Lett Math Phys (2017) 107:81–106
DOI 10.1007/s11005-016-0889-8

Compton scattering in the Buchholz–Roberts framework of relativistic QED

Sabina Alazzawi[1] · **Wojciech Dybalski**[1]

Received: 16 February 2016 / Accepted: 24 August 2016
Published online: 8 November 2016

Abstract We consider a Haag–Kastler net in a positive energy representation, admitting massive Wigner particles and asymptotic fields of massless bosons. We show that massive single-particle states are always vacua of the massless asymptotic fields. Our argument is based on the Mean Ergodic Theorem in a certain extended Hilbert space. As an application of this result, we construct the outgoing isometric wave operator for Compton scattering in QED in a class of representations recently proposed by Buchholz and Roberts. In the course of this analysis, we use our new technique to further simplify scattering theory of massless bosons in the vacuum sector. A general discussion of the status of the infrared problem in the setting of Buchholz and Roberts is given.

Keywords Quantum field theory · Scattering theory · Infrared problem

Mathematics Subject Classification 81T05 · 81U99 · 81V10

1 Introduction

In general, the term *infrared problems* can be understood as complications in mathematical description of quantum systems encountered at large spatiotemporal scales.

Dedicated to the memory of John E. Roberts.

✉ Wojciech Dybalski
dybalski@ma.tum.de

Sabina Alazzawi
sabina.alazzawi@tum.de

1 Zentrum Mathematik, Technische Universität München, 85747 Garching, Germany

However, its conventional definition is more specific and refers to difficulties in scattering theory of such systems in the presence of long range forces and/or massless particles. The simplest and well understood example is Coulomb scattering in quantum mechanics which requires the Dollard modifications of the wave operators. Infrared problems in quantum electrodynamics (QED) still evade a satisfactory solution and constitute an active field of research in mathematical physics. Among many advances of recent years [9,10,16,21], a particularly radical proposal was put forward by Buchholz and Roberts in the setting of algebraic quantum field theory (AQFT) [9]. In essence, these authors suggest that after restricting attention to measurements in some future lightcone V, infrared problems should disappear. Buchholz and Roberts adopt the general point of view on infrared problems and illustrate their ideas by results on superselection structure of QED. However, conventional infrared problems, understood as complications in scattering theory, are not treated in their work. It is therefore an open question if the appealing ideas of Buchholz and Roberts are helpful for the analysis of collision processes in QED. We give a partial answer in this work.

Infrared problems in QED can be traced back to the fact that the space-like asymptotic flux of the electric field,[1]

$$\phi(\boldsymbol{n}) = \lim_{r\to\infty} r^2 \boldsymbol{n}\boldsymbol{E}(r\boldsymbol{n}), \quad \boldsymbol{n} \in S^2, \tag{1.1}$$

commutes with all local observables [5]. Since this flux is an arbitrary function on the unit sphere S^2, restricted only by the Gauss Law, each value of the electric charge corresponds to uncountably many disjoint irreducible representations of the algebra of observables, which are of potential physical interest. This invalidates the standard Doplicher–Haag–Roberts (DHR) theory of superselection sectors. For non-zero charges none of these representations can be Poincaré covariant, since the existence of ϕ is not consistent with a unitary action of Lorentz transformations. For similar reasons, charged particles cannot have sharp masses [6]. This latter difficulty, called the *infraparticle problem*, invalidates the conventional Haag–Ruelle or Lehmann–Symanzik–Zimmermann (LSZ) scattering theory for electrically charged particles. In this situation, a charged particle is a composite object involving a soft photon cloud correlated with the particle's velocity. The cloud is needed for the purpose of 'fine-tuning the flux', that is, keeping it constant along the time evolution [5]. Such *infraparticles* have in fact been constructed in concrete models of non-relativistic QED [10].

The above discussion involves a tacit restriction to representations of the algebra of observables of QED in which the flux (1.1) exists. Buchholz and Roberts consider instead a class of representations in which this is not the case, i.e. the fluctuations of the electric field tend to infinity under large space-like translations. Thinking heuristically, one way to achieve this is to include highly fluctuating background radiation, emitted in the very distant past. Such radiation, which should not be confused with soft photon clouds mentioned above, will 'blur the flux', that is, prevent the existence of the limit in (1.1). On the other hand, it is clear from Fig. 1a and the Huygens principle that this background radiation will stay outside any future lightcone V. Thus, inside V one can

[1] For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, we denote by $\boldsymbol{x}\boldsymbol{y}$ the Euclidean scalar product. For $x, y \in \mathbb{R}^4$, we denote by xy the Minkowski scalar product with signature $(+, -, -, -)$.

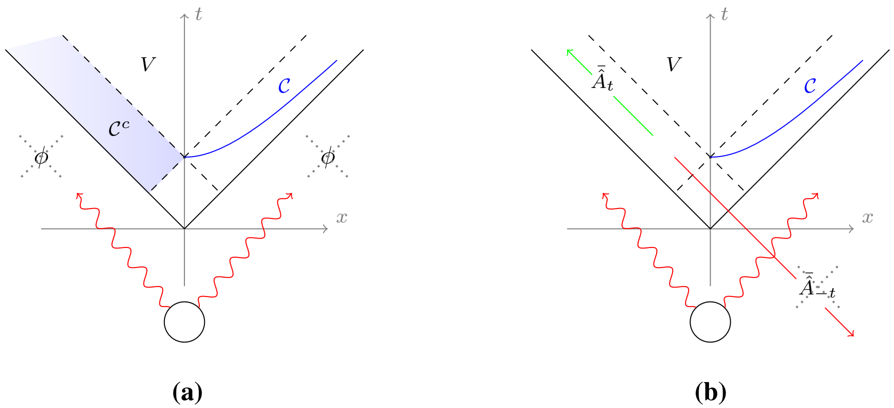

Fig. 1 **a** A hypercone localized representation of QED is equivalent to the vacuum representation in the causal complement $\mathcal{C}^c \subset V$ of any hypercone $\mathcal{C} \subset V$. This condition is consistent with the presence of highly fluctuating background radiation emitted in the distant past, which is needed to blur the flux ϕ. **b** If the approximating sequence $[1, \infty) \ni t \mapsto \bar{\hat{A}}_t$ of the *outgoing* asymptotic photon field is localized in $\mathcal{C}^c$, the existence of the limit $\hat{A}^{\text{out}}$ can be inferred from the corresponding result in the vacuum representation [3]. However, the *incoming* asymptotic field is not expected to exist, since its approximating sequence $[1, \infty) \ni t \mapsto \bar{\hat{A}}_{-t}$ collides with the background radiation

follow the usual DHR strategy to pass from the defining vacuum representation ι of the algebra of observables $\mathfrak{A}$ to an electrically charged positive energy representation π. To this end, consider a pair of opposite charges in a hypercone $\mathcal{C} \subset V$, which is a region depicted in Fig. 1a and defined precisely in Sect. 2. Next, transport one of the charges to lightlike infinity. As argued in [9], this process of charge creation in $\mathcal{C}$ should be only weakly correlated with operations performed in the spacelike complement of $\mathcal{C}$ in V, denoted $\mathcal{C}^c$. Therefore, the resulting charged representation π should satisfy the following property of *hypercone localization*

$$\pi \restriction \mathfrak{A}(\mathcal{C}^c) \simeq \iota \restriction \mathfrak{A}(\mathcal{C}^c), \tag{1.2}$$

where $\simeq$ denotes unitary equivalence and $\mathfrak{A}(\mathcal{C}^c)$ is the algebra of all observables measurable in $\mathcal{C}^c$. Since $\mathcal{C}^c \subset V$, this property is consistent with high fluctuations of the electric field at spacelike infinity, blurring the flux (1.1). (See again Fig. 1a.) As ϕ does not exist, we may require that π is covariant under Poincaré transformations and that charged particles have sharp masses.[2] We adopt these assumptions in this work and study their consequences.

The problem of verifying these assumptions in some concrete models of QED is outside the scope of this work. However, the above discussion reveals certain similarity of the Buchholz–Roberts ideas to the concept of *infravacua* [5, p. 59] [18,20], which result from adding to the vacuum a sufficiently strong background field. Concrete examples of such states were constructed in QED in the external current approximation

[2] Poincaré covariance is used in [9] at a technical level. The possibility of sharp masses of charged particles is only mentioned as a problem for future investigations.

by Kraus et al. [19]. It was conjectured already in [18] that in the infravacuum approach the electron is an 'ordinary particle' (and not an infraparticle), but this question was difficult to pose in the simple models of these early works. Nowadays, with more realistic models of QED under mathematical control, the problem of sharp mass of the electron in the infravacuum approach is an interesting and tractable research direction.

The purpose of this paper is to describe Compton scattering, i.e. collision processes involving one electron and some finite number of photons, in a hypercone localized representation $\pi : \mathfrak{A} \to B(\hat{\mathcal{H}})$. The Hilbert space of this representation should contain a subspace $\hat{\mathfrak{h}}_{\mathrm{el}}$ of single-electron states and the underlying vacuum representation should admit single-photon states. We construct asymptotic fields $\hat{A}^{\mathrm{out}}$ of photons via the LSZ prescription, exploiting the hypercone localization of π as illustrated in Fig. 1b. More precisely, we show that these fields exist as strong limits of their approximating sequences $t \mapsto \hat{\bar{A}}_t$ on a dense domain $D_{\hat{H}} \subset \hat{\mathcal{H}}$ of vectors of polynomially bounded energy and leave this domain invariant. Important technical ingredient here are energy bounds from [7,17]. In the next step, asymptotic creation and annihilation operators, denoted by $\hat{A}^{\mathrm{out}\pm}$, are defined such that vectors of the form

$$\Psi^{\mathrm{out}} := \hat{A}_1^{\mathrm{out}+} \dots \hat{A}_n^{\mathrm{out}+} \Psi_{\mathrm{el}}, \quad \Psi_{\mathrm{el}} \in \hat{\mathfrak{h}}_{\mathrm{el}}, \tag{1.3}$$

are natural candidates for Compton scattering states describing n photons and one electron. These states can now be used to construct the outgoing wave operator W^{out} which maps any configuration of one electron and n independent photons into the corresponding vector of the form (1.3). However, to show that W^{out} is well defined and isometric, two ingredients are needed. Firstly, the asymptotic creation and annihilation operators $\hat{A}^{\mathrm{out}\pm}$ must satisfy the standard canonical commutation relations. This can be shown by adapting results from [3,5] to a new geometric situation. Secondly, single-electron states must play a role of vacua of the asymptotic photon fields, i.e.

$$\hat{A}^{\mathrm{out}-} \Psi_{\mathrm{el}} = 0. \tag{1.4}$$

Our proof of this fact, given in Sect. 4, is the main new technical result of this paper and relies only on the Haag–Kastler postulates. In particular, the hypercone localization of π is not needed to show (1.4).

It has to be stressed that the technique used to verify (1.4) does also serve as a tool to simplify scattering theory of massless bosons in the vacuum sector. In particular, as shall be shown in Sect. 5, the proof of canonical commutation relations of the asymptotic photon fields in the vacuum sector can now be accomplished via a Pohlmeyer argument, without referring to the quadratic decay of the vacuum correlations of local observables. Thus, with the a priori information from [7,17] and the present paper, collision theory for massless bosons can be developed in a way completely parallel to the fermionic case [4].

Furthermore, since the argument used for the verification of (1.4) does not rely on strict locality, it may also be useful outside of the Haag–Kastler setting, e.g. in theories satisfying some kind of asymptotic Abelianess in spacelike directions. For example, it should help to remove Assumption 4 of [16] and Assumption 3 of [11]. It might also find applications in scattering theory of quantum spin systems satisfying the Lieb–Robinson bounds [2].

Our paper is organized as follows. In Sect. 2 we discuss Haag–Kastler nets and their representations. In Sect. 3, we introduce the asymptotic photon fields approximants and collect their representation-independent properties, such as uniform energy bounds and the decomposition into creation/annihilation operators. In Sect. 4, we give the proof of relation (1.4) which is our main technical result. In Sect. 5, we revisit and simplify the scattering theory of photons in the vacuum representation, construct asymptotic photon fields in hypercone localized representations and show that single-electron states induce vacuum representations of the Haag–Kastler net of asymptotic fields. More technical aspects of our discussion are postponed to the appendices.

2 Framework

Let $M = \mathbb{R}^4$ be the Minkowski spacetime. We denote by $\mathcal{K}$ the family of double cones $\mathcal{O} \subset M$ ordered by inclusion and write $\mathcal{O}_{\mathrm{c}}$ for the causal complement of $\mathcal{O}$ in M.[3] Furthermore, let $\widetilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ denote the covering group of the proper ortochronous Poincaré group $\mathcal{P}_+^\uparrow$. Its elements $\lambda = (x, \widetilde{\Lambda})$ act on M via $\lambda y = \Lambda y + x$, where $\Lambda \in \mathcal{L}_+^\uparrow$ is the Lorentz transformation corresponding to $\widetilde{\Lambda} \in SL(2,\mathbb{C})$.

