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Bounded Solutions of KdV and Non-Periodic One-Gap Potentials in Quantum Mechanics

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Abstract. We describe a broad new class of exact solutions of the KdV hierarchy. In general, these solutions do not vanish at infinity, and are neither periodic nor quasi-periodic. This class includes algebro-geometric finite-gap solutions as a particular case. The spectra of the corresponding Schrödinger operators have the same structure as those of N-gap periodic potentials, except that the reflectionless property holds only in the infinite band. These potentials are given, in a non-unique way, by $2N$ real positive functions defined on the allowed bands. In this letter we restrict ourselves to potentials with one allowed band on the negative semi-axis; however, our results apply in general. We support our results with numerical calculations.

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1. Introduction

In the 1970s, it became clear that certain nonlinear wave equations that are integrable by the inverse spectral transform (IST), such as the KdV, the NLS, and the sin-Gordon equations, are universal mathematical models and have wide applications in diverse areas of physics (see [1]). However, known exact methods for these equations only work under strong constraints on initial data, which limits their practical applications. For example, no analytic approach is known for the problem of integrable turbulence of waves on shallow water, or in optical fibers. In this letter, we describe a new exact method for the KdV equation, which is applicable to other equations as well.

We consider the KdV equation in the following form:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1}$$

The IST, and other methods for integrating KdV, is based on the observation that it is the consistency condition of the Schrödinger equation

$$-\psi_{xx}+u\psi=E\psi \tag{2}$$

and the evolution equation (see [2])

$$\psi_t+4\psi_{xxx}+6u\psi_x+3u_x\psi=0 \tag{3}$$

on an auxiliary function $\psi(x,t)$, for all values of E.

The spectrum of the Schrödinger operator

$$L=-\partial_x^2+u(x)$$

is the set of real values of E for which there exists a bounded solution of (2). The spectrum does not change if the potential evolves according to (1), hence solutions of KdV are classified by the structure of their spectra.

The simplest nontrivial potentials are the N-soliton solutions of KdV, whose spectra consist of the positive semi-axis and N points on the negative axis. These solutions were already known to Bargmann [3] and are expressed in terms of elementary functions. Another important class of solutions is the N-gap potentials. Their spectra consist of a union of N closed intervals and one half-infinite open interval, and they are explicitly given by the Matveev–Its formula in terms of Riemann theta functions of real hyperelliptic algebraic curves (see [1]). All these potentials are reflectionless for all allowed energy values (see [4]).

A generic periodic potential is a limit of N-gap potentials as $N\to\infty$ (see [5]), while N-soliton solutions are obtained from N-gap solutions by degenerating the spectral curve to a rational nodal curve. In addition, rapidly vanishing perturbations of finite-gap potentials admit a complete analytic description. These perturbations are no longer reflectionless. These methods also allow solving the initial value problem for KdV in the rapidly vanishing case. Despite the significance of these results, they do not solve the generic initial value problem, and hence are insufficient for many physical applications.

A key problem in the theory of nonlinear equations is the statistical description of solutions of KdV, also known as integrable turbulence [6]. An important case is the description of a "soliton gas" [7,8]. Suppose that the initial condition of (1) is a bounded function, which is neither rapidly vanishing nor periodic. What is its behavior under time evolution? Unfortunately, for a generic bounded potential little can be said about the spectrum of the corresponding Schrödinger operator, which can have an arbitrarily complicated structure. There is a known class of random, statistically homogeneous potentials whose spectra have a Cantor set-like structure and which display Anderson localization. Therefore, it is natural to pose the problem of describing the class of bounded potentials of the Schrödinger operator whose spectra nevertheless have the simple structure of an N-gap potential.

The simplest example of a finite-gap potential is an elliptic potential with spectrum $[-k_2^2, -k_1^2] \cup [0, \infty)$. By Hochstadt's theorem, a periodic (more generally, reflectionless, see [4]) potential $u(x)$ with such a spectrum has the form

$$u(x) = 2\wp(x + i\omega' - x_0) + e_3, \tag{4}$$

where $\wp$ is the Weierstrass function with periods 2ω, $2i\omega'$, where ω and ω' are real, and

$$k_1^2 = e_2 - e_3, \quad k_2^2 = e_1 - e_3, \quad e_1 + e_2 + e_3 = 0,$$

where e_i are the values of $\wp$ on the half-periods of the lattice, and x_0 is an arbitrary constant. Time evolution according to KdV is given by

$$x_0 \to x_0 + 6e_3 t,$$

which gives the well-known cnoidal traveling wave solution of KdV.

