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Estimates for Eigenvalues of Schrödinger Operators with Complex-Valued Potentials

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Abstract. New estimates for eigenvalues of non-self-adjoint multi-dimensional Schrödinger operators are obtained in terms of L_p-norms of the potentials. The results cover and improve those known previously, in particular, due to Frank (Bull Lond Math Soc 43(4): 745–750, 2011), Safronov (Proc Am Math Soc 138(6):2107–2112, 2010), Laptev and Safronov (Commun Math Phys 292(1):29–54, 2009). We mention the estimations of the eigenvalues situated in the strip around the real axis (in particular, the essential spectrum). The method applied for this case involves the unitary group generated by the Laplacian. The results are extended to the more general case of polyharmonic operators. Schrödinger operators with slowly decaying potentials and belonging to weak Lebesgue's classes are also considered.

Mathematics Subject Classifications. Primary 47F05; Secondary 35P15, 81Q12.

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1. Introduction

In this paper, we discuss estimates for eigenvalues of Schrödinger operators with complex-valued potentials. Among existing results on this problem regarding non-self-adjoint Schrödinger operators, we mention the works [1,9,11,19,24,25], and also [6] for an overview on certain aspects of spectral analysis of non-self-adjoint operators mainly needed for problems in quantum mechanics. In [1], it was observed that for the one-dimensional Schrödinger operator $H = -\mathrm{d}^2/\mathrm{d}x^2 + q$, where the potential q is a complex-valued function belonging to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, every its eigenvalue λ which does not lie on the non-negative semi-axis satisfies the following estimate

$$|\lambda|^{1/2} \leq \frac{1}{2} \int_{-\infty}^{\infty} |q(x)|\, \mathrm{d}x. \tag{1.1}$$

For the self-adjoint case, the estimate (1.1) was pointed out previously by Keller in [16]. In [9], related estimates are found for eigenvalues of Schrödinger operators on semi-axis with complex-valued potentials. Note that, as is pointed out in [9], the obtained estimates are in sense sharp for both cases of Dirichlet and Neumann boundary conditions. In [11,24] (see also [19,25]), the problem is considered for higher dimensions case. In particular, in [11], estimates for eigenvalues of Schrödinger operators with complex-valued potentials decaying at infinity, in a certain sense, are obtained in terms of appropriate weighted Lebesgue spaces norms of potentials.

In this paper, we mainly deal with the evaluation of eigenvalues of multi-dimensional Schrödinger operators. The methods which we apply allow us to consider the Schrödinger operators acting in one of the Lebesgue space $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$. We consider the formal differential operator $-\Delta + q$ on $\mathbb{R}^n$, where Δ is the n-dimensional Laplacian and q is a complex-valued measurable function. Under some reasonable conditions, ensuring, in a suitable averaged sense, decaying at infinity of the potential, there exists a closed extension H of $-\Delta + q$ in the space $L_p(\mathbb{R}^n)$ such that its essential spectrum $\sigma_{\text{ess}}(H)$ coincides with the semi-axis $[0, \infty)$, and any other point of the spectrum, i.e., not belonging to $\sigma_{\text{ess}}(H)$, is an isolated eigenvalue of finite (algebraic) multiplicity. We take the operator H as the Schrödinger operator corresponding to $-\Delta + q$ in above sense and we will be interested to find estimates of eigenvalues of H which lie outside of the essential spectrum. The problem reduces to estimation of the resolvent of the unperturbed operator H_0, that is defined by $-\Delta$ in $L_p(\mathbb{R}^n)$ on its domain the Sobolev space $W_p^2(\mathbb{R}^n)$, bordered by some suitable operators of multiplication (cf. reasoning in Section 2).

We begin with evaluation of perturbed eigenvalues belonging to the left half-plane $\operatorname{Re} z < 0$. Therewith, bounds of the negative eigenvalues for the self-adjoint case, mostly important in applications, are established. For this purpose, we make use the integral representation of the free Green function in the form

$$g(x-y;\lambda) = (4\pi)^{-n/2} \int_0^\infty e^{\lambda t} e^{-|x-y|^2/4t} t^{-n/2}\, dt, \qquad \operatorname{Re}\lambda < 0.$$

We assume that the potential q admits a factorization $q = ab$, where $a \in L_r(\mathbb{R}^n)$ and $b \in L_s(\mathbb{R}^n)$ for some $r, s, 0 < r, s \le \infty$, and prove that under conditions $1 < p < \infty$, $0 < r \le \infty$, $p \le s \le \infty$ and $r^{-1} + s^{-1} < 2n^{-1}$, for the eigenvalues λ with $\operatorname{Re}\lambda < 0$ of the Schrödinger operator H, the estimate

$$|\operatorname{Re}\lambda|^{1-n/2\alpha'} \le C\|a\|_r \|b\|_s \tag{1.2}$$

holds true with a positive constant $C = C(n, r, s)$ depending only on n, r and s; α' is conjugate exponent to $\alpha, \alpha = (1 - r^{-1} - s^{-1})^{-1}$.

The eigenvalues which are situated on the right half-plane behave in particular due to the presence in this side of the essential spectrum. In connection with this, the evaluation of the bordered resolvent of the unperturbed operator H_0 is made by applying a slightly modified approach. It involves somewhat *heat kernels* associated to the Laplacian. For it could be used the kernel $(4\pi it)^{-n/2}\exp(-|x-y|^2/4it)$, $-\infty<t<\infty$, representing the operator-group $U(t)=\exp(-itH_0)$, $-\infty<t<\infty$, and then making use of the formula expressing the resolvent $R(\lambda;H_0)$ as the Laplace transform of $U(t)$ (see [13]). In this way, we obtain a series of estimates for perturbed eigenvalues. In particular, supposing that $q=ab$, where $a\in L_r(\mathbb{R}^n)$, $b\in L_s(\mathbb{R}^n)$ for r,s satisfying $0<r\le\infty$, $p\le s\le\infty$, $r^{-1}-s^{-1}=1-2p^{-1}$, $2^{-1}-p^{-1}\le r^{-1}\le 1-p^{-1}$ and $r^{-1}+s^{-1}<2n^{-1}$, for any complex eigenvalue λ of the Schrödinger operator H with $\operatorname{Im}\lambda\neq 0$, we have

$$|\operatorname{Im}\lambda|^{\alpha}\le(4\pi)^{\alpha-1}\Gamma(\alpha)\|a\|_r\|b\|_s, \tag{1.3}$$

in which $\alpha:=1-n(r^{-1}+s^{-1})/2$ (Γ denotes the gamma function). An immediate consequence of this result (letting $r=s=2\gamma+n$, $\gamma>0$) is the estimate

$$|\operatorname{Im}\lambda|^{\gamma}\le(4\pi)^{-n/2}\Gamma\left(\frac{\gamma}{\gamma+n/2}\right)^{\gamma+n/2}\int_{\mathbb{R}^n}|q(x)|^{\gamma+n/2}\,\mathrm{d}x \tag{1.4}$$

for $\gamma>0$. The estimate (1.4) together with that corresponding to (1.2) (cf. Corollary 3.3) leads to an estimate like

$$|\lambda|^{\gamma}\le C\int_{\mathbb{R}^n}|q(x)|^{\gamma+n/2}\,\mathrm{d}x, \tag{1.5}$$

with an absolute constant $C=C(n,\gamma)$ depending only on n and γ. It should be emphasized that the estimate (1.5) concerns, however, eigenvalues λ lying only inside the left half-plane. In this context, we cite [19] for a conjecture concerning related estimate for eigenvalues of the Schrödinger operator considered acting on Hilbert space $L_2(\mathbb{R}^n)$.

Estimation of eigenvalues can be made representing a priori the resolvent of H_0 in terms of Fourier transform. The method leads, in particular, to the following result. Let $1<p<\infty$, and let $q=ab$ with $a\in L_r(\mathbb{R}^n)$, $b\in L_s(\mathbb{R}^n)$ for $0<r,s\le\infty$ satisfying $2^{-1}-p^{-1}\le r^{-1}\le 1-p^{-1}$, $-2^{-1}+p^{-1}<s^{-1}\le p^{-1}$, and $r^{-1}+s^{-1}<2n^{-1}$. Then, for any eigenvalue $\lambda\in\mathbb{C}\backslash[0,\infty)$ of the Schrödinger operator H, there holds

$$|\lambda|^{\alpha-n/2}\le C\|a\|_r^{\alpha}\|b\|_s^{\alpha}, \tag{1.6}$$

where $\alpha:=(r^{-1}+s^{-1})^{-1}$, and C being a constant of the potential (it is controlled; see Theorem 3.13). Notice that for the particular case $n=1$, $p=2$ and $r=s=2$, one has $\alpha=2$ and $C=1/2$, and the estimate (1.6) reduces to (1.1). From (1.6), it can be derived estimates for eigenvalues of Schrödinger operators with decaying potentials. So, for instance, taking $a(x)=(1+|x|^2)^{-\tau/2}$ ($\tau>0$), under suitable restrictions on r and τ, for an eigenvalue $\lambda\in\mathbb{C}\backslash[0,\infty)$ there holds

$$|\lambda|^{r-n}\le C\int_{\mathbb{R}^n}|(1+|x|^2)^{\tau/2}|q(x)|^{r}\,\mathrm{d}x. \tag{1.7}$$

In connection with (1.7), we note the related results obtained in [11,24] (see also [7,25]).

