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On the Road Map of Vogel's Plane

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Abstract. We define "population" of Vogel's plane as points for which universal character of adjoint representation is regular in the finite plane of its argument. It is shown that they are given exactly by all solutions of seven Diophantine equations of third order on three variables. We find all their solutions: classical series of simple Lie algebras (including an "odd symplectic" one), $D_{2,1,\lambda}$ superalgebra, the line of sl(2) algebras, and a number of isolated solutions, including exceptional simple Lie algebras. One of these Diophantine equations, namely $knm = 4k + 4n + 2m + 12$, contains all simple Lie algebras, except so$(2N+1)$. Among isolated solutions are, besides exceptional simple Lie algebras, so called $\mathfrak{e}_{7\frac{1}{2}}$ algebra and also two other similar unidentified objects with positive dimensions. In addition, there are 47 isolated solutions in "unphysical semiplane" with negative dimensions. Isolated solutions mainly belong to the few lines in Vogel plane, including some rows of Freudenthal magic square. Universal dimension formulae have an integer values on all these solutions at least for first three symmetric powers of adjoint representation.

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1. Introduction

In 1974 't Hooft introduced [7] now famous $1/N$ expansion of perturbative SU(N) gauge theories as a step towards gauge/string duality. This requires extension of quantum averages of gauge-invariant quantities to continuous values of rank N. It appears that similar extension of SO(N) and Sp(N) theories leads to their duality [15] under $N \to -N$ transformation (which is a symmetry of SU(N) theory). This raises the question whether this duality can be included into some wider connection between gauge theories with different groups. This remained an unproven hypothesis until in 1995 Pierre Vogel in his study [23] (about which we became aware in 2011) of Vassiliev's finite invariants in knot theory derived a "universal" expression for dimensions of simple Lie algebras $\mathfrak{g}$:

$$\dim \mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} \quad (1)$$

$$t = \alpha + \beta + \gamma \quad (2)$$

Here α, β, γ are projective parameters of Vogel's plane which parameterize simple Lie algebras according to Vogel's Table I. Vogel plane is defined [13,23] as projective plane factorized over all permutations of its three projective parameters: P^2/S_3.

Evidently, (1) points in desired direction of "unification" of simple Lie algebras and perhaps on a unification of their applications. Let's give more detailed definition of parameters and their values for simple Lie (super)algebras.

Let $2t$ be the value of (arbitrarily normalized) second Casimir operator on an adjoint representation ad of some simple Lie algebra over, say, complex numbers field. The symmetric square of ad has decomposition [23]:

$$\mathrm{Sym}^2 \mathrm{ad} = 1 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma), \tag{3}$$

$$4t - 2\alpha, \quad 4t - 2\beta, \quad 4t - 2\gamma \tag{4}$$

where the last row contains values of the same Casimir operator on representations $Y_2(\alpha), Y_2(\beta), Y_2(\gamma)$ respectively. Actually (4) is definition of parameters, and one can show that

$$\alpha + \beta + \gamma = t \tag{5}$$

With these definitions Vogel's result is Table I. For exceptional line $\mathrm{Exc}(n)$ $n = -1, -2/3, 0, 1, 2, 4, 8$ for $\mathfrak{sl}(2), \mathfrak{g}_2, \mathfrak{so}(8), \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ algebras, respectively. Actually definitions can be extended [23] to simple superalgebras. It appears that superalgebra $D_{2,1,\lambda}$ occupies line $\alpha + \beta + \gamma = 0$, but other superalgebras don't give new points on Vogel's plane, e.g. superalgebras $\mathfrak{sl}(p|q)$ have parameters $(-2, 2, p - q)$, etc., (see also Table 2 in [20]). Note that values of parameters agree with $N \to -N$ duality: for $\mathfrak{sl}(N)$ they are self-dual, if multiplied on (-1) and α, β switched, parameters for $\mathfrak{so}(N)$ under change of sign of N go into those of $\mathfrak{sp}(N)$ if multiplied on $(-1/2)$ and α, β switched. So, due to this relation $\mathfrak{sp}(N) = \mathfrak{so}(-N)$ (see [1,2,9,15,18]) we shall use below only the notation of orthogonal algebras $\mathfrak{so}$. Note also that all exceptional simple Lie algebras in Table I belong to the line $\gamma = 2(\alpha + \beta)$. This was the basis of the Deligne's hypothesis [3,5] that they combine into series of Lie algebras. He assumed an existence of tensor category, corresponding to the line $\gamma = 2(\alpha + \beta)$, such that the tensor products of corresponding adjoint

Table I. Vogel's parameters for simple Lie algebras

Algebra/parameters	α	β	γ	$t = \alpha + \beta + \gamma$	Line
$\mathfrak{sl}(N)$	-2	2	N	N	$\alpha + \beta = 0$
$\mathfrak{so}(N)$	-2	4	$N - 4$	$N - 2$	$2\alpha + \beta = 0$
$\mathfrak{sp}(2n)$	-2	1	$n + 2$	$n + 1$	$\alpha + 2\beta = 0$
$\mathrm{Exc}(n)$	-2	$n + 4$	$2n + 4$	$3n + 6$	$\gamma = 2(\alpha + \beta)$
$D_{2,1,\lambda}$	-2	β	$-\beta + 2$	0	$\alpha + \beta + \gamma = 0$

representations decomposes into irreducible modules in a uniform way. Formulas of the dimensions of these modules are given by ratio of the products of linear functions over the parameter on this line (see also description of Deligne's hypothesis in [14,23]).

We shall call *universal* those quantities in simple Lie algebras (e.g. dimensions of representations, eigenvalues of Casimir operators, etc.) which are given by some "reasonable" functions (rational, analytical, etc.) of universal parameters such that at their values from Table I they give values of that quantity for a corresponding simple Lie algebra. An example of universal quantity is an eigenvalue of the abovementioned second Casimir operator on adjoint representations, which is given by $2t$ function. An important universal quantity is the dimension of adjoint representation (1).

The abovementioned hypothesis on gauge theories (now formulated as "universality" of gauge theories) has been partially confirmed in works [16,19], where some quantities in Chern–Simons theory on 3d sphere have been shown to be universal: central charge, perturbative [19] and non-perturbative [16] partition functions, etc. Also a universal expression has been established [20] for the eigenvalues of higher Casimir operators on adjoint representation. In [17] the connection of universal partition function with number theory functions (Barnes' multiple gamma functions) is established. Another connection with the number theory will be revealed below, where the parameters of Table I are expressed in terms of solutions of certain Diophantine equations.

In this paper we continue to investigate different aspects of universality in the simple Lie algebras and gauge theories. Our focus is the "road map" of Vogel's plane, which consists of a "population" ("interesting" points on Vogel's plane, see definition below) and "roads", the term that we coin for the lines, to which large number of the populated points belong to.

It is already known, that Vogel's approach reveals some objects, which behave like simple Lie algebras, in some respects. These are an "odd symplectic algebras" [21], i.e. $\mathfrak{sp}(N)$ points $(-2,1,N/2+2)$ with odd N and $\mathfrak{e}_{7\frac{1}{2}}$ [12,25,26] point $(-1,5,8)$. Corresponding points give integers, when substituted in universal dimension formulae [13]. In the present paper we have discovered more interesting points on Vogel's plane, and their connections, as well as lines, passing through many of these points. Last feature is making contact with Freudenthal's magic square.

Our main tool will be one of the key ingredients in universal gauge theories, namely a universal expression $f(x)$ for character of adjoint representation of simple Lie algebra, restricted to the line $x\rho$ (see e.g. [6], where ρ is Weyl vector in roots space, half of the sum of all positive roots), derived in [19]:

$$\chi_{\mathrm{ad}}(x\rho)=r+\sum_{\mu\in R}\mathrm{e}^{x(\mu,\rho)}\equiv f(x) \tag{6}$$

$$f(x)=\frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(\frac{x\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} \tag{7}$$

We shall call this expression the universal character for adjoint representation.