In this letter we describe a much wider class of potentials of (2) that have the same spectrum as (4). These potentials are the initial data of a new class of exact solutions of KdV. They are parametrized, in a non-unique way, by a pair of positive functions R_1 and R_2 defined on the interval $[k_1, k_2]$. Functions R_1 and R_2 that vanish on subintervals of $[k_1, k_2]$ describe N-gap potentials. These potentials are reflectionless for positive values of energy, but not in general for negative values.

2. *N*-Soliton Potentials Via Dressing Method

We construct new solutions of KdV as limits of N-soliton solutions, which for fixed moments of time are reflectionless Bargmann potentials [3]. An elegant construction of N-soliton solutions can be carried out via the dressing method, as described in [9]. This method is local both in x and in t, so to save space we set $t = 0$ and consider time evolution only at the end of the letter. We note that the described method is more flexible than the traditional IST.

Following [9], we consider the following $\bar{\partial}$-problem on the complex k-plane:

$$\frac{\partial \chi}{\partial \bar{k}} = i e^{2ikx} T(k) \chi(x, -k), \quad \chi \to 1 \text{ as } |k| \to \infty, \tag{5}$$

where $T(k)$ is a compactly supported distribution called the *dressing function* of the $\bar{\partial}$-problem. The solution satisfies the following integral equation:

$$\chi(x, k) = 1 + \frac{i}{\pi} \iint \frac{T(-q)\chi(x, q)e^{-2iqx}}{k + q} \, dq d\bar{q} \tag{6}$$

where we regularize the integral in the following way:

$$\frac{1}{k} = \lim_{\varepsilon \to 0} \frac{\bar{k}}{k^2 + \varepsilon^2} \quad \frac{\partial}{\partial \bar{k}} \frac{1}{k} = \pi \delta(k),$$

where $\delta(k)$ is the two-dimensional δ-function. Suppose that the dressing function $T(k)$ has the property that equation (6) has a unique solution. Then

$$\chi(x,k) \to 1 + \frac{i\chi_0(x)}{k} + \cdots \qquad \text{as } |k| \to \infty \tag{7}$$

and χ is a solution of the equation:

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0, \quad u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}\chi_0(x). \tag{8}$$

To construct N-soliton potentials, we consider the dressing function

$$T(k) = \pi \sum_{n=1}^{N} T_n \delta(k - i\kappa_n), \tag{9}$$

where T_n and κ_n are nonzero real numbers satisfying $|\kappa_n| \in [k_1, k_2]$ and $T_n/\kappa_n > 0$ for all n. Then χ is a rational function:

$$\chi(x,k) = 1 + i\sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n}, \quad \chi_n(x) \text{ real.} \tag{10}$$

The corresponding potential

$$u = 2\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{N}\chi_n(x) \tag{11}$$

is an N-soliton potential. The function $\psi = \chi e^{-ikx}$ satisfies (2). The corresponding potential is rapidly vanishing and has a finite discrete spectrum $\{-\kappa_1^2, \ldots, -\kappa_N^2\}$, and $\psi_n(x) = \chi_n(x)e^{\kappa_n x}$ are the corresponding eigenfunctions.

The $\bar{\partial}$-problem (5) is equivalent to the following linear system on the eigenfunctions:

$$\psi_n + T_n \sum_{m=1}^{N} \frac{e^{-(\kappa_n+\kappa_m)x}}{\kappa_n + \kappa_m}\psi_m = T_n e^{-\kappa_n x}. \tag{12}$$

If $T_n/\kappa_n > 0$ the determinant A of system (12) is positive, so it has a unique solution, and moreover

$$u(x) = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln A. \tag{13}$$

The results of [10] imply that these potentials are strictly negative and satisfy $u(x) \geq -2k_2^2$, so they are bounded uniformly in N.

In the standard IST, it is always assumed that all T_n and κ_n are positive, in which case

$$\int_{-\infty}^{\infty} |\psi_n|^2 \mathrm{d}x = T_n.$$

Weakening these assumptions allows us to effectively construct a broader class of potentials.

