, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

□

Complete the space $Y_0 = P \otimes \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$ to the space Y. Let e_j be a homogeneous basis of $\mathcal{M}$ such that $\delta(e_j) = e_k$ for some k depending on j. Define $Q_0 = \sum_j (-1)^{1+d_j} e_j^* \delta(e_j)$ as in Lemma 8. Since $\{Q_0, Q_0\} = \sum_{j,k} \pm e_j^* \{\delta(e_j), \delta(e_k)\} e_k^* = 0$, the construction of Sect. 3 yields the BRST charge $S = Q_0$. Hence, (Y, S) is a BFV model for $(P, J) = (\mathbb{R}, 0)$. BFV models arising from this construction are called *trivial*.

Lemma 26 *For trivial BFV models, d_S equals the induced map of $\Delta = \delta \oplus \delta^* : \mathcal{M} \oplus \mathcal{M}^* \to \mathcal{M} \oplus \mathcal{M}^*$ on Y, where the dual differential $\delta^* : \mathcal{M}^* \to \mathcal{M}^*$ is given by $\delta^*(u) = (-1)^{\deg u} u \circ \delta$, i.e. $\delta^*(a \oplus b) = (-1)^j 0 \oplus a$ on $(\mathcal{M}^*)^j$. Conversely, the map d_S induces a differential on Y_0 which coincides with the induced map of $\delta \oplus \delta^*$ on Y_0.*

Proof By acting on the generators e_j and e_j^* defined above, one sees that the induced differential d_S on Y_0 equals $\Delta = \delta \oplus \delta^*$. Since Y_0 is dense in Y and both maps are continuous, the first claim follows. The second claim follows from the observation that the sum $d_S(x) = \sum_j \pm\{e_j \delta(e_j), x\}$ is effectively finite if $x \in Y_0$. □

Lemma 27 *We have $H^j(Y, d_S) = 0$ for $j \neq 0$ and $H^0(Y, d_S) = \mathbb{R}$. The same statement holds if we replace Y by Y_0.*

Proof Since the cohomology of the complex $(\mathcal{M} \oplus \mathcal{M}^*, \Delta)$ is trivial, there is a map $s : \mathcal{M} \oplus \mathcal{M}^* \to \mathcal{M} \oplus \mathcal{M}^*$ of degree -1 with $\Delta s + s\Delta = \text{id}$. Its extension to Y_0 as a derivation thus satisfies

$$s\Delta + \Delta s = j \,\text{id} \quad \text{on } Y_0^{n,j}. \tag{2}$$

Since both maps Δ and s preserve form degree, we obtain the statement about $H^j(Y_0, d_S)$.

Let $x \in Y^n$ with $d_S x = 0$. By Lemma 84, there are $x_j \in Y^n$ of form degree j with $\sum_j x_j = x$. By continuity of the bracket, $0 = \sum_j d_S x_j$. By Lemma 85, $d_S x_j = 0$ for all j, since d_S preserves form degree. Lift Δ and s to Y. Then, Eq. 2 is still valid on $Y^{n,j}$. For $j > 0$, there are $y_j \in Y^{n-1,j}$ with $d_S y_j = x_j$. By Lemma 83, the element $y = \sum_{j>0} y_j$ is well defined and

$$x = \sum_{j>0} x_j + x_0 = d_S y + x_0$$

with $x_0 \in Y^{n,0}$. For $n \neq 0$, this is the empty set and hence x is exact. For $n = 0$, this set is $\mathbb{R}$. We are left to show that two distinct d_S-closed elements of $\mathbb{R}$ always define the distinct cohomology classes. This follows from the fact that each summand in $d_s y = \sum_j (\pm\delta(e_j)\{e_j^*, y\} \pm e_j^*\{\delta(e_j), y\})$ is zero or has a factor of nonzero degree since $\delta(e_j) = e_k$ for some k depending on j. □

5.4 Stable equivalence

Let P be a unital, Noetherian Poisson algebra and $J \subset P$ a coisotropic ideal. Let (X, R) be a BFV model for (P, J) and (Y, S) be a trivial BFV model. Let $\mathcal{M}$ and $\mathcal{N}$ be the corresponding vector spaces. Define Z as the completion of $Z_0 := X_0 \otimes Y_0 = P \otimes \text{Sym}(\mathcal{U} \oplus \mathcal{U}^*)$, where $\mathcal{U} = \mathcal{M} \oplus \mathcal{N}$ and $L = R \otimes 1 + 1 \otimes S$. Both X and Y naturally sit inside Z as Poisson subalgebras, since the inclusions $X_0 \to Z_0$ and $Y_0 \to Z_0$ preserve the respective filtrations.

Lemma 28 *The pair (Z, L) defines another BFV model for (P, J).*

Proof Since the bracket between elements of X and elements of Y is zero, the element L solves the master equation. The Künneth formula implies together with Lemma 27 the conditions on the cohomology. □

We call Z the product of X and Y and write $Z = X\hat{\otimes}Y$. Adding the new variables in $\mathcal{N}$ does not change the cohomology of the BRST complex X:

Lemma 29 *The natural map $X \to Z$ defines a quasi-isomorphism of differential graded commutative algebras.*

Proof We define the maps $\iota : X_0 \to Z_0$ as the natural map and $p : Z_0 \to X_0$ as the map taking $x \otimes y$ to $x\pi(y)$, where $\pi : Y_0 \to Y_0$ is the projection onto $\mathbb{R} = \mathrm{Sym}^0(\mathcal{M} \oplus \mathcal{M}^*)$ along the $\mathrm{Sym}^j(\mathcal{M} \oplus \mathcal{M}^*)$ with $j > 0$. Both maps extend to the respective completion. We claim that they define mutual inverses on cohomology.

From Eq. 2, we infer that there exists a map t on Y_0 such that $d_S t + t d_S = \mathrm{id} - \pi$. In particular, $\pi d_S = d_S \pi = 0$. We have $t = \frac{1}{j}s$ on form degree $j > 0$ and $t = s$ on $\mathbb{R}$. Hence, t preserves the filtration up to degree shift. Hence, $\mathrm{id} \otimes t$ extends from Z_0 to the completion Z. By tensoring the other maps too, we obtain the identity

$$
\begin{aligned}
d_{R+S}(\mathrm{id} \otimes t) + (\mathrm{id} \otimes t)d_{R+S} &= (\mathrm{id} \otimes d_S)(\mathrm{id} \otimes t) + (\mathrm{id} \otimes t)(\mathrm{id} \otimes d_S) = \mathrm{id} \otimes (\mathrm{id} - \pi) \\
&= \mathrm{id} - \iota \circ p
\end{aligned}
$$

on Z. The first equality is true since t shifts degree by one. We are left to show that both maps ι and p descend to cohomology. For ι this is trivial. For p note that for homogeneous x of degree k,

$$
\begin{aligned}
p(d_{R+S}(x \otimes y)) &= p((d_R x) \otimes y + (-1)^k x \otimes d_S y) \\
&= \pi(y) d_R x + (-1)^k x \pi d_S y = d_R(x \pi y) = d_R(p(x \otimes y)).
\end{aligned}
$$

□

Now, we are ready to formulate the notion of stable equivalence introduced in [8]:

Definition 30 Let (X, R) and (X', R') be two simple BFV models for (P, J). We say that (X, R) and (X', R') are stably equivalent if there exist trivial BFV models (Y, S) and (Y', S') and a Poisson isomorphism $X\hat{\otimes}Y \longrightarrow X'\hat{\otimes}Y'$ taking $R + S$ to $R' + S'$.

5.5 Relating Tate resolutions

Now, we want to consider BFV models (R, X) and (R', X') whose Tate resolutions $(X/I, d_R)$ and $(X'/I', d_{R'})$ are not equal. We have the notion of stable equivalence. Our aim is to prove that any two such BFV models are stably equivalent and that stably equivalent BFV models are quasi-isomorphic. As a tool, we need the following lifting statement:

Lemma 31 *Let $P = \mathbb{R}[V]$ with bracket induced by a symplectic structure of a finite-dimensional vector space V and consider $T = P \otimes Sym(\mathcal{M})$ and $T' = P \otimes Sym(\mathcal{M}')$. Assume there is an isomorphism $\phi : T \to T'$ of graded commutative algebras which is the identity in degree zero. Let X be the completion of $X_0 = P \otimes Sym(\mathcal{M} \oplus \mathcal{M}^*)$. Construct analogously the space X'. Then, ϕ lifts to a Poisson isomorphism $\Phi : X \to X'$.*

Proof Since T and T' are negatively graded and isomorphic as graded algebras, we have $\mathcal{M} \cong \mathcal{M}'$ as graded vector spaces. Hence, we may assume $\mathcal{M} = \mathcal{M}'$ and thus $T = T'$ and $X = X'$.

Pick standard coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ on the space V for the symplectic structure, so that $\mathbb{R}[V] = \mathbb{R}[x_i, y_j]$ and $\{x_i, y_j\} = \delta_{ij}$. Let $\{e_j^{(l)}\}_j$ be a basis of $\mathcal{M}^{-l}$ and $\{e_j^{(l)*}\}_j$ be the respective dual bases. Then there are elements $a_{j_1 \dots j_k}^{l_1 \dots l_k}(j, l)(x_i, y_i) \in \mathbb{R}[x_i, y_i]$ and invertible matrices $a_{jk}^{(l)} \in \mathbb{R}[x_i, y_i]$ such that

$$\phi(e_j^{(l)}) = \sum_k a_{jk}^{(l)}(x_i, y_i) e_k^{(l)} + \sum a_{j_1 \dots j_k}^{l_1 \dots l_k}(j, l)(x_i, y_i) e_{j_1}^{(l_1)} \cdots e_{j_k}^{(l_k)},$$

where the sum runs over all integers $k \geq 2$ and $(j_1, l_1), \dots (j_k, l_k)$ with $l_1 + \dots + l_k = l$ and is thus finite. Consider indeterminats $Y_i, E_j^{(l)*} \in X_0$ of degree 0 and l, respectively, defining

$$\begin{aligned} S(x_i, Y_i, e_j^{(l)}, E_j^{(l)*}) = & \sum_i x_i Y_i + \sum_{j,k,l} a_{jk}^{(l)}(x_i, Y_i) E_j^{(l)*} e_k^{(l)} \\ & + \sum_{(j,l)} \sum a_{j_1 \dots j_k}^{l_1 \dots l_k}(j, l)(x_i, Y_i) E_j^{(l)*} e_{j_1}^{(l_1)} \cdots e_{j_k}^{(l_k)}. \end{aligned}$$

Consider the equations

$$\frac{\partial S}{\partial x_i} = y_i \qquad \frac{\partial S}{\partial Y_i} = X_i \qquad \frac{\partial S}{\partial e_j^{(l)}} = (-1)^l e_j^{(l)*} \qquad \frac{\partial S}{\partial E_j^{(l)*}} = E_j^{(l)},$$

which read

$$\begin{aligned} y_i = & Y_i + \sum_{j,k,l} \frac{\partial a_{jk}^{(l)}(x_i, Y_i)}{\partial x_i} E_j^{(l)*} e_k^{(l)} + \sum_{(j,l)} \sum \frac{\partial a_{j_1 \dots j_k}^{l_1 \dots l_k}(j, l)(x_i, Y_i)}{\partial x_i} \\ & \times E_j^{(l)*} e_{j_1}^{(l_1)} \cdots e_{j_k}^{(l_k)}, \\ X_i = & x_i + \sum_{j,k,l} \frac{\partial a_{jk}^{(l)}(x_i, Y_i)}{\partial Y_i} E_j^{(l)*} e_k^{(l)} + \sum_{(j,l)} \sum \frac{\partial a_{j_1 \dots j_k}^{l_1 \dots l_k}(j, l)(x_i, Y_i)}{\partial Y_i} \\ & \times E_j^{(l)*} e_{j_1}^{(l_1)} \cdots e_{j_k}^{(l_k)}, \end{aligned}$$

$$e_j^{(l)*} = \sum_k a_{kj}^{(l)}(x_i, Y_i) E_k^{(l)*} + \sum_{(j',l')} \sum a_{j_1\ldots j_k}^{l_1\ldots l_k}(j', l')(x_i, Y_i)(-1)^{l(l'+1)} E_{j'}^{(l')*}$$
$$\times \frac{\partial(e_{j_1}^{(l_1)} \cdots e_{j_k}^{(l_k)})}{\partial e_j^{(l)}},$$
$$E_j^{(l)} = \sum_k a_{jk}^{(l)}(x_i, Y_i) e_k^{(l)} + \sum a_{j_1\ldots j_k}^{l_1\ldots l_k}(j, l)(x_i, Y_i) e_{j_1}^{(l_1)} \cdots e_{j_k}^{(l_k)}.$$

The linear part is invertible. Hence, we can solve the equations for $(X_i, Y_i, E_j^{(l)}, E_j^{(l)*})$ in terms of $(x_i, y_i, e_j^{(l)}, e_j^{(l)*})$ (and vice versa) and hence also for $(x_i, Y_i, e_j^{(l)}, E_j^{(l)*})$ in terms of $(x_i, y_i, e_j^{(l)}, e_j^{(l)*})$ (and vice versa) in the completion X. Hence, the function S generates a Poisson automorphism $\Phi : X \to X$ by Lemma 86. Let I be the ideal generated by positive elements as defined previously. We have $\Phi(x_i) = X_i \equiv x_i = \phi(x_i) \pmod I$ and $\Phi(y_i) = Y_i \equiv y_i = \phi(y_i) \pmod I$; thus also $\Phi(e_j^{(l)}) = E_j^{(l)} \equiv \phi(e_j^{(l)}) \pmod I$. Hence, Φ is a lift of ϕ. □

Theorem 32 *Consider $P = \mathbb{R}[V]$ with bracket induced by a symplectic structure on a finite-dimensional vector space V. Any two BFV models for (P, J) are stably equivalent.*

Proof Let (X, R) and (X', R') be BFV models with associated Tate resolutions $T := X/I \cong P \otimes \mathrm{Sym}(\mathcal{M})$ and $T' := X'/I' \cong P \otimes \mathrm{Sym}(\mathcal{M}')$. By [8, Theorem A.2], there exist negatively graded vector spaces $\mathcal{N}$ and $\mathcal{N}'$ with finite-dimensional homogeneous components, differentials $\delta_{\mathcal{N}} : \mathrm{Sym}(\mathcal{N}) \to \mathrm{Sym}(\mathcal{N})$, $\delta_{\mathcal{N}'} : \mathrm{Sym}(\mathcal{N}') \to \mathrm{Sym}(\mathcal{N}')$ with cohomology $\mathbb{R}$, and an isomorphism ϕ of differential graded commutative algebras

$$P \otimes \mathrm{Sym}(\mathcal{M} \oplus \mathcal{N}) \to P \otimes \mathrm{Sym}(\mathcal{M}' \oplus \mathcal{N}')$$

restricting to $\mathrm{id}_P : P \to P$ in degree 0. Let Y and Y' be the trivial BFV models corresponding to $\mathcal{N}$ and $\mathcal{N}'$ with BRST charges S and S', respectively. Consider the spaces $Z = X\hat{\otimes}Y$ and $Z' = X'\hat{\otimes}Y'$. Together with the operators $L = R + S$ and $L' = R' + S'$, they form BFV models (Z, L) and (Z', L') for (P, J) by Lemma 28.