With the help of the universal character $f(x)$ we define "population" as all points for which $f(x)$ is non-singular in finite x plane, exactly as it happens for simple Lie algebras, as it is evident from the definition of character (6). Denominators for corresponding points cancel with numerators, and $f(x)$ becomes the sum of exponents in (6).

This requirement appears to be strong enough to restrict points on Vogel's plane mainly to those from Table I. However, there are some very interesting additions and details of this description. The most interesting one is its equivalence to some Diophantine equations.

Parameters in (7) should be non-zero, so if one of the numbers $2t-\alpha, 2t-\beta, 2t-\gamma$ is zero then the character is zero. Due to the permutation-invariance all these points belong to the same line, say $\alpha+2\beta+2\gamma=0$ on the Vogel plane which we denote as 0d line. When $2t-\alpha, 2t-\beta, 2t-\gamma$ are non-zero each zero of sines in the denominator should be canceled by zero of numerator, which means that for each value of $\sigma=\alpha,\beta,\gamma$ at least one of the ratios $(2t-\alpha)/\sigma$, $(2t-\beta)/\sigma$, $(2t-\gamma)/\sigma$ should be integer.

The complete matrix of these ratios

$$R_{\sigma,\kappa}=(2t-\kappa)/\sigma,$$

where $\kappa,\sigma=\alpha,\beta,\gamma$, can be easily calculated for any given simple Lie algebras, examples for $\mathfrak{sl}(n)$ and $\mathfrak{g}_2$ are given in Tables II and III.

We see that in each row there is an integer, as stated. The same happens for all other simple Lie algebras. We shall call a "pattern" the set of places, one in the each row, of matrix $R_{\kappa,\sigma}$ where integers are present. The same algebra can appear in different patterns. For example, exceptional algebra $\mathfrak{g}_2$ appears only in one abovementioned pattern, but $\mathfrak{e}_7$ in this one and two others, also, see below. It is easy to deduce, that there are 7 different, up to permutations, patterns of matrix R to have integers in each row.

In principle, the specified property is necessary, but not sufficient, for the character to be regular. Extra care should be taken for the cases when the ratio of some

Table II. Matrix R for $\mathfrak{sl}(N)$

$$\begin{vmatrix} -(N+1) & 1-N & -\frac{N}{2} \\ N+1 & N-1 & \frac{N}{2} \\ 2+\frac{2}{N} & 2-\frac{2}{N} & 1 \end{vmatrix}$$

Table III. Matrix R for $\mathfrak{g}_2$

$$\begin{vmatrix} -5 & -\frac{7}{3} & -\frac{8}{3} \\ 3 & \frac{7}{5} & \frac{8}{5} \\ \frac{15}{4} & \frac{7}{4} & 2 \end{vmatrix}$$

parameters, say α/β, becomes integer. If that happens in the pattern with integer $R_{\alpha,\kappa}$ and $R_{\beta,\kappa}$ (same κ) then the denominator in $f(x)$ can have a second-order zero, which cannot be canceled by the one first order zero in numerator. However, this does not happen. More exactly, below we find solutions for all patterns and explicitly check that for all of them $f(x)$ is regular, i.e. in all cases some other numerator "automatically" becomes integer. Partial explanation of this phenomena is given in the Section 3.3.

The abovementioned seven patterns are the following, all others can be obtained by permutations, and neither two of these seven are connected by permutation. To list them we indicate three values of κ, i.e. $(\kappa_1, \kappa_2, \kappa_3)$, for which R_{α,κ_1}, R_{β,κ_2}, R_{γ,κ_3} are integers (denote them k, n, m respectively):

$$(\alpha,\alpha,\alpha), (\alpha,\alpha,\beta), (\alpha,\alpha,\gamma), (\alpha,\beta,\gamma), (\alpha,\gamma,\beta), (\beta,\alpha,\alpha), (\beta,\gamma,\alpha)$$

We shall call them 1aaa, 2aab, 3aag, 4abg, 5agb, 6baa and 7bga respectively.

Consider for example the fourth, most symmetric, pattern 4abg. One ha?

$$(2t-\alpha)=k\alpha, \quad (2t-\beta)=n\beta, \quad (2t-\gamma)=m\gamma \tag{8}$$

or in matrix form in Table IV.

This is a system of three linear equations on three variables α, β, γ, with constant terms equal to zero, so non-trivial solution can exist only if the corresponding determinant is zero. That determinant is a third order polynomial over k, n, m and equation is

$$knm = kn + nm + km + 3n + 3k + 3m + 5 \tag{9}$$

We shall call this equation a Diophantine equation (or condition) for a given pattern.

This equation can be presented in a an elegant and more memorable form, provided all integers are not equal to -1:

$$\frac{2}{k+1} + \frac{2}{n+1} + \frac{2}{m+1} = 1 \tag{10}$$

$$\alpha = \frac{2t}{k+1}, \quad \beta = \frac{2t}{n+1}, \quad \gamma = \frac{2t}{m+1} \tag{11}$$

where we also present the solution for universal parameters in that case. In such form this Diophantine condition looks classical, although we could not find discussion in literature concerning exactly this equation. Let's stress that we present

Table IV. Matrix form of Equation (8)

$$\begin{vmatrix} 1-k & 2 & 2 \\ 2 & 1-n & 2 \\ 2 & 2 & 1-m \end{vmatrix} \begin{vmatrix} \alpha \\ \beta \\ \gamma \end{vmatrix} = 0$$

this form of equation just for its simplicity, with the correct form being (9), which makes sense at all values of integers k, n, m.

Another, normalized form, i.e. that without second order terms, can be derived by variables shift $k=\tilde{k}+1, n=\tilde{n}+1, m=\tilde{m}+1$, which leads to the equation $\tilde{k}\tilde{n}\tilde{m}=4\tilde{m}+4\tilde{n}+4\tilde{k}+16$.

In the next section we list all seven Diophantine equations in both initial and normalized forms. Next we prove Proposition 1, which provides an effective tool for finding all solutions of these equations. All solutions are presented, according to the Proposition 2, in Tables VII, VIII, IX, X, XI, XII, XIII and XIV. Section 2 contains also Proposition 3 which states that each solution of Diophantine equations provides the solution of initial problem of finding points on Vogel's plane for which the character $f(x)$ is a regular function on the entire finite x plane. In Section 3 we discuss some features of the solutions obtained: lines on Vogel's plane, to which many of these solutions belong to, Z_2 symmetry of 3aag pattern, dimension formulae, universal subgroup, etc. Most of these topics require further investigation. Section 4 contains discussion on the possible directions of development.

2. Diophantine Equations

In a similar way one can deduce Diophantine conditions for all other patterns. In each case one obtains the third order equation, which we present in the Table V (corresponding row of the second column). In the third column we present transformation (shift) of variables, which leads to a normalized form of equation, given in the last column.

We solve all these equations in the normalized form with the help of the following proposition.