Definition 2.1 We say that $\mathcal{K} \ni \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset B(\mathcal{H})$ is a Haag–Kastler net of von Neumann algebras if the following properties hold:

(a) (Isotony) $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$.
(b) (Locality) $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$ for $\mathcal{O}_1 \subset \mathcal{O}_{2,\mathrm{c}}$.
(c) (Covariance) There is a continuous unitary representation U of $\widetilde{\mathcal{P}}_+^\uparrow$ such that

$$U(\lambda)\mathfrak{A}(\mathcal{O})U(\lambda)^* = \mathfrak{A}(\lambda\mathcal{O}) \quad \text{for} \quad \lambda \in \widetilde{\mathcal{P}}_+^\uparrow. \tag{2.1}$$

(d) (Positivity of energy) The joint spectrum of the generators of translations, denoted $\mathrm{Sp}\,(U \restriction \mathbb{R}^4)$, is contained in the closed future lightcone $\overline{V}_+$.

A Haag–Kastler net will be denoted by $(\mathfrak{A}, U)$.

Definition 2.2 We say that a Haag–Kastler net describes Wigner particles of mass $m \geq 0$ if there is a subspace $\mathfrak{h} \subset \mathcal{H}$ on which $U(\lambda)$, $\lambda \in \widetilde{\mathcal{P}}_+^\uparrow$, acts like a representation of mass m.

Further useful definitions are as follows. For any region $\mathcal{U} \subset M$, we set

$$\mathfrak{A}_{\mathrm{loc}}(\mathcal{U}) := \bigcup_{\mathcal{O} \subset \mathcal{U}} \mathfrak{A}(\mathcal{O}) \quad \text{and} \quad \mathfrak{A}(\mathcal{U}) := \overline{\mathfrak{A}_{\mathrm{loc}}(\mathcal{U})}^{\|\cdot\|}. \tag{2.2}$$

[3] Note the distinction between the causal complements in M and V, which is indicated by, respectively, lower and upper indices.

In particular, we refer to $\mathfrak{A}_{\mathrm{loc}} := \mathfrak{A}_{\mathrm{loc}}(M)$ as the algebra of strictly local operators and to $\mathfrak{A} := \mathfrak{A}(M)$ as the global algebra of the net.

Now, let $\pi : \mathfrak{A} \to B(\mathcal{H}_\pi)$ be a (unital) representation. We say that π is (Poincaré) covariant, if there exists a strongly continuous unitary representation U_π of $\widetilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_\pi$ such that

$$U_\pi(\lambda)\pi(A)U_\pi(\lambda)^* = \pi(U(\lambda)AU(\lambda)^*), \quad A \in \mathfrak{A}. \tag{2.3}$$

Moreover, we say that π has positive energy if $\mathrm{Sp}\,(U_\pi \restriction \mathbb{R}^4) \subset \overline{V}_+$. It is easy to see that if π is a covariant, positive energy representation, then,

$$\mathcal{O} \mapsto \mathfrak{A}_\pi(\mathcal{O}) := \pi(\mathfrak{A}(\mathcal{O}))'' \tag{2.4}$$

is again a Haag–Kastler net which will be denoted $(\mathfrak{A}_\pi, U_\pi)$.

Definition 2.3 If π is an irreducible, covariant, positive energy representation and $\mathcal{H}_\pi$ contains a unique (up to a phase) unit vector Ω, invariant under U_π, then we say that π is a vacuum representation.

To proceed to charged representations, we choose an open future lightcone V and denote for any region $\mathcal{U} \subset V$ by $\mathcal{U}^{\mathrm{c}}$ its causal complement in V. Next, we define a class of regions in V which are called *hypercones* in [9]. We recall here briefly their definition referring to [9] for more details: Choose coordinates so that $V = \{x \in \mathbb{R}^4 \,|\, x_0 > |\boldsymbol{x}|\}$ and fix a hyperboloid $\mathsf{H}_{\bar\tau} = \{x \in V \,|\, x_0 = \sqrt{\boldsymbol{x}^2 + \bar\tau^2}\}$ for some $\bar\tau > 0$. Project $\mathsf{H}_{\bar\tau}$ through the origin onto the plane $x_0 = 1$, so as to identify it with the open unit ball $\mathsf{B} \subset \mathbb{R}^3$. This projection is the Beltrami–Klein model of hyperbolic geometry. Consider the family of (truncated) pointed convex Euclidean cones K in B with elliptical bases. It gives rise to a Lorentz invariant family of hyperbolic cones $\mathsf{C} = \mathsf{C}(\mathsf{K})$ in $\mathsf{H}_{\bar\tau}$. A hypercone $\mathcal{C} = \mathcal{C}(\mathsf{K})$ is the causal completion of such C, i.e. $\mathcal{C} = \mathsf{C}^{\mathrm{cc}}$, and the family of all hypercones as described above is denoted by $\mathcal{F}_V$.

Definition 2.4 Let $(\mathfrak{A}, U)$ be a Haag–Kastler net in a vacuum representation and let π be a covariant positive energy representation. We say that π is hypercone localized if for any future lightcone V and $\mathcal{C} \in \mathcal{F}_V$ there exists a unitary $W_{\mathcal{C}} : \mathcal{H} \to \mathcal{H}_\pi$ such that

$$\pi(A) = W_{\mathcal{C}} A W_{\mathcal{C}}^* \quad \text{for} \quad A \in \mathfrak{A}(\mathcal{C}^{\mathrm{c}}). \tag{2.5}$$

Remark 2.5 It is easy to see that the morphisms $\sigma_{\mathcal{C},M} : \mathfrak{A} \to B(\mathcal{H})$ from [9] are irreducible, hypercone localized representations.

Note that for any hypercone localized representation π and any $\mathcal{O} \in \mathcal{K}$, we have $\pi(\mathfrak{A}(\mathcal{O})) = \pi(\mathfrak{A}(\mathcal{O}))''$ and, therefore, $\pi(\mathfrak{A}) = \mathfrak{A}_\pi$. It is also easy to see that any hypercone localized representation is faithful.

3 Asymptotic photon fields

In this section, the pair $(\mathfrak{A}, U)$ refers to an arbitrary Haag–Kastler net. For the unitary representation of translations $U \restriction \mathbb{R}^4$ we shall write $U(x) = e^{i(Hx^0 - \boldsymbol{P}\boldsymbol{x})}$ and the joint spectral measure of the energy–momentum operators $(H, \boldsymbol{P})$ shall be denoted by $E(\,\cdot\,)$. For translated observables $A \in \mathfrak{A}$, the notations $\alpha_x(A) := A(x) := U(x)AU(x)^*$ are used. We also define for any $B \in \mathfrak{A}$ the smeared operators

$$B(g) := \begin{cases} \int d^3x\, B(\boldsymbol{x})g(\boldsymbol{x}) & \text{for } g \in S(\mathbb{R}^3), \\ \int d^4x\, B(x)g(x) & \text{for } g \in S(\mathbb{R}^4), \end{cases} \tag{3.1}$$

which are elements of $\mathfrak{A}$ since local algebras are von Neumann. Moreover, we have

$$\mathfrak{A}_{\mathrm{loc},0} := \{A \in \mathfrak{A}_{\mathrm{loc}} \,|\, x \mapsto A(x) \text{ smooth in norm}\}. \tag{3.2}$$

Next, we introduce the following Poincaré invariant subset of $S(\mathbb{R}^4)$

$$S_*(\mathbb{R}^4) := \left\{(n_\mu \partial^\mu)^5 g \,|\, g \in S(\mathbb{R}^4),\; n_0 = \sqrt{1+\boldsymbol{n}^2}\right\}, \qquad \mu = 0, \dots, 3. \tag{3.3}$$

Note that the summation convention is used in (3.3) and in the following. Moreover, the power 5 appearing in (3.3) is due to technical reasons which become obvious with regard to Propositions 3.2 and 3.4. Furthermore, we introduce certain Poincaré invariant subsets of $\mathfrak{A}$, namely

$$\mathfrak{A}_{S_*} := \{\, B(g) \,|\, B \in \mathfrak{A}_{\mathrm{loc},0},\; g \in S_*(\mathbb{R}^4)\,\}, \tag{3.4}$$
$$\mathfrak{A}^{S_*} := \mathrm{Span}\, \mathfrak{A}_{S_*}, \tag{3.5}$$

where Span denotes finite linear combinations. For any $A \in \mathfrak{A}^{S_*}$, $f \in C^\infty(S^2)$, we set as in [5]

$$A_t\{f\} := -2\,t \int d\omega(\boldsymbol{n})\, f(\boldsymbol{n})\, \partial_0 A(t, t\boldsymbol{n}). \tag{3.6}$$

Here, $d\omega(\boldsymbol{n}) = \frac{\sin\nu\, d\nu d\varphi}{4\pi}$ is the normalized, invariant measure on S^2 and $\partial_0 A := \partial_s(e^{isH} A e^{-isH})|_{s=0}$. To improve the convergence in the limit of large t, we proceed to time averages of $A_t\{f\}$, namely

$$\bar{A}_t\{f\} := \int dt'\, h_t(t')\, A_{t'}\{f\}, \tag{3.7}$$

where for non-negative $h \in C_0^\infty(\mathbb{R})$, supported in the interval $[-1, 1]$ and normalized so that $\int dt\, h(t) = 1$, we set $h_t(t') = t^{-\bar{\varepsilon}} h(t^{-\bar{\varepsilon}}(t' - t))$ with $t \geq 1$ and $0 < \bar{\varepsilon} < 1$.

For the discussion of asymptotic creation and annihilation operators below, it is important to use Schwartz class functions in (3.3). Since strict locality plays a crucial role in the subsequent part of this paper, we also define the following sets:

$$C_*(\mathbb{R}^4) := \left\{ (n_\mu \partial^\mu)^5 g \mid g \in C_0^\infty(\mathbb{R}^4),\ n_0 = \sqrt{1+\boldsymbol{n}^2} \right\} \subset S_*(\mathbb{R}^4), \tag{3.8}$$

$$\mathfrak{A}_{C_*} := \{ B(g) \mid B \in \mathfrak{A}_{\mathrm{loc},0},\ g \in C_*(\mathbb{R}^4) \} \subset \mathfrak{A}_{S_*} \cap \mathfrak{A}_{\mathrm{loc},0}, \tag{3.9}$$

$$\mathfrak{A}^{C_*} := \mathrm{Span}\, \mathfrak{A}_{C_*}, \tag{3.10}$$

$$\mathfrak{A}_{C_*}(\mathcal{O}) := \mathfrak{A}_{C_*} \cap \mathfrak{A}(\mathcal{O}), \quad \mathfrak{A}^{C_*}(\mathcal{O}) := \mathfrak{A}^{C_*} \cap \mathfrak{A}(\mathcal{O}), \quad \mathcal{O} \in \mathcal{K}. \tag{3.11}$$

The linear structure of $\mathfrak{A}^{S_*}$ and $\mathfrak{A}^{C_*}$ will be important in the discussion of the Haag–Kastler net of asymptotic fields in Sect. 5.

Now, we note a convenient representation for $A_t\{f\}$. A similar result can be found in [11].

Lemma 3.1 *Let $A \in \mathfrak{A}_{S_*}$, i.e. $A = B(g)$, where $B \in \mathfrak{A}_{\mathrm{loc},0}$ and $g \in S_*(\mathbb{R}^4)$. Then, for $f \in C^\infty(S^2)$,*

$$A_t\{f\} = (\partial_0 B)(g *_3 f_t)(t), \quad \text{where} \quad f_t(\boldsymbol{x}) := -\frac{1}{4\pi} \frac{2}{|\boldsymbol{x}|} \delta(t - |\boldsymbol{x}|) f\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \tag{3.12}$$

where $*_3$ denotes convolution in space variables and $C(t) = U(t)CU(t)^*$ for $C \in \mathfrak{A}$. Moreover, the Fourier transform of $g *_3 f_t \in S(\mathbb{R}^4)$ has the following form:

$$\widetilde{(g *_3 f_t)}(p) = \frac{\widetilde{g}(p)}{\mathrm{i}|\boldsymbol{p}|} \left(f\left(\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right) \mathrm{e}^{-\mathrm{i}t|\boldsymbol{p}|} - f\left(-\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right) \mathrm{e}^{\mathrm{i}t|\boldsymbol{p}|} + \int_0^\pi \mathrm{d}\nu\, F(\boldsymbol{p}, \nu) \mathrm{e}^{-\mathrm{i}t|\boldsymbol{p}|\cos\nu} \right), \tag{3.13}$$

where F is a bounded measurable function depending on f. (In particular, $F = 0$ if $f = \mathrm{const.}$)

Proof The equality $A_t\{f\} = \partial_0 B(g *_3 f_t)(t)$, with f_t given by (3.12), is straightforward to check. Equation (3.13) follows likewise from an easy computation. □

Next, we proceed to uniform bounds on $t \mapsto \bar{A}_t\{f\}$. We recall that first bounds of this sort were proven in [7].