We now construct a Poisson isomorphism $\Phi : X\hat{\otimes}Y \to X'\hat{\otimes}Y'$ sending $R + S$ to $R' + S'$. By Lemma 31, the map ϕ lifts to a Poisson isomorphism $\Psi : X\hat{\otimes}Y \to X'\hat{\otimes}Y'$. Now, $L'' = \Psi(L)$ solves $\{-, -\} = 0$ in $X'\hat{\otimes}Y'$. Moreover, $\{L'', -\}$ induces δ' on $P \otimes \mathrm{Sym}(\mathcal{M}' \oplus \mathcal{N}')$. By Theorem 24, there exists a Poisson isomorphism χ of $X'\hat{\otimes}Y'$ with $L' = \chi(L'')$. Set $\Phi = \chi \circ \Psi$.

We are now in the situation

$$\begin{array}{ccc} X & & X' \\ \downarrow & & \downarrow \\ X\hat{\otimes}Y & \xrightarrow{\Phi} & X'\hat{\otimes}Y', \end{array}$$

where the vertical arrows represent natural maps which are quasi-isomorphisms by Lemma 29. ◻

Lemma 33 *The complexes of two stably equivalent BFV models are quasi-isomorphic. In particular, they have cohomologies which are isomorphic as graded commutative algebras.*

Proof Let (X, R) and (X', R') be two stably equivalent BFV models. Hence, we are in the situation

$$\begin{array}{ccc} X & & X' \\ \downarrow & & \downarrow \\ X\hat{\otimes}Y & \longrightarrow & X'\hat{\otimes}Y', \end{array}$$

where the downward arrows are quasi-isomorphisms of differential graded commutative algebras by Lemma 29 and the bottom arrow is a Poisson isomorphism $X\hat{\otimes}Y \to X'\hat{\otimes}Y'$ sending $R + S$ to $R' + S'$. ◻

From Theorem 32 and Lemma 33, we obtain results analogously to the treatment of the Lagrangian case in [8]

Corollary 34 *Let $P = \mathbb{R}[V]$ with bracket induced by a symplectic structure on a finite-dimensional vector space V. Any two BRST complexes arising from BFV models for the same coisotropic ideal $J \subset P$ are quasi-isomorphic. Hence, the BRST cohomology is uniquely determined by $(P = R[x_i, y_i], J)$ up to an isomorphism of graded commutative algebras.*

6 Cohomology

Let P be a unital, Noetherian Poisson algebra and J a coisotropic ideal. Let (X, R) be a BFV model for $J \subset P$. In this section, we analyse the cohomology of the complex (X, d_R). We follow the strategy from [8].

6.1 Cohomology and filtration

The associated graded of X is defined by $\mathrm{gr}^p X = \mathcal{F}^p X/\mathcal{F}^{p+1} X$. The differential d_R induces a map δ on $X/I = T = P \otimes \mathrm{Sym}(\mathcal{M})$ and the results from Sect. 4 apply.

Lemma 35 *$H^j(\mathrm{gr}^p X, d_R) \cong B^p \otimes_P P/J$ for $j = p$ and $H^j(\mathrm{gr}^p X, d_R) \cong 0$ for $j \neq p$.*

Proof Fix p. B^p is a free P-module. By Lemma 16, we have

$$\begin{aligned} H^j(\mathcal{F}^p X/\mathcal{F}^{p+1} X, d_R) &\cong H^j(B^p \otimes_P T^{\bullet - p}, 1 \otimes \delta) \\ &\cong B^p \otimes_P H^{j-p}(T, \delta) \cong B^p \otimes_P H^{j-p}(X/I, d_R). \end{aligned}$$

◻

Next, we want to prove that, to compute the cohomology in a fixed degree, one may disregard elements of high filtration degree.

Lemma 36 *Let $j < p$ be integers with $p \geqslant 0$. Then, $H^j(\mathcal{F}^p X, d_R) = 0$.*

Proof Let $x \in \mathcal{F}^p X^j$ be a cocycle representing a cohomology class in $H^j(\mathcal{F}^p X, d_R)$. Then, $x + \mathcal{F}^{p+1} X^j$ defines a cocycle in $H^j(\mathcal{F}^p X / \mathcal{F}^{p+1} X, d_R)$. By Lemma 35, there is $y_0 \in \mathcal{F}^p X^{j-1}$ with $x - d_R y_0 \in \mathcal{F}^{p+1} X^j$. Hence, this element defines a cocycle in $H^j(\mathcal{F}^{p+1} X / \mathcal{F}^{p+2} X, d_R) = 0$. Hence, there is $y_1 \in \mathcal{F}^{p+1} X^{j-1}$ with $x - d_R y_0 - d_R y_1 \in \mathcal{F}^{p+2} X^j$. Iterating this procedure, we find a sequence $y_0, y_1, \ldots$ of elements $y_j \in \mathcal{F}^{p+j} X^{j-1}$ with $x - d_R(y_0 + \cdots + y_j) \in \mathcal{F}^{j+1} X^j$. By Lemma 77, the element $y := y_0 + \cdots \in X^{j-1}$ is well defined and $y_0 + \cdots + y_j \to y$. Since all y_j are in $\mathcal{F}^p X^{j-1}$ and this set is closed by Lemma 75, we have $y \in \mathcal{F}^p X^{j-1}$. Finally, for n fixed, and all j,

$$d_R y_0 + \cdots + d_R y_n + \cdots + d_R y_{n+j} - x \in \mathcal{F}^{n+1} X^j.$$

Since $d_R = \{R, -\}$ is continuous (Lemma 74), we have $d_R y - x \in \mathcal{F}^{n+1} X^j$. Since n was arbitrary, $d_R y = x$. □

Corollary 37 *The cohomology of (X, d_R) is concentrated in a non-negative degree.*

Corollary 38 *The natural map $H^j(X, d_R) \to H^j(X / \mathcal{F}^{p+1} X, d_R)$ is an isomorphism for $j < p$ and injective for $j = p$.*

Proof The short exact sequence $0 \to \mathcal{F}^{p+1} X \to X \to X / \mathcal{F}^{p+1} X \to 0$ defines the long exact sequence

$$\begin{aligned} \cdots \to H^j(\mathcal{F}^{p+1} X, d_R) \to H^j(X, d_R) \to H^j(X / \mathcal{F}^{p+1} X, d_R) \\ \to H^{j+1}(\mathcal{F}^{p+1} X, d_R) \to \cdots . \end{aligned}$$

For $j \leqslant p$, the first term is zero and for $j < p$, and both the first and the last terms are zero by Lemma 36. □

6.2 Spectral sequences

Lemma 39 *Let $E_r^{p,q}$ be the spectral sequence corresponding to the filtered complex $\mathcal{F}^p X^{p+q}$ with differential d_R. We have $H^\bullet(X, d_R) \cong E_2^{\bullet,0}$ as graded commutative algebras.*

Proof Begin with $E_0^{p,q} := \mathcal{F}^p X^{p+q} / \mathcal{F}^{p+1} X^{p+q}$. It is concentrated in degree $p \geqslant 0$, $q \leqslant 0$. By Lemma 35, we have the following isomorphism of differential bi-graded algebras:

$$\begin{aligned} E_1^{p,q} &= H^q(E_0^{p,\bullet}, d_R) = H^q(\mathcal{F}^p X^{p+\bullet} / \mathcal{F}^{p+1} X^{p+\bullet}, d_R) \\ &= H^{p+q}(\mathcal{F}^p X / \mathcal{F}^{p+1} X, d_R) \cong \begin{cases} B^p \otimes_P P/J, & \text{if } q = 0 \\ 0, & \text{if } q \neq 0. \end{cases} \end{aligned}$$

Hence, $E_1^{p,q}$ is concentrated in degree $p \geqslant 0$ and $q = 0$. Moreover, $d_1^{p,q}$ maps $E_1^{p,q}$ to $E_1^{p+1,q}$. Hence, also $E_2^{p,q}$ is concentrated in $p \geqslant 0$, $q = 0$. Since d_2 maps $E_2^{p,0}$ to $E_2^{p+2,-1}$, it is zero for degree reasons and hence the spectral sequence degenerates at E_2.

We are left to prove that the spectral sequence converges to the cohomology. By [6, chapter XV, proposition 4.1], this follows from Lemma 36. □

We could use this lemma to prove $H^0(X, d_R) \cong (P/J)^J$ as algebras. However, we want to consider an additional structure on the latter space.

6.3 The Poisson algebra structure on $(P/J)^J$

The following two remarks are well known and easily checked.

Remark 40 The Poisson algebra structure on P induces a Poisson algebra structure on $(P/J)^J$.

Remark 41 The graded Poisson algebra structure on X induces a Poisson algebra structure on the cohomology $H^0(X, d_R)$ in degree zero.

Those two structures are in fact isomorphic. We will explicitly construct a Poisson isomorphism. By Corollary 38, we have $H^0(X, d_R) \cong H^0(X/\mathcal{F}^2 X, d_R)$ as vector spaces.

Lemma 42 *Representatives in X^0 of cocycles in $X^0/\mathcal{F}^2 X^0$ defining elements in $H^0(X/\mathcal{F}^2 X, d_R)$ may be taken of the form*

$$x = x_0 + \sum_{i,j \in L} a_{ij} e_i^* e_j,$$

where $L = \{n \in \mathbb{N} : \deg(e_j^*) = 1\}$, $x_0 \in P$, the $\{e_j\}$ are a homogeneous basis of $\mathcal{M}$, and the $a_{ij} \in P$ are chosen such that

$$\{\delta(e_j), x_0\} = \sum_{i \in L} a_{ji} \delta(e_i).$$

Conversely, every such element defines a cohomology class.

Proof We have

$$\begin{aligned} X^0/\mathcal{F}^2 X^0 &= P \oplus (P \otimes (\mathcal{M}^*)^{-1} \otimes \mathcal{M}^{-1}) \\ X^{-1}/\mathcal{F}^2 X^{-1} &= (P \otimes \mathcal{M}^{-1}) \oplus (P \otimes (\mathcal{M}^*)^1 \otimes \mathcal{M}^{-2}) \\ &\quad \oplus (P \otimes (\mathcal{M}^*)^1 \otimes (\mathcal{M}^{-1} \wedge \mathcal{M}^{-1})). \end{aligned}$$

Hence, an arbitrary cochain may be taken to be of the form

$$x = x_0 + \sum_{i,j \in L} a_{ij} e_i^* e_j$$

for some $x_0, a_{ij} \in P$. We compute with the help of Lemma 18,

$$\begin{aligned} d_R x &= \{R, x_0\} + \sum_{i,j \in L} (\{R, a_{ij}\} e_i^* e_j + \{R, e_i^*\} e_j a_{ij} - \{R, e_j\} e_i^* a_{ij}) \\ &\equiv \{R, x_0\} - \sum_{i,j \in L} \{R, e_i\} e_j^* a_{ji} \\ &\equiv \sum_{j \in L} \left((-1)^{1+d_j} \{\delta(e_j), x_0\} - \sum_{i \in L} a_{ji} \delta(e_i) \right) e_j^* \pmod{\mathcal{F}^2 X^1}. \end{aligned}$$

□

Theorem 43 *$H^0(X, d_R) \cong (P/J)^J$ are Poisson algebras.*

Proof Let $\pi : X \to P = X/(I + I_-)$ denote the projection onto all monomials which contain no factors of nonzero degree. Here, $I_- \subset X$ denotes the ideal generated by all elements of negative degree. Define the map $\Phi : H^0(X, d_R) \to P/J$ by $\Phi([x]) := \pi(x) + J$. This map is well defined: Let $x = d_R y$ be exact. Consider again the differential δ on $T = X/I$ that is induced by d_R and its representation as an element $Q_0 \in X^1$. Also, pick a homogeneous basis e_j of $\mathcal{M}$ as done before. By Lemma 18, we obtain

$$d_R(y) \equiv \{Q_0, y\} \equiv \sum_i \{(-1)^{1+d_i} \delta(e_i) e_i^*, y\} \equiv \sum_{i : \deg e_i = -1} \delta(e_i) \{e_i^*, y\} \pmod{I + I_-}.$$

The last sum is finite. Hence,

$$\pi(x) = \pi(d_R y) = \sum_{i : \deg e_i = -1} \delta(e_i) \pi \{e_i^*, y\} \in J.$$

By Lemma 38, we have $H^0(X, d_R) \cong H^0(X/\mathcal{F}^2 X, d_R)$ as vector spaces. Hence, we have a corresponding linear map $X/\mathcal{F}^2 X \to P/J$.