3. Closure of Bargmann Potentials

To describe the closure of the set of N-soliton solutions, we replace the finite dressing function (9) with one supported on two cuts $[ik_1, ik_2]$ and $[-ik_2, -ik_1]$ on the imaginary axis:

$$T(k) = \pi \int_{k_1}^{k_2} R_1(\kappa)\delta(k - i\kappa)\,\mathrm{d}\kappa + \pi \int_{k_1}^{k_2} R_2(\kappa)\delta(k + i\kappa)\,\mathrm{d}\kappa. \tag{14}$$

Here R_1 and R_2 are functions on $[k_1, k_2]$. Formula (10) for the solution χ transforms into the following representation:

$$\chi(x,k) = 1 + i \int_{k_1}^{k_2} \frac{f(x,p)}{k - ip}\,\mathrm{d}p + i \int_{k_1}^{k_2} \frac{g(x,p)}{k + ip}\,\mathrm{d}p \tag{15}$$

where $f(x,p)$ and $g(x,p)$ are real-valued functions. The function χ is analytic away from the cuts. The potential $u(x)$ is given by

$$u(x) = 2\frac{\mathrm{d}\chi_0}{\mathrm{d}x}, \quad \chi_0 = \int_{k_1}^{k_2} [f(x,p) + g(x,p)]\,\mathrm{d}p \tag{16}$$

We plan to describe the details of the transition from the discrete to the continuous case in a future paper. The algebraic system (12) transforms into a system of two integral equations on χ, which is equivalent to the following Riemann–Hilbert problem:

$$\chi^+(x, ik) - \chi^-(x, ik) = i\pi R_1(k) e^{-2kx} \left[\chi^+(x, -ik) + \chi^-(x, -ik)\right], \tag{17}$$

$$\chi^+(x, -ik) - \chi^-(x, -ik) = -i\pi R_2(k) e^{2kx} \left[\chi^+(x, ik) + \chi^-(x, ik)\right], \tag{18}$$

where $\chi^{\pm}$ are the boundary values of χ along the cuts:

$$\chi^{\pm}(x,k) = \lim_{\varepsilon \to 0} \chi(x, k \pm i\varepsilon), \quad k \in [-ik_2, -ik_1] \cup [ik_1, ik_2].$$

This is a scalar, but non-local, Riemann–Hilbert problem, and it is equivalent to a local vector Riemann–Hilbert problem. Denote $\Xi(k) = [\chi(k) \;\; \chi(-k)]$, and let Ξ^+ and Ξ^- be the right and left values of Ξ on the cuts. Then Eqs. (17) and (18) are equivalent to

$$\Xi^+(i\kappa) = M(\kappa)\Xi^-(i\kappa), \quad \Xi^+(-i\kappa) = M^T(\kappa)\Xi^-(-i\kappa) \tag{19}$$

for $\kappa \in [k_1, k_2]$, where the transition matrix is

$$M(x,\kappa) = \frac{1}{1 + R_1 R_2} \begin{bmatrix} 1 - R_1 R_2 & 2i R_1 e^{-2\kappa x} \\ 2i R_2 e^{2\kappa x} & 1 - R_1 R_2 \end{bmatrix}$$

We show that if R_1 and R_2 are positive and α-Hölder for a positive α, then system (17) and (18) has a unique solution. The spectrum of the corresponding potential $u(x)$ is $[-k_2^2, -k_1^2] \cup [0, \infty)$ on the negative axis. The functions

$$\varphi(x,k) = f(x,k)e^{ikx} \quad \text{and} \quad \psi(x,k) = g(x,k)e^{-ikx} \tag{20}$$

are real bounded eigenfunctions of (2) when $k = i\kappa$, where $k_1 < \kappa < k_2$, so the spectral multiplicity is two. These eigenfunctions form an orthogonal system:

$$\int_{-\infty}^{\infty} \varphi(x,\kappa)\varphi(x,\kappa')\,\mathrm{d}\kappa = R_1\delta(\kappa - \kappa'), \tag{21}$$

$$\int_{-\infty}^{\infty} \psi(x,\kappa)\psi(x,\kappa')\,\mathrm{d}\kappa = R_2\delta(\kappa - \kappa'), \tag{22}$$

$$\int_{-\infty}^{\infty} \psi(x,\kappa)\varphi(x,\kappa')\,\mathrm{d}\kappa = 0. \tag{23}$$

We prove these nontrivial statements by approximating the integrals in (15) by finite sums. The one-gap potential is approximated by N-soliton potentials, and Eqs. (17) and (18) turn into the linear system (12), which we know to be solvable. We note that $R_1 = R_2$ implies that $u(x)$ is even.

A fundamental difference between the ISM and the dressing method is that in the former, the scattering data can be uniquely restored from the potential, while in the latter, the same potential can be constructed using a variety of different dressings. For example, in (9) we can change the signs of κ_n, and make an appropriate change to T_n, without changing $u(x)$, so each N-soliton solution is constructed using 2^N different dressing functions of the form (9). Hence, a given bounded potential is not determined by a unique choice of R_1 and R_2. Below we describe a class of dressings leading to one-gap periodic potentials.