Note that estimates of type (1.7) can be obtained by choosing other weight functions, also frequently occurred in concrete situations, as, for instance, $e^{\tau|x|}$, $e^{\tau|x|^2}$, $|x|^\sigma e^{\tau|x|^\alpha}$, etc.

Further, estimates obtained for Schrödinger operators can be successfully extended to polyharmonic operators

$$H_{q,m} = (-\Delta)^m + q,$$

in which (the potential) q is a complex-valued measurable function, and m is an arbitrary positive real number. For the eigenvalues $\lambda \in \mathbb{C}\backslash[0,\infty)$ of an operator of this class, it can be proved that

$$|\lambda|^\gamma \leq C \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2m}\,\mathrm{d}x \tag{1.8}$$

for $\gamma > 0$ if $n \geq 2m$ and $\gamma \geq 1 - n/2m$ for $n < 2m$. The estimate (1.8) is in fact a result analogous to the already mentioned (1.5) for Schrödinger operators.

Finally, it should be remarked that the methods used as the basic tools to carry our results are available in slightly more general situations where the potentials are considered belonging to the so-called weak Lebesgue's spaces. In particular, by applying the same methods, we prove that if the potential q belongs to the weak space $L_{\gamma+n/2m,w}(\mathbb{R}^n)$, where $\gamma > 0$ for $n > 2m$ and $\gamma \geq 1 - n/2m$ for $n < 2m$, then any eigenvalue lying outside of essential spectrum of the polyharmonic operator $H_{m,q}$ satisfies

$$|\lambda|^\gamma \leq C \sup_{t>0}(t^{\gamma+n/2m}\lambda_q(t))$$

with a constant $C = C(n,m,\gamma,\theta)(\theta := arg\lambda, 0 < \theta < 2\pi)$.

The paper is organized as follows. Section 2 contains a preliminary material needed for the further exposition. It is pointed out the setting of the problem and, in particular, defined the Schrödinger operators in a fashion suitable for main purposes. Section 3 is concerned with Schrödinger operators with Lebesgue power-summable potentials. This section is divided into four subsections. In the first two subsections, estimates are obtained for the eigenvalues located on the left half-plane $\mathrm{Re}\,\lambda < 0$. In the third one, there are established evaluations for the imaginary part of the possible eigenvalues. Thereby, the strip around the real axis (in particular, the essential spectrum) containing possible eigenvalues is determined. In the fourth subsection, evaluations are obtained via the Fourier transform. In Section 4, we discus the problem for the general case of polyharmonic operators. In Section 5, we treat the case of potentials belonging to weak Lebesgue's type spaces.

2. Preliminaries: Setting of the Problem

Consider, in the space $L_p(\mathbb{R}^n)$ $(1<p<\infty)$, the Schrödinger operator

$$-\Delta+q(x) \tag{2.1}$$

with a potential q being in general a complex-valued measurable function on $\mathbb{R}^n$. We assume that the potential q admits a factorization $q=ab$ with a,b belonging to some Lebesgue type spaces (appropriate spaces will be indicated in relevant places). We denote by H_0 the operator defined by $-\Delta$ in $L_p(\mathbb{R}^n)$ on its domain the Sobolev space $W_p^2(\mathbb{R}^n)$, and let A, B denote, respectively, the operators of multiplication by a,b defined in $L_p(\mathbb{R}^n)$ with their maximal domains. Thus, the differential expression (2.1) defines in the space $L_p(\mathbb{R}^n)$ an operator expressed as the perturbation of H_0 by AB. In order to determine the operator, being a closed extension of H_0+AB, suitable for our purposes, we need to require certain assumptions on the potential. For we let a and b be functions of Stummel classes [28] (see also [14,27]), namely

$$M_{\nu,p'}(a)<\infty, \quad 0<\nu<p', \tag{2.2}$$

$$M_{\mu,p}(b)<\infty, \quad 0<\mu<p, \tag{2.3}$$

(p' is the conjugate exponent to $p: p^{-1}+p'^{-1}=1$), where it is denoted

$$M_{\nu,p}(u)=\sup_x \int\limits_{|x-y|<1} |u(y)|^p \; |x-y|^{\nu-n}\,\mathrm{d}y$$

for functions $u\in L_{p,loc}(\mathbb{R}^n)$. If also the potential q decays at infinity, for instance, like

$$\int\limits_{|x-y|<1} |q(y)|\,\mathrm{d}y\to 0 \quad \text{as} \quad |x|\to\infty, \tag{2.4}$$

then the operator $H_0+AB \quad (=-\Delta+q)$ admits a closed extension H having the same essential spectrum as unperturbed operator H_0, i.e.,

$$\sigma_{\text{ess}}(H)=\sigma_{\text{ess}}(H_0) \quad (=\sigma\,(H_0)=[0,\infty)).$$

Note that the conditions (2.2) and (2.3) are used to derived boundedness and also, together with (2.4), compactness domination properties of the perturbation (reasoning are due to Rejto [23] and Schechter [26], cf. also [27]; Theorem 5.1, p. 116). To be more precise, due to conditions (2.2) and (2.3), the bordered resolvent $\mathrm{B}R(z;H_0)A$ $(R(z;H_0):=(H_0-zI)^{-1}$ denotes the resolvent of H_0) for some (or, equivalently, any) regular point z of H_0 represents a densely defined operator having a (unique) bounded extension, further on we denote it by $Q(z)$. If, in addition, (2.4), $Q(z)$ is a compact operator and, moreover, it is small with respect to the operator norm for sufficiently large $|z|$.

From now on, we let H denote the Schrödinger operator realized in this way in $L_p(\mathbb{R}^n)$ by the differential expression $-\Delta + q(x)$. Notice that constructions related to that mentioned above are widely known in the perturbation theory. In Hilbert case space $p = 2$, H, where the potential q is a real function, represents a self-adjoint operator presenting mainly interest for spectral and scattering problems.

It turns out that there is a constraint relation between the discrete part of the spectrum of H and that of $Q(z)$ (recall $Q(z)$ is the bounded extension of the bordered resolvent $BR(z; H_0)A$), namely, a regular point λ of H_0 is an eigenvalue for the extension H, the Schrödinger operator, if and only if -1 is an eigenvalue of $Q(\lambda)$. This fact, which will play a fundamental role in our arguments, can be deduced essentially, by corresponding accommodation to the situation of Banach space case, using similar arguments as in the proof of Lemma 1 [17]. Consequently, for an eigenvalue λ of the Schrödinger operator H, λ being a regular point of the unperturbed operator H_0, the operator norm of $Q(\lambda)$ must be no less than 1, i.e., $\|Q(\lambda)\| \geq 1$. Namely from this operator norm evaluation, we will derive estimates for eigenvalues of the Schrödinger operator H.

Throughout the paper, there will be always assumed (tacitly) that the conditions (2.2), (2.3) and (2.4) are satisfied.

3. Schrödinger Operators

1. Let H denote a Schrödinger operator defined in a space $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$, as was mentioned before, by $-\Delta + q(x)$ with a potential q admitting a factorization $q = ab$, where $a \in L_r(\mathbb{R}^n)$, $b \in L_s(\mathbb{R}^n)$ $(0 < r, s \leq \infty)$. For the general case of an arbitrary dimension, the fundamental solution $\Phi(x)$ of the Laplacian $-\Delta$, and therefore the kernel of the resolvent $R(\lambda; H_0)$ of $H_0 (= -\Delta)$, is expressed by Bessel's functions (see, for instance [5]). Of course, the asymptotic formula

$$\Phi(x) = c|x|^{-(n-1)/2} e^{-\mu|x|} \; (1 + o(1)), \quad |x| \to \infty,$$

$c > 0$ and $\mathrm{Re}\,\mu > 0$, will be useful for our purposes, however, we have not use this fact. Instead of that, we will use the following integral representation of the free Green function

$$g(x - y; \lambda) = (4\pi)^{-n/2} \int_0^\infty e^{\lambda t} e^{-|x-y|^2/4t} t^{-n/2} \, dt, \quad \mathrm{Re}\,\lambda < 0. \tag{3.1}$$

In other words we use the fact that the resolvent $R(\lambda; H_0)$ can be represented as a convolution integral operator with the kernel $g(x; \lambda)$, that will make useful in evaluation of the bordered resolvent.

There holds the following result.

THEOREM 3.1. *Let* $1<p<\infty$ *and let* $q=ab$, *where* $a\in L_r(\mathbb{R}^n), b\in L_s(\mathbb{R}^n)$ *with* $0<r\le\infty$, $p\le s\le\infty$ *and* $r^{-1}+s^{-1}<2n^{-1}$. *Then, for any eigenvalue* λ *with* $\operatorname{Re}\lambda<0$ *of the Schrödinger operator* H, *considered acting in the space* $L_p(\mathbb{R}^n)$, *there holds*

$$|\operatorname{Re}\lambda|^{1-n/2\alpha'}\le C(n,r,s)\|a\|_r\|b\|_s, \tag{3.2}$$

where $C(n,r,s)=(4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}\Gamma(1-n/2\alpha')$, $\alpha=(1-r^{-1}-s^{-1})^{-1}$.