The image of either of those maps is J-invariant: Let $[x] \in H^0(X/\mathcal{F}^2 X, d_R)$. According to Lemma 42, we may pick a representative $x_0 = \pi(x_0) + \sum_{i,j \in L} a_{ij} e_i^* e_j$ of x, where $a_{ij} \in P$ satisfy $\{\delta(e_j), \pi(x_0)\} = \sum_{i \in L} a_{ji} \delta(e_i)$. In particular, $\{\delta(e_j), \pi(x_0)\} \in J$. Fix $b \in J$. Then there exist $b_j \in P$ with $b = \sum_{i \in L} b_i \delta(e_i)$ and thus $\{b, \pi(x_0)\} = \sum_{i \in L} \big(b_i \{\delta(e_i), \pi(x_0)\} + \delta(e_i) \{b_i, \pi(x)\}\big) \in J$.

Hence, we have two linear maps:

$$\begin{aligned} \phi : H^0(X/\mathcal{F}^2 X, d_R) &\to (P/J)^J, \\ \Phi : H^0(X, d_R) &\to (P/J)^J, \end{aligned}$$

given by projection onto the P component followed by modding out J, which correspond to each other under the isomorphism $H^0(X, d_R) \cong H^0(X/\mathcal{F}^2 X, d_R)$.

The map ϕ is surjective: Let $p \in P$ with $\{J, p\} \subset J$. By Lemma 42, the element $x = p + \sum_{ij \in L} a_{ij} e_i^* e_j$ is a cocycle if $\{\delta(e_j), p\} = \sum_{i \in L} a_{ji} \delta(e_i)$. But those $a_{ij} \in P$ exist since $\{\delta(e_j)\}_{j \in L}$ generate J. Hence, also the map Φ is surjective.

The map Φ is injective: Let $x \in X^0$ represent $[x] \in H^0(X, d_R)$ with $\pi(x) \in J$. We claim that there exist $y_j \in \mathcal{F}^j X^{-1}$ with $x - d_R(y_0 + \cdots + y_n) \in \mathcal{F}^{n+1} X^0$. By Lemma 35, we know that $H^j(\mathcal{F}^p X/\mathcal{F}^{p+1} X, d_R)$ is concentrated in degree zero with $H^0(X/\mathcal{F}^1 X, d_R) \cong P/J$ via the natural map. Now, $x + \mathcal{F}^1 X^0$ defines the zero cohomology class in $H^0(X/\mathcal{F}^1 X, d_R)$, since $\pi(x) \in J$. Hence there exists $y_0 \in \mathcal{F}^0 X^{-1}$ with $x - d_R y_0 \in \mathcal{F}^1 X^0$. Again, $x - d_R y_0 + \mathcal{F}^2 X^0$ defines the zero cohomology class in $H^0(\mathcal{F}^1 X/\mathcal{F}^2 X, d_R) = 0$. Hence, there exists $y_1 \in \mathcal{F}^1 X^{-1}$ with $x - d_R(y_0 + y_1) \in \mathcal{F}^2 X^0$ and so on. Hence, y_j exist and their sum converges to an element $y \in X^{-1}$ by Lemma 77, which satisfies $x - d_R y = 0$ by Lemma 75.

Hence, the map Φ is an isomorphism of vector spaces. This map also respects the product structure

$$
\begin{aligned}
\Phi([x][y]) &= \Phi([xy]) = \pi(xy) + J = \pi(x)\pi(y) + J = (\pi(x) + J)(\pi(y) + J) \\
&= \Phi([x])\Phi([y])
\end{aligned}
$$

and is hence an isomorphism of algebras. Finally, map Φ respects the bracket:

$$
\begin{aligned}
\Phi(\{[x], [y]\}) = \Phi([\{x, y\}]) &= \pi(\{x, y\}) + J = \pi(\{\pi(x), y\}) + J \\
&= \pi(\{\pi(x), \pi(y)\}) + J = \{\pi(x), \pi(Y)\} + J = \{\pi(x) + J, \pi(y) + J\} \\
&= \{\Phi(x), \Phi(y)\},
\end{aligned}
$$

since $\{\pi(x) - x, X^0\} \subset \{(I + I_0) \cap X^0, X^0\} \subset \ker \pi$, where $\ker \pi = I + I_0 \subset X$ is the ideal generated by all elements of nonzero degree. The last inclusion holds by the Leibniz rule since all summands of elements in $I + I_0$ that are of degree zero contain at least two factors of nonzero degree. □

7 Examples

We present two well-known examples. More interesting examples can be found in [13].

7.1 Rotations of the Plane

Here, we present an example, where the cohomology in degree zero has a nontrivial bracket and the cohomology in degree 1 does not vanish. It is obtained by considering the symplectic lift of the rotations of the plane to the cotangent bundle of the plane.

Consider $P = \mathbb{R}[x_1, x_2, y_1, y_2]$ with $\{x_i, y_j\} = \delta_{ij}$. The ideal $J \subset P$ generated by $\mu = x_1 y_2 - x_2 y_1$ is coisotropic. A Tate resolution of J is given by

$$0 \to P \cdot e \to P \to P/J \to 0$$

where the differential δ is the P-linear derivation defined by $\delta(e) = \mu$. Indeed, this complex is a Koszul complex which is exact, since $\mu \neq 0$ defines a regular sequence. Hence, $X = \big(P \cdot e\big) \oplus \big(P \oplus P \cdot e^*e\big) \oplus \big(P \cdot e^*\big)$. We now apply the construction from Sect. 3. We obtain $Q_0 = e^*\mu$ and $R = Q_0$, since $\{Q_0, Q_0\} = 0$. One easily calculates

$$H^0(X, d_R) = \frac{\{a + be^*e : \{\mu, a\} = \mu b, a, b \in P\}}{\{\mu c + \{\mu, c\}e^*e : c \in P\}}.$$

Notice that the isomorphism $H^0(X, d_R) \to (P/J)^J$ given by projection onto P is evident here. Moreover, the bracket on this space does not vanish: $x_1^2 + x_2^2$ and $y_1^2 + y_2^2$ define cohomology classes, for which $\{x_1^2 + x_2^2, y_1^2 + y_2^2\} = 4(x_1y_1 + x_2y_2)$ is not in J. Furthermore,

$$H^1(X, d_R) = \frac{\{ae^* : a \in P\}}{\{d_R(a + be^*e) : a, b \in P\}} \cong \frac{P}{\{\{\mu, a\} + \mu b : a, b \in P\}}$$

does not vanish, since $\deg_0\{\mu, a\} \geq 1$ and $\deg_0(\mu b) \geq 2$. Here, $\deg_0$ denotes the degree in $P = \mathbb{R}[x_i, y_i]$.

7.2 Rotations of space

Let $X = \mathbb{R}^3$ and $M = T^*X \cong X \oplus X^*$. Consider the group $G = SO(3)$ acting on X via the standard representation $\rho_0 : G \to \operatorname{End} X$. The symplectic lift is given by $\rho : G \to \operatorname{End} M$, $\rho(A)(x, p) = (Ax, p \circ A^{-1})$. Mapping the standard basis of $X = \mathbb{R}^3$ to its dual basis, we obtain an isomorphism $\iota : X \to X^*$. A possible moment map is the angular momentum mapping $\mu : M \to \mathbb{R}^3$, $\mu(x, p) = x \times \iota p$. Here, $\times$ refers to the vector product, and we identified $\mathfrak{g} \cong \mathbb{R}^3$ using the basis

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We define $M_0 = \mu^{-1}(0) = \{(x, p) \in X \oplus X^* : \iota p \parallel x\}$. This is not a manifold. If it was one, it had dimension 4. However, all $(x, 0)$, $x \in \mathbb{R}^3$ and all $(0, p)$, $p \in (\mathbb{R}^3)^*$ belong to M_0. Hence, they also belong to the tangent space at the origin, provided M_0 was a manifold. Since the tangent space at the origin is linear, it would have dimension $6 > 4$. Since the constraint surface M_0 is not a manifold, results from [9,14] do not apply.

We take the Tate resolution T of the vanishing ideal J of M_0 in $P = \mathbb{R}[M] = \mathbb{R}[x_1, x_2, x_3, p_1, p_2, p_3]$ with $\{x_i, p_j\} = \delta_{ij}$. It exists since P is Noetherian. We obtain the existence and uniqueness of a BRST charge R as described in the previous sections.

8 Quantization

In this section, we discuss quantization. In Sect. 8.1, we define a quantum algebra quantizing the Poisson algebra X from the previous part of this note. We rigorously define multiplication via normal ordering in the presence of infinitely many ghost variables.

In Sect. 8.2, we construct a solution of the quantum master equation associated with a given solution of the classical master equation. This means we construct an element of the quantum algebra that agrees with the quantization of the classical solution up to an error of order $\hbar$ and squares to zero.

In Sect. 8.3, we discuss the uniqueness of such solutions of the quantum master equation. We parallel our discussion in the classical case. In Sect. 8.3.1, we define the notion of a quantum gauge equivalence. In Sect. 8.3.2, we prove that two solutions of the quantum master equation that agree up to an error of order $\hbar$ are related via an automorphism of associative algebras. In Sect. 8.3.3, we show that the solutions of the quantum master equation associated with two BRST models associated with the same Tate resolution are also related by an automorphism of associative algebras. In Sect. 5.5, we have shown that any two BRST models associated with the same coisotropic ideal J are stably equivalent. We would like to find a quantum analogue of this theorem. We were able to prove in Sect. 8.3.5 that the process of adding extra variables yields a quasi-isomorphism of differential graded algebras on the quantum level. However, as discussed in Sect. 8.3.6, we were unable to quantize the general Poisson isomorphism of Lemma 31.

8.1 Quantum algebra

Assumption 44 Assume that $P = \mathbb{R}[V]$ where V is a finite-dimensional real symplectic vector space and the bracket on P is induced by the symplectic structure.

Pick a decomposition $V = L \oplus L^*$ where L is a Lagrangian subspace of V and $\{v, \lambda\} = \lambda(v)$ for $v \in L$ and $\lambda \in L^*$. Set $\mathcal{N} := \mathcal{M} \oplus L$ so that we may write $X_0 = \mathrm{Sym}(\mathcal{N} \oplus \mathcal{N}^*)$. We define $G_0 = G_- \otimes G_+$ where $G_- = \mathrm{Sym}(\mathcal{N})$ and $G_+ = \mathrm{Sym}(\mathcal{N})$ as *vector spaces*. We introduce a formal parameter $\hbar$ and want to define a product on $G_0[\hbar]$ quantizing $X_0[\hbar]$. Intuitively, we want to define the product $(x_-^1 \otimes x_+^1)(x_-^2 \otimes x_+^2)$ of monomials $x_-^i \otimes x_+^i \in G_0 \subset G_0[\hbar]$ by commuting x_+^1 to the right of x_-^2 using the canonical commutation relations we obtain from the split $V = L \oplus L^*$. A rigorous definition works as follows:

Let $(W[\hbar], \star)$ be the deformation quantization [15] of X_0 defined by the (graded, see e.g. [7]) *Wick-type*, or *normal ordered-type* [20] Moyal product $\star$, defined by the split $V = L \oplus L^*$. Since X_0 contains only polynomials, the Moyal product of two elements of $W[\hbar]$ is a polynomial in $\hbar$. Since naturally $G_0[\hbar] \cong X_0[\hbar]$ as $\mathbb{R}[\hbar]$-modules and, by construction of the deformation quantization, $X_0[\hbar] \cong W[\hbar]$ as $\mathbb{R}[\hbar]$-modules, the Moyal product defines a product on $G_0[\hbar]$ turning $G_0[\hbar]$ into a unital associative algebra.

We do the construction involving $G_0[\hbar]$ instead of $X_0[\hbar]$, since we want to avoid convergence issues. For example, the relation $\sum_j e_j^* e_j = \sum_j (e_j e_j^* + \hbar)$ is problematic in the completion of $X_0[\hbar]$. Using a suitable completion of $G_0[\hbar]$, we avoid expressions involving infinite sums of terms that are not normally ordered.

Next, we define the completion and extend the product to it. We introduce the filtration on G_0 defined by the subspaces $\mathcal{F}^p\, G_0 = \bigoplus_{q \geq p} G_- \otimes G_+^p$. Set $\mathcal{F}^p\, G_0^n = \mathcal{F}^p\, G_0 \cap G_0^n = \bigoplus_{q\geq p} G_-^{n-p} \otimes G_+^p$. We complete G_0 to the graded vector space $G = \bigoplus_n G^n$ where

$$G^n = \lim_{\leftarrow p} \frac{G_0^n}{\mathcal{F}^p\, G_0^n}.$$

This graded vector space is again filtered by the subspaces $\mathcal{F}^p\, G^n = \lim_{\leftarrow q} \frac{\mathcal{F}^p\, G_0^n}{\mathcal{F}^{p+q}\, G_0^n}$.

Define the graded vector space $G_\hbar$ by its homogeneous components $H_\hbar^n = G^n[[\hbar]]$. We have a family of projections $p_j : G_\hbar \to G$ mapping $\sum_{k\geq 0} x_k \hbar^k \mapsto x_j$. The space $G_0[\hbar]$ can be considered a graded subspace of $G_\hbar$. We want to extend the algebra structure on $G_0[\hbar]$ to $G_\hbar$. For this task, we need to analyse the compatibility of the product on $G_0[\hbar]$ with the filtration on G_0.

Lemma 45 *We have for all $j, p \geq 0$ and $n, m \in \mathbb{Z}$,*

(1) $p_j(G_0^n \cdot \mathcal{F}^p\, G_0^m) \subset \mathcal{F}^p\, G_0^{n+m}$,
(2) $p_j(\mathcal{F}^p\, G_0^n \cdot G_0^m) \subset \mathcal{F}^{p+m}\, G_0^{n+m}$.