Proposition 3.2 *Let $A \in \mathfrak{A}_{S_*}$, i.e. $A = B((n_\mu \partial^\mu)^5 g')$, $B \in \mathfrak{A}_{\mathrm{loc},0}$ and $g' \in S(\mathbb{R}^4)$. Then, for $f \in C^\infty(S^2)$,*

$$\sup_{t \in [1,\infty)} \|\bar{A}_t\{f\}(1+H)^{-1}\| \leq c \sup_{\ell=0,1} \||\boldsymbol{p}|^{-1} \partial_0^\ell \big((n_\mu p^\mu)^2 \widetilde{g'}\big)\|_2 < \infty. \tag{3.14}$$

The constant c above is independent of g' but depends on B and f.

Proof Let $A \in \mathfrak{A}_{\mathrm{loc},0}$ and $n \in \mathbb{R}^4$ be a unit future oriented timelike vector, i.e. $n_0 = \sqrt{1+\boldsymbol{n}^2}$. Then results from [7,17] give for any $g \in S(\mathbb{R}^4)$

$$\|A((n_\mu \partial^\mu)^3 g)(1+H)^{-1}\| \leq c \sup_{\ell=0,1} \|\partial_0^\ell \widetilde{g}\|_2, \tag{3.15}$$

where the constant c is independent of g. Application of Lemma 3.1 immediately gives (3.14). □

In the following, we shall be interested in the convergence of $\bar{A}_t\{f\}$, $A \in \mathfrak{A}^{S_*}$, $f \in C^\infty(S^2)$, to the limit $A^{\text{out}}\{f\}$ as $t \to \infty$. To start with, we define $A^{\text{out}}\{f\}$ as an operator on the domain

$$D_{\max}(A, f) := \left\{ \Psi \in \mathcal{H} \mid A^{\text{out}}\{f\}\Psi := \lim_{t\to\infty} \bar{A}_t\{f\}\Psi \text{ exists} \right\}. \tag{3.16}$$

For $f \equiv 1$, we will abbreviate $D_{\max}(A, f)$ by $D_{\max}(A)$. Note that $D_{\max}(A, f)$ may depend on A and f, may not be Poincaré invariant and a priori may even be trivial. Another domain we shall be interested in is

$$D_H := \bigcap_{n\geq 1} D(H^n), \tag{3.17}$$

where $D(H^n)$ is the domain of self-adjointness of the n-th power of the Hamiltonian H. It is easy to see that D_H is dense and Poincaré invariant. The next result can be inferred from the discussion in Appendix B. We note here that Proposition 3.2 and smoothness of the relevant observables under translations ensures that properties (b), (c) from Definition B.1 are met.

Proposition 3.3 *Let $i = 1, \ldots, n$ and suppose that the domains $D_{\max}(A_i, f_i)$ and $D_{\max}(A_i^*, \bar{f}_i)$ are dense. Then, we have*

(a) $D_H \subset D_{\max}(A_i, f_i)$, $D_H \subset D_{\max}(A_i^*, \bar{f}_i)$,
(b) $A_i^{\text{out}}\{f_i\} D_H \subset D_H$,
(c) $A_1^{\text{out}}\{f_1\} \ldots A_n^{\text{out}}\{f_n\}\Psi = \lim_{t\to\infty} \bar{A}_{1,t}\{f_1\} \ldots \bar{A}_{n,t}\{f_n\}\Psi$ *for* $\Psi \in D_H$.

The operators $A_i^{\text{out}}\{f_i\} \restriction D_H$ are closable and uniquely determined by the values of $A_i^{\text{out}}\{f_i\}$ on any dense subspace of $D_{\max}(A_i, f_i)$.

Another consequence of the uniform bounds is the existence of asymptotic creation and annihilation operators under the assumptions of Proposition 3.3. In fact, a similar observation was made in [11]. To construct these operators, we proceed as follows. Let $\theta \in C^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, be supported in $(0, \infty)$ and equal to one on $(1, \infty)$. Moreover, let $\beta \in C_0^\infty(\mathbb{R}^4)$, $0 \leq \beta \leq 1$, be equal to one in some neighbourhood of zero and satisfy $\beta(-p) = \beta(p)$. Furthermore, for a parameter $1 \leq r < \infty$ and a future oriented timelike unit vector n, we define

$$\widetilde{\eta}_{\pm,r}(p) := \theta(\pm r(n_\mu p^\mu))\beta(r^{-1}p). \tag{3.18}$$

As $r \to \infty$ these functions approximate the characteristic functions of the positive/negative energy half planes $\{p \in \mathbb{R}^4 \mid \pm n_\mu p^\mu \geq 0\}$. We also have $\bar{\eta}_{\pm,r} = \eta_{\mp,r}$. Note that the family of functions $\eta_{\pm,r}$, as specified above, is invariant under Lorentz transformations. The following result is easily verified using Propositions 3.2, 3.3.

Proposition 3.4 *Let $A \in \mathfrak{A}_{S_*}$, $f \in C^\infty(S^2)$. Suppose that $D_{\max}(A, f)$, $D_{\max}(A^*, \bar{f})$ are dense and the timelike unit vectors n entering the definition of A and of $\eta_{\pm,r}$ coincide. Then*:

(a) *The limits $A^{\mathrm{out}}\{f\}^{\pm}\Psi := \lim_{r\to\infty} A^{\mathrm{out}}\{f\}(\eta_{\pm,r})\Psi$, $\Psi \in D_H$, exist and define the creation and annihilation parts of $A^{\mathrm{out}}\{f\}$ as operators on D_H. $A^{\mathrm{out}}\{f\}^{\pm}$ do not depend on the choice of the functions θ and β in (3.18) within the specified restrictions.*
(b) *$(A^{\mathrm{out}}\{f\}^{\pm})^* \restriction D_H = A^{*\mathrm{out}}\{\bar{f}\}^{\mp}$. In particular, $A^{\mathrm{out}}\{f\}^{\pm}$ are closable operators.*
(c) *$A^{\mathrm{out}}\{f\}^{\pm} D_H \subset D_H$.*
(d) *$A^{\mathrm{out}}\{f\} = A^{\mathrm{out}}\{f\}^{+} + A^{\mathrm{out}}\{f\}^{-}$ on D_H.*

Remark 3.5 The proposition can be generalized to $A \in \mathfrak{A}^{S_*}$ as follows. Consider a decomposition $A = \sum_{i=1}^{\ell} A_i$, $A_i \in \mathfrak{A}_{S_*}$ and assume that $D_{\max}(A_i, f)$ and $D_{\max}(A_i^*, \bar{f})$ are dense. Define $A^{\mathrm{out}}\{f\}^{\pm} := \sum_{i=1}^{\ell} A_i^{\mathrm{out}}\{f\}^{\pm}$ on D_H. Then it is easy to see that $A^{\mathrm{out}}\{f\}^{\pm}$ satisfy the properties (b), (c) and (d) of the proposition.

Proof (a) Making use of Propositions 3.2 and 3.3, we compute for $1 \le r_1 \le r_2$ and $\Psi \in D_H$ that

$$\|A^{\mathrm{out}}\{f\}(\eta_{\pm,r_1} - \eta_{\pm,r_2})\Psi\| \le c \sup_{\ell=0,1} \int_{r_1}^{r_2} \mathrm{d}r \, \| |\boldsymbol{p}|^{-1} \partial_0^{\ell} \big((n_\mu p^\mu)^2 (\partial_r \widetilde{\eta}_{\pm,r})(p) \widetilde{g'} \big) \|_2, \tag{3.19}$$

where $\widetilde{g}' \in S(\mathbb{R}^4)$ is defined as in Proposition 3.2 and the functions of p appearing in (3.19) are to be understood as multiplication operators acting on $\widetilde{g'}$. Using the fact that $\partial\theta$ is compactly supported, and therefore $|n_\mu p^\mu| \le c r^{-1}$ when multiplied by $\partial\theta(\pm r(n_\mu p^\mu))$, it is easy to check that

$$\big| \partial_0^{\ell} \big((n_\mu p^\mu)^2 \partial_r \widetilde{\eta}_{\pm,r}(p) \big) \big| \le \frac{c}{r^2}(1 + |p|^3), \quad \ell = 0, 1, \tag{3.20}$$

for c independent of p and r. This completes the proof of convergence. Independence of the choice of the functions θ and β is shown by a similar computation.
(b) This part is straightforward.
(c) This follows from the smoothness of A under translations and Proposition 3.2.
(d) We choose a function $\gamma \in C_0^\infty(\mathbb{R})$, $0 \le \gamma \le 1$, such that

$$\theta(-k) + \gamma(k) + \theta(k) = 1, \quad k \in \mathbb{R}, \tag{3.21}$$

and set $\widetilde{\eta}_r(p) := \gamma_r(r(n_\mu p^\mu))\beta(r^{-1}p)$. Since γ is compactly supported, we have for $\Psi \in D_H$

$$\|A^{\mathrm{out}}\{f\}(\eta_r)\Psi\| \le c \sup_{\ell=0,1} \| |\boldsymbol{p}|^{-1} \partial_0^{\ell} \big((n_\mu p^\mu)^2 \widetilde{\eta}_r(p) \widetilde{g'} \big) \|_2 \le c' r^{-1}. \tag{3.22}$$

Hence, $\lim_{r\to\infty} A^{\mathrm{out}}\{f\}(\eta_r)\Psi = 0$, which completes the proof. □

4 Asymptotic vacuum structure

In this section, $(\mathfrak{A}, U)$ still refers to an arbitrary Haag–Kastler net. In this setting, we state and prove our main technical result which is Theorem 4.2 below. As a preparation, we recall some concepts and facts from [13]. Let Ψ be a vector of bounded energy, i.e. $\Psi = E(\Delta)\Psi$ for some compact Δ. Let $B = B_0(g_0)$, where $B_0 \in \mathfrak{A}_{\mathrm{loc}}$ and $g_0 \in S(\mathbb{R}^4)$ is such that supp $\widetilde{g}_0$ is compact and supp $\widetilde{g}_0 \cap \bar{V}_+ = \emptyset$. Then, by [7, Lemma 2.2], the function

$$(a_B\Psi)(\boldsymbol{x}) := B(\boldsymbol{x})\Psi, \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{4.1}$$

is square-integrable. A closer inspection shows that formula (4.1) defines a linear map $a_B : \mathcal{H}_{\mathrm{c}} \to \mathcal{H} \otimes L^2(\mathbb{R}^3)$, where $\mathcal{H}_{\mathrm{c}}$ is the domain of vectors of bounded energy, with the following useful properties:

Proposition 4.1 [13] *Let B be as above and Δ compact. Then*:

(a) $a_B E(\Delta) : \mathcal{H} \to \mathcal{H} \otimes L^2(\mathbb{R}^3)$ *is bounded.*
(b) $a_B E(\Delta) \circ f(\boldsymbol{P}) = f(\boldsymbol{P} + D_{\boldsymbol{x}}) \circ a_B E(\Delta)$ *for any* $f \in L^\infty(\mathbb{R}^3)$.

Here, we set $D_{\boldsymbol{x}} = -\mathrm{i}\nabla_{\boldsymbol{x}}$ and use the shorthand notation $\boldsymbol{P} + D_{\boldsymbol{x}}$ for $\boldsymbol{P} \otimes 1_{L^2(\mathbb{R}^3)} + 1_{\mathcal{H}} \otimes D_{\boldsymbol{x}}$.

Next, we define for $g \in L^2(\mathbb{R}^3)$ the functionals $(1_{\mathcal{H}} \otimes \langle \overline{g}|) : \mathcal{H} \otimes L^2(\mathbb{R}^3) \to \mathcal{H}$ by

$$(1_{\mathcal{H}} \otimes \langle \overline{g}|)\Psi = \int \mathrm{d}^3x\, g(x)\Psi(x), \tag{4.2}$$

where on the right-hand side we identified $\mathcal{H} \otimes L^2(\mathbb{R}^3) \simeq L^2(\mathbb{R}^3; \mathcal{H})$. The identity

$$B(g)E(\Delta) = (1_{\mathcal{H}} \otimes \langle \overline{g}|) \circ a_B E(\Delta) \tag{4.3}$$

is first checked for $g \in S(\mathbb{R}^3)$ and then extended to $L^2(\mathbb{R}^3)$ for compact Δ using Proposition 4.1. These concepts can now be used to prove the following theorem.