PROPOSITION 1. *Let us have the following equation for integer variables* k, n, m *with the integer coefficients* a, b, c, d:

$$knm=ak+bn+cm+d$$

Table V. Diophantine equations

Pattern	Diophantine Eq.: $kmn=$	Shift: $(k,n,m)=$	Normalized form: $\tilde{k}\tilde{n}\tilde{m}=$
1aaa	$mn+2kn+2km$	$(\tilde{k},\tilde{n},\tilde{m})+(1,2,2)$	$4\tilde{k}+2\tilde{m}+2\tilde{n}+8$
2aab	$mn+2kn+2km+2n-2k$	$(\tilde{k},\tilde{n},\tilde{m})+(1,2,2)$	$2\tilde{k}+4\tilde{n}+2\tilde{m}+10$
3aag	$mn+kn+2km+3n+2k$	$(\tilde{k},\tilde{n},\tilde{m})+(1,2,1)$	$4\tilde{k}+4\tilde{n}+2\tilde{m}+12$
4abg	$nm+km+kn+3m+3n+3k+5$	$(\tilde{k},\tilde{n},\tilde{m})+(1,1,1)$	$4\tilde{m}+4\tilde{n}+4\tilde{k}+16$
5agb	$mn+2kn+2km+2n+2m-3k-5$	$(\tilde{k},\tilde{n},\tilde{m})+(1,2,2)$	$4\tilde{m}+4\tilde{n}+\tilde{k}+8$
6baa	$2mn+2kn+2km-2n-3m$	$(\tilde{k},\tilde{n},\tilde{m})+(2,2,2)$	$\tilde{m}+2\tilde{n}+4\tilde{k}+6$
7bga	$2mn+2kn+2km-2n-2m-2k+5$	$(\tilde{k},\tilde{n},\tilde{m})+(2,2,2)$	$2\tilde{m}+2\tilde{n}+2\tilde{k}+9$

Each of its solutions belongs to either series solutions (first two cases below), or to isolated solutions (the third case):

1. **Classical series:** $knm = 0$.
2. **Non-classical series:** $(kn - c)(km - b)(nm - a) = 0$.
3. **Isolated solutions:**

$$knm \neq 0$$
$$(kn - c)(km - b)(nm - a) \neq 0$$
$$|k| \leq K, \quad |n| \leq N, \quad |m| \leq M$$

where

$$K = (|a| + 1)(|b| + |c|) + |d|$$
$$N = (|b| + 1)(|a| + |c|) + |d|$$
$$M = (|c| + 1)(|a| + |b|) + |d|$$

Proof. It is sufficient to prove that if some solution (k, n, m) does not belong to the first two cases, i.e. classical and non-classical series, then it belongs to the third case.

We have

$$k(nm - a) = bn + cm + d$$

Next, $x = nm - a$ is non-zero (otherwise solution belongs to the non-classical series), so minimal value of $|x|$ is 1. For the fixed x maximal value of $|n|$ and $|m|$ is not greater than $|x| + |a|$, (since $m \neq 0$ and $n \neq 0$, otherwise we have the classical series). From this we obtain an upper bound for $|k|$: $|k| = |(bn + cm + d)/x| \leq (|b|(|x| + |a|) + |c|(|x| + |a|) + |d|)/|x| \leq (|b| + |c| + |ab| + |ac| + |d|) = K$. Similarly we get $|n| \leq N$, $|m| \leq M$.

Few comments will be pertinent here. The first is on designations of three cases. As we shall see below, solutions belonging to the first case correspond to the classical algebras: $\mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}$, hence the name. Solutions from the second case correspond to the superalgebra $D_{2,1,\lambda}$, lines 3d and 0d (see below), and depend on the arbitrary parameter(s), that is why we call them series. Solutions of the third type particularly contain exceptional algebras.

Second, Proposition 1 provides an effective way for finding all solutions. Indeed, first case leads to the easily solvable linear equations on two integers with the integer coefficients. Second case gives the system of two equations, linear and quadratic, over two variables, obviously easily solvable. The third case restricts possible remaining solutions to finitely many points, which can be checked by the computer. In our case bounds are very low, not greater than 56, so the number of points to be checked is about one million, which requires few minutes of the computing time.

Third, same algebras can appear both in series and as an isolated solution. I.e. different solutions of Diophantine equations can give the same points in Vogel's plane. E.g. algebra $\mathfrak{so}(8)$ with $(\alpha, \beta, \gamma) \propto (-2, 4, 4)$ appears in pattern 3aag as a classical series solution $(k, n, m) = (2, 2, -7)$ (i.e. $(\tilde{k}, \tilde{n}, \tilde{m}) = (1, 0, -8)$) and as an isolated solution $(k, n, m) = (2, -4, 2)$, (i.e. $(\tilde{k}, \tilde{n}, \tilde{m}) = (1, -6, 1)$), which doesn't belong to the classical or non-classical series.

Using Proposition 1, we get

PROPOSITION 2. *All essentially different solutions of Diophantine equations* (V) *are listed in Tables* VII (*series solutions*) *and Tables* VIII, IX, X, XI, XII, XIII *and* XIV (*isolated solutions*). "*Essentially different*" *means that for patterns* 1*aaa*, 4*abg*, 5*agb and* 7*bga*, *as described below*, *from solutions with permuted* k, n, m *giving the same points on Vogel's plane*, *we present in the tables only one representative.*

Finally, the following Proposition 3 shows that we completely solve an initial problem of finding all points on Vogel's plane with the regular character: they are given exactly by all solutions of seven Diophantine equations:

PROPOSITION 3. *Equation* (V) *are both necessary and sufficient for the regularity of* $f(x)$, *which for each solution becomes a finite sum of exponents* e^{Ax}.

Necessity was discussed above, sufficiency is proved by the direct case-by-case check.

In the Table VII we list all series solutions for the initial parameters (k, n, m) of all patterns, in Tables VIII, IX, X, XI, XII, XIII and XIV isolated solutions are listed. The solutions for normalized parameters can be easily obtained by the shift given in Table V. Besides the solution (k, n, m), Tables contain corresponding solution for the projective parameters (in some convenient normalizations, usually when they are integers without common divisor), dimension (1) of that solution (column "Dim"), column "Rank" is equal to the constant term in function $f(x)$ represented as a finite sum of exponents (cf. (6)). Column "Lines" for the isolated solutions contains notation of lines, to which given solution belongs. Corresponding lines are described in the next section.

Below we briefly comment solutions of all seven Diophantine equations.

Equations for **pattern 1aaa** are

$$2t - \alpha = k\alpha, \quad 2t - \alpha = n\beta, \quad 2t - \alpha = m\gamma \tag{12}$$

They are symmetric w.r.t. the transposition of n and m with simultaneous transposition of α, β, so we present isolated solutions with $n \geq m$, only. Similarly, in the series solutions we present only one solution among those, connected by symmetry transformation (the same for similarly symmetric cases below). Pattern 1aaa contains classical series $\mathfrak{sl}(2N), \mathfrak{so}(2k+4)$ and non-classical series 0d, i.e. entire line

$\alpha+2\beta+2\gamma=0$ with zero dimension and zero character function $f(x)$. Among isolated solutions, Table VIII, there is a solution $(\alpha,\beta,\gamma)\propto(1,-3,-5)$ of dimension 99 and rank 7, which we denote as X_2.

Beside these solutions, which have positive dimensions according to the dimension formula (1), we observe an appearance of other solutions. We call them Y-objects and numerate according to their dimensions in descending order. They have all projective parameters of the same sign and correspondingly negative dimension. The total number of them is 47, their nature is not known, but they have some features of the simple Lie algebras. In particular, they give integers when substituted into the universal dimension formulae [13], at least in the first three orders, and probably in all cases when it makes sense. Below we shall call the semiplane of the Vogel's plane with the different signs of parameters the physical semiplane, and that with same signs of parameters an unphysical semiplane. Dimension (1) can be positive for points in the physical semiplane only.

Pattern 2aab has an equations

$$2t-\alpha=k\alpha, \quad 2t-\alpha=n\beta, \quad 2t-\beta=m\gamma \tag{13}$$

and contains classical series $\mathfrak{sl}(2N), \mathfrak{so}(2k+4), \mathfrak{so}(2N+1)$ and points $(-n,3,2n-3)$ as non-classical series, which belong to the 3d line $\alpha+\beta+2\gamma=0$. Actually all points on the 3d line have $\dim=3, \text{rank}=1$, and character function

$$f(x)=\frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(\frac{x\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})} \tag{14}$$

$$=\frac{\sinh(x\frac{3\alpha}{4})}{\sinh(\frac{x\alpha}{4})}\frac{\sinh(x\frac{-\gamma}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{-\beta}{4})}{\sinh(x\frac{\gamma}{4})} \tag{15}$$

$$=e^{\frac{x\alpha}{2}}+1+e^{-\frac{x\alpha}{2}} \tag{16}$$

i.e. it is the whole line of $\mathfrak{sl}(2)$ algebras. All points on the 3d line appear as solutions in some patterns below.