Proof Consider $x = a \otimes u \cdot b \otimes v \in G_0^n \cdot G_0^m$ where $a \in G_-^{n-l}, u \in G_+^l, b \in G_-^{m-k}$ and $v \in G_+^k$ for some $l, k \geq 0$. Then,

$$p_j(a \otimes u \cdot b \otimes v) = \sum_{b'u'} \pm ab' \otimes u'v,$$

where u' and b' arise from u and b by deleting j matching pairs (e, e^*), in which $e \in \mathcal{N}$ is a factor in b and $e^* \in \mathcal{N}^*$ is a factor in u of opposite degree. The sum is finite.

To prove the first statement, it suffices to note that $\deg u' \geq 0$ and $k \geq p$. For the second statement, we note that $\deg u' + \deg b' = \deg u + \deg b$ and hence we can estimate $\deg u' + \deg v \geq \deg u' + \deg b' + \deg v = \deg u + \deg b + \deg v = l + m - k + k \geq p + m$. □

Lemma 46 *The product* (*and hence also the commutator*) *extend to the completion turning $(G_\hbar, \cdot)$ into a graded algebra.*

Proof First, we consider $x = (x_p + \mathcal{F}^p\, G_0^n)_p \in G^n$ and $y = (y_p + \mathcal{F}^p\, G_0^m)_p \in G^m$. By Lemma 45, the limit $\lim_{p\to\infty} x_p \cdot y_p \in H^{n+m}$ is well defined and the definition $x \cdot p = \lim_{p\to\infty} x_p \cdot y_p$ does not depend on the choice of representatives x_p, y_p. The multiplication extends to $G_\hbar$ by bi-linearity in $\mathbb{R}[[\hbar]]$. □

Lemma 47 *The product on $G_\hbar^n \times G_\hbar^m$ is continuous in each entry.*

Proof Let $x_r, x \in G^n_\hbar$ with $x_r \to x$ and $y \in G^m_\hbar$. Write $p_j(x_r) = (x^{(j)}_{r,p} + \mathcal{F}^p\, G^n_0)_p$, $p_j(x) = (x^{(j)}_p + \mathcal{F}^p\, G^n_0)_p$, and $p_j(y) = (y^{(j)}_p + \mathcal{F}^p\, G^m_0)_p$. Fix $j, p \in \mathbb{N}_0$. Take r_0 such that for all $0 \leq k \leq j$ and all $r \geq r_0$, we have $x^{(k)}_{r,p-m} \equiv x^{(k)}_{p-m} \pmod{\mathcal{F}^{p-m}\, G^n_0}$. Such r_0 exists since $x_r \to x$. For $r \geq r_0$, we have by Lemma 45

$$\begin{aligned} p_j(x_r \cdot y) - p_j(x \cdot y) &= \left(\sum_{l=0}^{j} p_l \Big(\sum_{k=0}^{j-l} (x^{(k)}_{r,p-m} - x^{(k)}_{p-m}) \cdot y^{(j-l-k)}_p \Big) + \mathcal{F}^p\, G^{n+m}_0 \right)_p \\ &= 0. \end{aligned}$$

Continuity in the other entry follows analogously. □

Now that we have set up the algebra, we define the quantization mapping. We have the canonical graded vector space isomorphism $q_0 : \mathrm{Sym}(\mathcal{N} \oplus \mathcal{N}^*) \to \mathrm{Sym}(\mathcal{N}) \otimes \mathrm{Sym}(\mathcal{N}^*)$. Since this map respects the respective filtrations, we can extend it to

$$q : X \to G \subset G_\hbar.$$

Since the inverse of q_0 also respects the filtration and thus extends, the map $q : X \to G$ is an isomorphism of graded vector spaces.

To relate the multiplicative structure on X to the one on $G_\hbar$, we set $A = G_\hbar/(\hbar)$ where $(\hbar) \subset G_\hbar$ is the two-sided ideal generated by $\hbar$. Hence, $A \cong G$ act as graded vector spaces, but not as graded algebras since G is not closed under multiplication.

Remark 48 We have $[G_\hbar, G_\hbar] \subset (\hbar)$. Hence, $(\hbar)$ is the two-sided ideal generated by the commutator. Hence, the map $\frac{1}{\hbar}[-,-] : G_\hbar \otimes G_\hbar \to G_\hbar$ is well defined and turns $G_\hbar$ into a graded noncommutative Poisson algebra. This map descends to A, turning it into a graded commutative Poisson algebra.

Theorem 49 *The graded (commutative) Poisson algebras X and A are isomorphic via the map $\phi = \pi \circ \iota \circ q : X \to G \to G_\hbar \to A$, where $\pi : G_\hbar \to A = G_\hbar/(\hbar)$ is the canonical projection and $\iota : G \to G_\hbar$ is the inclusion. In particular, for all $x, y \in X$*

$$\frac{1}{\hbar}[q(x), q(y)] \equiv q(\{x, y\}) \pmod{(\hbar)}. \tag{3}$$

Proof We already know that ϕ is an isomorphism of graded vector spaces. We have to prove the compatibility with the product and bracket structures. By density of $X_0 \subset X$ and continuity of all maps involved, it suffices to consider X_0. Without the completion the statement is standard; see e.g. [20]. □

Corollary 50 *If $R \in X^1$ solves the classical master equation, then* $\frac{1}{\hbar}[q(R), q(R)] \equiv 0 \pmod{(\hbar)}$. *Conversely, if $r = q(R) + \hbar(\cdots)$ solves the quantum master equation $[r, r] = 0$, then R solves the classical master equation.*

8.2 Solving the quantum master equation

We want to construct a solution $r \in G^1_\hbar$ of the quantum master equation $[r, r] = 0$. We seek a solution of the form

$$r = q(R) + \hbar q(R_1) + \hbar^2 q(R_2) + \cdots$$

for some $R_j \in X^1$ where $R \in X^1$ is a given solution of the classical master equation.

Assumption 51 We assume $H^2(X, d_R) = 0$.

8.2.1 A differential on the quantum algebra

Define $D = \frac{1}{\hbar}[q(R), -]$. This defines a map $G_\hbar \to G_\hbar$ by Remark 48. It preserves the ideal $(\hbar)$ and hence descends to a derivation D_0 on $A = G_\hbar/(\hbar)$. We calculate

$$D^2(x) = \frac{1}{\hbar^2}[q(R), [q(R), x]] = \frac{1}{\hbar^2}[[q(R), q(R)], x] - \frac{1}{\hbar^2}[q(R), [q(R), x]].$$

Hence by Corollary 50 and Remark 48,

$$D^2(x) = \frac{1}{2\hbar}\left[\frac{1}{\hbar}[q(R), q(R)], x\right] \equiv 0 \qquad (\text{mod } \hbar), \tag{4}$$

so D_0 is a differential on A.

Theorem 52 *We have $D_0 \circ \phi = \phi \circ d_R$. In particular, $\phi : X \to A$ is an isomorphism of differential graded commutative algebras and $H^\bullet(X, d_R) \cong H^\bullet(A, D_0)$.*

Proof Let $x \in X$. By Theorem 49,

$$\begin{aligned} D_0(\phi(x)) = D_0(\pi(q(x))) = \pi(D(q(x))) = \pi\left(\frac{1}{\hbar}[q(R), q(x)]\right) \\ = \pi(q(\{R, x\})) = \pi(q(d_R(x))) = \phi(d_R(x)). \end{aligned}$$

□

Corollary 53 *Under Assumption 51, $H^2(A, D_0) = 0$.*

8.2.2 Construction of a solution of the quantum master equation

Theorem 54 *Let $n \geq 0$ be an integer. For $n \geq 1$, assume we have constructed $R_1, R_2, \ldots, R_n \in X^1$ such that $r_n := q(R) + \sum_{l=1}^{n} \hbar^l q(R_l)$ satisfies $\frac{1}{\hbar}[r_n, r_n] \equiv 0$ (mod $(\hbar^{n+1})$). For $n = 0$, set $r_n = q(R)$ which also satisfies this assumption by Corollary 50. We claim that there exists $R_{n+1} \in X^1$ such that*

$$\frac{1}{\hbar}[r_n + \hbar^{n+1} q(R_{n+1}), r_n + \hbar^{n+1} q(R_{n+1})] \equiv 0 \qquad (\text{mod } (\hbar^{n+2})).$$

Proof We compute for any $R_{n+1} \in X^1$, using Corollary 50,

$$\begin{aligned}
&\frac{1}{\hbar}[r_n + \hbar^{n+1}q(R_{n+1}), r_n + \hbar^{n+1}q(R_{n+1})] \\
&\quad = \frac{1}{\hbar}[r_n, r_n] + 2\hbar^{n+1}\frac{1}{\hbar}[r_n, q(R_{n+1})] + \hbar^{2n+2}\frac{1}{\hbar}[q(R_{n+1}), q(R_{n+1})] \\
&\quad \equiv \frac{1}{\hbar}[r_n, r_n] + 2\hbar^{n+1}\frac{1}{\hbar}[q(R), q(R_{n+1})] \qquad (\text{mod } (\hbar^{n+2})).
\end{aligned}$$

By the induction assumption, we can write the right hand side as $\hbar^{n+1}$ times

$$\frac{1}{\hbar^{n+1}}\frac{1}{\hbar}[r_n, r_n] + 2D(q(R_{n+1})) \in H$$

and we want this to be a multiple of $\hbar$. This means we need to show that we can pick $R_{n+1} \in X^1$, such that $\pi(\frac{1}{\hbar^{n+1}}\frac{1}{\hbar}[r_n, r_n]) + 2D_0(\phi(R_{n+1}))$ vanishes in A. By Corollary 53 and the fact that ϕ is surjective, it suffices to prove that $\pi(\frac{1}{\hbar^{n+1}}\frac{1}{\hbar}[r_n, r_n])$ is D_0-closed. By the Jacobi identity,

$$0 = \frac{1}{\hbar^2}[r_n, [r_n, r_n]] = D\left(\frac{1}{\hbar}[r_n, r_n]\right) + \sum_{l=1}^{n} \hbar^l \frac{1}{\hbar}[q(R_l), \frac{1}{\hbar}[r_n, r_n]].$$

By the induction assumption, we may divide this equation by $\hbar^{n+1}$ to arrive at

$$D\left(\frac{1}{\hbar^{n+1}}\frac{1}{\hbar}[r_n, r_n]\right) = -\sum_{l=1}^{n} \hbar^l \frac{1}{\hbar}\left[q(R_l), \frac{1}{\hbar^{n+1}}\frac{1}{\hbar}[r_n, r_n]\right] \equiv 0 \quad (\text{mod } \hbar).$$

□

Hence, we have constructed $r = q(R) + \hbar q(R_1) + \cdots$ with $[r, r] = 0$.

8.3 Uniqueness of the solution

In this paragraph, we consider questions of uniqueness of solutions of the quantum master equation that arise from quantization of a solution of the classical master equation.

8.3.1 Quantum gauge equivalences

We define the subspace $K = \{x \in G_\hbar^0 : p_0(x) \in q(I^{(2)})\}$.

Lemma 55 *K is closed under the commutator.*

Proof Let $x, y \in K$. Theorem 49 allows us to calculate

$$\frac{1}{\hbar}[x, y] \equiv \frac{1}{\hbar}[p_0(x), p_0(y)] \equiv q(\{q^{-1}(p_0(x)), q^{-1}(p_0(x))\}) \pmod{\hbar}.$$

By Lemma 76, the claim follows. □

We call elements of K generators of quantum gauge equivalences. Typical elements of K are quantizations of generators of classical gauge equivalences or any degree zero multiple of $\hbar$. To exponentiate the Lie algebra K to a group acting on $G_\hbar$ by isomorphisms of associative algebras, we show that in each degree in $\hbar$ the Lie algebra L acts pro-nilpotent with respect to the filtration $\mathcal{F}^p G^n$.

Lemma 56 *We define* $ad_a\, b = \frac{1}{\hbar}[a, b]$ *for* $a \in K$ *and* $b \in G_\hbar$. *The Lie algebra ad* $K \subset End(G_\hbar)$ *acts pro-nilpotently in each degree of* $\hbar$. *In particular, it exponentiates to a group of automorphisms of associative algebras.*

Proof Fix integers $j \geq 0$ and $k \geq 1$. For $i = 1, \ldots, k$, take $u_i = u_{i0} + \hbar u_{i1} + \cdots$ where $u_{i0} \in q(I^{(2)})$ and $u_{ij} \in G^0$. Let $x \in G^n$. Fix $l = l_1 + l_k$ with integers $l_i \geq 0$. Then,

$$\begin{aligned}
&p_j(\mathrm{ad}_{u_1}^{l_1} \cdots \mathrm{ad}_{u_k}^{l_k} x)\\
&\quad = p_j\left(\sum_{\substack{j_i:\{1,\ldots,l_i\}\to\mathbb{N}_0\\ i=1,\ldots,k}} \hbar^{\sum_{i=1}^k \sum_{s=1}^{l_i} j_i(s)}\, \mathrm{ad}_{u_{1j_1(1)}} \circ \cdots \circ \mathrm{ad}_{u_{1j_1(l_1)}} \circ \cdots \circ \mathrm{ad}_{u_{kj_k(1)}} \circ \cdots \circ \mathrm{ad}_{u_{kj_k(l_k)}}(x)\right)\\
&\quad = \sum_{\substack{j_i:\{1,\ldots,l_i\}\to\mathbb{N}_0\\ i=1,\ldots,k\\ n=\sum_{i=1}^k \sum_{s=1}^{l_i} j_i(s)\leq j}} p_{j-n}(\mathrm{ad}_{u_{1j_1(1)}} \circ \cdots \circ \mathrm{ad}_{u_{1j_1(l_1)}} \circ \cdots \circ \mathrm{ad}_{u_{kj_k(1)}} \circ \cdots \circ \mathrm{ad}_{u_{kj_k(l_k)}}(x)).
\end{aligned}$$

This is a finite sum. We now write out each argument of p_{j-n} as a sum of products of the $u_{pq} \in G$. Each such product satisfies the following conditions. It contains $(l+1)$ factors. The factor x appears once. The number of factors u_{pq} with $q \geq 1$ is bounded above by n. Hence, the number of factors $u_{i0} \in q(I^{(2)})$ is bounded below by $(l-n)$. Thus, the number of positive factors before normal ordering is bounded from below by $2(l-n)$. After normal ordering and applying p_{j-n}, the number of positive factors that still remain are bounded from below by $2(l-n) - (j-n) \geq 2(l-j)$. Hence, $p_j(\mathrm{ad}_{u_1}^{l_1} \cdots \mathrm{ad}_{u_k}^{l_k} x)$ is a finite sum of elements in G^n, which contain at least $2(l-j)$ factors of positive degree. This bound is independent of x.