Theorem 4.2 *Let $\eta \in S(\mathbb{R}^4)$ be such that $\widetilde{\eta}$ is supported outside of $\overline{V}_+$. Let $A \in \mathfrak{A}_{S_*}$ and $f \in C^\infty(S^2)$. Then, for $\Psi \in E(\mathsf{H}_m)\mathcal{H} \cap D_H$ and $\mathsf{H}_m = \{\, p \in \mathbb{R}^4 \,|\, p^0 = \sqrt{\boldsymbol{p}^2 + m^2}\,\}$, we have*:

(a) *For $m = 0$, $\lim_{t\to\infty}(1 - E(\{0\}))\bar{A}_t\{f\}(\eta)\Psi = 0$.*
(b) *For $m > 0$, $\lim_{t\to\infty} \bar{A}_t\{f\}(\eta)\Psi = 0$.*

Proof Without loss of generality, we can assume that $\widetilde{\eta}$ is compactly supported and $\Psi = E(\Delta)\Psi$ for some compact Δ. Making use of Lemma 3.1, we have $A_t\{f\}(\eta) = (\partial_0 B(\eta))(g *_3 f_t)(t)$, where

$$(g *_3 f_t)(x) = (2\pi)^{-2} \int_0^\pi \mathrm{d}\mu(\nu) \int \mathrm{d}^4p\ \widetilde{f}_\nu(p)\, \mathrm{e}^{-\mathrm{i}px} \mathrm{e}^{-\mathrm{i}\cos\nu|\boldsymbol{p}|t}. \tag{4.4}$$

Here, $\mathrm{d}\mu(\nu) := \mathrm{d}\nu + \delta(\nu)\mathrm{d}\nu + \delta(\nu - \pi)\mathrm{d}\nu$, $(p, \nu) \mapsto \widetilde{f}_\nu(p)$ is absolutely integrable, smooth in p^0 and

$$\sup_{\nu\in[0,\pi]} (\|\widetilde{f}_\nu\|_2 + \|\partial_0 \widetilde{f}_\nu\|_2) < \infty. \tag{4.5}$$

We set $B' := \partial_0 B(\eta)$ and note that it is of the form $B_0(g_0)$ as specified in the above formula (4.1). Setting $\widetilde{f}^t_\nu(p) := \widetilde{f}_\nu(p)\mathrm{e}^{-\mathrm{i}\cos\nu|\boldsymbol{p}|t}$, we have

$$\begin{aligned}\bar{A}_t\{f\}(\eta)\Psi &= \int \mathrm{d}t'\, h_t(t')\mathrm{e}^{\mathrm{i}t'H} B'(g *_3 f_{t'})\mathrm{e}^{-\mathrm{i}t'\omega_m(\boldsymbol{P})}\Psi \\ &= \int_0^\pi \mathrm{d}\mu(\nu) \int \mathrm{d}t'\, h_t(t')\mathrm{e}^{\mathrm{i}t'H} B'(f^{t'}_\nu)\mathrm{e}^{-\mathrm{i}t'\omega_m(\boldsymbol{P})}\Psi. \end{aligned}\tag{4.6}$$

Now, we take $B'_{x^0}(\boldsymbol{x}) := B'(x^0,\boldsymbol{x})$, $f^t_{\nu,x^0}(\boldsymbol{x}) := f^t_\nu(x^0,\boldsymbol{x})$ and $f_{\nu,x^0}(\boldsymbol{x}) := f_\nu(x^0,\boldsymbol{x})$. Making use of (4.3) and Proposition 4.1, we obtain

$$\begin{aligned}&\bar{A}_t\{f\}(\eta)\Psi \\ &= \int \mathrm{d}x^0 \int_0^\pi \mathrm{d}\mu(\nu) \int \mathrm{d}t'\, h_t(t')\mathrm{e}^{\mathrm{i}t'H}(1_{\mathcal{H}} \otimes \langle \overline{f}^{t'}_{\nu,x^0}|) \circ a_{B'_{x^0}} \mathrm{e}^{-\mathrm{i}t'\omega_m(\boldsymbol{P})}\Psi \\ &= \int \mathrm{d}x^0 \int_0^\pi \mathrm{d}\mu(\nu)(1_{\mathcal{H}} \otimes \langle \overline{f}_{\nu,x^0}|) \circ \int \mathrm{d}t'\, h_t(t')\mathrm{e}^{\mathrm{i}t'(H-\cos\nu|D_{\boldsymbol{x}}|-\omega_m(\boldsymbol{P}+D_{\boldsymbol{x}}))} \circ a_{B'_{x^0}}\Psi. \end{aligned}\tag{4.7}$$

By means of the Dominated Convergence Theorem, the bound (4.5) and the Mean Ergodic Theorem (Theorem A.1), we obtain

$$\lim_{t\to\infty} \bar{A}_t\{f\}(\eta)\Psi = \int \mathrm{d}x^0 \int_0^\pi \mathrm{d}\mu(\nu)(1_{\mathcal{H}} \otimes \langle \overline{f}_{\nu,x^0}|) \circ F_S(\{0\}) \circ a_{B'_{x^0}}\Psi, \tag{4.8}$$

where F_S is the spectral measure of the operator $S := H - \cos\nu|D_{\boldsymbol{x}}| - \omega_m(\boldsymbol{P} + D_{\boldsymbol{x}})$ on $L^2(\mathbb{R}^3;\mathcal{H})$. To determine $F_S(\{0\})$, we diagonalize $D_{\boldsymbol{x}}$ with the help of the Fourier transform. We further note that $\|S\Phi\|^2 = 0$, for some $\Phi = \{\Phi_\xi\}_{\xi\in\mathbb{R}^3} \in L^2(\mathbb{R}^3;\mathcal{H})$, implies that $S_\xi\Phi_\xi = 0$ for almost all ξ w.r.t. the Lebesgue measure,[4] where

$$H - \cos\nu|\boldsymbol{\xi}| - \omega_m(\boldsymbol{P} + \boldsymbol{\xi}). \tag{4.9}$$

Suppose now that $m = 0$. Then, Proposition A.2 gives that $\Phi_\xi \in \mathrm{Ran}\, E(\{0\})$ for $\boldsymbol{\xi} = 0$ or $\nu = \pi$ and $\Phi_\xi = 0$ otherwise. Since $\boldsymbol{\xi} = 0$ is of zero Lebesgue measure, only $\nu = \pi$ contributes and we obtain

$$\lim_{t\to\infty} \bar{A}_t\{f\}(\eta)\Psi = \int \mathrm{d}x^0 (E(\{0\}) \otimes \langle \overline{f}_{\pi,x^0}|) \circ a_{B'_{x^0}}\Psi = E(\{0\})B'(f_\pi)\Psi. \tag{4.10}$$

For $m > 0$, a similar and simpler reasoning gives that the above limit is zero. □

[4] See [25, Section IV.7] for definition and basic properties of $L^2(\mathbb{R};\mathcal{H})$ for non-separable $\mathcal{H}$.

Corollary 4.3 *Let $A \in \mathfrak{A}_{S_*}$ and $f \in C^\infty(S^2)$. Suppose further that $D_{\max}(A, f)$ and $D_{\max}(A^*, \bar{f})$ are dense. Then, for $\Psi \in E(\mathsf{H}_m)\mathcal{H} \cap D_H$, we have:*

(a) *For $m = 0$, $(1 - E(\{0\}))A^{\mathrm{out}}\{f\}^- \Psi = 0$.*
(b) *For $m > 0$, $A^{\mathrm{out}}\{f\}^- \Psi = 0$.*

Remark 4.4 The result immediately generalizes to $A \in \mathfrak{A}^{S_*}$, cf. Remark 3.5.

5 Compton scattering in hypercone localized representations

In this section $(\mathfrak{A}, U)$ refers to a Haag–Kastler net in a vacuum representation admitting massless Wigner particles ('photons') and the single-photon subspace is denoted $\mathfrak{h}_{\mathrm{ph}}$. In this setting, we provide several applications of Theorem 4.2 and Corollary 4.3. First, in Proposition 5.1, we simplify scattering theory of photons in the vacuum representation, which was established in [3] and recently revisited in [11,26]. Second, in Theorem 5.4, we construct Compton scattering states in hypercone localized representations of $(\mathfrak{A}, U)$. Third, in Theorem 5.6, we verify that single-electron states induce vacuum representations of the Haag–Kastler net of asymptotic photon fields, defined in (5.21).

As a preliminary, we recall that the domain D_H is Poincaré invariant and denote by $g_\Lambda : S^2 \to S^2$ the action of the Lorentz transformation Λ on the unit sphere given by $\Lambda(1, \boldsymbol{n}) = c_\Lambda(\boldsymbol{n})(1, g_\Lambda(\boldsymbol{n}))$, where $c_\Lambda(\boldsymbol{n})$ is a normalization constant. The relevant properties of asymptotic photon fields in the vacuum sector are then summarized as follows.

Proposition 5.1 *Let $A, A', A_i \in \mathfrak{A}^{C_*}$ and $f, f', f_i \in C^\infty(S^2)$, $i = 1, \ldots, n$. Then:*

(a) *For any $\Psi \in D_H$, the limit $\lim_{t\to\infty} \bar{A}_t\{f\}\Psi$ exists and defines a closable operator $A^{\mathrm{out}}\{f\} \restriction D_H$. This operator is uniquely determined by the vector $A^{\mathrm{out}}\{f\}\Omega$.*
(b) *$A^{\mathrm{out}}\{f\}D_H \subset D_H$.*
(c) *$A_1^{\mathrm{out}}\{f_1\} \ldots A_n^{\mathrm{out}}\{f_n\}\Psi = \lim_{t\to\infty} \bar{A}_{1,t}\{f_1\} \ldots \bar{A}_{n,t}\{f_n\}\Psi$ for $\Psi \in D_H$.*
(d) *$U(\lambda)A^{\mathrm{out}}\{f\}U(\lambda)^* = A_\lambda^{\mathrm{out}}\{f \circ g_{\Lambda^{-1}}\}$ on D_H, where $A_\lambda = U(\lambda)AU(\lambda)^*$.*
(e) *$\Box_x A^{\mathrm{out}}\{f\}(x)\Psi = 0$ for $\Psi \in D_H$.*
(f) *$[A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}] = \langle \Omega, [A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]\Omega\rangle 1_{\mathcal{H}}$ as operators on D_H.*

Proof We will first discuss briefly properties (a)–(e), and then provide a novel proof of (f) which uses Theorem 4.2. First, by a standard computation using Lemma 3.1 and Proposition A.2 (a), one obtains

$$\lim_{t\to\infty} \bar{A}_t\{f\}\Omega = P_{\mathrm{ph}} f\left(\frac{\boldsymbol{P}}{|\boldsymbol{P}|}\right) A\Omega, \tag{5.1}$$

where P_{ph} is the projection on $\mathfrak{h}_{\mathrm{ph}}$. Next, denote by $\mathcal{O}_+$ the future tangent of a double cone $\mathcal{O}$, i.e. the cone of all points that have a positive timelike separation from $\mathcal{O}$. Following the arguments of [3], based on the Huygens principle, we have that the limit

$$A^{\mathrm{out}}\{f\}\Psi = \lim_{t\to\infty} \bar{A}_t\{f\}\Psi \tag{5.2}$$

exists for Ψ in the dense domain $D(\mathcal{O}) := \{ B\Omega \mid B \in \mathfrak{A}_{\mathrm{loc}}(\mathcal{O}_+) \}$. Moreover, $A^{\mathrm{out}}\{f\}$ depends only on the single-particle state $A^{\mathrm{out}}\{f\}\Omega$ within the above restrictions. Thus in view of Proposition 3.2, we obtain parts (a)–(c) of the proposition. Parts (d),(e) are checked first on $D(\mathcal{O})$ with the help of formula (5.1) and then extended to D_H by approximation arguments.

