

The Universal C^* -Algebra of the Electromagnetic Field

To the memory of Daniel Kastler and John E. Roberts

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Received: 19 June 2015 / Revised: 29 August 2015 / Accepted: 31 August 2015

Published online: 16 October 2015 – © Springer Science+Business Media Dordrecht 2015

Abstract. A universal C^* -algebra of the electromagnetic field is constructed. It is represented in any quantum field theory which incorporates electromagnetism and expresses basic features of the field such as Maxwell’s equations, Poincaré covariance and Einstein causality. Moreover, topological properties of the field resulting from Maxwell’s equations are encoded in the algebra, leading to commutation relations with values in its center. The representation theory of the algebra is discussed with focus on vacuum representations, fixing the dynamics of the field.

Mathematics Subject Classification. 81V10, 81T05, 14F40.

Keywords. electromagnetic field, de Rham theory, local algebras, dynamical ideals, Haag–Kastler axioms.

1. Introduction

The general structural analysis of the electromagnetic field is a long-standing research topic in quantum field theory. Originally, this analysis was based on the Borchers algebra approach to quantum field theory, cf. [3] and references quoted there. But this setting, involving unbounded field operators, has its mathematical shortcomings because of notorious domain problems. In particular, it does not allow for a thorough discussion of non-regular representations, appearing for example in the presence of constraints [16], or of omnipresent infrared problems [17]. This fact triggered attempts to reformulate the theory in terms of C^* -algebras [14]. For the non-interacting electromagnetic field, this step can be accomplished by proceeding to the Weyl algebra of the field, cf. [22] and references quoted there.

This approach works also with slight modifications for the electromagnetic field coupled to c-number currents [24]. But the case of paramount physical interest, describing the coupling of the electromagnetic field to quantized relativistic matter, is not covered by these approaches.

In this article, we exhibit a C*-algebra describing basic features of the electromagnetic field which may be taken for granted in any relativistic quantum field theory which includes electromagnetism. Whereas the ensuing algebra does not incorporate any dynamical law, it has a sufficiently rich structure to identify in its dual space states describing the vacuum. The resulting GNS representations fix dynamical ideals corresponding to specific theories. The algebra thus provides a concrete C*-algebraic framework for the general structural analysis and physical interpretation of the electromagnetic field. The ideas underlying its construction may be known to experts; but, to the best of our knowledge, this approach has not yet been put on record.

To motivate the relations encoded in the algebra let us briefly recall the basic properties of the electromagnetic field. We use units where $c = \hbar = 1$ and consider four-dimensional Minkowski space \mathbb{R}^4 with metric fixed by the Lorentz scalar product $x_\mu y^\mu = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}$. The proper description of the electromagnetic field requires the introduction of spaces of tensor valued test functions (differential forms). We denote by $\mathcal{D}_r(\mathbb{R}^4)$, $r = 0, \dots, 4$, the spaces of real test functions $x \mapsto f^{\mu_1 \dots \mu_r}(x)$ which have compact support in \mathbb{R}^4 and are skew symmetric in $\mu_1 \dots \mu_r$, $r \geq 2$. They are stable under Poincaré transformations $P \doteq (y, L) \in \mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$, given by $f \mapsto f_P$, where

$$f_P^{\mu_1 \dots \mu_r}(x) \doteq L^{\mu_1}_{\nu_1} \dots L^{\mu_r}_{\nu_r} f^{\nu_1 \dots \nu_r}(P^{-1}x).$$