Proof. We have to show the boundedness of the operator $Q(\lambda)=BR(\lambda;H_0)A$ and evaluate its norm. (A, B denote the operators of multiplications by a, b, respectively). Note that $Q(\lambda)$ is an integral operator with kernel

$$g(x-y;\lambda)a(y)b(x).$$

To evaluate this integral operator we first observe that, under supposed conditions, the operator of multiplication A is bounded viewed as an operator from $L_p(\mathbb{R}^n)$ to $L_\beta(\mathbb{R}^n)$ with some $\beta\ge1$. In fact, since $a\in L_r(\mathbb{R}^n)$, for any $u\in L_p(\mathbb{R}^n)$, by Hölder's inequality, we have

$$\|au\|_\beta\le\|a\|_r\|u\|_p,\quad \beta^{-1}=r^{-1}+p^{-1}. \tag{3.3}$$

Similarly, one can choose a γ with $p\le\gamma\le\infty$, for which

$$\|bv\|_p\le\|b\|_s\|v\|_\gamma,\quad \gamma^{-1}+s^{-1}=p^{-1}, \tag{3.4}$$

for $v\in L_\gamma(\mathbb{R}^n)$, that means that B represents a bounded operator from $L_\gamma(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$.

Now, we take an $\alpha\ge1$ such that

$$\alpha^{-1}+\beta^{-1}=\gamma^{-1}+1 \tag{3.5}$$

and find conditions under which the kernel function $g(x;\lambda)$ belongs to the space $L_\alpha(\mathbb{R}^n)$. By Minkowski's inequality, we have

$$\begin{aligned}\|g(\cdot;\lambda)\|_\alpha&=\left(\int_{\mathbb{R}^n}\left|(4\pi)^{-n/2}\int_0^\infty e^{\lambda t}e^{-|x|^2/4t}\,t^{-n/2}\,dt\right|^\alpha dx\right)^{1/\alpha}\\&\le(4\pi)^{-n/2}\int_0^\infty\left(\int_{\mathbb{R}^n}\left|e^{\lambda t}e^{-|x|^2/4t}\,t^{-n/2}\right|^\alpha dx\right)^{1/\alpha}dt\\&=(4\pi)^{-n/2}\int_0^\infty\left(\int_{\mathbb{R}^n}e^{-\alpha|x|^2/4t}\,dx\right)^{1/\alpha}e^{(\operatorname{Re}\lambda)t}t^{-n/2}\,dt,\end{aligned}$$

and since

$$\int_{\mathbb{R}^n}e^{-\alpha|x|^2/4t}\,dx=(4\pi t/\alpha)^{n/2},$$

it follows

$$\|g(\cdot;\lambda)\|_\alpha = (4\pi)^{-n/2}\int_0^\infty (4\pi t/\alpha)^{n/2\alpha} t^{-n/2} e^{(\operatorname{Re}\lambda)t}\,dt$$
$$= (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}\int_0^\infty t^{-n/2\alpha'} e^{(\operatorname{Re}\lambda)t}\,dt.$$

If α is chosen so that

$$-\frac{n}{2\alpha'}+1>0, \quad i.e., \quad \alpha' > \frac{n}{2}, \tag{3.6}$$

it can be applied the formula (see [12]; 3.381.4., p.331)

$$\int_0^\infty x^{\nu-1}e^{-\mu x}\,dx = \mu^{-\nu}\Gamma(\nu), \quad \operatorname{Re}\nu>0, \quad \operatorname{Re}\mu>0 \tag{3.7}$$

and we obtain

$$\|g(\cdot;\lambda)\|_\alpha \le (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}|\operatorname{Re}\lambda|^{-1+n/2\alpha'}\Gamma(1-n/2\alpha').$$

By Young's Inequality (see for instance, [4], Theorem 1.2.2, or also [10]; Proposition 8.9a) the operator $R(\lambda;H_0)$, representing an integral operator of convolution type (with the kernel $g(x-y;\lambda)$), is bounded as an operator from $L_\beta(\mathbb{R}^n)$ into $L_\gamma(\mathbb{R}^n)$ provided (4.3), and moreover,

$$\|R(\lambda;H_0)v\|_\gamma \le \|g(\cdot;\lambda)\|_\alpha\ \|v\|_\beta, \quad v\in L_\beta(\mathbb{R}^n). \tag{3.8}$$

Note that (3.5) indeed follows immediately from the relations between p, q, r and s given by (3.3) and (3.4):

$$1-\beta^{-1}+\gamma^{-1} = 1-r^{-1}-p^{-1}+p^{-1}-s^{-1} = 1-r^{-1}-s^{-1} = \alpha^{-1}.$$

The evaluations (3.3), (3.4) and (3.8) made above imply that

$$\|Q(\lambda)u\|_p = \|BR(\lambda;H_0)Au\|_p \le \|a\|_r\ \|b\|_s\ \|g\|_\alpha\|u\|_p$$

for each $u\in L_p(\mathbb{R}^3)$. Thus, under supposed conditions, we obtain the following estimation

$$\|Q(\lambda)\| \le (4\pi)^{-n/2\alpha'}\alpha^{-n/2\alpha}|\operatorname{Re}\lambda|^{-1+n/2\alpha'}\Gamma(1-n/2\alpha')\|a\|_r\ \|b\|_s,$$

and, therefore, for each $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda<0$ such that $\|Q(\lambda)\|\ge 1$, in particular, for an eigenvalue of the Schrödinger operator H, the desired estimation (3.2) holds true, where, as was seen, $\alpha=(1-r^{-1}-s^{-1})^{-1}$, and, due to (3.6), with the restriction $r^{-1}+s^{-1}<2n^{-1}$. □

From the obtained result, it can be derived many particular estimates useful in applications. We begin with the situation when $a, b \in L_r(\mathbb{R}^n)$ with $r > n$ if $1 < p \leq n$ and $p \leq r \leq \infty$ if $p > n$. In (3.2), we take $s = r$, then $r^{-1} + s^{-1} = 2r^{-1} (< 2n^{-1})$ and $\alpha = r/(r-2)$. In view of Theorem 3.1, we have the following result.

COROLLARY 3.2. *Suppose $q = ab$, where $a \in L_r(\mathbb{R}^n)$ with $r > n$ if $1 < p \leq n$ and $p \leq r \leq \infty$ if $p > n$. Then every eigenvalue λ with* $\operatorname{Re}\lambda < 0$ *of the Schrödinger operator H, considered acting in $L_p(\mathbb{R}^n)$, satisfies*

$$|\operatorname{Re}\lambda|^{r-n} \leq C(n, r) \|a\|_r^r \|b\|_r^r, \tag{3.9}$$

where $C(n, r) = (4\pi)^{-n} (1 - 2r^{-1})^{n(r-2)/2} \Gamma(1 - nr^{-1})^r$.

The following estimate is especially worthy to be mentioned. For related results see [20] (cf. also the estimate conjectured, but for the case of Hilbert space $L_2(\mathbb{R}^n)$, by Laptev and Safronov [19], and the discussion undertaken in this respect in [9]; see Remark 1.6 [9]).

COROLLARY 3.3. *Let $\gamma > 0$ if $1 < p \leq n$ and $2\gamma \geq p - n$ if $p > n$. Suppose*

$$q \in L_{\gamma+n/2}(\mathbb{R}^n).$$

Then every eigenvalue λ with $\operatorname{Re}\lambda < 0$ *of the Schrödinger operator H, considered acting in $L_p(\mathbb{R}^n)$, satisfies*

$$|\operatorname{Re}\lambda|^{\gamma} \leq C(n, \gamma) \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2} \, \mathrm{d}x, \tag{3.10}$$

where

$$C(n, \gamma) = \frac{1}{(4\pi)^{n/2}} \left(\frac{\gamma + n/2 - 1}{\gamma + n/2} \right)^{n(\gamma+n/2-1)/2} \Gamma \left(\frac{\gamma}{\gamma + n/2} \right)^{\gamma+n/2}.$$

Proof. It suffices to let $r = 2\gamma + n$ in (3.9) and take $a(x) = |q(x)|^{1/2}$, $b(x) = (\operatorname{sgn} q(x))|q(x)|^{1/2}$, where $\operatorname{sgn}\, q(x) = q(x)/|q(x)|$ if $q(x) \neq 0$ and $\operatorname{sgn}\, q(x) = 0$ if $q(x) = 0$. □

Frank [11] also obtains a result similar to that already mentioned by Corollary 3.3, but for the case of the Hilbert space $L_2(\mathbb{R}^3)$ and with restriction $0 < r \leq 3/2$. The proofs in [11] are based on a uniform Sobolev inequality due to Kenig et al. [18].