Proceeding to the proof of part (f), we first observe as in [4, Lemma 3] that

$$[A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}(x)] = 0 \tag{5.3}$$

in the sense of quadratic forms on $D_H \times D_H$, provided that $A \in \mathfrak{A}^{C_*}(\mathcal{O})$, $A' \in \mathfrak{A}^{C_*}(\mathcal{O}')$ and $\mathcal{O}' + x \subset \mathcal{O}_+$. Next, we use a method of Pohlmeyer [22] (applied also in the collision theory of massless fermions [4]) to show that

$$[A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]\Omega = c\,\Omega, \qquad c \in \mathbb{C}. \tag{5.4}$$

To this end, we take any vector Φ such that $\Phi = E(K_\Phi)\Phi$ for a compact set K_Φ in the interior of the future light cone and consider the function

$$F(x, y) = \langle \Phi, [A^{\mathrm{out}}\{f\}(x), A'^{\mathrm{out}}\{f'\}(y)]\Omega \rangle. \tag{5.5}$$

Making use of part (e) and the energy–momentum transfer relation, we get that the support of the Fourier transform of F is contained in the compact set

$$\{ p, q \in \mathbb{R}^4 \mid p_0^2 = |\boldsymbol{p}|^2,\ q_0^2 = |\boldsymbol{q}|^2,\ p + q \in K_\Phi \}. \tag{5.6}$$

Therefore, F is an entire analytic function and since it vanishes on an open subset of $\mathbb{R}^8$ by (5.3), it vanishes everywhere. Hence,

$$[A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]\Omega = c\,\Omega + \Psi_{\mathrm{ph}}, \tag{5.7}$$

where $\Psi_{\mathrm{ph}} \in \mathfrak{h}_{\mathrm{ph}}$. Thus to prove (5.4), it remains to show that $\Psi_{\mathrm{ph}} = 0$.[5] For this purpose, we choose $\Phi_1, \Phi_2 \in \mathfrak{h}_{\mathrm{ph}} \cap D_H$ and compute by means of Proposition 3.4 and Corollary 4.3 that

$$\langle \Phi_1, A^{\mathrm{out}}\{f\}\Phi_2 \rangle = \langle \Phi_1, A^{\mathrm{out}}\{f\}^+\Phi_2 \rangle + \langle \Phi_1, A^{\mathrm{out}}\{f\}^-\Phi_2 \rangle = 0. \tag{5.8}$$

[5] In collision theory of massless fermions, $\Psi_{\mathrm{ph}} = 0$ is automatic in the corresponding expression, since a bosonic operator cannot create a fermionic single-particle state from the vacuum [4, Lemma 4]. In the present bosonic case, we can conclude using Theorem 4.2.

Given (5.4), we complete the proof of part (f) as follows. Let $\mathcal{O}$ be a double cone such that $A, A' \subset \mathfrak{A}^{C_*}(\mathcal{O})$. Then, for any $B \in \mathfrak{A}_{\mathrm{loc}}(\mathcal{O}_+)$ and $\Psi \in D_H$,

$$\begin{aligned}\langle \Psi, [A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]B\Omega\rangle &= \lim_{t\to\infty} \langle \Psi, [\bar{A}_t\{f\}, \bar{A}'_t\{f'\}]B\Omega\rangle \\ &= \lim_{t\to\infty} \langle \Psi, B[\bar{A}_t\{f\}, \bar{A}'_t\{f'\}]\Omega\rangle \\ &= \langle \Psi, B\Omega\rangle\langle \Omega, [A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]\Omega\rangle, \end{aligned} \tag{5.9}$$

where in the first step Proposition 5.1 (c) and in the second step the localization properties of the approximating sequences are used. Equation (5.9) extends by continuity from $D(\mathcal{O}_+)$ to D_H, since Ψ is in the domain of $([A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}])^*$. □

Now, we consider a representation π of $(\mathfrak{A}, U)$ which is hypercone localized w.r.t. the vacuum representation and describes Wigner particles of mass $m > 0$ ('electrons'). For brevity, we will write $(\hat{\mathfrak{A}}, \hat{U})$ for the resulting net $(\mathfrak{A}_\pi, U_\pi)$ and $\hat{\mathcal{H}} := \mathcal{H}_\pi$. Furthermore, we set $\hat{A} := \pi(A)$ for $A \in \mathfrak{A}$ and denote by $(\hat{H}, \hat{\boldsymbol{P}})$ the energy–momentum operators in the representation π.

Given $\hat{A} \in \hat{\mathfrak{A}}^{C_*}(\mathcal{O})$ and $\mathrm{supp}\, f \subset \Theta \subset S^2$, the asymptotic field approximants $t \mapsto \bar{\hat{A}}_t\{f\}$ are localized in

$$\mathcal{O}_t := \bigcup_{\tau \in t + t^{\bar{\varepsilon}} \mathrm{supp} h} \{\mathcal{O} + \tau(1, \Theta)\}, \quad t \geq 1, \tag{5.10}$$

where h is the time-averaging function which appeared in (3.7). To ensure Poincaré covariance of our constructions, one also needs to consider $t \mapsto \bar{\hat{A}}_{\lambda,t}\{f \circ g_{\Lambda^{-1}}\}$, $\lambda = (0, \widetilde{\Lambda}) \in \widetilde{\mathcal{P}}_+^\uparrow$, whose localization regions are

$$\mathcal{O}_t^\Lambda := \bigcup_{\tau \in t + t^{\bar{\varepsilon}} \mathrm{supp} h} \{\Lambda\mathcal{O} + \tau(1, g_\Lambda(\Theta))\}, \quad t \geq 1. \tag{5.11}$$

The following geometric lemma is at the basis of the proof of Proposition 5.3 below. Its proof can be found in Appendix C.

Lemma 5.2 *For any $\mathcal{O} \in \mathcal{K}$ and any open $\Theta \subset S^2$ such that $\overline{\Theta} \subsetneq S^2$, there is a future lightcone V, a hypercone $\mathcal{C} \subset \mathcal{F}_V$ and a neighbourhood N of unity in the Lorentz group such that*

$$\Lambda O_t \subset \mathcal{C}^{\mathrm{c}}, \quad O_t^\Lambda \subset \mathcal{C}^{\mathrm{c}}, \quad t \geq 1, \tag{5.12}$$

for all $\Lambda \in N$.

After this preparation, we reestablish in hypercone localized representations all the properties of asymptotic photon fields listed in Proposition 5.1. The geometric idea behind the proof is similar as in the case of spacelike cone localized representations of [5], so we can be brief.

Proposition 5.3 *Let* $\hat{A}, \hat{A}', \hat{A}_i \in \mathfrak{A}^{C_*}$ *and* $f, f', f_i \in C^\infty(S^2)$, $i = 1, \dots, n$. *Then*:

(a) *For any* $\Psi \in D_{\hat{H}}$, *the limit* $\lim_{t\to\infty} \bar{\hat{A}}_t\{f\}\Psi$ *exists and defines a closable operator* $\hat{A}^{\mathrm{out}}\{f\} \restriction D_{\hat{H}}$. *This operator is uniquely determined by the vector* $\hat{A}^{\mathrm{out}}\{f\}\Omega$.
(b) $\hat{A}^{\mathrm{out}}\{f\}D_{\hat{H}} \subset D_{\hat{H}}$.
(c) $\hat{A}_1^{\mathrm{out}}\{f_1\}\dots\hat{A}_n^{\mathrm{out}}\{f_n\}\Psi = \lim_{t\to\infty} \bar{\hat{A}}_{1,t}\{f_1\}\dots\bar{\hat{A}}_{n,t}\{f_n\}\Psi$ *for* $\Psi \in D_{\hat{H}}$.
(d) $\hat{U}(\lambda)\hat{A}^{\mathrm{out}}\{f\}\hat{U}(\lambda)^* = \hat{A}_\lambda^{\mathrm{out}}\{f \circ g_{\Lambda^{-1}}\}$ *on* $D_{\hat{H}}$, *where* $\hat{A}_\lambda = \hat{U}(\lambda)\hat{A}\hat{U}(\lambda)^*$.
(e) $\Box_x \hat{A}^{\mathrm{out}}\{f\}(x)\Psi = 0$ *for* $\Psi \in D_{\hat{H}}$.
(f) $[\hat{A}^{\mathrm{out}}\{f\}, \hat{A}'^{\mathrm{out}}\{f'\}] = \langle\Omega, [A^{\mathrm{out}}\{f\}, A'^{\mathrm{out}}\{f'\}]\Omega\rangle 1_{\hat{\mathcal{H}}}$ *as operators on* $D_{\hat{H}}$.

Proof Let us first assume that $f \in C^\infty(S^2)$ and supp $f \subset \Theta$, with Θ as in Lemma 5.2. Let $\mathcal{O}$ be the localization region of $\hat{A}$, and $\mathcal{O}_t$ be given by (5.10). By Lemma 5.2, there exists a future lightcone V and $\mathcal{C} \subset \mathcal{F}_V$ such that $O_t \subset \mathcal{C}^{\mathrm{c}}$. Hence, by hypercone localization of π, there is a unitary $W_{\mathcal{C}}$ such that for all $t \geq 1$,

$$\bar{\hat{A}}_t\{f\} = \pi(\bar{A}_t\{f\}) = W_{\mathcal{C}}(\bar{A}_t\{f\})W_{\mathcal{C}}^*. \tag{5.13}$$

Now by Proposition 5.1 (a), the right-hand side converges on $W_{\mathcal{C}}D_H$ to an operator which is uniquely determined by $A^{\mathrm{out}}\{f\}\Omega$. Then, by Proposition 3.3, the left-hand side converges on $D_{\hat{H}}$ to an operator which is uniquely determined by $A^{\mathrm{out}}\{f\}\Omega$.

To remove the restriction on the support of f, we choose a partition of unity on S^2 consisting of $f^j \in C^\infty(S^2)$, $j = 1, 2$, such that supp $f^j \subsetneq S^2$. Thus, we may write

$$\bar{\hat{A}}_t\{f\} = \sum_{j=1,2} \bar{\hat{A}}_t\{ff^j\}. \tag{5.14}$$

By the above discussion and Proposition 3.3, $\hat{A}^{\mathrm{out}}\{f\} = \lim_{t\to\infty} \bar{\hat{A}}_t\{f\}$ exists on $D_{\hat{H}}$ and has the properties specified in parts (a)–(c) of the proposition.

Properties (d)–(f) are concluded from the corresponding parts of Proposition 5.1 by a repetitive use of relations similar to (5.13), (5.14). In part (d), it suffices to consider transformations from a small neighbourhood of unity in $SL(2, \mathbb{C})$, as this group is generated by any such neighbourhood. □

Now, we proceed to the construction of scattering states of one electron and a finite number of photons, i.e. Compton scattering. It suffices to consider $f \in C^\infty(S^2)$ which are identically equal to one, in which case we write $\hat{A}^{\mathrm{out}}$ for $\hat{A}^{\mathrm{out}}\{f\}$. Proposition 5.3 (f) gives

$$[\hat{A}^{\mathrm{out}-}, \hat{A}'^{\mathrm{out}+}] = \langle A^{*\mathrm{out}+}\Omega, A'^{\mathrm{out}+}\Omega\rangle, \quad [\hat{A}^{\mathrm{out}-}, \hat{A}'^{\mathrm{out}-}] = [\hat{A}^{\mathrm{out}+}, \hat{A}'^{\mathrm{out}+}] = 0. \tag{5.15}$$

Recalling that by Proposition 3.4 (c) $\hat{A}^{\mathrm{out}+}D_{\hat{H}} \subset D_{\hat{H}}$, scattering states are constructed in a straightforward manner.

Theorem 5.4 *The states* $\Psi^{\mathrm{out}} := \hat{A}_1^{\mathrm{out}+} \dots \hat{A}_n^{\mathrm{out}+}\Psi_{\mathrm{el}}$, $\Psi_{\mathrm{el}} \in \hat{\mathfrak{h}}_{\mathrm{el}} \cap D_{\hat{H}}$, *have the following properties*:

(a) Ψ^{out} *depends only on the single-photon states* $\Phi_i := A_i^{\mathrm{out}}\Omega$ *and the single-electron state* $\Psi_{\mathrm{el}} \in \hat{\mathfrak{h}}_{\mathrm{el}} \cap D_{\hat{H}}$. *Thus, we can write* $\Psi^{\mathrm{out}} = \Phi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_n \overset{\mathrm{out}}{\times} \Psi_{\mathrm{el}}$.

(b) *Given* $\Psi^{\mathrm{out}}, \Psi'^{\mathrm{out}}$ *as above,*

$$\langle \Psi^{\mathrm{out}}, \Psi'^{\mathrm{out}} \rangle = \delta_{n,n'} \langle \Psi_{\mathrm{el}}, \Psi'_{\mathrm{el}} \rangle \sum_{\sigma \in \mathfrak{S}_n} \langle \Phi_1, \Phi'_{\sigma_1} \rangle \cdots \langle \Phi_n, \Phi'_{\sigma_n} \rangle, \tag{5.16}$$

where $\mathfrak{S}_n$ *is the set of all permutations of* $(1, \dots, n)$.

(c) $\hat{U}(\lambda)(\Phi_1 \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} \Phi_n \overset{\mathrm{out}}{\times} \Psi_{\mathrm{el}}) = (U(\lambda)\Phi_1) \overset{\mathrm{out}}{\times} \cdots \overset{\mathrm{out}}{\times} (U(\lambda)\Phi_n) \overset{\mathrm{out}}{\times} (\hat{U}(\lambda)\Psi_{\mathrm{el}})$, $\lambda \in \widetilde{\mathcal{P}}_+^{\uparrow}$.