Finally, fix $p, j \geq 0$ and let $x = \sum_j x_j \hbar^h \in G^n_\hbar$. We have

$$p_j(\mathrm{ad}_{u_1}^{l_1} \cdots \mathrm{ad}_{u_k}^{l_k} x) = \sum_{k=0}^{j} p_{j-k}(\mathrm{ad}_{u_1}^{l_1} \cdots \mathrm{ad}_{u_k}^{l_k} x_k).$$

Pick l_0 such that for all $m = 0, \ldots, j$, for all $r \geq 0$ and $l = l_1 + \cdots + l_k \geq l_0$ we have $p_m(\mathrm{ad}_{u_1}^{l_1} \cdots \mathrm{ad}_{u_k}^{l_k} x_r) \in \mathcal{F}^p G^n$. Then for all $l = l_1 + \cdots + l_k \geq l_0$, we have $p_j(\mathrm{ad}_{u_1}^{l_1} \cdots \mathrm{ad}_{u_k}^{l_k} x) \in \mathcal{F}^p G^n$. Hence, ad K acts pro-nilpotently with respect to this filtration and thus ad K exponentiates to a group of vector space automorphisms $\{\exp \mathrm{ad}_u : G_\hbar \to G_\hbar, u \in K\}$. These maps preserve the multiplicative structure since ad_u is a derivation for the product. □

8.3.2 Ambiguity for a given solution of the classical master equation

Let $R \in X^1$ be a solution of the classical master equation. Throughout this paragraph, we assume

Assumption 57 We have $H^1(X, d_R) = 0$. Thus, $H^1(A, D_0) = 0$.

Let

$$r = q(R) + \hbar q(R_1) + \cdots,$$
$$r' = q(R) + \hbar q(R_1') + \cdots$$

be two solutions to the quantum master equation, so that $r \equiv r' \pmod{\hbar}$.

Lemma 58 *Let $n \in \mathbb{N}_0$. Assume that for $l = 1, \ldots, n$, we have $R_l = R_l'$. Then there exists a generator $c \in (\hbar^{n+1}) \subset K$ of a quantum gauge equivalence such that $\exp ad_c\, r \equiv r' \pmod{(\hbar^{n+2})}$.*

Proof Let $v = q(R_{n+1}) - q(R_{n+1}') \in G^1$. Then, $0 = [r + r', r - r']$ since r, r' solve the quantum master equation. Moreover, $r - r' \equiv \hbar^{n+1} v \pmod{\hbar^{n+2}}$. Hence,

$$0 = \frac{1}{\hbar}[r + r', v + \hbar \cdots] \equiv \frac{1}{\hbar}[2q(R), v] \equiv 2Dv \pmod{\hbar}.$$

Thus, $D_0 \pi v = 0$. Hence by Assumption 57, $\pi v = D_0 \pi u$ for some $u \in G_\hbar^0$, so $v \equiv Du \pmod{\hbar}$. Since $v \in G^1$ is constant in $\hbar$, we may also assume that $u \in G^0$. Set $c = \hbar^{n+1} u \in K$. We check that

$$\exp \mathrm{ad}_c\, r - r' = r - r' + \frac{1}{\hbar}[c, r] + \sum_{l=2}^{+\infty} \frac{1}{l!} \mathrm{ad}_c^l\, r$$
$$\equiv \hbar^{n+1}\left(v - \frac{1}{\hbar}[r, u]\right) \equiv \hbar^{n+1}(v - Du) \equiv 0 \pmod{(\hbar^{n+2})}.$$

□

Theorem 59 *Under Assumption 57, there is a quantum gauge equivalence mapping r to r'.*

Proof By Lemma 58, there exists a sequence of generators $c_j \in (\hbar^{j+1}) \subset K$ of quantum gauge equivalences $\exp \mathrm{ad}_{c_j}$ which define a sequence $r_{(j)}$ of solutions of the quantum master equation via $r_{(0)} = r$ and $r_{(j+1)} = \exp \mathrm{ad}_{c_j} r_{(j)}$, so that $r_{(j+1)} \equiv r' \pmod{\hbar^{j+2}}$. We have

$$r_{(j+1)} = \exp \mathrm{ad}_{c_j} \exp \mathrm{ad}_{c_{j-1}} \cdots \exp \mathrm{ad}_{c_0}\, r = \exp \mathrm{ad}_{\gamma_j}\, r,$$

for some $\gamma_j \in K$. We are left to show that $\gamma_j \to \gamma \in K$ and $\exp \mathrm{ad}_\gamma r = r'$. By the Campbell–Baker–Hausdorff formula, we have $\gamma_0 = c_0$ and $\gamma_{j+1} = \gamma_j + c_{j+1} + \cdots$, where the terms we have dropped involve sums of nested commutators $\frac{1}{\hbar}[-,-]$, each of which contain at least one $c_{j+1} \in (\hbar^{j+2})$. Hence, $\gamma_{j+1} \equiv \gamma_j \pmod{\hbar^{j+2}}$ and thus $\lim \gamma_j = \gamma \in K$ exists.

Finally, fix $k \in \mathbb{N}_0$. We will prove that $p_k(\exp \mathrm{ad}_\gamma r - r') = 0$. We already know that $p_k(\exp \mathrm{ad}_{\gamma_{k-1}} r - r') = 0$, since $\exp \mathrm{ad}_{\gamma_{k-1}} r = r_{(k)}$. Hence, it suffices to prove that $p_k(\exp \mathrm{ad}_{\gamma_{k-1}} r - \exp \mathrm{ad}_\gamma r) = 0$. We show by induction in $l \in \mathbb{N}_0$ that $\mathrm{ad}^l_{\gamma_{k-1}} r \equiv \mathrm{ad}^l_\gamma r \pmod{\hbar^{k+1}}$. The case $l = 0$ is trivial. Now, suppose the statement holds for some $l \in \mathbb{N}_0$. Then,

$$\begin{aligned}\mathrm{ad}^{l+1}_{\gamma_{k-1}} r - \mathrm{ad}^{l+1}_\gamma r &= \mathrm{ad}_{\gamma_{k-1}}(\mathrm{ad}^l_{\gamma_{k-1}} r - \mathrm{ad}^l_\gamma r + \mathrm{ad}^l_\gamma r) - \mathrm{ad}^{l+1}_\gamma r \\ &\equiv \mathrm{ad}_{\gamma_{k-1}} \mathrm{ad}^l_\gamma r - \mathrm{ad}^{l+1}_\gamma r = \mathrm{ad}_{\gamma_{k-1}-\gamma} \mathrm{ad}^l_\gamma r \equiv 0 \pmod{\hbar^{k+1}},\end{aligned}$$

since $\gamma \equiv \gamma_{k-1} \pmod{\hbar^{k+1}}$. □

8.3.3 Ambiguity for two classical solutions corresponding to the same Tate resolution

Let R, R' be two solutions of the classical master equation associated with the same Tate resolution. Let

$$\begin{aligned} r &= q(R) + \hbar q(R_1) + \cdots, \\ r' &= q(R') + \hbar q(R'_1) + \cdots \end{aligned}$$

be two solutions of the quantum master equation.

Theorem 60 *If either of the two solutions R, R' of the classical master equation satisfy Assumption 57, then there is a quantum gauge equivalence mapping r to r'.*

Proof By Theorem 24, there exists a classical gauge equivalence $g = \exp \mathrm{ad}_u$ mapping R to R'. In particular, Assumption 57 is satisfied for both solutions. We have $c = q(u) \in K$. Set $r'' = \exp \mathrm{ad}_c r$. It is a solution of the quantum master equation. We first prove that $r'' \equiv r' \pmod{\hbar}$. We have

$$r'' - r' = \exp \mathrm{ad}_c r - r' \equiv \exp \mathrm{ad}_{q(u)} q(R) - q(R') \pmod{\hbar}.$$

Now, we prove by induction in $l \in \mathbb{N}_0$ that $\mathrm{ad}^l_{q(u)} q(R) \equiv q(\mathrm{ad}^l_u R) \pmod{\hbar}$. For $l = 0$, this is obvious. Suppose it holds for some $l \in \mathbb{N}_0$. Then, by Eq. 3,

$$\mathrm{ad}^{l+1}_{q(u)} q(R) = \mathrm{ad}_{q(u)} \mathrm{ad}^l_{q(u)} q(R) \equiv \mathrm{ad}_{q(u)} q(\mathrm{ad}^l_u R) \equiv q(\mathrm{ad}^{l+1}_u R) \pmod{\hbar}.$$

We now have

$$\sum_{l=0}^{L} \frac{1}{l!} \mathrm{ad}^l_{q(u)} q(R) - q(R') \equiv q\left(\sum_{l=0}^{L} \frac{1}{l!} \mathrm{ad}^l_{c_0} R - R'\right) \pmod{\hbar}.$$

For $L \to +\infty$, the left hand side converges to $\exp \operatorname{ad}_{q(u)} q(R) - q(R')$, and the argument of q on the right hand side converges to zero. By continuity of q and $(\hbar)$ being closed, we conclude that $r'' \equiv r' \pmod{\hbar}$.

We are now in the situation

$$\begin{aligned} r &= q(R) + \hbar q(R_1) + \cdots, \\ r' &= q(R') + \hbar q(R'_1) + \cdots, \\ r'' &= \exp \operatorname{ad}_c r = q(R') + \hbar q(R''_1) + \cdots. \end{aligned}$$

By Theorem 59, there exists a quantum gauge equivalence $\exp \operatorname{ad}_v$ with $\exp \operatorname{ad}_v r'' = r'$. In particular, $r' = \exp \operatorname{ad}_v \exp \operatorname{ad}_c r$. □

8.3.4 Quantization of trivial BRST models

Let (Y, S) be a trivial BRST model, so $Y = \operatorname{Sym}(\mathcal{N} \oplus \mathcal{N}^*)$ for some negatively graded vector space $\mathcal{N}$ with finite-dimensional homogeneous components and $S = \sum_j e_j^* \delta(e_j)$ with $\delta(e_j) = e_k$ for some k depending on j. Let $q : Y \to G$ denote the quantization map. Then, $s = q(S)$ solves the quantum master equation. Moreover, $D_s = \frac{1}{\hbar}[s, -]$ maps G_0 to G_0 and G to G, as can be seen using the Leibnitz rule. Hence, both $q_0 : Y_0 \to G_0$ and $Y \to G$ are isomorphisms of differential graded vector spaces, in particular, $H^j(G_0, D_s) = 0$ for $j \neq 0$ and $H^0(G_0, D_s) = \mathbb{R}$ by Lemma 27.

8.3.5 Quantization of products with trivial BRST models

Let (X, R) be a BRST model and (Y, S) a trivial BRST model. Consider quantizations $q : X \to F \subset F_\hbar$ and $q : Y \to G \subset G_\hbar$ with associated solutions of the quantum master equation $s = q(S)$ and $r = q(R) + \hbar \cdots$, respectively. Write $F_0 = \operatorname{Sym}(\mathcal{N}) \otimes \operatorname{Sym}(\mathcal{N}*)$ for some non-positively graded vector space $\mathcal{N}$ and $G_0 = \operatorname{Sym}(\mathcal{U}) \otimes \operatorname{Sym}(\mathcal{U}^*)$ for some negatively graded vector space $\mathcal{U}$. Let $Z = X \hat{\otimes} Y$ and $q : Z \to H \subset H_\hbar$ be the quantization obtained from the splitting $H_0 = \operatorname{Sym}(\mathcal{N} \oplus \mathcal{U}) \otimes \operatorname{Sym}(\mathcal{N}^* \oplus \mathcal{U}^*)$.

Lemma 61 *The natural map $F_\hbar \to H_\hbar$ is a quasi-isomorphism of graded associative algebras.*

Proof The natural map is a morphism of graded associative algebras, since adding new variables from $\mathcal{U}$ and $\mathcal{U}^*$ does not change the rules defining normal ordering in $F_\hbar$.

Consider the isomorphism $\phi_0 : F_0 \otimes G_0 \to H_0$ of graded vector spaces. It extends $\hbar$-linearly to an isomorphism

$$\phi : F_0[\hbar] \otimes_{\mathbb{R}[\hbar]} G_0[\hbar] \to H_0[\hbar]$$

of graded associative algebras. On the left hand side, we may take the tensor product of algebras, since elements of F_0 and G_0 commute.

Using this isomorphism, we construct the $\hbar$-linear maps

$$\begin{aligned}\iota : & F_0[\hbar] \to F_0[\hbar] \otimes_{\mathbb{R}[\hbar]} G_0[\hbar] \to H_0[\hbar],\\ p : & H_0[\hbar] \to F_0[\hbar] \otimes_{\mathbb{R}[\hbar]} G_0[\hbar] \to F_0[\hbar],\end{aligned}$$

where the last arrow takes $f \otimes g \in F_0 \otimes G_0$ to $f\pi(g)$. Here, $\pi : G_0 \to G_0$ is the projection onto $\mathbb{R}$ along elements of nonzero form degree.