Another type of estimates can be obtained directly from (3.9) by involving decaying potentials. So, for instance, if we take $a(x) = (1 + |x|^2)^{-\tau/2}$ and $b(x) = (1 + |x|^2)^{\tau/2} q(x)$ with $\tau r > n$, then $a \in L_r(\mathbb{R}^n)$ and,

$$\|a\|_r^r = \pi^{n/2} \Gamma\left((\tau r - n)/2\right) / \Gamma\left((\tau r)/2\right).$$

In view of Corollary 3.2, the following result holds true.

COROLLARY 3.4. *Suppose*

$$(1+|x|^2)^{\tau/2} q \in L_r(\mathbb{R}^n),$$

where $\tau r > n$, *and* $r > n$ *if* $1 < p \le n$ *and* $p \le r \le \infty$ *if* $p > n$. *Then every eigenvalue* λ *with* $\operatorname{Re}\lambda < 0$ *of the Schrödinger operator* H, *considered acting in* $L_p(\mathbb{R}^n)$, *satisfies*

$$|\operatorname{Re}\lambda|^{r-n} \le C(n,r,\tau) \int_{\mathbb{R}^n} |(1+|x|^2)^{\tau/2} q(x)|^r \, \mathrm{d}x, \tag{3.11}$$

where

$$C_1(n,r,\tau) = (16\pi)^{-n/2}(1-2r^{-1})^{n(r-2)/2}\Gamma(1-nr^{-1})^r \Gamma((r\tau-n)/2)\Gamma(r\tau/2).$$

It stands to reason that estimates of type (3.11) can be given choosing other (weight) functions, used frequently for diverse proposes, as, for instance, $\mathrm{e}^{\tau|x|}$, $|x|^\sigma$ $\mathrm{e}^{\tau|x|}$, $\mathrm{e}^{\tau|x|^2}$, etc. We cite [11] (see also [24] and [25] for some related results involving weight functions as in Corollary 3.4).

Remark 3.5. The estimate (3.2) can be improved up to a factor $(A_\alpha A_\beta A_{\gamma'})^n$ if in proving of Theorem 3.1 it would be used the sharp form of Young's convolution inequality due to Beckner [3], where A_α, A_β and $A_{\gamma'}$ are defined in accordance with the notation $A_p = (p^{1/p}/p'^{1/p'})^{1/2}$. If it turns out that $A_\alpha A_\beta A_{\gamma'} < 1$ as, for instance, in case $1 < \alpha, \beta, \gamma' < 2$, one has indeed an improvement of (3.2). So, it happens in the case of a Schrödinger operator considered in the following example.

EXAMPLE 3.6. Let $p=2$, and suppose $q \in L_n(\mathbb{R}^n)$. Put $r=s=2n$, and let $a(x) = |q(x)|^{1/2}$, $b(x) = q(x)/|q(x)|^{1/2}$. Then, by (3.9), for eigenvalues λ with $\operatorname{Re}\lambda < 0$ of H there holds

$$|\operatorname{Re}\lambda| \le C\|q\|_n^2 \tag{3.12}$$

with a constant C depending only on n, namely, $C = 4^{-1}(1-n^{-1})^{n-1}$. However, in this case, $\alpha = n/(n+1)$ and $\beta = \gamma' = 2n/(n+1)$, hence the estimate (3.12) also holds true with the constant $C = n(1-n)^{n-1}/(n+1)^{n+1}$ provided that

$$(A_\alpha A_\beta A_{\gamma'})^n = \frac{2}{\sqrt{n}}\left(\frac{n}{n+1}\right)^{(n+1)/2},$$

as is easily checked. Obviously, $A_\alpha A_\beta A_{\gamma'} < 1$.

2. In the previous argument somewhat it was involved the *heat kernel* associated to the Laplacian on $\mathbb{R}^n$. In fact, it could be equivalently used the kernel

$$h(x, y; t) = (4\pi t)^{-n/2} \mathrm{e}^{-|x-y|^2/4t}, \quad t > 0, \tag{3.13}$$

representing the (one-parameter) semi-group $\mathrm{e}^{-tH_0} (0 \le t < \infty)$. More exactly, e^{-tH_0} is represented by the integral operator with the kernel (3.13), i.e.,

$$(\mathrm{e}^{-tH_0} u)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|x-y|^2/4t} u(y)\, \mathrm{d}y, \quad t > 0. \tag{3.14}$$

The arguments similar to those used in proving Theorem 3.1 can be applied to obtain (under suitable conditions) the estimate

$$\| B\mathrm{e}^{-tH_0} Au \|_p \le (4\pi t)^{-n/2\alpha'} \alpha^{-n/2\alpha} \|a\|_r \|b\|_s \|u\|_p, \quad u \in L_p(\mathbb{R}^n).$$

Then, from the formula expressing the resolvent $R(\lambda; H_0)$ as the Laplace transform of the semi-group e^{-tH_0} (see, for instance [13]), i.e.,

$$R(\lambda; H_0) = \int_0^\infty \mathrm{e}^{\lambda t} \mathrm{e}^{-tH_0}\, \mathrm{d}t, \quad \operatorname{Re} \lambda < 0, \tag{3.15}$$

we can further estimate

$$\begin{aligned} \| BR(\lambda; H_0) A \| &\le \int_0^\infty \mathrm{e}^{(\operatorname{Re}\lambda)t} \| B\mathrm{e}^{-tH_0} A \|\, \mathrm{d}t \\ &\le (4\pi)^{-n/2\alpha'} \alpha^{-n/2\alpha} \|a\|_r \ \|a\|_s \int_0^\infty t^{-n/2\alpha'} \mathrm{e}^{(\operatorname{Re}\lambda)t}\, \mathrm{d}t \\ &\le (4\pi)^{-n/2\alpha'} \alpha^{-n/2\alpha} |\operatorname{Re}\lambda|^{-1+n/2\alpha'}\ \Gamma(1 - n/2\alpha') \|a\|_r \|b\|_s, \end{aligned}$$

i.e.,

$$\| BR(\lambda; H_0) A \| \le (4\pi)^{-n/2\alpha'} \alpha^{-n/2\alpha} |\operatorname{Re}\lambda|^{-1+n/2\alpha'}\ \Gamma(1 - n/2\alpha') \|a\|_r \|b\|_s,$$

and, thus, we come to the same estimate as in (3.2).

3. The next result concerns evaluation of the imaginary part for a complex eigenvalue λ of H.

THEOREM 3.7. *Let* $1 < p < \infty$, *and let* $q = ab$, *where* $a \in L_r(\mathbb{R}^n), b \in L_s(\mathbb{R}^n)$ *for* r, s *satisfying* $0 < r \le \infty, \quad p \le s \le \infty$, $r^{-1} - s^{-1} = 1 - 2p^{-1}, \quad 2^{-1} - p^{-1} \le r^{-1} \le 1 - p^{-1}$ *and* $r^{-1} + s^{-1} < 2n^{-1}$. *Then, for any complex eigenvalue* λ *with* $\operatorname{Im}\lambda \ne 0$ *of the Schrödinger operator* H, *considered acting in the space* $L_p(\mathbb{R}^n)$, *there holds*

$$|\operatorname{Im}\lambda|^\alpha \le (4\pi)^{\alpha-1} \Gamma(\alpha) \|a\|_r \|b\|_s, \tag{3.16}$$

where $\alpha = 1 - n(r^{-1} + s^{-1})/2$.

Proof. The proof will depend upon a modification of the argument used in proving the previous result. Instead of (3.15), it will be used the formula expressing the resolvent $R(\lambda; H_0)$ as the Laplace transform of the operator-group e^{-itH_0} $(-\infty < t < \infty)$, namely

$$R(\lambda; H_0) = i \int_0^\infty e^{i\lambda t} e^{-itH_0}\, dt \tag{3.17}$$

if, for instance, $\operatorname{Im} \lambda > 0$. First, we estimate the norm $\|Be^{-itH_0}A\|$ and then using the formula (3.17) we will derive estimation for $\operatorname{Im} \lambda$ (we preserve notations made above).

As is known (cf., for instance, [15,22], Ch.IX), for a fixed real t, e^{-itH_0} represents an integral operator with the heat kernel (cf. (3.13))

$$h(x, y; it) = (4\pi it)^{-n/2} e^{-|x-y|^2/4it}.$$

Writing

$$(e^{-itH_0}Au)(x) = (4\pi it)^{-n/2} e^{-|x|^2/4it} \int_{\mathbb{R}^n} e^{-i\langle x,y\rangle/2t} e^{-|y|^2/4it} a(y)u(y)\, dy, \tag{3.18}$$

we argue as follows.