We have so far assumed R_1 and R_2 to be positive. If one of the functions vanishes along an interval $[a,b]$ contained in $[k_1,k_2]$, then $u(x)$ is a reflecting potential, namely $u \to 0$ in one of the directions, and the spectral multiplicity on $[a,b]$ is equal to one. If both functions vanish on $[a,b]$, then this zone is a forbidden gap. In this way, by requiring R_1 and R_2 to vanish along $N-1$ disjoint intervals inside $[k_1,k_2]$, we can obtain N-gap potentials. We note that the spectral multiplicity is always equal to one on the boundary of a gap.

We remark that potentials of the Schrödinger operator defined by a Riemann–Hilbert problem (17, 18) with $R_2 = 0$ were considered by Krichever [11]. Such potentials may also be related to solutions of KdV with step-like initial data (see [12–14]), due to the similar behavior at $x \to \pm\infty$ (see Figure below).

4. Time Evolution

We now use a potential $u(x)$ constructed above as the initial condition for the KdV equation (1). The spectrum of $u(x)$ is preserved, while R_1 and R_2 evolve as follows:

$$R_1(k) \to R_1(k)e^{S(k)t}, \quad R_2(k) \to R_2(k)e^{-S(k)t}, \tag{24}$$

where $S(k) = 8k^3$. Evolution according to one of the higher KdV flows corresponds to a different odd function $S(k)$. Any choice of an odd function S defines a unitary transformation of the corresponding Schrödinger operator. Hence, the potentials that we construct are naturally partitioned into unitarily equivalent classes, with two potentials being equivalent if they are both solutions of an equation of the KdV hierarchy for different times.

Define $F(k) = \ln R_1(k)$ and $G(k) = \ln R_2(k)$, and assume that F and G extend analytically to the annulus $k_1 < |k| < k_2$. They are then given by convergent Laurent series

$$F(k) = \sum_{n=-\infty}^{\infty} f_n k^n, \quad G(k) = \sum_{n=-\infty}^{\infty} g_n k^n.$$

The above discussion implies that the even coefficients f_{2n} and g_{2n}, as well as the sums $f_{2n+1} + g_{2n+1}$ of the odd coefficients, are invariants of the unitary transformations. Alternatively, since R_1 and R_2 are defined for $-k_1 < k < -k_2$, the following functions are invariant under the unitary transformations:

$$S_1(k) = R_1(k)R_1(-k), \quad S_2(k) = R_2(k)R_2(-k), \quad S_3 = R_1(k)R_2(k).$$

These functions form a system of conserved quantities for the full KdV hierarchy.

5. Periodic One-Gap Potentials

All periodic one-gap potentials are given by formula (4). Applying a transformation of the form (24) only changes the constant x_0. To describe the entire class, we only need to construct one example and then apply transformation (4) with an arbitrary odd function s.

We now construct a dressing function of the form (14) that determines a periodic one-gap potential. We put $x_0 = \omega$ in (4) and map the k-plane to the period parallelogram as follows:

$$k^2 = e_3 - \wp(z) \quad z(k) \to -\frac{i}{k} \quad \text{as} \quad |k| \to \infty. \tag{25}$$

Then (2) becomes the Lamé equation

$$\varphi'' - \left[2\wp(x - \omega' - i\omega') + \wp(z)\right]\varphi = 0 \tag{26}$$

which has following solution:

$$\varphi(x,z)=\frac{\sigma(x-\omega-i\omega+z)\sigma(\omega+i\omega')}{\sigma(x-\omega-i\omega)\sigma(\omega+i\omega'-z)}\exp(-\zeta(z)x). \tag{27}$$

Now define the function:

$$\xi(x,k)=\sqrt{\frac{k-ik_1}{k-ik_2}}\varphi(x,z(k))e^{ikx}. \tag{28}$$

This function satisfies the equation:

$$\xi''-2ik\xi'-u(x)\xi=0, \quad \text{for} \quad \xi\to 1 \quad as \quad |k|\to\infty \tag{29}$$

The function ξ satisfies the RH problem (17) and (18), where:

$$R_1(q)=\frac{1}{\pi}h(q), \quad R_2(q)=\frac{1}{\pi h(q)}, \tag{30}$$

$$h(q)=\sqrt{\frac{(k_2-q)(q+k_1)}{(q-k_1)(q+k_2)}}, \quad k_1<q<k_2. \tag{31}$$

Since $h(-q)=1/h(q)$, all three invariant functions coincide:

$$R_1(q)R_1(-q)=R_2(q)R_2(-q)=R_1(q)R_2(q)=\frac{1}{\pi^2}.$$

Hence, we can also construct the potential $u(x)$ using the constant dressing functions

$$R_1(q)=R_2(q)=\frac{1}{\pi}.$$

We remark that it is already known that Riemann–Hilbert problems with constant jumps can be used to construct finite-gap potentials of the Schrödinger operator and other operators appearing in the theory of integrable systems (see [15,16]).