There exist two canonical mappings between these spaces: The exterior derivative $d: \mathcal{D}_r(\mathbb{R}^4) \rightarrow \mathcal{D}_{r+1}(\mathbb{R}^4)$ is defined by

$$(d f)^{\mu_1 \dots \mu_{r+1}}(x) \doteq -\partial^{[\mu_1} f^{\mu_2 \dots \mu_{r+1}]}(x),$$

where ∂^μ denotes the partial derivatives with respect to the coordinates of x and the square bracket indicates anti-symmetrization. The corresponding co-derivative $\delta: \mathcal{D}_r(\mathbb{R}^4) \rightarrow \mathcal{D}_{r-1}(\mathbb{R}^4)$ is given by

$$(\delta f)^{\mu_1 \dots \mu_{r-1}}(x) \doteq -r \partial_\nu f^{\nu \mu_1 \dots \mu_{r-1}}(x);$$

it is related to d by $\delta = -\star d \star$, where $\star: \mathcal{D}_r \rightarrow \mathcal{D}_{4-r}$ is the Hodge operator,

$$(\star f)^{\mu_1 \dots \mu_{4-r}}(x) \doteq (1/r!) \epsilon_{\nu_1 \dots \nu_r}^{\mu_1 \dots \mu_{4-r}} f^{\nu_1 \dots \nu_r}(x),$$

and $\epsilon_{\mu_1 \dots \mu_4}$ the Levi-Civita tensor. The particular choice of signs in these definitions is convenient here.

Making use of this notation, the electromagnetic field F can be presented as an operator valued real linear map from the space of real test functions $\mathcal{D}_2(\mathbb{R}^4)$ to the symmetric (hermitean) generators of some polynomial *-algebra \mathfrak{P} ,

$$f \mapsto F(f) = F_{\mu\nu}(f^{\mu\nu}), f \in \mathcal{D}_2(\mathbb{R}^4).$$

The homogeneous Maxwell equation for the field reads $F(\delta h) = 0$ for $h \in \mathcal{D}_3(\mathbb{R}^4)$ and the inhomogeneous Maxwell equation is given by $j(g) = j_\mu(g^\mu) \doteq F(dg)$ for $g \in \mathcal{D}_1(\mathbb{R}^4)$. The latter relation is to be interpreted as the definition of the (identically conserved) current j since F is regarded as being given. Einstein causality is expressed by the condition of locality according to which the commutator of the electromagnetic field satisfies $[F(f_1), F(f_2)] = 0$ whenever the supports of $f_1, f_2 \in \mathcal{D}_2(\mathbb{R}^4)$ are spacelike separated. These relations are consistent with the automorphic action of the Poincaré group on the algebra \mathfrak{A} fixed by the mapping $F(f) \mapsto F(f_P), P \in \mathcal{P}_+^\uparrow$.

It is convenient to proceed from the electromagnetic field F to its intrinsic (gauge invariant) vector potential A , given by

$$A(\delta f) = A_\mu((\delta f)^\mu) \doteq F(f) \quad \text{for } f \in \mathcal{D}_2(\mathbb{R}^4).$$

Clearly, $\delta(\delta f) = 0, f \in \mathcal{D}_2(\mathbb{R}^4)$, that is, $\delta f \in \mathcal{D}_1(\mathbb{R}^4)$ is co-closed. Conversely, given any co-closed $g \in \mathcal{D}_1(\mathbb{R}^4), \delta g = 0$, Poincaré's lemma (cf. [18, Lem. 17.27] and the appendix) implies that there exists some $f \in \mathcal{D}_2(\mathbb{R}^4)$ such that $g = \delta f$, i.e., g is co-exact. Moreover, the ambiguities involved in the choice of f consist of additive terms of the form δh where $h \in \mathcal{D}_3(\mathbb{R}^4)$. Denoting by $\mathcal{C}_1(\mathbb{R}^4) \subset \mathcal{D}_1(\mathbb{R}^4)$ the real subspace of co-closed forms $g \in \mathcal{D}_1(\mathbb{R}^4), \delta g = 0$, one can therefore define for any $g \in \mathcal{C}_1(\mathbb{R}^4)$ and corresponding co-primitive $f \in \mathcal{D}_2(\mathbb{R}^4)$ the potential

$$A(g) \doteq F(f), \quad f \in \{f' \in \mathcal{D}_2(\mathbb{R}^4) : \delta f' = g\}.$$

This definition is consistent since $\delta f' = g = \delta f$ implies according to Poincaré's lemma that $f' = f + \delta h$ for some $h \in \mathcal{D}_3(\mathbb{R}^4)$ and consequently $F(f') = F(f)$ by the homogeneous Maxwell equation.