We already know that

$$\|Au\|_\beta \le \|a\|_r \|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}.$$

It follows that for any $u \in L_p(\mathbb{R}^n)$ the function v defined by $v(y) = e^{-|y|^2/4it} a(y)$ $u(y)$ belongs to $L_\beta(\mathbb{R}^n)$, and

$$\|v\|_\beta \le \|a\|_r \|u\|_p. \tag{3.19}$$

Further, the integral on the right-hand side in (3.18) represents the function $(2\pi)^{n/2} \hat{v}(x/2t)$, where $\hat{v}$ denotes the Fourier transform of v. According to the Hausdorff–Young theorem (see, for instance, [4], Theorem 1.2.1), the Fourier transform represents a bounded operator from $L_\beta(\mathbb{R}^n)$ to $L_{\beta'}(\mathbb{R}^n)$ with $1 \le \beta \le 2$, and its norm is bounded by $(2\pi)^{-n/2+n/\beta'}$, i.e.,

$$\|\hat{v}\|_{\beta'} \le (2\pi)^{-n/2+n/\beta'} \|v\|_\beta. \tag{3.20}$$

It follows that $\hat{v} \in L_{\beta'}(\mathbb{R}^n)$ and, since

$$(e^{-itH_0}Au)(x) = (4\pi it)^{-n/2} e^{-|x|^2/4it} (2\pi)^{n/2} \hat{v}(x/2t),$$

the function $\mathrm{e}^{-itH_0}Au$ belongs to $L_{\beta'}(\mathbb{R}^n)$. Moreover, in view of (3.19) and (3.20),

$$\begin{aligned}\|\mathrm{e}^{-itH_0}Au\|_{\beta'} &= (4\pi t)^{-n/2}(2\pi)^{n/2}\left(\int_{\mathbb{R}^n}|\hat{v}(x/2t)|^{\beta'}\,\mathrm{d}x\right)^{1/\beta'}\\ &= (4\pi t)^{-n/2}(2\pi)^{n/2}(2t)^{n/\beta'}\,\|\hat{v}\|_{\beta'}\\ &\le (4\pi t)^{-n/2}(2\pi)^{n/2}(2t)^{n/\beta'}(2\pi)^{-n/2+n/\beta'}\,\|v\|_{\beta}\\ &\le (4\pi t)^{-n/2+n/\beta'}\,\|a\|_r\;\|u\|_p,\end{aligned}$$

so that

$$\|\mathrm{e}^{-itH_0}Au\|_{\beta'} \le (4\pi t)^{-n/2+n/\beta'}\,\|a\|_r\|u\|_p,\quad u\in L_p(\mathbb{R}^n).$$

On the other hand, since $r^{-1}-s^{-1}=1-2p^{-1}$, and since $\beta^{-1}=r^{-1}+p^{-1}$, one has $s^{-1}+\beta'^{-1}=p^{-1}$ that guarantees the boundedness of the operator of multiplication B regarded as an operator acting from $L_{\beta'}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$. Moreover,

$$\|Bv\|_p \le \|b\|_s\|v\|_{\beta'},\quad v\in L_{p'}(\mathbb{R}^n),$$

It is seen that for any $u\in L_p(\mathbb{R}^n)$ the element $B\mathrm{e}^{-itH_0}Au$ belongs to $L_p(\mathbb{R}^n)$, and

$$\|B\mathrm{e}^{-itH_0}Au\|_p \le (4\pi t)^{-n/2+n/\beta'}\,\|a\|_r\|b\|_s\|u\|_p,\quad u\in L_p(\mathbb{R}^n).$$

Now, we apply (3.17) and for $\operatorname{Im}\lambda>0$, we find

$$\begin{aligned}\|BR(\lambda;H_0)Au\|_p &\le \int_0^\infty \mathrm{e}^{-(\operatorname{Im}\lambda)t}\|B\mathrm{e}^{-itH_0}Au\|_p\,\mathrm{d}t\\ &\le (4\pi)^{-n/2+n/\beta'}\,\|a\|_r\|b\|_s\|u\|_p\int_0^\infty t^{-n/2+n/\beta'}\mathrm{e}^{-(\operatorname{Im}\lambda)t}\,\mathrm{d}t.\end{aligned}$$

Next, we observe $1-n/2+n/\beta'=\alpha$ that was assumed to be positive, and thus we can apply the formula (3.7), due to which, we have

$$\int_0^\infty t^{-n/2+n/\beta'}\mathrm{e}^{-(\operatorname{Im}\lambda)t}\,\mathrm{d}t = (\operatorname{Im}\lambda)^{-\alpha}\Gamma(\alpha).$$

Therefore,

$$\|BR(\lambda;H_0)A\| \le (4\pi)^{\alpha-1}(\operatorname{Im}\lambda)^{-\alpha}\Gamma(\alpha)\|a\|_r\|b\|_s.$$

For an eigenvalue λ of H, it should be

$$1\le(4\pi)^{\alpha-1}(\operatorname{Im}\lambda)^{-\alpha}\Gamma(\alpha)\|a\|_r\;\|b\|_s,$$

that is (3.16).

The estimate for the case $\operatorname{Im}\lambda < 0$ is treated similarly coming from the formula

$$R(\lambda;\, H_0) = -i \int_{\infty}^{0} e^{i\lambda t} e^{-iH_0 t}\, dt, \quad \operatorname{Im}\lambda < 0.$$

□

Notice that if $r = s$ in Theorem 3.7, it must be only $p = 2$ and $r > n$. For this case, we have the following result.

COROLLARY 3.8. *Let $r > n$, and suppose $q \in L_{r/2}(\mathbb{R}^n)$. Then any complex eigenvalue λ with $Im\lambda \neq 0$ of the Schrödinger operator H defined in the space $L_2(\mathbb{R}^n)$ satisfies*

$$|\operatorname{Im}\lambda|^{1-n/r} \leq (4\pi)^{-n/r}\Gamma(1-n/r)\|q\|_{r/2}. \tag{3.21}$$

For the particular case, when $r = 2\gamma + n$, we have the following result (an analogous result to that given by Corollary 3.2).

COROLLARY 3.9. *Let $\gamma > 0$ and suppose that $q \in L_{\gamma+n/2}(\mathbb{R}^n)$. Then for any complex eigenvalue λ with* $\operatorname{Im}\lambda \neq 0$ *of the Schrödinger operator defined in $L_2(\mathbb{R}^n)$ there holds*

$$|\operatorname{Im}\lambda|^{\gamma} \leq (4\pi)^{-n/2}\Gamma\left(\frac{2\gamma}{2\gamma+n}\right)^{\gamma+n/2} \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2}\, dx. \tag{3.22}$$

Remark 3.10. The estimate given by Theorem 3.7 can be improved upon a constant less than 1. The point is that in proving Theorem 3.7 it can be applied the sharp form of the Hausdorff–Young theorem which is due to K. I. Babenko [2] (see also W. Beckner [3] for the general case relevant for our purposes). According to Babenko's result estimation (3.20), and hence (3.16) as well, can be refined upon a constant less than 1, namely

$$\|\hat{v}\|_{\beta'} \leq (2\pi)^{-n/2+n/\beta'} A\|v\|_{\beta},$$

where $A = (\beta^{1/\beta}/\beta'^{1/\beta'})^{n/2}$. It is always $A \leq 1$ provided of $1 \leq \beta \leq 2$, and it is strictly less than 1 if β is chosen such that $1 < \beta < 2$. The same concerns estimate (3.21) and (3.22).

4. The norm evaluation for the operators $BR(\lambda;\, H_0)A$ for $\lambda \in \mathbb{C}\setminus[0,\infty)$ can be carried out representing the resolvent of H_0 in terms of the Fourier transform. Namely, it can use the following equality

$$BR(\lambda;\, H_0)A = BF^{-1}\widehat{R(\lambda;\, H_0)}FA, \tag{3.23}$$

where it is denoted

$$\widehat{R(\lambda; H_0)} = FR(\lambda; H_0)F^{-1}$$

(F, F^{-1} denote the Fourier operators). Clearly, $\widehat{R(\lambda; H_0)}$ represents the multiplication operator by $(|\xi|^2 - \lambda)^{-1}$, i.e.,

$$\widehat{R(\lambda; H_0)}\hat{u}(\xi) = (|\xi|^2 - \lambda)^{-1}\hat{u}(\xi), \quad \xi \in \mathbb{R}^n.$$

On $L_2(\mathbb{R}^n)$, the mentioned relations are obviously true. However, we will use them for the spaces $L_p(\mathbb{R}^n)$ with $p \neq 2$, as well, preserving the same notations as in the Hilbert space case $p = 2$.