The quantization map $q : Y_0 \to G_0$ is compatible with the decompositions $Y_0 = \mathbb{R} \oplus \bigoplus_{j>0} \mathrm{Sym}^j(\mathcal{U} \oplus \mathcal{U}^*)$ and $G_0 = \mathbb{R} \oplus \bigoplus_{p+q>0} \mathrm{Sym}^p(\mathcal{U}) \otimes \mathrm{Sym}^q(\mathcal{U})$. Moreover, it intertwines the differentials d_S on Y_0 and D_s on G_0. Hence, the situation of the proof of Lemma 29 is established in the quantum version as well. We conclude that ι and p extend $\hbar$-linearly to the respective completions, descend to cohomology, and induce mutual inverses on cohomology. □

8.3.6 Relating arbitrary BRST models

To relate the quantizations of two BRST charges defining BRST models for the same ideal, we need to quantize general automorphisms of the space X that are the identity on $P = \mathbb{R}[V]$ modulo I (see Lemma 31). Since the quantization procedure is not functorial, we do not directly obtain an isomorphism of differential graded algebras on the quantum level. We were unable to rigorously define a quantum analog to such an isomorphism.

Acknowledgements I am very grateful to my supervisor Giovanni Felder for many stimulating discussions and mathematical support. I also want to thank the referee for valuable comments on the first version of the manuscript. This research was partially supported by the NCCR SwissMAP, funded by the Swiss National Science Foundation.

Appendix A: Graded Poisson algebras

Let $(P, [-,-]_0)$ be a unital Poisson algebra over $\mathbb{K} = \mathbb{R}$. Let $\mathcal{M}$ be a negatively graded vector space with finite-dimensional homogeneous components. Let $\mathcal{M}^*$ be the positively graded vector space with homogeneous components $(M^*)^i = (M^{-i})^*$. Define the graded algebra $X_0 = P \otimes \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$.

Lemma 62 *The bracket $\{-,-\}_0$ on P naturally extends to a skew-symmetric, bilinear map $\{-,-\}$ on X_0 via the natural pairing of $\mathcal{M}$ and $\mathcal{M}^*$. This map has degree zero. Moreover, it is a derivation for the product on X_0 and satisfies the Jacobi identity. Thus, it turns X_0 into a graded Poisson algebra.*

Proof First, we define a bracket on $\mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$. For $x \in \mathcal{M}$ and $\alpha \in \mathcal{M}^*$, we set

$$\{x, x\}_1 = 0, \quad \{\alpha, \alpha\}_1 = 0, \quad \{x, \alpha\}_1 = \alpha(x), \quad \{\alpha, x\}_1 = -(-1)^{\deg\alpha \deg x}\alpha(x)$$

and extend this definition as a bi-derivation to all of $\mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$. It is then a bilinear, skew-symmetric map $\{-,-\}_1 : \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*) \times \mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*) \to$

$\mathrm{Sym}(\mathcal{M} \oplus \mathcal{M}^*)$ of degree zero which is by definition a derivation for the product. The expression

$$\zeta(a, b, c) := (-1)^{\deg a \deg c}\{a, \{b, c\}\} + \text{ cyclic permutations}$$

satisfies $\zeta(a_1a_2, b, c) = (-1)^{\deg a_1 \deg c} a_1\zeta(a_2, b, c) + (-1)^{\deg a_2 \deg b}\zeta(a_1, b, c)a_2$ and similar derivation-like statements for the other entries. Let $\{e_j\}$ be a homogeneous basis of $\mathcal{M}$ and $\{e_j^*\}$ its dual basis. Then the bracket of any two of those generators is a scalar, whence ζ is zero on generators. By the above derivation-type property, ζ vanishes identically, proving the graded Jacobi identity.

Now, set $\{-, -\} = \{-, -\}_0 + \{-, -\}_1$ on X_0. □

The grading on $\mathcal{M}$ induces a grading on X_0. We obtain a descending filtration indexed by nonnegative integers n: $\mathcal{F}^n X_0$ is defined to be the ideal generated by elements of X_0 of degree at least n. Note that $\mathcal{F}^0 X_0 = X_0$ is the whole algebra. We set $\mathcal{F}^n X_0^m = \mathcal{F}^n X_0 \cap X_0^m$. We also define $I_0 := \mathcal{F}^1 X_0$ and $I_0^{(n)} := I_0 \cdots I_0$ to be the n-fold product ideal.

A.1: Compatibility of filtration and bracket on X_0

We use the derivation properties of the bracket on X_0 to derive the compatibility relations between the filtration and the bracket.

Lemma 63 *For $m, n \in \mathbb{Z}$ and $p, q \in \mathbb{N}_0$, we have $\{\mathcal{F}^p X_0^n, \mathcal{F}^q X_0^m\} \subset \mathcal{F}^{r_{n,m}(p,q)} X_0^{m+n}$ where*

$$r_{n,m}(p, q) = \max\{m + n, \min\{\max\{p, q + n\}, \max\{q, p + m\}\}\}. \tag{5}$$

Proof Let $a, b, u, v \in X_0$ be homogeneous elements with $\deg a + \deg u = n$, $\deg b + \deg v = m$, $\deg u = p$, and $\deg v = q$. Suppose without loss of generality that $p \geqslant n$ and $q \geqslant m$. Using the Leibniz rule we see that $\{au, bv\}$ is in $\mathcal{F}^r X_0$, where $r = \min\{p+q, \max\{p, q+n\}, \max\{q, p+m\}\} = \min\{\max\{p, q+n\}, \max\{q, p+m\}\}$. □

Corollary 64 *We obtain for $p, q \in \mathbb{N}_0$ and $m, n \in \mathbb{Z}$,*

(1) $\{\mathcal{F}^p X_0^1, \mathcal{F}^q X_0^1\} \subset \mathcal{F}^{l(p,q)} X_0$, *where* $l(p, q) = \begin{cases} \max\{p, q\}, & if p \neq q \\ p + 1, & if p = q \end{cases}$.

(2) $\{\mathcal{F}^p X_0, X_0^m\} \subset \mathcal{F}^p X_0$ *provided* $m \geqslant 0$.

(3) $\{\mathcal{F}^p X_0^n, X_0^m\} \cup \{X_0^n, \mathcal{F}^p X_0^m\} \subset \mathcal{F}^{t_{n,m}(p)} X_0^{n+m}$, *where* $t_{n,m}(p) = p - \max\{|n|, |m|\}$.

Lemma 65 *We have* $\{X_0^1 \cap I_0^{(2)}, \mathcal{F}^m X_0\} \subset \mathcal{F}^{m+1} X_0$.

Proof Let $a, u_1, u_2 \in X_0$ with $\deg(a) = 1 - n$, $\deg(u_1) + \deg(u_2) = n$, and $\deg(u_1), \deg(u_2) > 0$. Let $b, v \in X_0$ with $\deg(v) = m$. Expand $\{au_1u_2, bv\}$ using the Leibniz rule. □

Lemma 66 *The ideal I_0 is closed under the bracket.*

Proof Let $a, u, b, v \in X_0$ with $\deg(u) = \deg(v) = 1$. Expand $\{au, bv\}$ using the Leibniz rule. □

A.2: Completion

For each j, we use the filtration on X_0^j to complete this space to the space

$$X^j = \lim_{\leftarrow p} \frac{X_0^j}{\mathcal{F}^p X_0 \cap X_0^j}.$$

The sum and scalar multiplication on X_0^j extend to this space, turning $X = \bigoplus_j X^j$ into a graded vector space. The product of two elements $(x_p + \mathcal{F}^p X_0^j)_p \in X^j$ and $(y_p + \mathcal{F}^p X_0^k)_p \in X^k$ is defined to be $(x_p y_p + \mathcal{F}^p X_0^{j+k})_p \in X^{j+k}$. This definition does not depend on the choice of representatives, since the product is compatible with the filtration. Moreover, it defines an element of X^{j+k} since for $p \leqslant q$ we have $x_p y_p \equiv x_q y_q \pmod{\mathcal{F}^p X_0^{j+k}}$, and we may shift the representatives of x and y. The multiplication is compatible with the addition turning X into a graded commutative algebra.

Endow $X_0^j / \mathcal{F}^p X_0^j$ with the discrete topology and $\prod_p X_0^j / \mathcal{F}^p X_0^j$ with the product topology. Equip $\lim_{\leftarrow} X_0^j / \mathcal{F}^p X_0^j \subset \prod_p X_0^j / \mathcal{F}^p X_0^j$ with the subspace topology. Finally, equip $X = \bigoplus_j X^j$ with the product topology. Hence, a sequence $\{x_l\}_l \subset X^j$, with $x_l = (x_{l,p} + \mathcal{F}^p X_0^j)_p$, converges to an element $x = (x_p + \mathcal{F}^p X_0^j)_p \in X^j$ if and only if for all $p \in \mathbb{N}_0$ there exists a l_0 such that for all $l \geqslant l_0$ we have $x_{p,l} \equiv x_p \pmod{\mathcal{F}^p X_0^j}$. A sequence $\{x_l\}_l \subset X$ converges to an element $x \in X$ if and only if all homogeneous components converge. Since X is first-countable, continuity is characterized by the convergence of sequences. We immediately obtain:

Lemma 67 *The sum $X \times X \to X$ is continuous.*

For the product, only a weaker statement holds in general:

Lemma 68 *The product $X \to X$ is continuous in each entry. For each pair $(j, k) \in \mathbb{Z}^2$, the product $X^j \times X^k \to X^{j+k}$ is continuous.*

Proof Consider a sequence $\{x_i\}_i$ in X converging to $x \in X$ and fix $y \in X$. Denote the homogeneous components of x_i by $x_i^j = (x_{i,p}^j + \mathcal{F}^p X_0^j)_p$ and similarly for x and y. Fix $l \in \mathbb{Z}$ and $p \in \mathbb{N}_0$. The l-th homogeneous component of $x_i y$ has a p-th component with representative $\sum_{j \in C} x_{i,p}^{l-j} y_p^j$ where the $C \subset \mathbb{Z}$ is the finite set for which $y^j \neq 0$. It does not depend on i. (Such a finite set which is independent of i only exists in general when one entry of the product remains fixed.) We have

$$\sum_{j \in C} x_{i,p}^{l-j} y_p^j = \sum_{j \in C} \Big((x_{i,p}^{l-j} - x_p^{l-j}) y_p^j + x_p^{l-j} y_p^j \Big).$$

For each $j \in C$, pick a number $i_{0,j}$ such that for $i_j \geqslant i_{0,j}$ we have $x_{i_j,p}^{l-j} \equiv x_p^{l-j} \pmod{\mathcal{F}^p X_0^{l-j}}$ and let i_0 be their maximum. Now, for $i \geqslant i_0$, we have $\sum_{j \in C} x_{i,p}^{l-j} y_p^j \equiv \sum_{j \in C} x_p^{l-j} y_p^j \pmod{\mathcal{F}^p X_0^l}$. The second statement follows similarly. □

Next, we approximate elements in X by elements in X_0.

Lemma 69 *The map* $\iota : X_0^n \longrightarrow X^n$ *sending* $x \in X_0^n$ *to* $(x + \mathcal{F}^p X_0^n)_p$ *is injective.*

Proof Since ι is linear, the claim follows from $\bigcap_p \mathcal{F}^p X_0^n = 0$. □

Lemma 70 *For* $x = (x_p + \mathcal{F}^p X_0^n)_p \in X_0^n$, *we have* $\lim_{m \to \infty} \iota(x_m) = x$.

Proof Fix p. For $m \geqslant p$, we have that the p-th component of $\iota(x_m) - x$ is $x_m - x_p \in \mathcal{F}^p X_0^n$. □

Corollary 71 X_0 *can be considered a dense subset of* X.

Now, we turn to the extension of the bracket to the completion. Let $x = (x_p + \mathcal{F}^p X_0^j)_p \in X^j$ and $y = (y_p + \mathcal{F}^p X_0^k)_p \in X^k$. We define $\{x, y\} \in X^{j+k}$ to be the element

$$(\{x_{s_{j,k}(p)}, y_{s_{j,k}(p)}\} + \mathcal{F}^p X_0^{j+k})_p,$$

where $s_{j,k}(p) := p + \max\{|j|, |k|\}$. This definition does not depend on the representatives of x and y by Corollary 64, since $t_{j,k}(s_{j,k}(p)) = s_{j,k}(t_{j,k}(p)) = p$. Moreover, it defines an element of X^{j+k}: for $p \leqslant q$ we have $\{x_{s_{j,k}(p)}, y_{s_{j,k}(p)}\} \equiv \{x_{s_{j,k}(q)}, y_{s_{j,k}(q)}\} \pmod{\mathcal{F}^{s_{j,k}(p)}}$, since we may shift the representatives of x and y. We extend this bracket as a bilinear map to $X \times X$.

Lemma 72 *The extension of the bracket on* X_0 *is a skew-symmetric, bilinear degree zero map on* X *that satisfies the graded Jacobi identity* (*i.e. the bracket is an odd derivation for itself*).

Proof It is trivial that the extended bracket is a skew-symmetric, bilinear degree zero map. These properties follow directly from the definitions.