In view of these facts, one can express the properties of the electromagnetic field in terms of its intrinsic vector potential A . This potential defines a real linear map $g \mapsto A(g)$ from $\mathcal{C}_1(\mathbb{R}^4)$ to the symmetric generators of the *-algebra \mathfrak{A} . The homogeneous Maxwell equation is satisfied by construction and the inhomogeneous Maxwell equation now reads $j(g) \doteq A(\delta dg), g \in \mathcal{D}_1(\mathbb{R}^4)$. Noticing that the subspace $\mathcal{C}_1(\mathbb{R}^4) \subset \mathcal{D}_1(\mathbb{R}^4)$ is stable under the action of Poincaré transformations, the automorphic action of the Poincaré group on the algebra \mathfrak{A} is fixed by the mappings $A(g) \mapsto A(g_P), P \in \mathcal{P}_+^\uparrow$.

The formulation of Einstein causality in terms of the intrinsic vector potential is more subtle, however, since one may not assume from the outset that A can be extended to the space $\mathcal{D}_1(\mathbb{R}^4)$ as a local and covariant field. These non-observable extensions require the consideration of indefinite metric spaces [26,27] or of modifications of the *-operation [28], so they do not fit into the present setting. To avoid these auxiliary constructs we make use of some pertinent geometrical facts. Given any $g \in \mathcal{C}_1(\mathbb{R}^4)$ that has support in some open double cone $\mathcal{O} \subset \mathbb{R}^4$ it follows from a local version of Poincaré's lemma (cf. Appendix) that there is some co-primitive $f \in \mathcal{D}_2(\mathbb{R}^4)$, satisfying $\delta f = g$, that has its support in \mathcal{O} as well. The

locality of the electromagnetic field then implies

$$[A(g_1), A(g_2)] = [F(f_1), F(f_2)] = 0$$

whenever $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ have their supports in spacelike separated double cones. If $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ have their supports in spacelike separated but topologically non-trivial regions one can, however, no longer conclude that the commutators vanish. Yet, as is shown in the appendix, all of these commutators are invariant under spacetime translations, viz.

$$[A(g_1), A(g_2)] = [A(g_{1y}), A(g_{2y})] \text{ for any } y \in \mathbb{R}^4.$$

Hence, because of locality, they commute with all elements of the algebra \mathfrak{A} , i.e., they are elements of its center. Note that the customary Gupta–Bleuler potential of the electromagnetic field, restricted to the test function space $\mathcal{C}_1(\mathbb{R}^4)$, provides a concrete representation of the algebra \mathfrak{A} where these central elements vanish; yet this does not hold true within the abstract algebra.

These basic properties of the intrinsic vector potential A can be recast in terms of unitary operators that may heuristically be interpreted as its exponentials, viz., $V(a, g) \hat{=} \exp(iaA(g))$ where $a \in \mathbb{R}$, $g \in \mathcal{C}_1(\mathbb{R}^4)$. As a matter of fact, this correspondence can rigorously be established in regular representations of the algebra \mathfrak{A} generated by these unitaries. This algebra is defined in the subsequent section, where its C^* -property is also established. In the third section, the incorporation of dynamics into the framework is explained, based on the choice of vacuum states on the algebra \mathfrak{A} . This approach is illustrated by examples, clarifying its relation to standard field theoretic treatments. The article concludes with a brief summary and an appendix containing specific local and causal versions of Poincaré’s lemma which are of relevance in the present context.