As before, by assuming that $a \in L_r(\mathbb{R}^n)$ and $b \in L_s(\mathbb{R}^n)$ $(0 < r, s \leq \infty)$, we choose $\beta > 0$ and $\gamma > 0$ such that

$$\|Au\|_\beta \leq \|a\|_r \ \|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}, \tag{3.24}$$

$$\|Bv\|_p \leq \|b\|_s \ \|v\|_\gamma, \quad p^{-1} = s^{-1} + \gamma^{-1}. \tag{3.25}$$

According to the Hausdorff–Young theorem, if $1 \leq \beta \leq 2$, the Fourier transform F represents a bounded operator from $L_\beta(\mathbb{R}^n)$ to $L_{\beta'}(\mathbb{R}^n)$ the norm of which is bounded by $(2\pi)^{-n/2+n/\beta'}$, i.e.,

$$\|Ff\|_{\beta'} \leq (2\pi)^{-n/2+n/\beta'} \|f\|_\beta. \tag{3.26}$$

The same concerns the inverse Fourier transform F^{-1} considered as an operator acting from $L_{\gamma'}(\mathbb{R}^n)$ to $L_\gamma(\mathbb{R}^n)$. If $1 \leq \gamma' \leq 2$, that is equivalent to $2 \leq \gamma \leq \infty$, we have

$$\|F^{-1}g\|_\gamma \leq (2\pi)^{-n/2+n/\gamma} \|g\|_{\gamma'}. \tag{3.27}$$

Now, we take $\alpha, 0 < \alpha \leq \infty$, such that

$$\gamma'^{-1} = \alpha^{-1} + \beta'^{-1}, \tag{3.28}$$

equivalently, $\alpha^{-1} = r^{-1} + s^{-1}$, and evaluate the L_α-norm of the function $h(\cdot; \lambda)$ defined by

$$h(\xi; \lambda) = (|\xi|^2 - \lambda)^{-1}, \quad \xi \in \mathbb{R}^n.$$

For $\alpha \neq \infty$, we have

$$\|h(\cdot; \lambda)\|_\alpha^\alpha = \int_{\mathbb{R}^n} ||\xi|^2 - \lambda|^{-\alpha} \, d\xi = \int_0^\infty \int_{S^{n-1}} \rho^{n-1}|\rho^2 - \lambda|^{-\alpha} \, d\rho \, d\omega?$$

$$= mes(S^{n-1}) \int_0^\infty \rho^{n-1}|\rho^2 - \lambda|^{-\alpha} \, d\rho,$$

where $mes(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$ is the surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. Therefore,

$$\|h(\cdot;\lambda)\|_\alpha^\alpha = 2\pi^{n/2}/\Gamma(n/2)\int_0^\infty \rho^{n-1}|\rho^2-\lambda|^{-\alpha}\,\mathrm{d}\rho. \tag{3.29}$$

If, we are particularly interesting in estimation of negative eigenvalues, we let that $\operatorname{Re}\lambda < 0$ and evaluate the integral in (3.29) as follows. First, we observe that

$$|\rho^2-\lambda|^{-1} \le (\rho^2 - \operatorname{Re}\lambda)^{-1},$$

and then by setting $\rho^2 = t$, we obtain

$$\|h(\cdot;\lambda)\|_\alpha^\alpha \le \frac{\pi^{n/2}|\operatorname{Re}\lambda|^{-\alpha}}{\Gamma(n/2)}\int_0^\infty \frac{t^{n/2-1}}{(|\operatorname{Re}\lambda|^{-1}t+1)^\alpha}\,\mathrm{d}t.$$

By supposing $\alpha > n/2$, the formula ([12], 3.194.3.)

$$\int_0^\infty \frac{x^{\mu-1}}{(1+\beta x)^\nu}\,\mathrm{d}x = \beta^{-\mu}B(\mu,\nu-\mu), \qquad |arg\beta| < \pi, \qquad \operatorname{Re}\nu > \operatorname{Re}\mu > 0$$

($B(x,y)$ denotes the beta function), can be applied. We get

$$\|h(\cdot;\lambda)\|_\alpha^\alpha \le \pi^{n/2}(\Gamma(n/2))^{-1}|\operatorname{Re}\lambda|^{n/2-\alpha}B(n/2,\alpha-n/2),$$

or, in view of the functional relation between beta and gamma functions,

$$\|h(\cdot;\lambda)\|_\alpha^\alpha \le \pi^{n/2}|\operatorname{Re}\lambda|^{n/2-\alpha}\ \Gamma(\alpha-n/2)/\Gamma(\alpha). \tag{3.30}$$

Thus, for $\alpha > n/2$, the function $h(\cdot;\lambda)$ belongs to the space $L_\alpha(\mathbb{R}^n)$ and, since (3.28), it follows that the operator of multiplication $\widehat{R(\lambda;H_0)}$ is bounded as an operator acting from $L_{\beta'}(\mathbb{R}^n)$ to $L_{\gamma'}(\mathbb{R}^n)$, and, due to of (3.30), there holds

$$\|\widehat{R(\lambda;H_0)}f\|_{\gamma'} \le \pi^{n/2\alpha}|\operatorname{Re}\lambda|^{n/2\alpha-1}\ (\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|f\|_{\beta'}. \tag{3.31}$$

In this way, we obtain (cf. (3.24)–(3.27), (3.31))

$$\|BR(\lambda;H_0)A\| \le (2\pi)^{-n/\alpha}\pi^{n/2\alpha}|\operatorname{Re}\lambda|^{n/2\alpha-1}\ (\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|a\|_r\|b\|_s.$$

Therefore, for an eigenvalue λ of H, it should by fulfilled

$$1 \le (2\pi)^{-n/\alpha}\pi^{n/2\alpha}|\operatorname{Re}\lambda|^{n/2\alpha-1}(\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|a\|_r\|b\|_s,$$

or, equivalently,

$$|\operatorname{Re}\lambda|^{1-n/2\alpha} \le (4\pi)^{-n/2\alpha}(\Gamma(\alpha-n/2)/\Gamma(\alpha))^{1/\alpha}\|a\|_r\|b\|_s. \tag{3.32}$$

In the extremal case $\alpha = \infty$, that is only happen if $r = s = \infty$ (recall that $\alpha^{-1} = r^{-1} + s^{-1}$), there holds

$$\|h(\cdot;\lambda)\|_\infty = \sup_{\xi\in\mathbb{R}^n} ||\xi|^2 - \lambda|^{-1} \le \sup_{\rho>0}(\rho^2 - \operatorname{Re}\lambda)^{-1} = |\operatorname{Re}\lambda|^{-1},$$

i.e.,

$$\|h(\cdot;\lambda)\|_\infty \le |\operatorname{Re}\lambda|^{-1}.$$

In accordance with this evaluation, one follows

$$|\operatorname{Re}\lambda| \le \|a\|_\infty \|b\|_\infty, \tag{3.33}$$

a natural estimate for eigenvalues occurred outside of the continuous spectrum of H_0 by bounded perturbations.

Note that, the restriction $1 \le \beta \le 2$ is equivalent to $2^{-1} - p^{-1} \le r^{-1} \le 1 - p^{-1}$, while $1 \le \gamma' \le 2$ to $2^{-1} + p^{-1} \le s^{-1} \le p^{-1}$, and $\alpha > n/2$ to $r^{-1} + s^{-1} < 2n^{-1}$.

We have proved the following result.

THEOREM 3.11. *Let $1 < p < \infty$, and let $q = ab$, where $a \in L_r(\mathbb{R}^n)$, $b \in L_s(\mathbb{R}^n)$ for r, s satisfying $0 < r \le \infty$, $0 < s \le \infty$, $2^{-1} - p^{-1} \le r^{-1} \le 1 - p^{-1}$, $-2^{-1} + p^{-1} \le s^{-1} \le p^{-1}$, and $r^{-1} + s^{-1} < 2n^{-1}$. Then, for any eigenvalue λ with $\operatorname{Re}\lambda < 0$ of the Schrödinger operator H, considered acting in the space $L_p(\mathbb{R}^n)$, there holds*

$$|\operatorname{Re}\lambda|^{\alpha - n/2} \le C(n,\alpha)\|a\|_r^\alpha \ \|b\|_s^\alpha, \tag{3.34}$$

where $C(n,\alpha) = (4\pi)^{-n/2}\Gamma(\alpha - n/2)/\Gamma(\alpha)$, $\alpha = (r^{-1} + s^{-1})^{-1}$.

For $r = s = \infty$ there holds (3.33).

For the particular case $n = 1, p = 2$ and $r = s = 2$, one has $\alpha = 1$ and $C = 1/2$, hence, in view of (3.34), the following estimate

$$|\operatorname{Re}\lambda|^{1/2} \le \frac{1}{2}\|V\|_1 \quad \left(= \frac{1}{2}\int_{-\infty}^{\infty} |V(x)|\,\mathrm{d}x \right) \tag{3.35}$$

holds true for any eigenvalue λ of H with $\operatorname{Re}\lambda < 0$.

The obtained evaluation (3.35) corresponds to the well-known result of L. Spruch (mentioned in [16]) concerning negative eigenvalues of the one-dimensional self-adjoint Schrödinger operator considered in $L_2(\mathbb{R})$. For other related results, see [1,7–9,19,24].

Theorem 3.11 implies more general result (cf. also Corollary 3.3).

