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Nonexistence of Large Nuclei in the Liquid Drop Model

RUPERT L. FRANK[1], ROWAN KILLIP[2] and PHAN THÀNH NAM[3]
[1] *Mathematics 253-37, Caltech, Pasadena, CA 91125, USA. e-mail: rlfrank@caltech.edu*
[2] *UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA. e-mail: killip@math.ucla.edu*
[3] *Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria. e-mail: pnam@ist.ac.at*

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Abstract. We give a simplified proof of the nonexistence of large nuclei in the liquid drop model and provide an explicit bound. Our bound is within a factor of 2.3 of the conjectured value and seems to be the first quantitative result.

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We consider the minimization problem

$$E(A) = \inf\{\mathcal{E}(\Omega) \colon |\Omega| = A\}$$

over all measurable set $\Omega \subset \mathbb{R}^3$ with the energy functional

$$\mathcal{E}[\Omega] = \operatorname{Per}\Omega + \frac{1}{2}\iint_{\Omega\times\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|}.$$

Here Ω can be interpreted as a nucleus in the liquid drop model with density 1, and hence the volume $|\Omega| = A$ is the number of nucleons (protons and neutrons) in the nucleus. Mathematically, A is not necessarily an integer. The perimeter $\operatorname{Per}\Omega$ is taken in the sense of De Giorgi, namely

$$\operatorname{Per}\Omega = \sup\left\{\int_\Omega \operatorname{div} F(x)\,\mathrm{d}x \mid F \in C_0^1(\mathbb{R}^3, \mathbb{R}^3),\ |F| \le 1\right\},$$

which boils down to the surface area of Ω when the boundary is smooth. The Coulomb term describes the proton repulsion in the nucleus, where the proton charge has been normalized appropriately. The liquid drop model goes back to the pioneering works of Gamow [9], von Weizsäcker [16] and Bohr [2] in 1930s, and recently it has gained renewed interest from many authors, see for instance [1,3–6,8,10–12].

It is well known that among all measurable sets of a given volume, balls minimize the perimeter (by the isoperimetric inequality [7]) and maximize the Coulomb self-interaction energy (by the Riesz rearrangement inequality [15]). This energy competition makes the liquid drop model highly nontrivial. It is generally assumed in the physics literature and conjectured in the mathematics literature [6] that $E(A)$ is minimized by a ball up to

$$A_c = \frac{2-2^{2/3}}{2^{2/3}-1} \cdot \frac{|B| \ \mathrm{Per} B}{\frac{1}{2}\iint_{B\times B} |x-y|^{-1} \, \mathrm{d}x \, \mathrm{d}y} = 5 \cdot \frac{2-2^{2/3}}{2^{2/3}-1} \approx 3.518,$$

(see also [8]) and that for $A > A_c$ there is no minimizer.

The fact that there is no minimizer for large A has been shown only recently in remarkable works of Knüpfer–Muratov [10] and Lu–Otto [12]. Their methods are inspired by techniques from geometric measure theory and seem to lead to rather large constants. In the present paper, we will provide a direct and simple proof of the nonexistence and give an explicit bound on the maximal size of a nucleus. Our main result is

THEOREM. *If* $A > 8$, *then* $E(A)$ *has no minimizer.*

This is within a factor of 2.3 of the conjectured value and seems to be the first quantitative result. We also emphasize that balls are locally stable up to $A = 10$ [3]. Our proof builds on ideas in [12,14], which were originally developed to deal with the nonexistence in the Thomas–Fermi–Dirac–von Weisäcker theory.

For every $\nu \in \mathbb{S}^2$ and $\ell \in \mathbb{R}$ we consider the plane

$$H_{\nu,\ell} := \{x \in \mathbb{R}^3 \mid \nu \cdot x = \ell\}$$

as well as the half-spaces

$$H^+_{\nu,\ell} := \{x \in \mathbb{R}^3 \mid \nu \cdot x > \ell\}, \quad H^-_{\nu,\ell} := \{x \in \mathbb{R}^3 \mid \nu \cdot x < \ell\}.$$

For a set $\Omega \subset \mathbb{R}^3$ we denote

$$\Omega^\pm_{\nu,\ell} = \Omega \cap H^\pm_{\nu,\ell}.$$

We use the following simple result from the theory of sets of finite perimeter.