2. The Universal Algebra

The construction of the C^* -algebra of the electromagnetic field proceeds from the $*$ -algebra \mathfrak{A}_0 generated by the elements of the set $\{V(a, g) : a \in \mathbb{R}, g \in \mathcal{C}_1(\mathbb{R}^4)\}$ which satisfy the relations given below. Denoting by the symbol \perp pairs of spacelike separated subsets of \mathbb{R}^4 and by $[X, Y] \doteq XYX^*Y^*$ the group theoretic commutator of unitary operators X, Y , these relations read

$$V(a_1, g)V(a_2, g) = V(a_1 + a_2, g), \quad V(a, g)^* = V(-a, g), \quad V(0, g) = 1 \tag{2.1}$$

$$V(a_1, \delta f_1)V(a_2, \delta f_2) = V(a_1\delta f_1 + a_2\delta f_2) \quad \text{if } \text{supp } f_1 \perp \text{supp } f_2 \tag{2.2}$$

$$[V(a, g), [V(a_1, g_1), V(a_2, g_2)]] = 0 \quad \text{if } \text{supp } g_1 \perp \text{supp } g_2, \tag{2.3}$$

where $a, a_1, a_2 \in \mathbb{R}$, $g, g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ and $f, f_1, f_2 \in \mathcal{D}_2(\mathbb{R}^4)$. That is, we start with the unitary group \mathfrak{G}_0 generated by $\{V(a, g) : a \in \mathbb{R}, g \in \mathcal{C}_1(\mathbb{R}^4)\}$, subject to these relations, and proceed to the complex linear span of the elements of \mathfrak{G}_0 to obtain the $*$ -algebra \mathfrak{A}_0 .

Relation (2.1) encodes the algebraic properties of unitary one-parameter groups $a \mapsto V(a, g)$, expressing the idea that one deals with the exponential function of the smeared potential. Relation (2.2) combines the information that the electromagnetic field is linear on $\mathcal{D}_2(\mathbb{R}^4)$ and local: since the functional equation for the exponential function of operators holds only if the operators commute, the additivity of the field manifests itself in this restricted form. Relation (2.3) embodies the information that the commutator $[A(g_1), A(g_2)]$ of the potential commutes with any other $A(g)$ whenever g_1, g_2 have spacelike separated supports. In the special case where the supports of g_1, g_2 are contained in spacelike separated double cones, it follows from the local Poincaré lemma and relation (2.2) that one has

$$V(a_1, g_1)V(a_2, g_2) = V(1, a_1 g_1 + a_2 g_2) = V(a_2, g_2)V(a_1, g_1),$$

hence $[V(a_1, g_1), V(a_2, g_2)] = 1$.

The algebra \mathfrak{A}_0 can be equipped with a C*-norm by a standard construction which we recall for the convenience of the reader. It relies on the fact that each state (viz., positive, linear and normalized functional) on a *-algebra gives rise to a Hilbert space representation by the GNS construction, cf. [14, Sec. III.2]. Since the present algebra consists of linear combinations of unitary operators, their respective Hilbert space representatives are bounded and their Hilbert space norm defines a C*-seminorm on this algebra. If the underlying state is faithful, this seminorm is even a norm. The existence of such states is established in the subsequent lemma. There we make use of the fact that \mathfrak{A}_0 is the complex linear span of the elements of the unitary group \mathfrak{G}_0 .

LEMMA 1. *Let ω be the functional on the unitary group \mathfrak{G}_0 given by $\omega(V) = 0$ for $V \in \mathfrak{G}_0 \setminus \{1\}$ and $\omega(1) = 1$. The canonical extension of this functional to the complex linear span of \mathfrak{G}_0 is a faithful state on \mathfrak{A}_0 .*

Proof. Since the elements of \mathfrak{G}_0 form a basis of \mathfrak{A}_0 , the linear extension of ω to \mathfrak{A}_0 is consistently defined by $\omega(c_0 1 + \sum_n c_n V_n) = c_0$, where $V_n \in \mathfrak{G}_0 \setminus \{1\}$. Assuming without loss of generality that the unitaries V_n are different one also obtains

$$\omega\left((c_0 1 + \sum_n c_n V_n)^* (c_0 1 + \sum_{n'} c_{n'} V_{n'})\right) = |c_0|^2 + \sum_n |c_n|^2 \geq 0$$

because the terms for $n \neq n'$ contain operators in $\mathfrak{G}_0 \setminus \{1\}$ and therefore vanish. This shows that the linear and normalized functional ω on \mathfrak{A}_0 is positive on positive elements, hence it is a state. Moreover, since the equality sign in the above relation holds only for the zero element of \mathfrak{A}_0 , this state is faithful, completing the proof of the statement. \square