COROLLARY 3.12. *Let $\gamma > 0$ for $n \geq 2$ and $\gamma \geq 1/2$ for $n = 1$. If $q \in L_{\gamma+n/2}(\mathbb{R}^n)$, then every eigenvalue λ with* $\operatorname{Re}\lambda < 0$ *of the Schrödinger operator H defined in $L_2(\mathbb{R}^n)$ satisfies*

$$|\operatorname{Re}\lambda|^\gamma \leq (4\pi)^{-n/2} \frac{\Gamma(\gamma)}{\Gamma(\gamma+n/2)} \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2}\, dx. \tag{3.36}$$

A rigorous evaluation of the integral on the right-hand side of (3.29) leads to more exact estimates for the perturbed eigenvalues. To this end, we let $\lambda = |\lambda| e^{i\theta} (0 < \theta < 2\pi)$ and put $\rho^2 = |\lambda| t$. Then

$$\int_0^\infty \frac{\rho^{n-1}}{|\rho^2 - \lambda|^\alpha}\, d\rho = \frac{1}{2} |\lambda|^{n/2-\alpha} \int_0^\infty \frac{t^{n/2-1}}{(t^2 - 2t\cos\theta + 1)^{\alpha/2}}\, dt.$$

If $n/2 < \alpha$, it can be applied the formula ([12]; 3.252.10.)

$$\int_0^\infty \frac{x^{\mu-1}}{(x^2 + 2x\cos t + 1)^\nu}\, dx = (2\sin t)^{\nu-1/2} \Gamma(\nu+1/2) B(\mu, 2\nu-\mu) P^{1/2-\nu}_{\mu-\nu-1/2}(\cos t)$$

$$(-\pi < t < \pi, \quad 0 < \operatorname{Re}\mu < \operatorname{Re}\ 2\nu),$$

where $P^\nu_\mu(z)(-1 \leq z \leq 1)$ denote for the Gegenbauer polynomials ([12]; 8.7–8.8). As a result we have

$$\|h(\cdot;\lambda)\|_\alpha = \pi^{n/2\alpha} |\lambda|^{n/2\alpha-1} I(n,\alpha,\theta), \tag{3.37}$$

where

$$I(n,\alpha,\theta) = (2\sin\theta)^{1/2-1/2\alpha} \left(\frac{\Gamma(\alpha/2+1/2)\Gamma(\alpha-n/2)}{\Gamma(\alpha)} P^{1/2-\alpha/2}_{n/2-\alpha/2-1/2}(-\cos\theta) \right)^{1/\alpha},$$

and hence

$$\|BR(\lambda; H_0)Au\|_p \leq (4\pi)^{-n/2\alpha} |\lambda|^{n/2\alpha-1} I(n,\alpha,\theta) \|a\|_r \|b\|_s \|u\|_p$$

(note that $(-n/2 + n/\beta') + (-n/2 + n/\gamma) + n/2\alpha = -n/2\alpha$).

Therefore, we obtain the following result.

THEOREM 3.13. *Under the same assumptions as in Theorem 3.11 for any eigenvalue $\lambda \in \mathbb{C}\backslash[0,\infty)$ of the Schrödinger operator H, considering acting in the space $L_p(\mathbb{R}^n)$, there holds the estimation*

$$|\lambda|^{\alpha-n/2} \leq C(n,\alpha,\theta) \|a\|_r^\alpha \|b\|_s^\alpha, \tag{3.38}$$

where $C(n,\alpha,\theta) = (4\pi)^{-n/2} I(n,\alpha,\theta)^\alpha$ and $I(n,\alpha,\theta)$ as in (3.37).

Remark 3.14. The estimate (3.34) and, of course, (3.38) as well can be improved upon the constant $A_\beta A_{\gamma'} (= (\beta^{1/\beta} \gamma'^{1/\gamma'} / \beta'^{1/\beta'} \gamma^{1/\gamma})^{n/2})$ due to the sharp form of the Hausdorff–Young theorem [2] (cf. Remark 3.10).

4. Polyharmonic Operators

We will extend the estimates established previously to the operators of the form

$$H=(-\Delta)^m+q$$

in which (the potential) q is a complex-valued function, and m is an arbitrary positive real number. Unperturbed operator

$$H_0=(-\Delta)^m$$

can be comprehend, as

$$(H_0u)(x)=\int_{\mathbb{R}^n}|\xi|^{2m}\hat{u}(\xi)e^{-i\langle x,\xi\rangle}\,d\xi$$

defined, for instance, in $L_2(\mathbb{R}^n)$ on its maximal domain consisting of all functions $u\in L_2(\mathbb{R}^n)$ such that $H_0u\in L_2(\mathbb{R}^n)$ (or, what is the same, $\hat{v}$ determined by $\hat{v}(\xi)=|\xi|^{2m}\hat{u}(\xi)$ belongs to $L_2(\mathbb{R}^n)$); $\hat{u}$ denotes the Fourier transform of u. H_0 can be treated upon a unitary equivalence (by the Fourier transform) as the operator of multiplication by $|\xi|^{2m}$.

In the space $L_p(\mathbb{R}^n)$ $(1<p<\infty)$ the operator H can be viewed as an elliptic operator of order $2m$ defined on its domain the Sobolev space $W_p^{2m}(\mathbb{R}^n)$. As in preceding sections, we assume that the potential q admits a factorization $q=ab$ with a,b for which conditions (2.2), (2.3), but with $0<\nu<p'\kappa$ and $0<\mu<p(m-\kappa)$ for some $0<\kappa<m$, and (2.4) are satisfied. Under these conditions, the operator $(-\Delta)^m+q$ admits a closed extension H, let us denote it by $H_{m,q}$, to which the approach for the evaluation of perturbed eigenvalues proposed in Section 2 is applied.

Thus, to obtain estimation for the norm of $BR(\lambda;H_0)A$ (the operators A,B are defined as in previous subsections), we can use the relation (3.23), where

$$\widehat{R(\lambda;H_0)}\hat{u}(\xi)=(|\xi|^{2m}-\lambda)^{-1}\hat{u}(\xi),\quad \xi\in\mathbb{R}^n.$$

The arguments used in proving Theorems 3.11 and 3.13 can be applied, and as is seen we have only to evaluate, for appropriate $\alpha>0$, the L_α-norm of the function $h_m(\cdot;\lambda)$ defined by

$$h_m(\xi;\lambda)=(|\xi|^{2m}-\lambda)^{-1},\quad \xi\in\mathbb{R}^n.$$

For any $\alpha, 0<\alpha<\infty,$ we have

$$\|h_m(\cdot;\lambda)\|_\alpha^\alpha=\int_{\mathbb{R}^n}\frac{d\xi}{||\xi|^{2m}-\lambda|^\alpha}=\frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty\frac{\rho^{n-1}}{|\rho^{2m}-\lambda|^\alpha}\,d\rho.$$

Writing $\lambda = |\lambda| e^{i\theta} (0 < \theta < 2\pi)$ and making the substitution $\rho^{2m} = |\lambda| t$, we obtain

$$\int_0^\infty \frac{\rho^{n-1}}{|\rho^{2m} - \lambda|^\alpha} \, d\rho = \frac{1}{2m} |\lambda|^{n/2m-\alpha} \int_0^\infty \frac{t^{n/2m-1}}{(t^2 - 2t\cos\theta + 1)^{\alpha/2}} \, dt.$$

Assuming $n/2m < \alpha$, we apply again the formula ([12]; 3.252.10.), and obtain Hence,

$$\|h_m(\cdot;\lambda)\|_\alpha^\alpha = \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \frac{1}{2m} |\lambda|^{n/2m-\alpha} I_m(n,\alpha,\theta), \tag{4.1}$$

where

$$I_m(n,\alpha,\theta) = (2\sin\theta)^{\alpha/2-1/2} \Gamma(\alpha/2 + 1/2) B(n/2m, \alpha - n/2m) P^{1/2-\alpha/2}_{n/2m-\alpha/2-1/2}(-\cos\theta).$$

Collecting all evaluations, we obtain the following result.

THEOREM 4.1. *Let $1 < p < \infty, m > 0$, and let $q = ab$, where $a \in L_r(\mathbb{R}^n)$, $b \in L_s(\mathbb{R}^n)$ for r, s satisfying $0 < r \leq \infty$, $0 < s \leq \infty$, $2^{-1} - p^{-1} \leq r^{-1} \leq 1 - p^{-1}$, $-2^{-1} + p^{-1} \leq s^{-1} \leq p^{-1}$, and $r^{-1} + s^{-1} < 2m/n$. Then, for any eigenvalue $\lambda \in \mathbb{C} \setminus [0,\infty)$ of the operator $H_{m,q}$, considered acting in $L_p(\mathbb{R}^n)$, there holds*

$$|\lambda|^{\alpha - n/2m} \leq C(n,m,\alpha,\theta) \|a\|_r^\alpha \|b\|_s^\alpha, \tag{4.2}$$

where $C(n,m,\alpha,\theta) = (4\pi)^{-n/2} (m\Gamma(n/2))^{-1} I_m(n,\alpha,\theta)$, $I_m(n,\alpha,\theta)$ is determined as in (4.1), and $\alpha = (r^{-1} + s^{-1})^{-1}$.

As a consequence of Theorem 4.1, we have a result analogous to that given by Corollary 3.12.