Proof Parts (a) and (b) follow directly from (5.15) and Corollary 4.3 which gives $\hat{A}^{\mathrm{out}-}\Psi_{\mathrm{el}} = 0$ for $\Psi_{\mathrm{el}} \in \hat{\mathfrak{h}}_{\mathrm{el}} \cap D_{\hat{H}}$. Part (c) follows from Proposition 5.3 (d). □

Let $\Gamma(\mathfrak{h}_{\mathrm{ph}})$ be the symmetric Fock space over $\mathfrak{h}_{\mathrm{ph}}$ and we denote by $a^*(\,\cdot\,)$ and $a(\,\cdot\,)$ the corresponding creation and annihilation operators. Using Theorem 5.4 (a), (b), we define the *outgoing wave operator* of Compton scattering

$$W^{\mathrm{out}}(\Gamma(\mathfrak{h}_{\mathrm{ph}}) \otimes \hat{\mathfrak{h}}_{\mathrm{el}}) \to \hat{\mathcal{H}} \tag{5.17}$$

as the unique linear isometry, satisfying

$$W^{\mathrm{out}}(a^*(\Phi_1)\dots a^*(\Phi_n)\Omega \otimes \Psi_{\mathrm{el}}) = \hat{A}_1^{\mathrm{out}+} \dots \hat{A}_n^{\mathrm{out}+}\Psi_{\mathrm{el}}, \tag{5.18}$$

for $\Phi_i = A_i^{\mathrm{out}}\Omega$. Setting $U_{\mathrm{ph}}(\lambda) := \Gamma(U(\lambda) \restriction \mathfrak{h}_{\mathrm{ph}})$ and $\hat{U}_{\mathrm{el}}(\lambda) := \hat{U}(\lambda) \restriction \hat{\mathfrak{h}}_{\mathrm{el}}$, we obtain from Theorem 5.4 (c)

$$\hat{U}(\lambda) \circ W^{\mathrm{out}} = W^{\mathrm{out}} \circ (U_{\mathrm{ph}}(\lambda) \otimes \hat{U}_{\mathrm{el}}(\lambda)), \quad \lambda \in \widetilde{\mathcal{P}}_+^{\uparrow}, \tag{5.19}$$

which amounts to the Poincaré covariance of the wave operator.

To conclude this section, we construct the Haag–Kastler net of asymptotic photon fields in the hypercone localized representation π and show that single-electron states induce vacuum representations of this net. To this end, we need the following technical lemma which summarizes and extends the information about the asymptotic fields and their domains.

Lemma 5.5 *Let* $\hat{A}, \hat{A}_1, \hat{A}_2 \in \hat{\mathfrak{A}}^{C_*}$ *be self-adjoint. Then*:

(a) $D(\hat{H}) \subset D_{\max}(\hat{A})$ *and* $\hat{A}^{\mathrm{out}} \restriction D(\hat{H})$ *is a symmetric operator uniquely determined by* $A^{\mathrm{out}}\Omega$.

(b) $\|\hat{A}^{\mathrm{out}}\Psi\| \leq c\|(1+\hat{H})\Psi\|$, $\Psi \in D(\hat{H})$.

(c) $|\langle \hat{H}\Psi, \hat{A}^{\mathrm{out}}\Psi \rangle - \langle \hat{A}^{\mathrm{out}}\Psi, \hat{H}\Psi \rangle| \leq c\|(1+H)^{1/2}\Psi\|^2$, $\Psi \in D(\hat{H})$.

(d) $\hat{A}_1^{\mathrm{out}}, \hat{A}_2^{\mathrm{out}}$ *are essentially self-adjoint on* $D(\hat{H})$ *and their self-adjoint extensions* $\hat{A}_1^{\mathrm{out}\bullet}, \hat{A}_2^{\mathrm{out}\bullet}$ *satisfy*

$$e^{i(\hat{A}_1+\hat{A}_2)^{\mathrm{out}\bullet}} = e^{\frac{1}{2}\langle \Omega,[A_1^{\mathrm{out}},A_2^{\mathrm{out}}]\Omega\rangle} e^{i\hat{A}_1^{\mathrm{out}\bullet}} e^{i\hat{A}_2^{\mathrm{out}\bullet}}. \tag{5.20}$$

(e) $[\hat{A}_1^{\mathrm{out}}, \hat{A}_2^{\mathrm{out}}] = \langle \Omega, [A_1^{\mathrm{out}}, A_2^{\mathrm{out}}]\Omega\rangle 1_{\hat{\mathcal{H}}}$ *as quadratic forms on* $D(\hat{H}) \times D(\hat{H})$.

(f) $\langle \Omega, [A_1^{\mathrm{out}}, A_2^{\mathrm{out}}]\Omega\rangle = 0$ *if* A_1, A_2 *are localized in spacelike separated double cones.*

Proof Parts (a), (b) follow from Proposition 5.3 (a), the energy bounds and approximation arguments. Part (c) is shown by interpolation (cf. [23, Appendix to IX.4]). Part (d) follows from (a) to (c), the Nelson Commutator Theorem and standard results on integrating Heisenberg commutation relations to Weyl relations from [14]. Part (e) is a consequence of Proposition 5.3 (f) and part (f) is a known consequence of the JLD representation (cf. [3, p. 160]). □

Now, we are in a position to define the net of asymptotic photon fields. For any $\mathcal{O} \in \mathcal{K}$, we, thus, introduce the von Neumann algebra

$$\hat{\mathfrak{A}}^{\mathrm{out}}(\mathcal{O}) := \{ e^{i\hat{A}^{\mathrm{out}\bullet}} \mid \hat{A} \in \hat{\mathfrak{A}}^{C_*}(\mathcal{O}),\ \hat{A}^* = \hat{A} \}''. \tag{5.21}$$

It is easy to see that $(\hat{\mathfrak{A}}^{\mathrm{out}}, \hat{U})$ is a Haag–Kastler net in the sense of Definition 2.1. For example, locality follows from Lemma 5.5 (d), (f). In the following theorem, we consider representations of $(\hat{\mathfrak{A}}^{\mathrm{out}}, \hat{U})$ induced by vector states from $\hat{\mathfrak{h}}_{\mathrm{el}}$.

Theorem 5.6 *Let* $\Psi_{\mathrm{el}} \in \hat{\mathfrak{h}}_{\mathrm{el}} \cap D_{\hat{H}}$, $\|\Psi_{\mathrm{el}}\| = 1$, $\omega_{\mathrm{el}}(\,\cdot\,) := \langle \Psi_{\mathrm{el}}, \cdot\, \Psi_{\mathrm{el}}\rangle$ *be the corresponding state on* $\hat{\mathfrak{A}}^{\mathrm{out}}$ *and* $(\pi_{\mathrm{el}}, \mathcal{H}_{\pi_{\mathrm{el}}}, \Omega_{\pi_{\mathrm{el}}})$ *its GNS representation. Then,* π_{el} *is a vacuum representation of* $(\hat{\mathfrak{A}}^{\mathrm{out}}, \hat{U})$ *in the sense of Definition 2.3.*

Remark 5.7 It is easy to see that the above theorem also holds if π is the original vacuum representation and Ψ_{el} is replaced with Ω. This gives a different proof of a result from [3].

Proof It suffices to show for self-adjoint $\hat{A} \in \hat{\mathfrak{A}}^{C_*}$ that

$$\langle \Psi_{\mathrm{el}}, e^{i\hat{A}^{\mathrm{out}\bullet}} \Psi_{\mathrm{el}}\rangle = e^{-\frac{1}{2}\|A^{\mathrm{out}}\Omega\|^2}, \tag{5.22}$$

as the statement of the theorem follows from this relation by standard arguments [1,12]. Consider the function $f(s) := \langle \Psi_{\mathrm{el}}, e^{is\hat{A}^{\mathrm{out}\bullet}} \Psi_{\mathrm{el}}\rangle$. Since $\Psi_{\mathrm{el}} \in D_{\hat{H}}$ is contained in the domain of $\hat{A}^{\mathrm{out}\bullet}$, we have by the Stone theorem

$$(-i)\partial_s f(s) = \langle \Psi_{\mathrm{el}}, e^{is\hat{A}^{\mathrm{out}\bullet}} \hat{A}^{\mathrm{out}\bullet} \Psi_{\mathrm{el}}\rangle = \langle \Psi_{\mathrm{el}}, e^{is\hat{A}^{\mathrm{out}\bullet}} \hat{A}^{\mathrm{out}} \Psi_{\mathrm{el}}\rangle. \tag{5.23}$$

As $\hat{A}^{\mathrm{out}} D_{\hat{H}} \subset D_{\hat{H}}$, we can iterate. This gives in particular

$$(-i)^n \partial_s^n f(s)|_{s=0} = \langle \Psi_{\mathrm{el}}, (\hat{A}^{\mathrm{out}})^n \Psi_{\mathrm{el}}\rangle. \tag{5.24}$$

Now, we use Proposition 3.4 to decompose $\hat{A}^{\mathrm{out}} = \hat{A}^{\mathrm{out}+} + \hat{A}^{\mathrm{out}-}$ on $D_{\hat{H}}$ while keeping in mind that $\hat{A}^{\mathrm{out}\pm} D_{\hat{H}} \subset D_{\hat{H}}$. Due to the canonical commutation relations (5.15), the fact that $\hat{A}^{\mathrm{out}-}\Psi_{\mathrm{el}} = 0$ (Corollary 4.3) and standard combinatorics, we, moreover, have for even $n \geq 2$

$$\langle \Psi_{\mathrm{el}}, (\hat{A}^{\mathrm{out}})^n \Psi_{\mathrm{el}} \rangle = (n-1)!! \langle \Omega, (A^{\mathrm{out}})^2 \Omega \rangle^{n/2} \tag{5.25}$$

and zero for odd $n \geq 1$. Thus, we obtain

$$\sum_{n=0}^{\infty} \frac{\mathrm{i}^n s^n}{n!} \langle \Psi_{\mathrm{el}}, (\hat{A}^{\mathrm{out}})^n \Psi_{\mathrm{el}} \rangle = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell s^{2\ell}}{2^\ell \ell!} \langle \Omega, (A^{\mathrm{out}})^2 \Omega \rangle^\ell = \mathrm{e}^{-\frac{1}{2}s^2 \|A^{\mathrm{out}}\Omega\|^2}, \tag{5.26}$$

where we set $\ell = n/2$. Since the sum on the left-hand side above is absolutely convergent for any $s \in \mathbb{C}$, we conclude that f extends to an entire analytic function which coincides with the function on the right-hand side of (5.26). □

6 Conclusion

From Theorem 5.4 and the ensuing discussion, it may seem that the incoming wave operator W^{in} can be constructed analogously as W^{out} and the scattering matrix $S = (W^{\mathrm{out}})^* W^{\mathrm{in}}$ is available. Unfortunately, the situation is less satisfactory than that. As far as we can see, the incoming wave operator W^{in} is not at hand in a representation π which is hypercone localized in a future lightcone. While the hypercone localization property (1.2) allows us to establish the existence of the *outgoing* asymptotic photon fields, it is of no help for the *incoming* photon fields. This is due to the fact that the approximating sequences of the incoming asymptotic photon fields are localized in regions moving to infinity in negative lightlike directions. Heuristically speaking, such regions inevitably collide with the highly fluctuating background radiation, emitted in the very distant past, which must be present in π to prevent the existence of the flux (1.1). It is therefore reasonable to expect that also the incoming asymptotic photon fields are blurred by this radiation as depicted in Fig. 1b. As a possible solution, one can consider a representation π' hypercone localized in a *backward* lightcone in which by obvious modifications of our discussion only the incoming wave operator $(W')^{\mathrm{in}}$ exists. Since π and π' act on the same Hilbert space in the Buchholz–Roberts setting, the scattering matrix $S' = (W^{\mathrm{out}})^*(W')^{\mathrm{in}}$ can be defined. It may not be Poincaré invariant and its physical interpretation is obscured by the expected disjointness of π and π', but similar difficulties are encountered in the traditional infraparticle approach. One advantage of the present approach is that a tentative expression for the scattering matrix can be given in the Haag–Kastler setting.