As indicated, any state ω induces by the GNS construction a representation $(\pi, \mathcal{H}, \Omega)$ of \mathfrak{A}_0 , where π is a homomorphism mapping \mathfrak{A}_0 into the algebra of

all bounded operators on some Hilbert space \mathcal{H} and $\Omega \in \mathcal{H}$ is a unit vector such that $\langle \Omega, \pi(A)\Omega \rangle = \omega(A)$, $A \in \mathfrak{A}_0$. If ω is faithful, such as the state exhibited in the preceding lemma, the Hilbert space norm $\|\pi(A)\|_{\mathcal{H}}$ of the represented operators $A \in \mathfrak{A}_0$ defines a norm on \mathfrak{A}_0 which has the C^* -property. In order not to exclude from the outset any representations we proceed here to the largest C^* -norm on \mathfrak{A}_0 , given by

$$\|A\| \doteq \sup \|\pi(A)\|_{\mathcal{H}}, \quad A \in \mathfrak{A}_0,$$

where the supremum extends over all GNS representations of \mathfrak{A}_0 . Note that the supremum exists since the Hilbert space representatives of any given finite sum of unitary operators are uniformly bounded. The completion of \mathfrak{A}_0 with respect to this norm defines the universal C^* -algebra \mathfrak{A} of the electromagnetic field; it is represented in any theory incorporating electromagnetism.

We conclude this section by showing that the algebra \mathfrak{A} provides a physically significant example fitting into the general framework of observable algebras, established by Haag and Kastler [15]. To this end, we define for any given open double cone $\mathcal{O} \subset \mathbb{R}^4$ the subalgebra $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}$ that is generated by the unitaries $\{V(a, g) : a \in \mathbb{R}, g \in \mathcal{C}_1(\mathcal{O})\}$, where $\mathcal{C}_1(\mathcal{O})$ denotes the subspace of co-closed forms in $\mathcal{D}_1(\mathbb{R}^4)$ having support in \mathcal{O} . By definition, $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ whenever $\mathcal{O}_1 \subset \mathcal{O}_2$, so the assignment $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ defines an isotonus net of C^* -algebras on Minkowski space \mathbb{R}^4 with common identity. Since the unitaries underlying the construction of \mathfrak{A} are based on test functions with compact support, the inductive limit of this net coincides with \mathfrak{A} . Moreover, as has been explained, relation (2.2) implies that the operators assigned to spacelike separated double cones $\mathcal{O}_1, \mathcal{O}_2$ commute, in short $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$. So, the net satisfies the condition of locality.

In order to see that this net is also Poincaré covariant, we note that the relations (2.1) to (2.3) do not change if, for given $P \in \mathcal{P}_+^\uparrow$, one replaces all test functions by their respective Poincaré transforms. This implies that the invertible maps α defined on $\{V(a, g) : a \in \mathbb{R}, g \in \mathcal{C}_1(\mathbb{R}^4)\}$ by

$$\alpha_P(V(a, g)) \doteq V(a, g_P), \quad P \in \mathcal{P}_+^\uparrow,$$

extend to automorphisms of the group \mathfrak{G}_0 and thereon to its linear span \mathfrak{A}_0 . Composing these automorphisms yields a representation of the Poincaré group, that is $\alpha_{P_1} \circ \alpha_{P_2} = \alpha_{P_1 P_2}$ and $\alpha_P^{-1} = \alpha_{P^{-1}}$ for $P, P_1, P_2 \in \mathcal{P}_+^\uparrow$. Moreover, by continuity one can further extend these automorphisms to the C^* -algebra \mathfrak{A} . For the set of GNS representations of \mathfrak{A}_0 is stable under composition with any automorphism and consequently

$$\|\alpha_P(A)\| = \sup \|\pi(\alpha_P(A))\|_{\mathcal{H}} = \sup \|\pi \circ \alpha_P(A)\|_{\mathcal{H}} = \|A\| \quad \text{for } A \in \mathfrak{A}_0.$$