We prove the graded Jacobi identity. Consider elements

$$x = (x_p + \mathcal{F}^p X_0^j)_p \in X^j \quad y = (y_p + \mathcal{F}^p X_0^k)_p \in X^k \quad z = (z_p + \mathcal{F}^p X_0^l)_p \in X^l.$$

The p-th element of $\{y, z\}$ has representative $\{y_{s_{k,l}(p)}, z_{s_{k,l}(p)}\}$. Hence, the p-th element of $\{x, \{y, z\}\}$ has representative $\{x_{s_{j,k+l}(p)}, \{y_{s_{k,l}(s_{j,k+l}(p))}, z_{s_{k,l}(s_{j,k+l}(p))}\}$. We now want to bound the indices from above by a term which is invariant under cyclic permutations of (j, k, l). The function $r_{j,k,l}(p) := p + 2(|j| + |k| + |l|)$ does the job. So, $(-1)^{jl}\{x, \{y, z\}\} + (-1)^{kj}\{y, \{z, x\}\} + (-1)^{lk}\{z, \{x, y\}\}$ has representative

$$(-1)^{jl}\{x_{r_{j,k,l}(p)}, \{y_{r_{j,k,l}(p)}, z_{r_{j,k,l}(p)}\}\} + \text{cyclic permutations}$$

which vanishes by the graded Jacobi identity on X_0. □

Lemma 73 *The bracket on X is a derivation for the product.*

Proof Let

$$x = (x_p + \mathcal{F}^p X_0^j)_p \in X^j \quad y = (y_p + \mathcal{F}^p X_0^k)_p \in X^k \quad z = (z_p + \mathcal{F}^p X_0^l)_p \in X^l.$$

The p-th element of $\{xy, z\} - (x\{y, z\} + (-1)^{jk} y\{x, z\})$ has representative

$$\begin{aligned} &\{x_{s_{j+k,l}(p)} y_{s_{j+k,l}(p)}, z_{s_{j+k,l}(p)}\} - \left(x_p\{y_{s_{k,l}(p)}, z_{s_{k,l}(p)}\} + (-1)^{jk} y_p\{x_{s_{j,l}(p)}, z_{s_{j,l}(p)}\}\right) \\ &\equiv \{x_q y_q, z_q\} - (x_q\{y_q, z_q\} + (-1)^{jk} y_q\{x_q, z_q\}) \pmod{\mathcal{F}^p X_0^{j+k+l}}, \end{aligned}$$

where $q := p + |m| + |n| + |k|$ is a common upper bound of all indices appearing in the formula. The last line vanishes by the derivation property of the bracket on X_0. □

Lemma 74 *For each pair $(j, k) \in \mathbb{Z}^2$, the map $\{-, -\} : X^j \times X^k \to X$ is continuous. The map $\{-, -\} : X \times X \to X$ is continuous in each entry.*

Proof Let $x_n = (x_{n,p} + \mathcal{F}^p X_0^j)_p \in X^j$ and $y_n = (y_{n,p} + \mathcal{F}^p X_0^k)_p \in X^k$ define two sequences converging to the respective elements $x = (x_p + \mathcal{F}^p X_0^j)_p \in X^j$ and $y = (y_p + \mathcal{F}^p X_0^k) \in X^k$. Fix p, set $s = s_{j,k}(p)$, and pick n_0 such that for $n \geqslant n_0$,

$$x_{n,s} \equiv x_s \pmod{\mathcal{F}^s X_0^j} \qquad y_{m,s} \equiv y_s \pmod{\mathcal{F}^s X_0^k}.$$

Let $n \geqslant n_0$. The p-th element of $\{x_n, y_n\} - \{x, y\}$ has the representative $\{x_{n,s}, y_{n,s}\} - \{x_s, y_s\} \in \mathcal{F}^{t_{j,k}(s)} \subset \mathcal{F}^p X_0^{j+k}$ by Corollary 64.

Now, consider a sequence $\{x_i\}_i$ in X converging to $x \in X$ and fix $y \in X$. Denote the homogeneous components of x_i by $x_i^j = (x_{i,p}^j + \mathcal{F}^p X_0^j)_p$ and similarly for x and y. Fix $l \in \mathbb{Z}$ and $p \in \mathbb{N}_0$. Set $C = \{j \in \mathbb{Z} : y^j \neq 0\}$. This is a finite set. The l-th homogeneous component of $\{x_i, y\}$ has a p-th component with representative

$$\sum_{j \in C} \{x_{i, s_{l-j,j}(p)}^{l-j}, y_{s_{l-j,j}(p)}^j\}.$$

Set $s = \max\{s_{l-j,j}(p) : j \in C\}$ and pick n_0 such that for $n \geqslant n_0$ and all $j \in C$, we have $x_{n,s}^{l-j} \equiv x_s^{l-j} \pmod{\mathcal{F}^s X_0^{l-j}}$. For such n,

$$\sum_{j \in C} \{x_{n, s_{l-j,j}(p)}^{l-j}, y_{s_{l-j,j}(p)}^j\} \equiv \sum_{j \in C} \{x_{n,s}^{l-j}, y_s^j\} \equiv \sum_{j \in C} \{x_s^{l-j}, y_s^j\} \pmod{\mathcal{F}^p X_0^l}.$$

□

A.3: The filtration and the bracket on the completion

The filtration on X_0 induces a descending filtration on the completion with homogeneous components

$$\mathcal{F}^p X^n := \lim_{\leftarrow q} \frac{\mathcal{F}^p X_0^n}{\mathcal{F}^{p+q} X_0^n} = \{(x_q + \mathcal{F}^q X_0^n)_{q \geqslant p} \in X^n : x_q \in \mathcal{F}^p X_0^n\}.$$

This defines a homogeneous ideal $\mathcal{F}^p X = \bigoplus_n \mathcal{F}^p X^n$ in X. Here, p is a nonnegative integer and again $\mathcal{F}^0 X = X$. We set $I = \mathcal{F}^1 X$ and

$$I^{(n)} = \bigoplus_m \lim_{\leftarrow p} \frac{X_0^m \cap I_0^{(n)}}{\mathcal{F}^p X_0^m \cap I_0^{(n)}}.$$

Those are homogeneous ideals in X.

Lemma 75 *For each $j \in \mathbb{Z}$, the sets $\mathcal{F}^p X^j$ and $I^{(2)} \cap X^j$ are closed.*

Proof Consider the first statement. Since X is first-countable, it suffices to consider sequences $x_n = (x_{n,q} + \mathcal{F}^q X_0^j)_q$ converging to an $x = (x_q + \mathcal{F}^q X_0^j)_q$ in X with $x_{n,q} \in \mathcal{F}^p X_0^j$ and show that $x \in \mathcal{F}^p X^j$. So, fix $q \geqslant p$. Let n be an integer with $x_{n,q} \equiv x_q \pmod{\mathcal{F}^q X_0^j}$. Then, $x_q \equiv x_{n,q} \equiv 0 \pmod{\mathcal{F}^p X_0^j}$.

Now, let $x_n = (x_{n,p} + \mathcal{F}^p X_0^j)_p$ be a sequence converging to $x = (x_p + \mathcal{F}^p X_0^j)_p$ in X with $x_{n,p} \in I_0^{(2)}$. Fix p. For n large enough, we may replace x_p by $x_{n,p} \in I_0^{(2)}$. □

Lemma 76 *Fix $p \in \mathbb{N}_0$.*

(1) $\{I^{(2)}, I^{(2)}\} \subset I^{(2)}$.
(2) $\{I^{(2)}, I^{(p)}\} \subset I^{(p+1)} \subset \mathcal{F}^{p+1} X$.
(3) $\{I^{(p)}, X^1\} \subset I^{(p)}$.

Proof The first statement: Consider elements $u = (u_p + \mathcal{F}^p X_0^j)_p$ and $v = (v_p + \mathcal{F}^p X_0^k)_p$ of X with $u_p, v_p \in I_0^{(2)}$. Then the p-th element of $\{u, v\}$ has the representative $\{u_{s_{j,k}(p)}, v_{s_{j,k}(p)}\} \in \{I_0^{(2)}, I_0^{(2)}\} \subset I_0^{(2)}$ by the Leibniz rule.

Now, the second statement: First consider $p = 0$. Then by the Leibniz rule, $\{I_0^{(2)}, X_0\} \subset I_0\{I_0, X_0\} \subset I_0$. Now consider $p > 0$. By repeated use of the Leibniz rule,

$$\{I_0^{(2)}, I_0^{(p)}\} \subset \{I_0^{(2)}, I_0\} I_0^{(p-1)} \subset I_0\{I_0, I_0\} I_0^{(p-1)} \subset I_0^{(p+1)}$$

by Lemma 66. The statement generalizes to the completion, as in the case above.

The third statement follows analogously by picking representatives. □

Lemma 77 *Let $l \mapsto q(l)$ define an unbounded non-decreasing function $\mathbb{N} \to \mathbb{N}$. Let $x_l = (x_{l,p} + \mathcal{F}^p X_0^n)_p \in \mathcal{F}^{q(l)} X^n$ define a sequence of elements in X^n. Then, $\sum_{l=0}^{\infty} x_l$ converges to an element $x \in X^n$.*

Proof We may suppose $q(l) = l$. Define $x_p := \sum_{l=0}^{p-1} x_{l,p}$. Then, $x := (x_p + \mathcal{F}^p X_0^n)_p$ defines an element of X^n since, for $p \leqslant q$, we have

$$x_q - x_p = \sum_{l=0}^{q-1} x_{l,q} - \sum_{l=0}^{p-1} x_{l,p} = \sum_{l=0}^{p-1} (x_{l,q} - x_{l,p}) + \sum_{l=p}^{q-1} x_{l,q} \in \mathcal{F}^p X_0^n.$$

We claim that $\sum_{l=0}^{k} x_l$ converges to x as $k \to \infty$. Fix p. Let $k \geqslant k_0 := p$. Then the p-th element of $\sum_{l=0}^{k} x_l - x$ has representative $\sum_{l=0}^{k} x_{l,p} - x_p = \sum_{l=p}^{k} x_{l,p} \in \mathcal{F}^p X_0^n$. □

Lemma 78 *Each $H \in X^n$ can be expanded as $H = \sum_{p \geq 0} h_p$ with $h_p \in B^p \otimes_P T^{n-p}$.*

Proof Write $H = (x_p + \mathcal{F}^p X_0^n)_p$ with $x_0 = 0$. Pick a homogeneous basis e_i of the underlying graded vector space. Redefine x_p such that x_p does not contain a monomial in $\mathcal{F}^p X_0^n$. Set $h_p = x_{p+1} - x_p \in \mathcal{F}^p X_0^n$. It cannot contain a monomial of degree $(p+1)$ or higher. Hence, $h_p \in B^p \otimes_P T^{n-p}$. Then by Lemmas 70 and 77, $\sum_p h_p = H$. □

Lemma 79 *All statements from Sect. A.1 in Appendix are valid for X_0 replaced by X.*

Proof The bracket on X is defined by acting on representatives with the bracket of X_0 where the statements hold.

A.4: Extension of maps

Next, we consider the problem of extending maps on X_0 to X.

Remark 80 A linear map on X_0 of a fixed degree preserving the filtration up to a fixed shift naturally extends to a linear map on X preserving the filtration up to the same shift. This extension is continuous.

A.5: The associated graded

The associated graded gr X of X is defined as the graded algebra with homogeneous components $\mathrm{gr}^p X = \mathcal{F}^p X \big/ \mathcal{F}^{p+1} X$. We have $\mathrm{gr}^0 X = X/I$.

Lemma 81 *X/I is naturally identified with $P \otimes Sym(\mathcal{M})$.*

Proof Let $x = (x_p + \mathcal{F}^p X_0^n)_p \in X^n$. Let $u_p \in I_0$ and $z_p \in X_0^n$ such that $x_p = u_p + z_p$ and z_p does not contain a summand in I_0 (or is zero), i.e. $z_p \in \mathrm{Sym}_P(\mathcal{M})$. Then, $z_p - z_1 = x_p - x_1 - (u_p - u_1) \in I_0$. Hence, $z_p = z_1$ for all p. Therefore, $z := (z_1 + \mathcal{F}^p X_0^n)_p \in X^n$ and x define the same equivalence class in X/I. It is clear that different values of z_1 yield different equivalence classes. □

Lemma 82 *There is a natural isomorphism $gr^\bullet X \cong B^\bullet \otimes_P T$ of graded algebras.*

Proof The inclusions $B^p \hookrightarrow \mathcal{F}^p X$ induce a P-linear map $B \longrightarrow \operatorname{gr} X$. From this, we obtain a map $B \otimes_P T \rightarrow \operatorname{gr} X$ via $B^p \otimes_P T \ni b \otimes t \mapsto bt \in \operatorname{gr}^p X$. The claim follows since the monomials in B^p span the free T-module $\operatorname{gr}^p X$: The image of linearly independent monomials in B^p under the above map is obviously T-linearly independent. Now for a given $x \in \mathcal{F}^p X$, decompose it into homogeneous elements $x^n = (x_{n,q} + \mathcal{F}^q X_0^n)_q \in \mathcal{F}^p X^n$. Split $x_{n,q} = b_{n,q} + y_{n,q}$ with $y_{n,q} \in \mathcal{F}^{p+1} X_0^n$ and $b_{n,q} \in \mathcal{F}^p X_0^n$ does not contain a summand in $\mathcal{F}^{p+1} X_0^n$. Then $b_{n,q} - b_{n,p+1} \in \mathcal{F}^{p+1} X_0^n$ for $q > p$, and hence this difference vanishes. Set $b^n = (b_{n,p+1} + \mathcal{F}^q X_0^n)_q$. We have that b^n and x^n define the same equivalence class in $\operatorname{gr}^p X^n$ and hence $b = \sum_n b^n$ and x define the same equivalence class in $\operatorname{gr}^p X$. Each b^n is in the image of $B^p \otimes_P T \rightarrow \operatorname{gr}^p X$. □

A.6: Form degree

We can filter the algebra X by form degree. For $n \in \mathbb{Z}$ and $j \in \mathbb{N}_0$, we set $X_0^{n,j} = P \otimes \operatorname{Sym}^j(\mathcal{M} \oplus \mathcal{M}^*) \cap X_0^n$ and define the homogeneous components of $X^{(j)} = \bigoplus_n X^{n,j}$ to be

$$X^{n,j} = \lim_{\leftarrow p} \frac{X_0^{n,j}}{\mathcal{F}^p X_0^n \cap X_0^{n,j}}.$$

We have

Lemma 83 *If $x_j \in X^n$ have form degree j, then $\sum_j x_j$ converges in X.*