Isolated solutions, Table IX, contain, among others, the points X_2, which appears earlier in the pattern 1aaa, and X_1 with $\dim=156$ and $\text{rank}=8$, as well as the abovementioned point $(-8,1,-5)$ of $\mathfrak{e}_{7\frac{1}{2}}$.

Pattern 3aag has an equations

$$2t-\alpha=k\alpha, \quad 2t-\alpha=n\beta, \quad 2t-\gamma=m\gamma \tag{17}$$

which contain classical series $\mathfrak{sl}(N), \mathfrak{sl}(2N), \mathfrak{so}(2k+4)$, as non-classical series superalgebra $D_{2,1,\lambda}$, integer points $(-2,-2k,k+1)$ on the 3d line, and points $(2(m+1),-(m+3),2)$ on the 0d line. The novel element in the isolated solutions, Table X, is that both $\mathfrak{e}_6$ and $\mathfrak{e}_8$ appear twice, with the essentially different (k,l,m) but, of course, the same (α,β,γ).

We denote these solutions as $\mathfrak{e}_6(1), \mathfrak{e}_6(2)$ and $\mathfrak{e}_8(1), \mathfrak{e}_8(2)$. These are different descriptions of the same algebra. The descriptions are really different, which can be understood e.g. from the expression for the dimension of algebra in terms of k, n, m. These formulae are given in the Section 3.3, for 3aag pattern it is

$$\dim(\mathfrak{g}) = m(k - n - kn)$$

Thus, for e.g. $\mathfrak{e}_8$ the same dimension's integer 248 is represented as 4×62 for $\mathfrak{e}_8(1)$ and as 2×124 for $\mathfrak{e}_8(2)$.

Concluding with the pattern 3aag, one may remark that it contains all simple Lie algebras except the odd orthogonal $\mathfrak{so}(2n+1)$.

Pattern 4abg, discussed above, contains as solutions the classical series $\mathfrak{sl}(N)$, the superalgebra $D_{2,1,\lambda}$, as well as a number of isolated solutions, Table XI, particularly $\mathfrak{e}_8, \mathfrak{e}_6$ and $\mathfrak{so}(8)$. Since Diophantine equation is symmetric w.r.t. the permutations of k, n, m with corresponding permutation of projective parameters, we present isolated solutions with $k \geq n \geq m$ only.

For the fifth **pattern 5agb** with equations

$$2t - \alpha = k\alpha, \quad 2t - \gamma = n\beta, \quad 2t - \beta = m\gamma \tag{18}$$

we have a symmetry between interchange of n and m, which change α, β, so we list solutions with $n \geq m$ only. The classical series are $\mathfrak{sl}(N), \mathfrak{so}(4N)$, non-classical series are represented by 3d line, isolated solutions are combined in the Table XII.

For the sixth **pattern 6baa**

$$2t - \beta = k\alpha, \quad 2t - \alpha = n\beta, \quad 2t - \alpha = m\gamma \tag{19}$$

we have classical series $\mathfrak{so}(2N+1), \mathfrak{so}(4N)$ (twice), and non-classical 3d line ($(-m-2, m, 1)$ points) and 0d line ($(2, 2-2k, 2k-3)$ points). Isolated solution are given in the Table XIII.

Equations of the last, seventh **pattern 7bga**

$$2t - \beta = k\alpha, \quad 2t - \gamma = n\beta, \quad 2t - \alpha = m\gamma \tag{20}$$

are cyclic symmetric w.r.t. the permutation $k \to n \to m \to k$, which leads to $\alpha \to \beta \to \gamma \to \alpha$, so isolated solutions that we present contain one representative of each orbit of cyclic transformations. There are no series solutions for this pattern, and only few isolated ones are present, Table XIV.

We combine information about the isolated solutions with the positive dimension in the physical Vogel semiplane in the Table XV.

3. Properties and Relations of Population of Vogel's Plane

In this section we describe additional properties of solutions, as well as some hypotheses, which require (and worth) further investigation.

3.1. ROAD MAP: ROADS

In the previous section we have studied the population of Vogel's plane, i.e. points with the non-singular universal character (7). In the present section we are interested in the roads, connecting these populated points, by which we understand lines, connecting sufficiently large numbers of them. We say, following [13], that a collection of points lie on a line in Vogel's plane (which is a factor projective plane over group of all permutations of its projective parameters, P^2/S_3) if some lift of them to P^2 is a collinear set of points.

$\mathfrak{sl}(N)$ algebras are located along the line $\alpha+\beta=0$, which we shall call SL line, all $\mathfrak{so}(N)$ and $\mathfrak{sp}(N)$ algebras—along the line SO : $2\alpha+\beta=0$. All five exceptional algebras lie on one line Exc: $\gamma=2(\alpha+\beta)$.

The choice of lines is to a large extent arbitrary, we are free to put a line through any of two points. But the line becomes worth to mention when it contains a large number of populated points, in analogy with the classical series or exceptional algebras' lines. In our case we would like to note the following lines: T line with points satisfying $\alpha+2\beta=\gamma$, F line with the equation $\alpha=\beta+\gamma$, K line $\alpha+2\beta=2\gamma$, M line $3\alpha=2\beta+2\gamma$, D line $\alpha+\beta+\gamma=0$, 0d line $2\alpha+2\beta+\gamma=0$, 3d line $2\alpha+\beta+\gamma=0$. In the Tables XI, XII, XIII, XIV and XV it is shown for each point whether it belongs to one of these lines. We see that a lot of points belong to these lines, particularly many to the T line. This line T is of particular interest for us as a line containing the points X_1 and X_2, and it appears to coincide with the so-called subexceptional line in [13], denoted there as $F3_3$.

One can make a contact with the Freudenthal magic square. The line Exc is the $\mathfrak{e}_8$ line of the magic square, line T is the line of $\mathfrak{e}_7$ family, and line K is the line of $\mathfrak{f}_4$ family. Interestingly, the algebras of $\mathfrak{e}_6$ line of magic square do not constitute a line on Vogel's plane (it seems to be an F line, but actually does not coincide with it). All of this certainly worth further investigation.

One can consider the points of intersection of these lines. Many intersection points can be found from the Tables, namely in many cases one point belongs to more than one line, which means an intersection in that point. Actually, the tables completely answer the question of intersections on the isolated points. Note also that a "central" point $(\alpha,\beta,\gamma)=(1,1,1)$ with $(k,n,m)=(5,5,5)$ (in all patterns) does not belong to any of abovementioned lines.

3.2. X_1, X_2 AND $\mathfrak{e}_{7\frac{1}{2}}$

$\mathfrak{e}_{7\frac{1}{2}}$ was discovered in [12,26] as a point on an exceptional line, where the dimension formulae give an integer answer. It is proved to be a Lie algebra "between" $\mathfrak{e}_7$ and $\mathfrak{e}_8$ (hence notation) in a following sense: the six-dimensional algebra S ("sextonions") in between the quaternions and octonions is identified in [12,26] (see also [8,10]), and the triality construction of exceptional algebras is applied to S to get an algebra $\mathfrak{e}_{7\frac{1}{2}}$. The latter appears to be a semidirect product $\mathfrak{e}_7 \ltimes H_{56}$ where H is

Table VI. Dimensions

Pattern	Dim
1aaa	$k-3m-3km-3n-3kn+4mn$
2aab	$n(k+1)(m-1)+k$
3aag	$m(k-n-kn)$
4abg	$-knm$
5agb	$-knm$
6baa	$k(3n+2m-nm)$
7bga	$-knm$

$(56+1)$-dimensional Heisenberg algebra, where $\underline{56}$ is an irrep of $\mathfrak{e}_7$ with an invariant symplectic form.

The dimension formulae work for X_1, X_2 algebras as well. For example for X_2 we have $\dim Y_2(\alpha)=3927, \dim Y_2(\beta)=77, \dim Y_2(\gamma)=945$, for X_1 $\dim Y_2(\alpha)=10166, \dim Y_2(\beta)=90, \dim Y_2(\gamma)=1989$, etc.