Finally, noticing that for any $g \in \mathcal{C}_1(\mathcal{O})$ one has $g_P \in \mathcal{C}_1(P\mathcal{O})$, it is apparent that $\alpha_P(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(P\mathcal{O})$, $P \in \mathcal{P}_+^\uparrow$, proving the Poincaré covariance of the net.

Thus the universal algebra \mathfrak{A} generated by the electromagnetic field satisfies all Haag–Kastler axioms [15, p. 849] with one exception: it is not a primitive algebra since it does not have any faithful irreducible representation. In fact, it follows from relation (2.3) that \mathfrak{A} has a non-trivial center. This deficiency can be resolved, however, by identifying suitable irreducible representations $(\pi, \mathcal{H}, \Omega)$ of \mathfrak{A} . The kernel of a representation, denoted by $\ker \pi$, characterizes a two-sided ideal in \mathfrak{A} and if this kernel is stable under the action of the automorphisms $\alpha_P, P \in \mathcal{P}_+^\uparrow$, the corresponding quotient algebra $\mathfrak{A}/\ker \pi$ is by construction a primitive algebra which satisfies all Haag–Kastler axioms. Even more importantly, using this device one can in principle incorporate any dynamics into the quotient algebra which is compatible with the basic properties of the electromagnetic field. This issue will be discussed in the subsequent section.

3. Representations

All possible states of the electromagnetic field are described by elements of the dual space of the universal algebra \mathfrak{A} . We begin by characterizing those states and representations which are of primary physical interest, allowing it to recover from \mathfrak{A} the electromagnetic field, respectively the intrinsic vector potential as well defined observables.

DEFINITION 1. Let ω be a state on \mathfrak{A} . This state is regular if all functions

$$a_1, \dots, a_n \mapsto \omega(V(a_1, g_1) \cdots V(a_n, g_n)), \quad g_1, \dots, g_n \in \mathcal{C}_1(\mathbb{R}^4),$$

are continuous, $n \in \mathbb{N}$. It is strongly regular if all of these functions are smooth and their derivatives at $a_1 = \dots = a_n = 0$ are bounded by Schwartz norms of the underlying test functions (tempered).

It is not difficult to see that in the GNS representation $(\pi, \mathcal{H}, \Omega)$ induced by a regular state ω the unitary one-parameter groups $a \mapsto \pi(V(a, g))$ are continuous in the strong operator topology. Stone's theorem therefore implies that there exist densely defined selfadjoint operators $A_\pi(g)$ in the underlying Hilbert space \mathcal{H} such that $\pi(V(a, g)) = e^{iaA_\pi(g)}$ for $a \in \mathbb{R}$, $g \in \mathcal{C}_1(\mathbb{R}^4)$. So, one recovers in these representations the intrinsic vector potential. Moreover, if ω is strongly regular these operators have a common dense domain $\mathcal{D} \subset \mathcal{H}$, containing Ω , that is stable under their action and a core for each of them. In particular, the correlation functions $\langle \Omega, A_\pi(g_1) \dots A_\pi(g_n) \Omega \rangle$ are well defined for any $g_1, \dots, g_n \in \mathcal{C}_1(\mathbb{R}^4)$ and they are bounded by Schwartz norms of these test functions, $n \in \mathbb{N}$.

It follows from relation (2.2) that the operators A_π appearing in the GNS representation induced by a strongly regular state satisfy on their common domain \mathcal{D} a restricted form of linearity. Namely, $a_1 A_\pi(g_1) + a_2 A_\pi(g_2) = A_\pi(a_1 g_1 + a_2 g_2)$ whenever $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ have their supports in spacelike separated double cones and

$a_1, a_2 \in \mathbb{R}$. Linearity on all of $\mathcal{C}_1(\mathbb{R}^4)$ is ensured if the state satisfies the following stronger condition.