COROLLARY 4.2. *Let $\gamma > 0$ for $n \geq 2m$ and $\gamma \geq 1 - n/2m$ for $n < 2m$. If $q \in L_{\gamma+n/2m}(\mathbb{R}^n)$, then every eigenvalue $\lambda \in \mathbb{C} \setminus [0,\infty)$ of the operator $H_{m,q}$ defined in $L_2(\mathbb{R}^n)$ satisfies*

$$|\lambda|^\gamma \leq C(n,m,\alpha,\theta) \int_{\mathbb{R}^n} |q(x)|^{\gamma+n/2m} \, dx, \tag{4.3}$$

where $C(n,m,\alpha,\theta)$ is as in (4.2).

Remark 4.3. Similarly, as for estimates (3.34) and (3.38), the estimate (4.2) and hence (4.3) can be improved upon the constant $A_\beta A_{\gamma'}$ (see Remark 3.14).

5. Schrödinger Operators with Potentials of Weak Lebesgue's Classes

The methods used above for the evaluation of perturbed eigenvalues are available under slightly weakened conditions involving potentials belonging to weak Lebesgue's spaces. For we consider a Schrödinger operator H generated by $-\Delta + q(x)$, written as a product $q = ab$, however, with $a \in L_{r,w}(\mathbb{R}^n)$ and $b \in L_{s,w}(\mathbb{R}^n)$ (we will use $L_{r,w}$ to denote the so-called weak L_r-spaces). Recall that the weak $L_{r,w}(\mathbb{R}^n)$ space consists of all measurable almost everywhere finite complex-valued functions on $\mathbb{R}^n$ such that

$$\|f\|_{r,\infty} := \sup_{t>0} (t^r \lambda_f(t))^{1/r} < \infty,$$

where λ_f denotes the distribution function of $|f|$, namely,

$$\lambda_f(t) = \mu(\{x \in \mathbb{R}^n : |f(x)| > t\}), \quad 0 < t < \infty,$$

(μ is the standard Lebesgue measure on $\mathbb{R}^n$). The weak L_r-spaces are confined on the more general so-called Lorentz classes $L_{p,r}(\mathbb{R}^n)(0 < p < \infty, 0 < r \le \infty)$ (see, e.g., [4]) which will be also needed. We define $L_{p,r}(\mathbb{R}^n)$ to be the space of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{p,r}^r := \int_{\mathbb{R}^n} t^r (\lambda_f(t))^{r/p} \frac{dt}{t} < \infty.$$

Note that $L_{r,r}(\mathbb{R}^n) = L_r(\mathbb{R}^n)$, and it will be convenient to let $L_{\infty,r}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ $(0 < r \le \infty)$.

As before we let A, B to denote the operators of multiplication by a, b, respectively. In view of $a \in L_{r,w}(\mathbb{R}^n)$ and $b \in L_{s,w}(\mathbb{R}^n)$, as was assumed, we can apply a result of O'Neil [21] due to which there can be chosen $\beta > 0$ and $\gamma > 0$ such that the multiplication A to be bounded from $L_{p,p}(\mathbb{R}^n)(= L_p(\mathbb{R}^n))$ to $L_{\beta,p}(\mathbb{R}^n)$ and, respectively, B to be bounded from $L_{\gamma,p}(\mathbb{R}^n)$ to $L_{p,p}(\mathbb{R}^n)$ and, moreover,

$$\|Au\|_{\beta,p} \le C\|a\|_{r,w}\|u\|_p, \quad \beta^{-1} = r^{-1} + p^{-1}, \tag{5.1}$$

and

$$\|Bv\|_p \le C\|b\|_{s,w}\|v\|_{\gamma,p}, \quad p^{-1} = s^{-1} + \gamma^{-1}. \tag{5.2}$$

Note that in (5.1) and (5.2), the constants in general are distinct, but depending only on r, p and s, p, respectively. Further, following notations made in the Section 3, for $\lambda \in \mathbb{C} \backslash [0, \infty)$ we let

$$h(\xi; \lambda) = (|\xi|^2 - \lambda)^{-1}, \quad \xi \in \mathbb{R}^n.$$

It was shown that for $\alpha > n/2$ one has $h(\cdot; \lambda) \in L_\alpha(\mathbb{R}^n)$, and for its norm there holds (3.37). Then, by applying the just mentioned result of O'Neil [21] the operator of multiplication by $h(\cdot; \lambda)$, that is, $\widehat{R(\lambda; H_0)}$, is acting boundedly from $L_{\beta',p}(\mathbb{R}^n)$

to $L_{\gamma,p}(\mathbb{R}^n)$ provided that (3.28). Besides, by interpolation, the Fourier operator F is in turn bounded from $L_{\beta,p}(\mathbb{R}^n)$ to $L_{\beta',p}(\mathbb{R}^n)$ for $1 \le \beta \le 2$, and its bound is less than $(2\pi)^{-n/2+n/\beta'}C$, C being a constant depending only on r and p. Similarly, F^{-1} is a bounded operator acting $L_{\gamma',p}(\mathbb{R}^n)$ to $L_{\gamma,p}(\mathbb{R}^n)$ for $2 \le \gamma \le \infty$, its bound is less than $(2\pi)^{-n/2+n/\gamma}c$, where c is a constant depending only on s and p. Consequently, the resolvent operator $R(\lambda; H_0)(= F^{-1}\widehat{R(\lambda; H_0)}F)$ represents a bounded operator from $L_{\beta,p}(\mathbb{R}^n)$ to $L_{\gamma,p}(\mathbb{R}^n)$ and, moreover,

$$\|R(\lambda; H_0)f\|_{\gamma,p} \le CI(n,\alpha,\theta)|\lambda|^{n/2\alpha-1},$$

where $I(n,\alpha,\theta)(\theta := \arg\lambda, 0 < \theta < 2\pi)$ is as in (3.27) and c being a constant depending on r, s and p. Therefore, in view of (5.1) and (5.2), we have

$$\|BR(\lambda; H_0)A\| \le CI(n,\alpha,\theta)|\lambda|^{n/2\alpha-1}\|a\|_{r,\infty}\|b\|_{s,\infty},$$

and, in this way, for any eigenvalue $\lambda \in \mathbb{C}\backslash[0,\infty)$ of the Schrödinger operator H, we obtain the following estimate

$$|\lambda|^{1-n/2\alpha} \le CI(n,\alpha,\theta)\|a\|_{r,\infty}\|b\|_{s,\infty}. \tag{5.3}$$

where C ia a constant depending only on p, r and s.

We have proved the following result.

THEOREM 5.1. *Let p, r, s be as in Theorem 3.1, and suppose $q = ab$, where $a \in L_{r,w}(\mathbb{R}^n)$ and $b \in L_{s,w}(\mathbb{R}^n)$. Then, any eigenvalue $\lambda \in \mathbb{C}\backslash[0,\infty)$ of the Schrödinger operator H, considered acting in the space $L_p(\mathbb{R}^n)$, satisfies (5.3).*

By similar arguments it can be evaluated eigenvalues for the polyharmonic operator $H_{m,q}$ discussed in Section 4. For this case there holds the following result.

THEOREM 5.2. *Let $1 < p < \infty, m > 0$, and let $q = ab$, where $a \in L_{r,w}(\mathbb{R}^n)$ and $b \in L_{s,w}(\mathbb{R}^n)$ with r and s restricted as in Theorem 4.1. Then, for any eigenvalue $\lambda \in \mathbb{C}\backslash[0,\infty)$ of the operator $H_{m,q}$, considered acting in $L_p(\mathbb{R}^n)$, there holds*

$$|\lambda|^{\alpha-n/2m} \le CI_m(n,\alpha,\theta)\|a\|^{\alpha}_{r,w}\|b\|^{\alpha}_{s,w}, \tag{5.4}$$

where C is a constant depending on n, p, r and s and $I_m(n,\alpha,\theta)$ is determined by (4.1).

The following result is a version of that given by Corollary 3.3 for polyharmonic operators with weak Lebesgue's classes potentials.

COROLLARY 5.3. *Let $\gamma > 0$ for $n \geq 2m$ and $\gamma \geq 1 - n/2m$ for $n < 2m$. If $q \in L_{\gamma+n/2m,w}(\mathbb{R}^n)$, then every eigenvalue $\lambda \in \mathbb{C}\backslash[0, \infty)$ of the operator $H_{m,q}$, considered acting in $L_p(\mathbb{R}^n)$, satisfies*

$$|\lambda|^\gamma \leq C I_m(n, \alpha, \theta) \sup_{t>0}(t^{\gamma+n/2m}\lambda_q(t)). \tag{5.5}$$

Proof. In (5.4), we let $r = s = 2\gamma + n/m$ and take $a(x) = |q(x)|^{1/2}$, $b(x) = (sgnq(x))$ $|q(x)|^{1/2}$. Then, $\alpha = r/2 = \gamma + n/2m$ and also

$$\|a\|_{r,w} = \|b\|_{s,w} = \||q|^{1/2}\|_{r,w},$$

and since $\|q^{1/2}\|_{r,w} = \|q\|_{r/2,w}^{1/2}$, we have

$$|\lambda|^\gamma \leq C I_m(n, \alpha, \theta)\|q\|_{\gamma+n/2m,w}^{\gamma+n/2m}$$

that is, the desired estimate (5.5). □

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