We have not identified X_1 and X_2 as some Lie algebras. One can note however that their dimensions can be represented in a similar to $\mathfrak{e}_{7\frac{1}{2}}$ way: $\dim(X_2)=99=66+32+1$, where 66 is the dimension of an algebra $\mathfrak{so}(12)$, 32 is the dimension of its spinor representation, and $\dim(X_1)=156=91+64+1$, where 91 is the dimension of algebra $\mathfrak{so}(14)$, $\underline{64}$ is its spinor representation. Moreover, the spinor representations $\underline{32}$ of $\mathfrak{so}(12)$ has an invariant symplectic form, in full analogy with $\underline{56}$ of $\mathfrak{e}_{7\frac{1}{2}}$ (but $\underline{64}$ of $\mathfrak{so}(14)$ doesn't[1]). So, the natural hypothesis is that X_2 is the semidirect product $\mathfrak{so}(12) \rtimes H_{32}$, where H_{32} is the Heisenberg algebra based on an invariant symplectic form in spinor representation. Note that $\mathfrak{so}(12) \rtimes H_{32}$ (denoted there $D_6.H_{32}$) already appears in additional row/column of extended magic square in [12].

Point X_1 is also noted in [25], where it is suggested to be a centralizer of unipotent element of $\mathfrak{e}_8$.

3.3. DIMENSION FORMULAE

Since projective parameters can be expressed through integers k, n, m, we can express dimensions in terms of that integers, see Table VI.

It is beautiful that dimensions appear as explicitly integer numbers, without the denominators of dimension formula (1). This gives some (partial) explanation, why Diophantine conditions actually are sufficient to provide a regularity of character function $f(x)$.

We see from e.g. expressions for fourth, fifth and seventh patterns, that solutions of Diophantine equations directly give multipliers in some decomposition of dimension of algebra. For example, $\dim(\mathfrak{e}_6)=78=-knm=-3\times 2\times(-13)$, accord-

[1] I'm indebted to P. Deligne and A. Marrani for pointing this out to me.

ing to the pattern 4abg, same number is represented in pattern 3aag in two ways: as $78 = m(k-n-kn) = 3 \times 26$ and as $78 = m(k-n-kn) = 2 \times 39$.

Higher dimension formulae [13,23] probably gives an integers on all solutions, if not singular. This is directly checked for $Y_2(\alpha), Y_2(\beta), Y_2(\gamma)$ and $Y_3(\alpha), Y_3(\beta), Y_3(\gamma)$.

3.4. UNIVERSALLY CHARACTERIZED SUBALGEBRA AND $D_{2,1,\lambda}$, 3D AND 0D LINES

In their work on the universal dimension formulae Landsberg and Manivel [13] found a special simple subalgebra (simple factor of centralizer of principal $\mathfrak{sl}(2)$, see [13] for the exact definition) of simple Lie algebras, universal parameters of which $(\alpha', \beta', \gamma')$ are expressed by those of initial algebra in the simple way:

$$\alpha' = \alpha \tag{21}$$

$$\beta' = \gamma - \beta \tag{22}$$

$$\gamma' = \beta \tag{23}$$

$$\alpha < 0, \quad \beta > 0, \quad \gamma > \beta \tag{24}$$

where the initial parameters are supposed to be ordered in the way shown in the last line.

The application of this transformation to $D_{2,1,\lambda}$ superalgebra is quite interesting. For this superalgebra $\alpha + \beta + \gamma = 0$, for primed parameters one get $\alpha' + \beta' + 2\gamma' = \alpha + \beta + \gamma = 0$, which is a characteristic equation of 3d line from Table VII. This is an explanation of appearance of 3d line on Vogel's plane.

Similarly, applying transformation to the 3d line we get $2\alpha' + \beta' + 2\gamma' = 2\alpha + \beta + \gamma$(=0 on 3d line), which means that if point with initial parameters belongs to 3d line, then primed one belongs to 0d line, which is an explanation of the appearance of 0d line on Vogel's plane.

Similarly, applying this transformation to Y-objects, we obtain the whole tree of relations between Y-objects, e.g. $Y_{11} \rightarrow Y_1$.

However, not always this transformation gives other objects in Vogel's plane. E.g. from 0d line we get line $\alpha' + 2\beta' + 4\gamma' = 0$ which does not correspond to any of the solution above. Similarly Y_1 doesn't pass into any of the objects listed in this paper.

3.5. Z_2 SYMMETRY OF PATTERN 3AAG

In the normalized form of Diophantine equation of pattern 3aag we notice a symmetry under the interchange of $\tilde{k}$ and $\tilde{n}$. In the language of initial k and n it is a transformation $(k, n, m) \leftrightarrow (n-1, k+1, m)$. Under this transformation points in Vogel's plane do really change their places so this transformation in fact connects different algebras.

For the points in the physical semiplane we get transpositions $\mathfrak{sl}(2n) \leftrightarrow \mathfrak{so}(2n+2)$, $\mathfrak{sl}(k+1) \leftrightarrow \mathfrak{sl}(k)$, $X_1 \leftrightarrow \mathfrak{e}_8(1)$, $\mathfrak{f}_4 \leftrightarrow \mathfrak{e}_6(2)$, $\mathfrak{so}(10) \leftrightarrow \mathfrak{e}_6(1)$, $\mathfrak{e}_7 \leftrightarrow \mathfrak{e}_7$, $\mathfrak{g}_2 \leftrightarrow \mathfrak{so}(8)$, $\mathfrak{e}_{7\frac{1}{2}} \leftrightarrow$

$\mathfrak{e}_8(2), 0d_1 \leftrightarrow 0d_4, 0d_2 \leftrightarrow 0d_3, 0d_5 \leftrightarrow 0d_6$. It would be interesting to connect this transformation to the known one(s) in the theory of simple Lie algebras.

4. Conclusion

The main result of present paper is the established connection between the solutions of certain Diophantine equations, population of Vogel's plane (defined above) and simple Lie (super)algebras. Some additional objects, similar in some respects to the simple Lie algebras, appear in this classification.

The starting point of the research is the universal character $f(x)$ for the adjoint representations, (7). Such universal characters (i.e. character, restricted to Weyl line and expressed in terms of universal parameters) can be derived for all representations having a universal dimension formula. E.g. universal character for representation $Y_2(\alpha)$ is

$$\chi_{Y_2(\alpha)}(x\rho) = -\frac{\sinh[\frac{xt}{2}]\sinh[\frac{x(\beta-2t)}{4}]\sinh[\frac{x(\gamma-2t)}{4}]\sinh[\frac{x(\beta+t)}{4}]\sinh[\frac{x(\gamma+t)}{4}]\sinh[\frac{x(3\alpha-2t)}{4}]}{\sinh[\frac{x\alpha}{4}]\sinh[\frac{x\alpha}{2}]\sinh[\frac{x\beta}{4}]\sinh[\frac{x\gamma}{4}]\sinh[\frac{x(\alpha-\beta)}{4}]\sinh[\frac{x(\alpha-\gamma)}{4}]} \tag{25}$$

and permutations of this for $Y_2(\beta), Y_2(\gamma)$. Compare this with Vogel's expression for dimension of $Y_2(\alpha)$ [23]:

$$\dim Y_2(\alpha) = -\left(\frac{t\ (\beta-2t)\ (\gamma-2t)\ (\beta+t)\ (\gamma+t)\ (3\alpha-2t)}{\alpha^2\ (\alpha-\beta)\ \beta\ (\alpha-\gamma)\ \gamma}\right) \tag{26}$$

If for a given Lie algebra we consider decomposition of the product of representations into the sum of representations, the same relation holds for the corresponding characters. Deligne [4] suggested that if all characters involved are universal, then that relation holds for the arbitrary points in the Vogel's plane. E.g. for (3) it would mean

$$\chi_{\mathrm{Sym}^2(\mathrm{ad})}(x\rho) = \chi_{Y_2(\alpha)}(x\rho) + \chi_{Y_2(\beta)}(x\rho) + \chi_{Y_2(\gamma)}(x\rho) + 1 \tag{27}$$

for all points in Vogel's plane, not only for those from Table I. Deligne [4], in particular, completely checked the version of his hypothesis, restricted to $\mathfrak{sl}(N)$ line. Relation (27) holds, also, provided one expresses the l.h.s by $f(x)$:

$$\chi_{\mathrm{Sym}^2(\mathrm{ad})}(x\rho) = \frac{1}{2}(f^2(x) + f(2x)) \tag{28}$$

Next, derivations of universal dimension formulae in [13] and universal character of adjoint in [19] (and (25)) are based on the Weyl character formula and produce a unique answer. However, if we ask whether one can deform them in such way that their values at true simple Lie algebras remains unchanged, the answer would be positive. One can easily construct a symmetric polynomial over projective coordinates of order 16, which is zero on all of the lines from Table I.