DEFINITION 2. A state ω on \mathfrak{A} has property L if it is strongly regular and

$$\frac{d}{da} \omega(V_1 V(a, g_1) V(a, g_2) V(-a, g_1 + g_2) V_2) \Big|_{a=0} = 0$$

for every $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ and $V_1, V_2 \in \mathfrak{G}_0$.

We mention as an aside that in the GNS representations induced by states having property L the operator sums $a_1 A_\pi(g_1) + a_2 A_\pi(g_2)$ are essentially selfadjoint on \mathcal{D} since they coincide on this domain with $A_\pi(a_1 g_1 + a_2 g_2)$ for $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ and $a_1, a_2 \in \mathbb{R}$. Next, we recall the familiar characterization of vacuum states by their physically expected properties of Poincaré invariance and stability [14].

DEFINITION 3. Let ω be a pure state on \mathfrak{A} ; ω is interpreted as vacuum state if for every $A, B \in \mathfrak{A}$ (i) $\omega(\alpha_P(A)) = \omega(A)$ for $P \in \mathcal{P}_+^\uparrow$, (ii) $P \mapsto \omega(A \alpha_P(B))$ is continuous and (iii) the Fourier transforms of all functions $x \mapsto \omega(A \alpha_x(B))$ on the subgroup $\mathbb{R}^4 \subset \mathcal{P}_+^\uparrow$ of spacetime translations have support in the closed forward lightcone V_+ (relativistic spectrum condition).

As is well known [14], there exists in the GNS representation $(\pi, \mathcal{H}, \Omega)$ induced by a vacuum state a continuous unitary representation U_π of the Poincaré group such that (i) $U_\pi(P)\Omega = \Omega$, $P \in \mathcal{P}_+^\uparrow$, (ii) the generators of the subgroup $U_\pi \upharpoonright \mathbb{R}^4$ (energy–momentum) have joint spectrum in V_+ (i.e., Ω is a ground state in all Lorentz frames) and (iii) the unitaries U_π implement the action of the Poincaré transformations on observables, viz., $U_\pi(P)\pi(A)U_\pi(P)^{-1} = \pi(\alpha_P(A))$ for any $P \in \mathcal{P}_+^\uparrow$ and $A \in \mathfrak{A}$. The latter relation implies that the kernel of π is stable under Poincaré transformations. Moreover, since vacuum states are pure states by definition, the representation π is irreducible. Hence, proceeding to the quotient algebra $\mathfrak{A}/\ker \pi$, each vacuum state on \mathfrak{A} defines a consistent dynamical theory of the electromagnetic field which fulfills all Haag–Kastler axioms.

It follows from these remarks that the construction of theories involving the electromagnetic field amounts to the task of exhibiting vacuum states in the dual space of the algebra \mathfrak{A} . As a matter of fact, every vacuum state ω on \mathfrak{A} is determined by its generating function $g \mapsto \omega(V(1, g))$ on $\mathcal{C}_1(\mathbb{R}^4)$. For, due to the invariance of vacuum states under spacetime translations and the relativistic spectrum condition, the functions $x_1, \dots, x_n \mapsto \omega(\alpha_{x_1}(V(1, g_1)) \cdots \alpha_{x_n}(V(1, g_n)))$ extend continuously in the variables $(x_{m+1} - x_m)$, $m = 1, \dots, n - 1$, to the tube $(\mathbb{R}^4 + iV_+)^{n-1} \subset \mathbb{C}^{4(n-1)}$ and are analytic in its interior [25]. Moreover, for given functions $g_m \in \mathcal{C}_1(\mathbb{R}^4)$ there exist open sets of translations $x_m \in \mathbb{R}^4$ such that the supports of the shifted functions g_{m, x_m} are contained in spacelike separated double

cones, $m = 1, \dots, n$. Thus, the local Poincaré lemma and relation (2.2) imply that