As an alternative to this scattering matrix, one may try to construct inclusive collision cross sections. This idea, implemented in AQFT by Buchholz et al. [8], amounts in our situation to the preparation of incoming states using asymptotic observables of the form

$$C_t := \int \mathrm{d}^3x\, h(\boldsymbol{x}/t)(B^*B)(t,\boldsymbol{x}). \tag{6.1}$$

Here, $h \in C_0^\infty(\mathbb{R}^3)$ is supported on velocities of the desired particle and B is an almost local observable whose energy–momentum transfer is outside of the future lightcone. Due to this latter property, which cannot be imposed on strictly local observables $\hat{A}$ appearing in the definition of asymptotic photon fields, B is much less sensitive to the background radiation mentioned above. Thus, the tentative inclusive collision cross sections of the form

$$\lim_{t\to-\infty} \langle \Psi^{\mathrm{out}}, C_{1,t} \dots C_{\ell,t} \Psi^{\mathrm{out}} \rangle \tag{6.2}$$

are likely to exist. Although available methods allow to control such limits only in massive theories [13], their extension to the case of sharp masses embedded in continuous spectrum is thinkable. Another strategy may be to consider limits (6.2) in the framework of algebraic perturbative QFT. As a matter of fact, (6.2) bears some similarity to expressions studied in the book of Steinmann [24, formula (16.38)].

Acknowledgements W.D. would like to thank D. Buchholz, M. Duell, C. Gérard, A. Pizzo and Y. Tanimoto for interesting discussions on topics related to this paper. Financial support from the Emmy Noether Programme of the DFG, within the Grant DY 107/2-1, is gratefully acknowledged.

Appendix A: Mean Ergodic Theorem and invariant vectors

We pick h as in (3.7) and recall a variant of the abstract Mean Ergodic Theorem:

Theorem A.1 *Let S be a self-adjoint operator on (a domain in) $\mathcal{H}$ and F_S its spectral measure. Then,*

$$\text{s-lim}_{t\to\infty} \int \mathrm{d}t'\, h_t(t')\mathrm{e}^{\mathrm{i}t'S} = F_S(\{0\}). \tag{A.1}$$

Now, we determine the projection $F_S(\{0\})$ on the subspace of invariant vectors of $t \mapsto \mathrm{e}^{\mathrm{i}tS}$ for the relevant operators S.

Proposition A.2 *Let $(H, \boldsymbol{P})$ be the energy–momentum operators of a Haag–Kastler theory and E their joint spectral measure.*

(a) *Let $S_\nu := H - \cos\nu |\boldsymbol{P}|$ and F_{S_ν} be the spectral measure of S_ν. Then,*

$$F_{S_\nu}(\{0\}) = \begin{cases} E(\partial\overline{V}_+) & \text{for } \nu = 0, \\ E(\{0\}) & \text{for } \nu \in (0,\pi]. \end{cases} \tag{A.2}$$

(b) *Let $S_{\nu,\boldsymbol{\xi}} := H - |\boldsymbol{\xi}|\cos\nu - \omega_m(\boldsymbol{P}+\boldsymbol{\xi})$, where $\omega_m(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2+m^2}$, and $F_{S_{\nu,\boldsymbol{\xi}}}$ be the spectral measure of $S_{\nu,\boldsymbol{\xi}}$. Then, for $\boldsymbol{\xi} \neq 0$,*

$$F_{S_{\nu,\boldsymbol{\xi}}}(\{0\}) = \begin{cases} 0 & \text{for } \nu \in [0,\pi) \text{ or } m > 0, \\ E(\{0\}) & \text{for } \nu = \pi \text{ and } m = 0. \end{cases} \tag{A.3}$$

Proof (a) For $\Psi_0 \in \mathrm{Ran} F_{S_0}(\{0\})$, we have $(H - |\boldsymbol{P}|)\Psi_0 = 0$; hence, $\Psi_0 \in \mathrm{Ran} E(\partial \overline{V}_+)$. This gives the first part of (A.2). To check the second part, we note that for $\nu \in (0, \pi]$ the set

$$\Delta_\nu := \{ (p^0, \boldsymbol{p}) \mid p^0 = \cos\nu\, |\boldsymbol{p}| \} \tag{A.4}$$

intersects with $\overline{V}_+$ only at $\{0\}$.

(b) First, we note that the set

$$\Delta_{\nu,\boldsymbol{\xi}} := \{ (p^0, \boldsymbol{p}) \mid p^0 = |\boldsymbol{\xi}| \cos\nu + \omega_m(\boldsymbol{p} + \boldsymbol{\xi}) \} \tag{A.5}$$

describes a mass hyperboloid shifted by a spacelike or lightlike vector $(|\boldsymbol{\xi}| \cos\nu, -\boldsymbol{\xi})$. Thus, $\Delta_{\nu,\boldsymbol{\xi}}$ contains zero only if $m = 0$ and $\nu = \pi$. Hence, it suffices to show that the relation

$$(H - \omega_m(\boldsymbol{P} + \boldsymbol{\xi}))\Psi = |\boldsymbol{\xi}| \cos\nu\Psi, \tag{A.6}$$

where $\Psi = E(\Delta)\Psi$, Δ compact, can only hold for $\Psi \in E(\{0\})\mathcal{H}$. This is shown by generalizing an argument from Appendix of [4]. $\square$

Appendix B: Admissible propagation observables

Definition B.1 Let $[1, \infty) \ni t \mapsto A_t \in B(\mathcal{H})$ be a propagation observable, H a positive, self-adjoint operator on a domain $D(H)$ in $\mathcal{H}$, and $D, D^* \subset \mathcal{H}$ some dense domains. We say that A is admissible if:

(a) For any $\Psi \in D^{(*)}$, the limit $\lim_{t\to\infty} A_t^{(*)}\Psi$ exists.
(b) $\sup_{t\in[1,\infty)} \|A_t^{(*)}(1 + H)^{-1}\| < \infty$.
(c) Set $A_t(s) := \mathrm{e}^{\mathrm{i}sH} A_t \mathrm{e}^{-\mathrm{i}sH}$. All the derivatives $A_t^{(n)} = \partial_s^n A_t(s)|_{s=0}$ exist in norm and satisfy (a), (b).

Here, $(*)$ means that the statement holds with and without all $*$ symbols (correlated).

As shown in the next two propositions, limits of admissible propagation observables exist as closable operators on the following dense domain:

$$D_H := \bigcap_{n\geq 1} D(H^n). \tag{B.1}$$

Moreover, D_H is an invariant domain of these limits.

Proposition B.2 *Let A be an admissible propagation observable. Then*:

(a) *For any $\Psi \in D_H$, the limit* $\lim_{t\to\infty} A_t\Psi$ *exists and defines a closable operator A^{out} on D_H. This operator is uniquely specified by its values on D.*
(b) $A^{\mathrm{out}} D_H \subset D_H$.

Proof Exploiting part (c) of Definition B.1, we write

$$A_t\Psi = (1+H)^{-1}(-\mathrm{i})A_t^{(1)}\Psi + (1+H)^{-1}A_t(1+H)\Psi. \tag{B.2}$$

Vectors Ψ, $(1+H)\Psi$ appearing on the right-hand side of (B.2) can be approximated uniformly in t by elements of D (cf. Definition B.1 (b), (c)). By part (a) of Definition B.1, A_t, $A_t^{(1)}$ converge on D which gives the existence of A^{out} as an operator on D_H. Since the above reasoning applies also to A_t^*, the operator A^{out} is closable. To show that it is uniquely determined by its values on D, consider admissible propagation observables A_1 and A_2 such that $\lim_{t\to\infty} A_{1,t}\Phi = \lim_{t\to\infty} A_{2,t}\Phi$ for $\Phi \in D$. Then, it is clear from the above discussion that $A_1^{\mathrm{out}} = A_2^{\mathrm{out}}$ as operators on D_H. This completes the proof of (a).

To prove (b), we make use of a standard commutator formula (see e.g. [15])

$$[(1+H)^{\ell}, A_t] = \sum_{k=1}^{\ell} \binom{\ell}{k} \mathrm{ad}_H^k(A_t)(1+H)^{\ell-k}, \tag{B.3}$$

$$\mathrm{ad}_H^0(A_t) := A_t, \qquad \mathrm{ad}_H^k(A_t) := [H, \mathrm{ad}_H^{k-1}(A_t)], \tag{B.4}$$

which holds as an equality of quadratic forms on $D_H \times D_H$. Exploiting part (c) of Definition B.1, which ensures that $\mathrm{ad}_H^k(A_t) = (-\mathrm{i})^k A_t^{(k)}$ are bounded operators, we obtain for any $\Psi \in D_H$

$$A_t\Psi = (1+H)^{-\ell}\left(\sum_{k=0}^{\ell}\binom{\ell}{k}(-\mathrm{i})^k A_t^{(k)}(1+H)^{\ell-k}\right)\Psi, \tag{B.5}$$

where we set by convention $A_t^{(0)} = A_t$. Taking now the limit $t\to\infty$ on both sides of (B.5), we obtain (b). □

Proposition B.3 *Let A_i, $i = 1,\dots,n$, be admissible propagation observables. Then, for any $\Psi \in D_H$,*

$$A_1^{\mathrm{out}}\dots A_n^{\mathrm{out}}\Psi = \lim_{t\to\infty} A_{1,t}\dots A_{n,t}\Psi. \tag{B.6}$$

Proof For $n = 1$, the statement follows from Proposition B.2. We suppose now it holds for some $n > 1$ and prove it for $n+1$. Similarly as in (B.2), $\Psi \in D_H$,

$$\begin{aligned} A_{1,t}\dots A_{n+1,t}\Psi = {} & A_{1,t}(1+H)^{-1}(-\mathrm{i})\sum_{\ell=2}^{n+1} A_{2,t}\dots A_{\ell,t}^{(1)}\dots A_{n+1,t}\Psi \\ & + A_{1,t}(1+H)^{-1}A_{2,t}\dots A_{n+1,t}(1+H)\Psi. \end{aligned} \tag{B.7}$$

By the induction hypothesis and Proposition B.2 the above expression converges strongly as $t\to\infty$. Next, we pick $\Phi \in D_H$ and write

$$\begin{aligned}\langle \Phi, A_{1,t} \dots A_{n+1,t} \Psi \rangle &= \langle \Phi, A_{1,t} A_2^{\text{out}} \dots A_{n+1}^{\text{out}} \Psi \rangle + o(t^0) \\ &= \langle \Phi, A_1^{\text{out}} A_2^{\text{out}} \dots A_{n+1}^{\text{out}} \Psi \rangle + o(t^0), \end{aligned} \tag{B.8}$$

where in the first step we used the induction hypothesis, in the second step Proposition B.2 and $o(t^0)$ denotes terms which tend to zero as $t \to \infty$. This concludes the proof. □

Appendix C: Geometric argument

We refer to Sect. 2 and to [9, Appendix] for a brief summary of relevant geometric concepts.

Proof of Lemma 5.2 We prove the statement only for $\Lambda = I$, since the generalization to small Lorentz transformations then easily follows. First, we note that

$$\bigcup_{t \geq 1} O_t = \bigcup_{t \geq 1} \bigcup_{\tau \in t + t^{\bar{\varepsilon}} \operatorname{supp} h} \{\mathcal{O} + \tau(1, \Theta)\} \subset \bigcup_{\tau \in \mathbb{R}_+} \{\mathcal{O} + \tau(1, \Theta)\} =: \mathcal{U}. \tag{C.1}$$

We will show that for any double cone $\mathcal{O} \in \mathcal{K}$ and open $\Theta \subset S^2$ with $\overline{\Theta} \subsetneq S^2$, there is a future lightcone V and a hypercone $\mathcal{C} \subset \mathcal{F}_V$ such that the corresponding set $\mathcal{U}$ given by (C.1) is in $\mathcal{C}^c$. Such a situation is depicted in Fig. 2.