On the other side, it is not possible to get a unique definition [11,24] of the group weights of diagrams on entire Vogel's plane, based on Vogel's Λ-algebra, due to constraint found by Vogel [23]. That constraint restricts values of projective parameters at the most to few lines, including classical and exceptional ones.

One can imagine definition of unique universal characters, independent from that in [19], based on Deligne's hypothesis, so that characters' relations of that hypothesis partially play the role of definitions and partially the check for such defined characters.

These considerations worth further investigation and can be relevant for the problem of universality of Yang–Mills theory.

Another direction of development is an investigation of the role of Y_n points, which seem to be an interesting and unique objects. One may speculate on their connection with Rozansky–Witten non-Lie-algebraic weights in knot theory, in this connection see [22].

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Appendix: Tables of Solutions of Diophantine Equations

See Tables VII, VIII, IX, X, XI, XII, XIII, XIV and XV.

Table VII. Points in Vogel plane: series

$\mathfrak{g}$	α, β, γ	Patterns	k, n, m	Dim	Rank
$\mathfrak{sl}(N)$	$2, -2, N$	3aag	$N-1, -N+1, 1$	N^2-1	$N-1$
	$-2, 2, N$	4abg	$-N-1, N-1, 1$		
	$N, -2, 2$	5agb	$1, 1-N, N+1$		
$\mathfrak{sl}(2N)$	$N, 1, -1$	1aaa	$1, -N, N$	$(2N)^2-1$	$2N-1$
		2aab	$1, N, 1-2N$		
		3aag	$1, N, -2N-1$		

Table VII. continued

$\mathfrak{g}$	α, β, γ	Patterns	k, n, m	Dim	Rank
$\mathfrak{so}(2k+4)$	$2, k, -1$	3aag	$k, 2, -2k-3$	$(k+2)(2k+3)$	$2k+2$
	$2, -1, k$	1aaa	$k, -2k, 2$		
	$2, k, -1$	2aab	$k, 2, -2-k$		
$\mathfrak{so}(2N+1)$	$-2, 4, 2N-3$	2aab	$-2N, N, 2$	$N(2N+1)$	N
	$-4, 2, 3-2N$	6baa	$N, 3-2N, 2$		
$\mathfrak{so}(4N)$	$-1, 2, 2N-2$	5agb	$1-4N, N, 2$	$2N(4N-1)$	$2N$
	$2N-2, 2, -1$	6baa	$2, N, -2N$		
	$-1, 2, 2N-2$	6baa	$N, 2, 4-4N$		
$D_{2,1,\lambda}$	$\alpha+\beta+\gamma=0$	4abg	$-1, -1, -1$	1	1
	$-n, 1, (n-1)$	3aag	$-1, n, -1$		
3d	$2\alpha+\beta+\gamma=0$	5agb	$-3, 1, 1$	3	1
	$-n, 3, 2n-3$	2aab	$-3, n, 1$		
	$-2, -2k, k+1$	3aag	$k, 1, -3$		
	$-m-2, m, 1$	6baa	$1, 1, m$		
0d	$\alpha+2\beta+2\gamma=0$	1aaa	$0, 0, 0$	0	0
	$2(m+1), -(m+3), 2$	3aag	$0, 0, m$		
	$2, 2-2k, 2k-3$	6baa	$k, 0, 0$		
	$2m-2, -m-2, 3$	2aab	$0, 0, m$		

Table VIII. Isolated solutions of 1aaa pattern

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines
2 3 −12	−6 −4 1	133	7	$\mathfrak{e}_7$	Exc, T
−15 5 3	1 −3 −5	99	7	X_2	T
−6 4 3	2 −3 −4	21	3	$\mathfrak{so}(7)$	SO, T, K
−3 1 −3	−1 3 −1	3	1	$3d_1$	3d
−3 3 3	1 −1 −1	3	1	$\mathfrak{sl}(2)$	3d, T
−2 1 −4	−2 4 −1	3	1	$\mathfrak{so}(3)$	3d
5 5 5	1 1 1	−125	−19	Y_1	?
6 6 4	2 2 3	−132	−10	Y_4	K
4 8 4	2 1 2	−144	−14	Y_{10}	K, M
3 6 6	2 1 1	−147	−17	Y_{11}	F
10 5 4	2 4 5	−153	−7	Y_{13}	K
9 9 3	1 1 3	−189	−17	Y_{21}	T
3 12 4	4 1 3	−195	−11	Y_{23}	F, K
6 12 3	2 1 4	−195	−13	Y_{24}	T
12 8 3	2 3 8	−207	−7	Y_{26}	T
5 15 3	3 1 5	−221	−11	Y_{28}	T
2 8 8	4 1 1	−242	−18	Y_{31}	Exc
2 12 6	6 1 2	−272	−14	Y_{38}	Exc
21 7 3	1 3 7	−285	−11	Y_{39}	T
4 24 3	6 1 8	−319	−9	Y_{42}	T, M
2 20 5	10 1 4	−377	−11	Y_{45}	Exc

Table IX. Isolated solutions of 2aab pattern

k n m	α β γ	Dim	Rank	Algebra	Lines
2 −16 5	−8 1 −5	190	8	$\mathfrak{e}_{7\frac{1}{2}}$	Exc
4 −16 3	−4 1 −7	156	8	X_1	T, M
2 3 −14	−6 −4 1	133	7	$\mathfrak{e}_7$	Exc, T
−15 3 3	1 −5 −3	99	7	X_2	T
2 −6 4	−6 2 −5	52	4	$\mathfrak{f}_4$	Exc, K
3 −6 3	−2 1 −3	45	5	$\mathfrak{so}(10)$	SO, T, F
5 5 5	1 1 1	−125	−19	Y_1	?
4 5 6	10 8 7	−129	−1	Y_2	M
6 6 4	2 2 3	−132	−10	Y_4	K
7 4 5	4 7 5	−135	−3	Y_7	K
4 4 8	2 2 1	−144	−14	Y_{10}	K, M
3 6 7	2 1 1	−147	−17	Y_{11}	F
12 4 4	2 6 5	−168	−6	Y_{16}	K
4 12 4	6 2 7	−184	−6	Y_{17}	K, M
10 8 3	4 5 13	−186	−2	Y_{18}	T
9 3 7	1 3 1	−189	−17	$Y_{21}(1)$	T
9 9 3	1 1 3	−189	−17	$Y_{21}(2)$	T
3 4 13	4 3 1	−195	−11	$Y_{23}(2)$	F, K
3 12 5	4 1 3	−195	−11	$Y_{23}(1)$	F, K
6 3 10	2 4 1	−195	−13	Y_{24}	T
12 3 6	2 8 3	−207	−7	Y_{26}	T
15 6 3	2 5 9	−207	−5	Y_{27}	T
5 3 13	3 5 1	−221	−11	Y_{28}	T
7 14 3	2 1 5	−231	−13	Y_{30}	T
2 8 11	4 1 1	−242	−18	Y_{31}	Exc
2 9 10	18 4 5	−245	−3	Y_{33}	Exc
2 6 16	6 2 1	−272	−14	Y_{38}	Exc
21 3 5	1 7 3	−285	−11	Y_{39}	T
25 5 3	1 5 7	−285	−9	Y_{40}	T
2 14 8	14 2 5	−296	−6	Y_{41}	Exc
4 3 22	6 8 1	−319	−9	Y_{42}	T, M
6 24 3	4 1 9	−342	−10	Y_{44}	T
2 5 26	10 4 1	−377	−11	Y_{45}	Exc
2 24 7	12 1 5	−434	−10	Y_{46}	Exc