LEMMA. *Let* $\Omega \subset \mathbb{R}^3$ *have finite perimeter and* $\nu \in \mathbb{S}^2$. *Then for almost every* $\ell \in \mathbb{R}$,

$$\mathrm{Per}\,\Omega^+_{\nu,\ell} + \mathrm{Per}\,\Omega^-_{\nu,\ell} = \mathrm{Per}\,\Omega + 2\mathcal{H}^2(\Omega \cap H_{\nu,\ell}).$$

Here $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure.

Proof of Lemma. For a set $E \subset \mathbb{R}^3$ of finite perimeter and its characteristic function χ_E we consider the measure $\mu_E = -\nabla \chi_E$ and note that $\mathrm{Per} E = |\mu_E|(\mathbb{R}^3)$.

According to [13, Ex. 15.13], for almost every $\ell \in \mathbb{R}$,

$$\mu_{\Omega^-_{\nu,\ell}} = \mu_\Omega|_{H^-_{\nu,\ell}} + \nu\mathcal{H}^2|_{\Omega\cap H_{\nu,\ell}} .$$

As in the proof of [13, Lem. 15.12], the measures on the right side are mutually singular and therefore

$$|\mu_{\Omega^-_{\nu,\ell}}| = |\mu_\Omega||_{H^-_{\nu,\ell}} + \mathcal{H}^2|_{\Omega\cap H_{\nu,\ell}} .$$

Thus,

$$\operatorname{Per}\Omega^-_{\nu,\ell} = |\mu_\Omega|(H^-_{\nu,\ell}) + \mathcal{H}^2(\Omega\cap H_{\nu,\ell}) .$$

Adding this and the corresponding equality for $-\nu$ and $-\ell$ we obtain the lemma.

Proof of Theorem. Let Ω be a minimizer for $E(A)$ for some $A>0$. By minimality of Ω and subadditivity of E, we have for every $\nu\in\mathbb{S}^2$ and $\ell\in\mathbb{R}$,

$$\mathcal{E}(\Omega^+_{\nu,\ell}) + \mathcal{E}(\Omega^-_{\nu,\ell}) \geq E(|\Omega^+_{\nu,\ell}|) + E(|\Omega^-_{\nu,\ell}|) \geq E(A) = \mathcal{E}(\Omega) .$$

By the lemma, for almost every $\ell\in\mathbb{R}$ this is the same as

$$2\mathcal{H}^2(\Omega\cap H_{\nu,\ell}) \geq \iint_{H^+_{\nu,\ell}\times H^-_{\nu,\ell}} \frac{\chi_\Omega(x)\chi_\Omega(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y .$$

We integrate this inequality over $\ell\in\mathbb{R}$ and use the fact that

$$\int_{\mathbb{R}} \mathcal{H}^2(\Omega\cap H_{\nu,\ell})\,\mathrm{d}\ell = |\Omega| = A$$

(by Fubini's theorem) and

$$\int_{\mathbb{R}} \chi_{\{\nu\cdot x>\ell>\nu\cdot y\}}\,\mathrm{d}\ell = (\nu\cdot(x-y))_+ ,$$

to get

$$2A \geq \iint_{\mathbb{R}^3\times\mathbb{R}^3} \chi_\Omega(x)\frac{(\nu\cdot(x-y))_+}{|x-y|}\chi_\Omega(y)\,\mathrm{d}x\,\mathrm{d}y .$$

Finally, we average the bound with respect to $\nu\in\mathbb{S}^2$ and use the fact that, for any $a\in\mathbb{R}^3$,

$$(4\pi)^{-1}\int_{\mathbb{S}^2} (\nu\cdot a)_+\,\mathrm{d}\nu = \frac{|a|}{2}\int_0^{\pi/2} \cos\theta\sin\theta\,\mathrm{d}\theta = \frac{|a|}{4}$$

to conclude that

$$2A \geq \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \chi_\Omega(x)\chi_\Omega(y)\,\mathrm{d}x\,\mathrm{d}y = \frac{A^2}{4} .$$

Thus, $A\leq 8$, which proves the theorem.

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