6. Numerical Solution

We solve Eqs. (17), (18) numerically for $k_1=2$ and $k_2=4$. Denote $p=\kappa+3$, where $-1<p<1$. It is convenient to replace $\phi(x,k)$ and $\psi(x,k)$ with the following functions:

$$P(x,p)=\sqrt{1-p^2}\phi(x,p+3),$$
$$Q(x,p)=\sqrt{1-p^2}\psi(x,p+3),$$

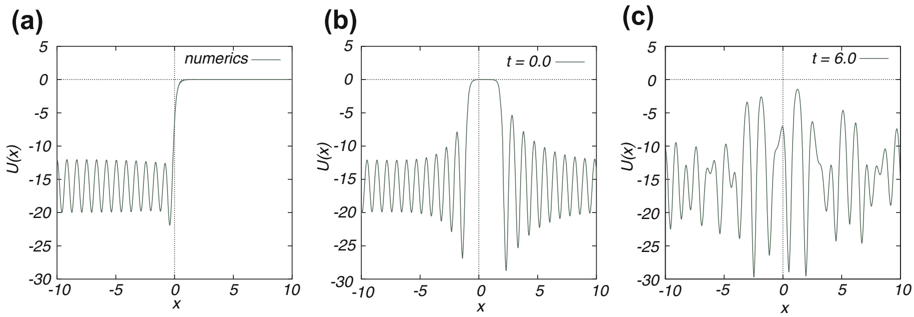

Figure 1. A potential $U(x)$ that appears as a result of dressing with: **a** $R_1 = 1/\pi$ and $R_2 = 0$, decaying for $x \to +\infty$; **b** $R_1 = \frac{1}{\pi} \times 10^{-3}$ and $R_2 = \frac{1}{\pi} \times 10^{-6}$; **c** $R_1 = \frac{1}{\pi} \times 10^{-3}$ and $R_2 = \frac{1}{\pi} \times 10^{-6}$ at moment of time $t = 6$ under KdV flow

which then satisfy the following integral equations:

$$P(x, p) + r_1(x, p)\left[\int_{-1}^{1} \frac{P(x, q)e^{-qx}\,\mathrm{d}q}{(6 + p + q)\sqrt{1 - q^2}} + \fint_{-1}^{1} \frac{Q(x, q)e^{qx}\,\mathrm{d}q}{(p - q)\sqrt{1 - q^2}}\right] = r_1(x, p), \tag{32}$$

$$Q(x, p) + r_2(x, p)\left[\fint_{-1}^{1} \frac{P(x, q)e^{-qx}\,\mathrm{d}q}{(p - q)\sqrt{1 - q^2}} + \int_{-1}^{1} \frac{Q(x, q)e^{qx}\,\mathrm{d}q}{(6 + p + q)\sqrt{1 - q^2}}\right] = -r_2(x, p), \tag{33}$$

where $r_j(x, p) = \sqrt{1 - p^2} R_j(p + 3)e^{(-1)^j 2(p+3)x}$ for $j = 1, 2$. Discretized at Chebyshev nodes $q_k = \cos\frac{(2k-1)\pi}{2M}$ with $k = 1, 2, \ldots, M$ the integrals are evaluated via Gauss–Chebyshev quadrature that is exact for polynomials of degree less than $2M - 1$. Note that each equation of the system contains a Cauchy principal value integral denoted by $\fint$, and that integration in the vicinity the of singularity at $q = p$ requires a shift from the real axis.

The spatial variable x appears as a parameter in (32) and (33) and the x-dependence of r_1 and r_2 becomes a major obstacle, since the condition number of the discretized system is exponential in x and requires using multiprecision arithmetics.

A mesh for the spatial parameter x is a Chebyshev grid with M nodes; a high-order polynomial interpolation by means of Lagrange interpolation is used for intermediate points. In a typical simulation an interpolating polynomial of degree 200 suffices to have an accurate approximation for $|x| < 10$.

Figure 1 shows the numerical solutions for three cases, the last two being related by the first KdV flow. Comparing Figure 1b with c, we see that a relatively ordered potential becomes chaotic under the KdV flow, giving an example of integrable turbulence, which we can interpret as a dense soliton gas. The local minima correspond to solitons having amplitude close to the limiting one, in this case equal to $2k_2^2 = 32$ (see bound on $u(x)$ in [10]). The next step for the theory would be to determine the correlation functions and the space-time spectrum of this turbulence, which can then be compared to experimental data.

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