Table X. Isolated solutions of 3aag pattern

k n m	α β γ	Dim	Rank	Algebra	Lines
2 −20 4	−10 1 −6	248	8	$\mathfrak{e}_8(1)$	Exc, M
4 −24 2	−6 1 −10	248	8	$\mathfrak{e}_8(2)$	Exc, M
−25 5 2	1 −5 −8	190	8	$\mathfrak{e}_{7\frac{1}{2}}$	Exc
−21 3 4	1 −7 −4	156	8	X_1	T, M
2 3 −19	−6 −4 1	133	7	$\mathfrak{e}_7$	Exc, T
2 −8 3	−4 1 −3	78	6	$\mathfrak{e}_6(1)$	Exc, F
3 −9 2	−3 1 −4	78	6	$\mathfrak{e}_6(2)$	Exc, F

Table X. continued

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines
−10 4 2	2 −5 −6	52	4	$\mathfrak{f}_4$	Exc, K
−9 3 3	1 −3 −2	45	5	$\mathfrak{so}(10)$	SO, T, F
2 −4 2	−2 1 −2	28	4	$\mathfrak{so}(8)$	SO, Exc
−5 3 2	3 −5 −4	14	2	$\mathfrak{g}_2$	Exc, T
−9 −3 0	−1 −3 8	0	0	$0d_1$	0d
−6 −4 0	−2 −3 10	0	0	$0d_2$	0d
−5 −5 0	−1 −1 4	0	0	$0d_3$	0d
−4 −8 0	−2 −1 6	0	0	$0d_4$	0d
−2 4 0	2 −1 −2	0	0	$0d_5$	K, 0d
3 −1 0	−1 3 −4	0	0	$0d_6$	F, 0d
5 5 5	1 1 1	−125	−19	Y_1	?
4 6 5	6 4 5	−130	−4	Y_3	M
6 4 6	2 3 2	−132	−10	Y_4	K
7 5 4	5 7 8	−132	−2	Y_5	M
5 4 7	4 5 3	−133	−2	Y_6	K
8 4 5	2 4 3	−140	−8	Y_9	K
4 4 9	2 2 1	−144	−14	$Y_{10}(1)$	K, M
4 8 4	2 1 2	−144	−14	$Y_{10}(2)$	K, M
3 6 7	2 1 1	−147	−17	$Y_{11}(1)$	F
7 7 3	1 1 2	−147	−17	$Y_{11}(2)$	F
3 7 6	7 3 4	−150	−4	$Y_{12}(1)$	F
6 8 3	4 3 7	−150	−4	$Y_{12}(2)$	F
3 5 9	5 3 2	−153	−7	$Y_{14}(1)$	F
9 6 3	2 3 5	−153	−7	$Y_{14}(2)$	F
3 9 5	3 1 2	−165	−13	$Y_{15}(1)$	F
5 10 3	2 1 3	−165	−13	$Y_{15}(2)$	F
14 4 4	2 7 6	−184	−6	Y_{17}	K, M
7 3 11	3 7 2	−187	−7	Y_{20}	T
9 3 9	1 3 1	−189	−17	Y_{21}	T
3 4 15	4 3 1	−195	−11	$Y_{23}(1)$	F, K
6 3 13	2 4 1	−195	−13	Y_{24}	T
15 5 3	1 3 4	−195	−11	$Y_{23}(2)$	F, K
11 3 8	3 11 4	−200	−4	Y_{25}	T
5 3 17	3 5 1	−221	−11	Y_{28}	T
3 15 4	5 1 4	−228	−10	$Y_{29}(1)$	F, M
4 16 3	4 1 5	−228	−10	$Y_{29}(2)$	F, M
15 3 7	1 5 2	−231	−13	Y_{30}	T
2 8 11	4 1 1	−242	−18	$Y_{31}(1)$	Exc
11 11 2	1 1 4	−242	−18	$Y_{31}(2)$	Exc
10 12 2	6 5 22	−244	−2	Y_{32}	Exc
2 7 13	14 4 3	−247	−5	Y_{34}	Exc
2 10 9	10 2 3	−252	−8	Y_{35}	Exc
15 9 2	3 5 16	−258	−4	Y_{37}	Exc
2 6 17	6 2 1	−272	−14	$Y_{38}(1)$	Exc
2 12 8	6 1 2	−272	−14	$Y_{38}(2)$	Exc
8 16 2	2 1 6	−272	−14	$Y_{38}(3)$	Exc

Table X. continued

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines
20 8 2	2 5 14	−296	−6	Y_{41}	Exc
4 3 29	6 8 1	−319	−9	Y_{42}	T, M
2 16 7	8 1 3	−322	−12	$Y_{43}(1)$	Exc
7 21 2	3 1 8	−322	−12	$Y_{43}(2)$	Exc
27 3 6	1 9 4	−342	−10	Y_{44}	T
2 5 29	10 4 1	−377	−11	Y_{45}	Exc
35 7 2	1 5 12	−434	−10	Y_{46}	Exc
2 28 6	14 1 6	−492	−10	$Y_{47}(1)$	Exc
6 36 2	6 1 14	−492	−10	$Y_{47}(2)$	Exc

Table XI. Isolated solutions of 4abg pattern

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines
4 2 −31	−6 −10 1	248	8	$\mathfrak{e}_8$	Exc, M
3 2 −13	−3 −4 1	78	6	$\mathfrak{e}_6$	Exc, F
2 2 −7	−2 −2 1	28	4	$\mathfrak{so}(8)$	SO, Exc
0 −5 −5	4 −1 −1	0	0	$0d_3$	0d
0 −4 −7	6 −2 −1	0	0	$0d_4$	0d
5 5 5	1 1 1	−125	−19	Y_1	?
9 4 4	1 2 2	−144	−14	Y_{10}	K, M
7 7 3	1 1 2	−147	−17	Y_{11}	F
11 5 3	1 2 3	−165	−13	Y_{15}	F
19 4 3	1 4 5	−228	−10	Y_{29}	F, M
11 11 2	1 1 4	−242	−18	Y_{31}	Exc
14 9 2	2 3 10	−252	−8	Y_{35}	Exc
17 8 2	1 2 6	−272	−14	Y_{38}	Exc
23 7 2	1 3 8	−322	−12	Y_{43}	Exc
41 6 2	1 6 14	−492	−10	Y_{47}	Exc

Table XII. Isolated solutions of 5agb pattern

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines
2 5 −19	−8 −5 1	190	8	$\mathfrak{e}_{7\frac{1}{2}}$	Exc
4 3 −13	−4 −7 1	156	8	X_1	T, M
3 3 −5	−2 −3 1	45	5	$\mathfrak{so}(10)$	SO, T, F
−1 −1 −1	2 −1 −1	1	1	$\mathfrak{so}(2)$	SO, $D_{2,1,\lambda}$
0 −5 −5	4 −1 −1	0	0	$0d_3$	0d
0 −3 −11	8 −3 −1	0	0	$0d_1$	0d
0 7 −1	−4 −1 3	0	0	$0d_6$	F, 0d
5 5 5	1 1 1	−125	−19	Y_1	?
5 7 4	3 2 4	−140	−8	Y_9	K
9 4 4	1 2 2	−144	−14	Y_{10}	K, M

Table XII. continued

k n m	α β γ	Dim	Rank	Algebra	Lines
3 7 7	2 1 1	−147	−17	Y_{11}	F
9 7 3	1 1 3	−189	−17	Y_{21}	T
3 13 5	4 1 3	−195	−11	Y_{23}	F, K
13 5 3	1 2 4	−195	−13	Y_{24}	T
7 11 3	2 1 5	−231	−13	Y_{30}	T
2 11 11	4 1 1	−242	−18	Y_{31}	Exc
21 4 3	1 4 6	−252	−10	Y_{36}	T, K
6 19 3	4 1 9	−342	−10	Y_{44}	T
2 31 7	12 1 5	−434	−10	Y_{46}	Exc