Proof Fix n. Let g denote the ghost degree and a the anti-ghost degree. This means that $g = \deg$ on (homogeneous elements in) $P \otimes \operatorname{Sym}(\mathcal{M}^*)$ and $g = 0$ on $\operatorname{Sym}(\mathcal{M})$. Similarly, $a = \deg$ on $P \otimes \operatorname{Sym}(\mathcal{M})$ and $a = 0$ on $\operatorname{Sym}(\mathcal{M}^*)$. Hence $g \geqslant 0$, $a \leqslant 0$ and $a + g = \deg$. We decompose a summand $s \in X_0^{n,j}$ of a representative of x_j as $s = a_j \otimes x_{j,1} \ldots x_{j,j} \in X_0^{n,j}$ according to form degree. Let l_j be the number of factors of positive degree in this decomposition. Then, $g(x_j) \geqslant l_j$ and $a(x_j) \leqslant -(j - l_j)$. Hence,

$$\begin{aligned} g(x_j) &= a(x_j) + (g(x_j) - a(x_j)) \geqslant a(x_j) + (l_j + (j - l_j)) = a(x_j) + j \\ &= n + j - g(x_j). \end{aligned}$$

So, $g(x_j) \geqslant \frac{1}{2}(n + j)$. Set $p(j) = \max\{k \in \mathbb{Z} : k \leqslant \frac{j+n}{2}\}$. We obtain $x_j \in \mathcal{F}^{p(j)} X^n$. Now apply Lemma 77. □

Lemma 84 *If $x \in X^n$, then there are $x_j \in X^n$ of form degree j with $x = \sum_j x_j$.*

Proof Write $x = \sum_l x^l$ with $x^l \in \mathcal{F}^l X_0^n$. Expand each $x^l = \sum_j x_j^l$ where the sum is finite with x_j^l being of form degree j. By Lemma 77, $x_j = \sum_l x_j^l$ converges to an element of X^n of form degree j. By Lemma 83, the sum $\sum_j x_j$ converges. We have $x = \sum_l \sum_j x_j^l = \sum_j \sum_l x_j^l$, which can be verified evaluating both sides modulo $\mathcal{F}^p$ for general p. □

Lemma 85 *For $\xi_j \in X^n$ of form degree j with $\sum_j \xi_j = 0$, we have $\xi_j = 0$.*

Proof Write $\xi_j = (\xi_{j,p} + \mathcal{F}^p X_0^n)_p$ with $\xi_{j,p} \in X_0^{n,j}$. The p-th component of $\sum_j \xi_j$ has representative $\sum_j \xi_{j,p} \in \mathcal{F}^p X_0^n$, where this sum is effectively finite by the proof of Lemma 83. We see that $\xi_{j,p} \in \mathcal{F}^p X_0^n$ by expanding in a P-basis of X_0 consisting of monomials in basis elements of the underlying vector space $\mathcal{M} \oplus \mathcal{M}^*$. □

A.7: Symplectic case

Consider the graded commutative algebra $X_0 = \mathbb{R}[x_i, y_i, e_j^{(l)}, e_j^{(l)*}]$ where x_i, y_i are of degree zero and $e_j^{(l)}$ and $e_j^{(l)*}$ are of degree $-l$ and l, respectively, and there are only finitely many generators of a given degree. Define a Poisson structure by setting $\{x_i, y_i\} = \delta_{ij}$, $\{e_j^{(l)}, e_m^{(n)}\} = \delta_{jm}\delta_{ln}$ and setting all other brackets between generators to zero. We complete the space X_0 to the space X as described above. The partial derivatives

$$\frac{\partial}{\partial x_i} = -\{y_i, -\} \quad \frac{\partial}{\partial y_i} = \{x_i, -\} \quad \frac{\partial}{\partial e_j^{(l)}} = (-1)^{l+1}\{e_j^{(l)*}, -\} \quad \frac{\partial}{\partial e_j^{(l)*}} = \{e_j^{(l)}, -\}$$

are defined via the bracket and hence are all well defined on X.

Lemma 86 *Let $X_i, Y_i, E_j^{(l)}, E_j^{(l)*}$ be elements of X such that the assignments*

$$(x_i, y_i, e_j^{(l)}, e_j^{(l)*}) \mapsto (x_i, Y_i, e_j^{(l)}, E_j^{(l)*})$$
$$(x_i, y_i, e_j^{(l)}, e_j^{(l)*}) \mapsto (X_i, Y_i, E_j^{(l)}, E_j^{(l)*})$$

both define automorphisms $X \to X$ of graded commutative algebras. Then the latter map is a Poisson automorphism if there exists an element $S(x_i, Y_i, e_j^{(l)}, E_j^{(l)*}) \in X$ such that

$$\frac{\partial S}{\partial x_i} = y_i \qquad \frac{\partial S}{\partial Y_i} = X_i \qquad \frac{\partial S}{\partial e_j^{(l)}} = (-1)^l e_j^{(l)*} \qquad \frac{\partial S}{\partial E_j^{(l)*}} = E_j^{(l)}.$$

Here, the partial derivatives with respect to the new variables are defined via the chain rule.

Proof Set $\xi_i^{(0)} = x_i$ and $\xi_j^{(l)} = e_j^{(l)}$ for $l > 0$ and similarly $\eta_i^{(0)} = y_i$ and $\eta_j^{(l)} = e_j^{(l)*}$. We use capital Greek letters Ξ and H for the corresponding transformed variables. We express the bracket as

$$\{f, g\} = \sum_l (-1)^{l \deg f} \sum_j \left((-1)^l \frac{\partial f}{\partial \xi_j^{(l)}} \frac{\partial g}{\partial \eta_j^{(l)}} - \frac{\partial f}{\partial \eta_j^{(l)}} \frac{\partial g}{\partial \xi_j^{(l)}} \right),$$

since both sides define derivations which agree on generators. The sums converge by Lemma 77. We have

$$\frac{\partial S}{\partial \xi_j^{(l)}} = (-1)^l \eta_j^{(l)} \quad \text{and} \quad \frac{\partial S}{\partial H_j^{(l)}} = \Xi_j^{(l)} \quad \text{so also} \quad \frac{\partial \eta_p^{(q)}}{\partial \xi_j^{(l)}} = \frac{\partial \eta_j^{(l)}}{\partial \xi_p^{(q)}}(-1)^{q+l+ql}.$$

There are functions $f_{j,l}(\xi, \eta) = \Xi_j^{(l)}$ and $g_{j,l}(\xi, \eta) = H_j^{(l)}$ realizing the change of coordinates so that

$$f_{j,l}(\xi, \eta(\xi, H)) = \frac{\partial S}{\partial H_j^{(l)}} \qquad g_{j,l}(\xi, \eta(\xi, H)) = H_j^{(l)}.$$

We obtain in the variables $(\xi_j^{(l)}, H_j^{(l)})$,

$$\frac{\partial f_{m,n}}{\partial \xi_j^{(l)}} + \sum_{p,q} \frac{\partial \eta_p^{(q)}}{\partial \xi_j^{(l)}} \frac{\partial f_{m,n}}{\partial \eta_p^{(q)}} = \frac{\partial^2 S}{\partial \xi_j^{(l)} \partial H_m^{(n)}},$$

$$\frac{\partial g_{m',n'}}{\partial \xi_j^{(l)}} + \sum_{pq} \frac{\partial \eta_p^{(q)}}{\partial \xi_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_p^{(q)}} = 0 \qquad \sum_{pq} \frac{\partial \eta_p^{(q)}}{\partial H_m^{(n)}} \frac{\partial g_{m',n'}}{\partial \eta_p^{(q)}} = \delta_{mm'}\delta_{nn'}.$$

These expressions make sense in the completion by Lemma 77 since (j, l, n, m) are fixed and the $\eta_p^{(q)}$ derivatives are of non-decreasing and unbounded degree. Using those equalities, we calculate in the variables $(\xi_j^{(l)}, H_j^{(l)})$ the bracket $\{f_{m,n}, g_{m',n'}\}$ as

$$\begin{aligned}
&\sum_l (-1)^{ln} \sum_j ((-1)^l \frac{\partial f_{m,n}}{\partial \xi_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} - \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \xi_j^{(l)}}) \\
&= \sum_{jl} (-1)^{l(n+1)} \frac{\partial^2 S}{\partial \xi_j^{(l)} \partial H_m^{(n)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \\
&\quad - \sum_{jl} (-1)^{ln} \left(\sum_{pq} (-1)^l \frac{\partial \eta_p^{(q)}}{\partial \xi_j^{(l)}} \frac{\partial f_{m,n}}{\partial \eta_p^{(q)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} + \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \xi_j^{(l)}} \right) \\
&= \sum_{jl} (-1)^l \frac{\partial^2 S}{\partial H_m^{(n)} \partial \xi_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \\
&\quad - \sum_{jl} (-1)^{ln} \sum_{pq} \left((-1)^l \frac{\partial \eta_p^{(q)}}{\partial \xi_j^{(l)}} \frac{\partial f_{m,n}}{\partial \eta_p^{(q)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} - \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial \eta_p^{(q)}}{\partial \xi_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_p^{(q)}} \right) \\
&= \sum_{jl} \frac{\partial \eta_j^{(l)}}{\partial H_m^{(n)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}}
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{jlpq}\left((-1)^{ln+l}\frac{\partial\eta_p^{(q)}}{\partial\xi_j^{(l)}}\frac{\partial f_{m,n}}{\partial\eta_p^{(q)}}\frac{\partial g_{m',n'}}{\partial\eta_j^{(l)}}-(-1)^{ql+qn+l}\frac{\partial\eta_p^{(q)}}{\partial\xi_j^{(l)}}\frac{\partial f_{m,n}}{\partial\eta_j^{(l)}}\frac{\partial g_{m',n'}}{\partial\eta_p^{(q)}}\right)\\
&=\delta_{mm'}\delta_{nn'}\\
&-\sum_{jlpq}\left((-1)^{ln+l}\frac{\partial\eta_p^{(q)}}{\partial\xi_j^{(l)}}\frac{\partial f_{m,n}}{\partial\eta_p^{(q)}}\frac{\partial g_{m',n'}}{\partial\eta_j^{(l)}}-(-1)^{qn+q}\frac{\partial\eta_j^{(l)}}{\partial\xi_p^{(q)}}\frac{\partial f_{m,n}}{\partial\eta_j^{(l)}}\frac{\partial g_{m',n'}}{\partial\eta_p^{(q)}}\right)\\
&=\delta_{mm'}\delta_{nn'}.
\end{aligned}
$$

□

References

1. Batalin, I., Fradkin, E.: A generalized canonical formalism and quantization of reducible gauge theories. Phys. Lett. **122B**(2), 157–164 (1983)
2. Batalin, I., Vilkovisky, G.: Relativistic s-matrix of dynamical systems with boson and fermion constraints. Phys. Lett. **69B**(3), 309–312 (1977)
3. Becchi, B., Rouet, A., Stora, R.: Renormalization of gauge theories. Ann. Phys. **98**, 287–321 (1976)
4. Bordemann, M., Herbig, H.-C., Pflaum, M.: A homological approach to singular reduction in deformation quantization. In: Cheniot, D., Dutertre, N., Murolo, C., Trotman, D., Pichon, A. (eds.) Proceedings of the 2005 Marseille Singularity School and Conference, pp. 443–462. World Scientific, Singapore (2007)
5. Bordemann, M., Herbig, H.-C., Waldmann, S.: Brst cohomology and phase space reduction in deformation quantization. Commun. Math. Phys. **210**(1), 107–144 (2000)
6. Cartan, H., Eilenberg, S.: Homological Algebra. Princeton University Press, Princeton (1956)
7. Deser, A.: Star products on graded manifolds and α'-corrections to double field theory. In: Kielanowski, P., Ali, T.S., Bieliavsky, P., Odzijewicz, A., Schlichenmaier, M., Voronov, T. (eds.) Geometric Methods in Physics: XXXIV Workshop, Białowieża, Poland, June 28 -July 4, 2015. pp. 311–320. Springer International Publishing (2016)
8. Felder, G., Kazhdan, D.: The classical master equation (with an appendix by Tomer M. Schlank). In: Etingof, P., Khovanov, M., Savage, A. (eds.) Contemporary Mathematics 610, Perspectives in representation theory, pp 79–137. American Mathematical Society (2014)
9. Fisch, J., Henneaux, M., Stasheff, J., Teitelboim, C.: Existence, uniqueness and cohomology of the classical brst charge with ghosts of ghosts. Commun. Math. Phys. **120**(3), 379–407 (1989)
10. Fradkin, E., Fradkina, T.: Quantization of relativistic systems with boson and fermion first- and second-class constraints. Phys. Lett. **72B**(3), 343–348 (1978)
11. Fradkin, E., Vilkovisky, G.: Quantization of relativistic systems with constraints. Phys. Lett. **55B**(2), 224–226 (1975)
12. Henneaux, M., Teitelboim, C.: Quantization of Gauge Systems. Princeton University Press, Princeton (1994)
13. Herbig, H.-C.: Variations on Homological Reduction. PhD thesis, University of Frankfurt, 2007. ArXiv e-prints. arXiv:0708.3598
14. Kostant, B., Sternberg, S.: Symplectic reduction, brs cohomology, and infinite-dimensional clifford algebras. Ann. Phys. **176**(1), 49–113 (1987)
15. Moyal, J.: Quantum mechanics as a statistical theory. Math. Proc. Camb. Philos. Soc. **45**(2), 99–124 (1949)
16. Schätz, F.: Invariance of the BFV complex. Pacific J. Math. **248**(2), 453–474 (2010)
17. Stasheff, J.: Homological reduction of constrained poisson algebras. J. Differ. Geom. **45**(1), 221–240 (1997)
18. Tate, J.: Homology of noetherian rings and local rings. Ill. J. Math. 1(1), 14–27, 03 (1957)
19. Tyutin, I.: FIAN preprint. (39) (1975)
20. Waldmann, S.: Poisson-Geometrie und Deformationsquantisierung. Springer, Berlin (2007)