We fix a future lightcone V so that $\overline{\mathcal{O}} \subset V$ and choose a coordinate frame in which the origin is at the apex of V. Next, we use the fact that there is an $\boldsymbol{\ell}_0 \in S^2$ and an $1 \geq \varepsilon_0 > 0$ such that the spherical cap

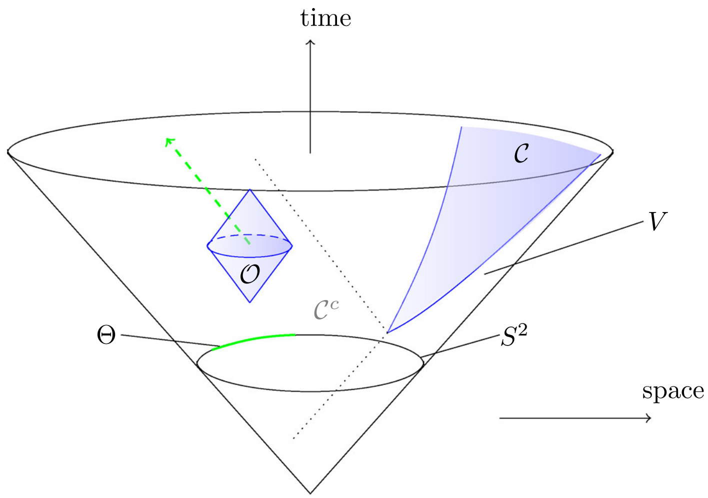

Fig. 2 Geometrical situation in Lemma 5.2 for the case $\Lambda = I$. The double cone $\mathcal{O}$ is shifted into lightlike directions determined by $\Theta \subset S^2$. The resulting union of shifted double cones gives the region $\mathcal{U}$ in accordance with Eq. (C.1). As shown below, $\mathcal{U}$ is in $\mathcal{C}^c$ which is indicated by the *dotted lines*

$$\Theta_\varepsilon := \{\boldsymbol{\ell} \in S^2 \,|\, 1 - \varepsilon \le \boldsymbol{\ell}\boldsymbol{\ell}_0 \le 1\} \tag{C.2}$$

is contained in $S^2 \backslash \overline{\Theta}$ for all $0 < \varepsilon \le \varepsilon_0$. Let, moreover, K_ε be a cone in the unit ball B with apex at $\boldsymbol{u}_\varepsilon := (1-\varepsilon)\boldsymbol{\ell}_0$ and the opening angle determined by Θ_ε. More precisely,

$$\mathsf{K}_\varepsilon := \{\boldsymbol{u} \in \mathsf{B} \,|\, \boldsymbol{u} = \boldsymbol{u}_\varepsilon + s\,(\boldsymbol{\ell} - \boldsymbol{u}_\varepsilon),\ 0 \le s < 1,\ \boldsymbol{\ell} \in \Theta_\varepsilon\}\,. \tag{C.3}$$

Using the Beltrami–Klein map $\boldsymbol{v} : \mathsf{H}_{\bar\tau} \to \mathsf{B}$ given by $\boldsymbol{v}(a) = \boldsymbol{a}/a^0$, the corresponding hyperbolic cone $\mathsf{C}(\mathsf{K}_\varepsilon) \subset \mathsf{H}_{\bar\tau}$ is given by

$$\mathsf{C}(\mathsf{K}_\varepsilon) = \left\{ \bar\tau \frac{(1,\boldsymbol{u})}{\sqrt{1-\boldsymbol{u}^2}} \in \mathsf{H}_{\bar\tau} \,\middle|\, \boldsymbol{u} = \boldsymbol{u}_\varepsilon + s\,(\boldsymbol{\ell} - \boldsymbol{u}_\varepsilon),\ 0 \le s < 1,\ \boldsymbol{\ell} \in \Theta_\varepsilon \right\}. \tag{C.4}$$

We note that as $\varepsilon \to 0$, the apex of $\mathsf{C}(\mathsf{K}_\varepsilon)$ tends to lightlike infinity in the direction of $\boldsymbol{\ell}_0$ and the opening angle tends to zero. In fact, for all $0 \le s < 1$ and $\boldsymbol{\ell} \in \Theta_\varepsilon$, we have

$$\boldsymbol{u}_\varepsilon(s,\boldsymbol{\ell}) := \boldsymbol{u}_\varepsilon + s\,(\boldsymbol{\ell} - \boldsymbol{u}_\varepsilon) = \boldsymbol{\ell}_0(1 - \varepsilon(1-s)) + s(\boldsymbol{\ell} - \boldsymbol{\ell}_0). \tag{C.5}$$

Noting that $(\boldsymbol{\ell} - \boldsymbol{\ell}_0)^2 = 2(1 - \boldsymbol{\ell}\boldsymbol{\ell}_0) \le 2\varepsilon$ and setting $\boldsymbol{h}_\varepsilon(s,\boldsymbol{\ell}) := -\varepsilon^{\frac12}\boldsymbol{\ell}_0(1-s) + s\varepsilon^{-\frac12}(\boldsymbol{\ell} - \boldsymbol{\ell}_0)$, we have

$$\boldsymbol{u}_\varepsilon(s,\boldsymbol{\ell}) = \boldsymbol{\ell}_0 + \varepsilon^{\frac12}\boldsymbol{h}_\varepsilon(s,\boldsymbol{\ell}), \tag{C.6}$$
$$|\boldsymbol{h}_\varepsilon(s,\boldsymbol{\ell})| \le 3. \tag{C.7}$$

Now, a simple computation using (C.5) gives

$$1 - \boldsymbol{u}_\varepsilon(s,\boldsymbol{\ell})^2 = \varepsilon(1-s)\big\{2 - \varepsilon(1-s) + 2s(1-\varepsilon)(1 - \boldsymbol{\ell}\boldsymbol{\ell}_0)\varepsilon^{-1}\big\}. \tag{C.8}$$

It is easy to see that $1 \le \{\ldots\} \le 4$ and, therefore, we can find a function $(s,\boldsymbol{\ell}) \mapsto g_\varepsilon(s,\boldsymbol{\ell})$ such that $\frac{\bar\tau}{2} \le g_\varepsilon(s,\boldsymbol{\ell}) \le \bar\tau$ and

$$\bar\tau \frac{1}{\sqrt{1 - \boldsymbol{u}_\varepsilon(s,\boldsymbol{\ell})^2}} = \frac{g_\varepsilon(s,\boldsymbol{\ell})}{\sqrt{\varepsilon(1-s)}}. \tag{C.9}$$

Thus, skipping the arguments of $g, \boldsymbol{h}$ and setting $M := \varepsilon^{-\frac12}$, $S := g(1-s)^{-\frac12}$, we have

$$\bar\tau \frac{(1, \boldsymbol{u}_\varepsilon(s,\boldsymbol{\ell}))}{\sqrt{1 - \boldsymbol{u}_\varepsilon(s,\boldsymbol{\ell})^2}} = MS\cdot(1,\boldsymbol{\ell}_0) + S\cdot(0,\boldsymbol{h}), \tag{C.10}$$

where M takes values in $[\varepsilon_0^{-1/2}, \infty)$ and S in $[\frac{\bar\tau}{2}, \infty)$. Thus, we found a convenient parametrization of $\mathsf{C}(\mathsf{K}_\varepsilon)$. We will use it to establish the relation (C.14) below, which ensures spacelike separation of $\mathsf{C}(\mathsf{K}_\varepsilon)$ and $\mathcal{U}$ for sufficiently small ε.

As a preparation, let us show that there is a $c > 0$ such that for sufficiently large M

$$(MS\cdot(1, \boldsymbol{\ell}_0) + S\cdot(0, \boldsymbol{h}) - x)^2 < -c, \tag{C.11}$$

for all $x \in \mathcal{O}$, $S \in [\frac{\bar{\tau}}{2}, \infty)$ and $\boldsymbol{h}$ within the above restrictions. Since $\overline{\mathcal{O}} \subset V$, there are constants $c_{\mathcal{O}}, c'_{\mathcal{O}}$ such that

$$0 < c_{\mathcal{O}} \leq (x^0 \pm |\boldsymbol{x}|) \leq c'_{\mathcal{O}}, \tag{C.12}$$

uniformly in $x \in \mathcal{O}$. Moreover, due to (C.10), we have $(MS\cdot(1, \boldsymbol{\ell}_0) + S\cdot(0, \boldsymbol{h}))^2 = \bar{\tau}^2$. Hence,

$$\begin{aligned}(MS\cdot(1, \boldsymbol{\ell}_0) + S\cdot(0, \boldsymbol{h}) - x)^2 &= \bar{\tau}^2 - 2MS(x^0 - \boldsymbol{x}\boldsymbol{\ell}_0) - 2S\cdot(0, \boldsymbol{h})x + x^2 \\ &\leq -2MSc_{\mathcal{O}} + 6Sc'_{\mathcal{O}} + (c'_{\mathcal{O}})^2 + \bar{\tau}^2, \end{aligned} \tag{C.13}$$

which proves (C.11).

Finally, let us show that there is a $c' > 0$ such that for sufficiently large M,

$$(MS\cdot(1, \boldsymbol{\ell}_0) + S\cdot(0, \boldsymbol{h}) - x - \tau(1, \boldsymbol{\ell}'))^2 < -c', \tag{C.14}$$

for all $\tau \in \mathbb{R}_+$, $\boldsymbol{\ell}' \in \Theta$, $x \in \mathcal{O}$, $S \in [\frac{\bar{\tau}}{2}, \infty)$ and $\boldsymbol{h}$ within the above restrictions. In view of (C.11), it suffices to note the estimate

$$\begin{aligned}\big(MS\cdot(1, \boldsymbol{\ell}_0) + S\cdot(0, \boldsymbol{h}) - x\big)(1, \boldsymbol{\ell}') &= S(M(1 - \boldsymbol{\ell}_0\boldsymbol{\ell}') - \boldsymbol{h}\boldsymbol{\ell}') - x(1, \boldsymbol{\ell}') \\ &\geq (\bar{\tau}/2)\big(M\varepsilon_0 - 3\big) - c'_{\mathcal{O}}. \end{aligned} \tag{C.15}$$

Thus, we have proven that $\mathcal{U} \subset \mathsf{C}(\mathsf{K}_\varepsilon)^{\mathrm{c}} = \mathcal{C}(\mathsf{K}_\varepsilon)^{\mathrm{c}}$ for ε sufficiently small, depending on $\mathcal{O}$ and Θ. $\square$

References

1. Araki, H.: Mathematical Theory of Quantum Fields. Oxford University Press, New York (1999)
2. Bachmann, S., Dybalski, W., Naaijkens, P.: Lieb–Robinson bounds, Arveson spectrum and Haag–Ruelle scattering theory for gapped quantum spin systems. To appear in Ann. Henri Poincaré. Preprint arXiv:1412.2970
3. Buchholz, D.: Collision theory for massless bosons. Commun. Math. Phys. **52**, 147–173 (1977)
4. Buchholz, D.: Collision theory for massless fermions. Commun. Math. Phys. **42**, 269–279 (1975)
5. Buchholz, D.: The physical state space of quantum electrodynamics. Commun. Math. Phys. **85**, 49–71 (1982)
6. Buchholz, D.: Gauss' law and the infraparticle problem. Phys. Lett. B **174**, 331–334 (1986)
7. Buchholz, D.: Harmonic analysis of local operators. Commun. Math. Phys. **129**, 631–641 (1990)
8. Buchholz, D., Porrmann, M., Stein, U.: Dirac versus Wigner: towards a universal particle concept in quantum field theory. Phys. Lett. B **267**, 377–381 (1991)
9. Buchholz, D., Roberts, J.E.: New light on infrared problems: sectors, statistics, symmetries and spectrum. Commun. Math. Phys **330**, 935–972 (2014)
10. Chen, T., Fröhlich, J., Pizzo, A.: Infraparticle scattering states in non-relativistic QED: I. The Bloch-Nordsieck paradigm. Commun. Math. Phys. **294**, 761–825 (2010)

11. Duch, P., Herdegen, A.: Massless asymptotic fields and Haag-Ruelle theory. Lett. Math. Phys. **105**, 245–277 (2015)
12. Dybalski, W.: A sharpened nuclearity condition and the uniqueness of the vacuum in QFT. Commun. Math. Phys. **283**, 523–542 (2008)
13. Dybalski, W., Gérard, C.: A criterion for asymptotic completeness in local relativistic QFT. Commun. Math. Phys. **332**, 1167–1202 (2014)
14. Fröhlich, J.: Application of commutator theorems to the integration of representations of Lie algebras and commutator relations. Commun. Math. Phys. **54**, 135–150 (1977)
15. Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field. Adv. Math. **164**, 349–398 (2001)
16. Herdegen, A.: Infraparticle problem, asymptotic fields and Haag-Ruelle theory. Ann. Henri Poincaré **15**, 345–367 (2014)
17. Herdegen, A.: On energy momentum transfer of quantum fields. Lett. Math. Phys. **104**, 1263–1280 (2014)
18. Kraus, K.: Aspects of the infrared problem in quantum electrodynamics. Found. Phys. **13**, 701–713 (1983)
19. Kraus, K., Polley, L., Reents, G.: Models for infrared dynamics. I. Classical currents. Ann. Inst. H. Poincaré t. **26**, 109–162 (1977)
20. Kunhardt, W.: On infravacua and the localization of sectors. J. Math. Phys. **39**, 6353 (1998)
21. Morchio, G., Strocchi, F.: The infrared problem in QED: a lesson from a model with Coulomb interaction and realistic photon emission. Preprint arXiv:1410.7289
22. Pohlmeyer, K.: The Jost-Schroer theorem for zero-mass fields. Commun. Math. Phys. **12**, 204–211 (1969)
23. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Fourier Analysis, Self-adjointness. Academic Press, New York, San Francisco, London (1975)
24. Steinmann, O.: Perturbative Quantum Electrodynamics and Axiomatic Field Theory. Springer, Berlin, Heidelberg, New York (2000)
25. Takesaki, M.: Theory of Operator Algebras I. Springer, Berlin, Heidelberg, New York (1979)
26. Tanimoto, Y.: Massless Wigner particles in conformal field theory are free. Forum Math. Sigma **2**, e21 (2014)