Table XIII. Isolated solutions of 6baa pattern

k n m	α β γ	Dim	Rank	Algebra	Lines
−11 5 3	−5 1 −3	99	7	$X_2(1)$	T
−9 3 5	−3 1 −5	99	7	$X_2(2)$	T
3 −9 3	−3 −5 1	99	7	$X_2(3)$	T
−3 3 4	−3 2 −4	21	3	$\mathfrak{so}(7)$	SO, T, K
−3 −3 1	3 −1 −1	3	1	$3d_1$	3d
−1 −5 1	5 −3 −1	3	1	$3d_2$	3d, T
−1 3 3	−1 1 −1	3	1	$\mathfrak{sl}(2)(1)$	3d, T
3 −1 1	−1 −1 1	3	1	$\mathfrak{sl}(2)(2)$	3d, T
5 5 5	1 1 1	−125	−19	Y_1	?
4 5 6	5 8 6	−132	−2	Y_6	K
6 6 4	3 2 2	−132	−10	Y_4	K
5 7 4	7 6 4	−135	−3	Y_8	K
4 4 8	1 2 2	−144	−14	Y_{10}	K, M
9 5 4	5 2 4	−153	−7	Y_{13}	K
4 10 4	5 6 2	−168	−6	Y_{16}	K
3 6 8	3 10 4	−186	−4	$Y_{19}(1)$	T
6 3 10	3 4 10	−186	−4	$Y_{19}(2)$	T
3 7 7	1 3 1	−189	−17	$Y_{21}(1)$	T
7 3 9	1 1 3	−189	−17	$Y_{21}(2)$	T
7 11 3	11 5 3	−189	−3	Y_{22}	T
9 9 3	3 1 1	−189	−17	$Y_{21}(3)$	T
3 5 10	1 4 2	−195	−13	$Y_{24}(1)$	T
5 3 12	1 2 4	−195	−13	$Y_{24}(2)$	T
3 9 6	3 8 2	−207	−7	$Y_{26}(1)$	T
9 3 8	3 2 8	−207	−7	$Y_{26}(2)$	T
3 4 16	1 6 4	−252	−10	$Y_{36}(1)$	T, K
4 3 18	1 4 6	−252	−10	$Y_{36}(2)$	T, K
3 15 5	3 7 1	−285	−11	$Y_{39}(1)$	T
5 21 3	7 5 1	−285	−9	Y_{40}	T
15 3 7	3 1 7	−285	−11	$Y_{39}(2)$	T
19 7 3	7 1 3	−285	−11	$Y_{39}(3)$	T

Table XIV. Isolated solutions of 7bga pattern

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines
3 3 −11	−3 −5 1	99	7	X_2	T
1 1 −3	1 −3 1	3	1	$3d_1$	3d
3 −1 1	−1 1 −1	3	1	$\mathfrak{sl}(2)$	3d
3 1 −1	1 −5 3	3	1	$3d_2$	3d, T
5 5 5	1 1 1	−125	−19	Y_1	
9 3 7	3 11 5	−189	−3	Y_{22}	T
9 7 3	1 1 3	−189	−17	Y_{21}	T
19 3 5	1 7 5	−285	−9	Y_{40}	T
19 5 3	1 3 7	−285	−11	Y_{39}	T

Table XV. Isolated solutions in physical semiplane

$k\ n\ m$	$\alpha\ \beta\ \gamma$	Dim	Rank	Algebra	Lines	Patterns
2 −20 4	−10 1 −6	248	8	$\mathfrak{e}_8(1)$	Exc, M	3aag
4 −24 2	−6 1 −10	248	8	$\mathfrak{e}_8(2)$	Exc, M	3aag
4 2 −31	−6 −10 1	248	8	$\mathfrak{e}_8$	Exc, M	4abg
2 −16 5	−8 1 −5	190	8	$\mathfrak{e}_{7\frac{1}{2}}$	Exc	2aab
−25 5 2	1 −5 −8	190	8	$\mathfrak{e}_{7\frac{1}{2}}$	Exc	3aag
2 5 −19	−8 −5 1	190	8	$\mathfrak{e}_{7\frac{1}{2}}$	Exc	5agb
4 −16 3	−4 1 −7	156	8	X_1	T, M	2aab
−21 3 4	1 −7 −4	156	8	X_1	T, M	3aag
4 3 −13	−4 −7 1	156	8	X_1	T, M	5agb
2 3 −12	−6 −4 1	133	7	$\mathfrak{e}_7$	Exc, T	1aaa
2 3 −14	−6 −4 1	133	7	$\mathfrak{e}_7$	Exc, T	2aab
2 3 −19	−6 −4 1	133	7	$\mathfrak{e}_7$	Exc, T	3aag
−15 5 3	1 −3 −5	99	7	X_2	T	1aaa
−15 3 3	1 −5 −3	99	7	X_2	T	2aab
−11 5 3	−5 1 −3	99	7	$X_2(1)$	T	6baa
−9 3 5	−3 1 −5	99	7	$X_2(2)$	T	6baa
3 −9 3	−3 −5 1	99	7	$X_2(3)$	T	6baa
3 3 −11	−3 −5 1	99	7	X_2	T	7bga
2 −8 3	−4 1 −3	78	6	$\mathfrak{e}_6(1)$	Exc, F	3aag
3 −9 2	−3 1 −4	78	6	$\mathfrak{e}_6(2)$	Exc, F	3aag
3 2 −13	−3 −4 1	78	6	$\mathfrak{e}_6$	Exc, F	4abg
2 −6 4	−6 2 −5	52	4	$\mathfrak{f}_4$	Exc, K	2aab
−10 4 2	2 −5 −6	52	4	$\mathfrak{f}_4$	Exc, K	3aag
3 −6 3	−2 1 −3	45	5	$\mathfrak{so}(10)$	SO, T, F	2aab
−9 3 3	1 −3 −2	45	5	$\mathfrak{so}(10)$	SO, T, F	3aag
3 3 −5	−2 −3 1	45	5	$\mathfrak{so}(10)$	SO, T, F	5agb
2 −4 2	−2 1 −2	28	4	$\mathfrak{so}(8)$	SO, Exc	3aag
2 2 −7	−2 −2 1	28	4	$\mathfrak{so}(8)$	SO, Exc	4abg
−6 4 3	2 −3 −4	21	3	$\mathfrak{so}(7)$	SO, T, K	1aaa

Table XV. continued

k n m	α β γ	Dim	Rank	Algebra	Lines	Patterns
−3 3 4	−3 2 −4	21	3	$\mathfrak{so}(7)$	SO, T, K	6baa
−5 3 2	3 −5 −4	14	2	$\mathfrak{g}_2$	Exc, T	3aag
−3 1 −3	−1 3 −1	3	1	$3d_1$	3d	1aaa
−3 3 3	1 −1 −1	3	1	$\mathfrak{sl}(2)$	3d, T	1aaa
−2 1 −4	−2 4 −1	3	1	$\mathfrak{so}(3)$	3d	1aaa
−3 −3 1	3 −1 −1	3	1	$3d_1$	3d	6baa
−1 −5 1	5 −3 −1	3	1	$3d_2$	3d, T	6baa
−1 3 3	−1 1 −1	3	1	$\mathfrak{sl}(2)(1)$	3d, T	6baa
3 −1 1	−1 −1 1	3	1	$\mathfrak{sl}(2)(2)$	3d, T	6baa
1 1 −3	1 −3 1	3	1	$3d_1$	3d	7bga
3 −1 1	−1 1 −1	3	1	$\mathfrak{sl}(2)$	3d	7bga
3 1 −1	1 −5 3	3	1	$3d_2$	3d, T	7bga
−1 −1 −1	2 −1 −1	1	1	$\mathfrak{so}(2)$	SO, D	5agb

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