$$\begin{aligned} \omega(\alpha_{x_1}(V(1, g_1)) \cdots \alpha_{x_n}(V(1, g_n))) &= \omega(V(1, g_{1,x_1}) \cdots V(1, g_{n,x_n})) \\ &= \omega(V(1, g_{1,x_1} + \dots + g_{n,x_n})), \end{aligned}$$

where the right hand side can be determined if the generating function is given. The left hand side can then be continued analytically to arbitrary configurations $x_m \in \mathbb{R}^4$, $m = 1, \dots, n$. Hence, the generating function fixes the expectation values of vacuum states ω on the unitary group \mathfrak{G}_0 , whence on all of \mathfrak{V} by linearity and continuity.

Note that if a vacuum state on \mathfrak{V} also satisfies condition L then the expectation values of the polynomials in the smeared field $F_\pi(f) = A_\pi(\delta f)$, $f \in \mathcal{D}_2(\mathbb{R}^4)$, are defined in this state and comply with all Wightman axioms [25]. We make use of this fact in the following simple example.

3.1. TRIVIAL CURRENTS

In this subsection, we determine all vacuum states ω on \mathfrak{V} which lead to theories with trivial current and have property L . The unique result is the theory of the free electromagnetic field. Since this result is obtained without any input from a classical Lagrangian, respectively quantization scheme, it illustrates the fact that the algebra \mathfrak{V} embodies fundamental physical information.

As outlined in the introduction, the current is related to the intrinsic vector potential by the formula $j(g) = A(\delta dg)$, $g \in \mathcal{D}_1(\mathbb{R}^4)$. Thus in the GNS representation induced by a vacuum state ω in which the current vanishes one has

$$\langle \Omega, A_\pi(\delta dg) A_\pi(\delta dg) \Omega \rangle = 0, \quad g \in \mathcal{D}_1(\mathbb{R}^4).$$

This equality implies $A_\pi(\delta dg) = 0$ since the vacuum vector Ω is separating for local operators by the Reeh–Schlieder theorem, cf. [14, Ch. II.5.3]. Since $\delta d + d\delta = \square$, where \square denotes the d’Alembertian, one has $\delta dg = \square g$ for $g \in \mathcal{C}_1(\mathbb{R}^4) \subset \mathcal{D}_1(\mathbb{R}^4)$, so the vector potential fulfills the wave equation $A_\pi(\square g) = 0$, $g \in \mathcal{C}_1(\mathbb{R}^4)$. It then follows from locality and Poincaré covariance of the potential by standard arguments (Källén–Lehmann representation) that its Wightman two-point function coincides with that of the free field,

$$\begin{aligned} W(g_1, g_2) &\doteq \langle \Omega, A_\pi(g_1) A_\pi(g_2) \Omega \rangle \\ &= c \int dp \theta(p_0) \delta(p^2) \widehat{g}_{1\mu}(-p) \widehat{g}_2^\mu(p), \quad g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4), \end{aligned} \tag{3.1}$$

where \widehat{g} denotes the Fourier transform of g . Rescaling the potential, one can adjust the constant in this equality to its conventional value $c = -(2\pi)^{-3}$, where the sign is dictated by the condition of positivity of states. It is a remarkable consequence of this result that in the given vacuum representation all commutators of the smeared potential are multiples of the identity, $[A_\pi(g_1), A_\pi(g_2)] = c(g_1, g_2) 1_{\mathcal{H}}$,

cf. the arguments in [19]; the value of $c(g_1, g_2)$ can be read off from the preceding two-point function, $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$. So, one recovers the intrinsic vector potential of the free electromagnetic field in the Fock representation. The corresponding generating function is well known,

$$g \mapsto \omega(V(1, g)) = e^{-W(g, g)/2}, \quad g \in \mathcal{C}_1(\mathbb{R}^4),$$

where W is the two-point function given above.

This special form of generating functions, depending only on a two-point function, is distinctive of quasifree states on \mathfrak{A} . Making use of the general Källén–Lehmann representation of two-point functions, it is not difficult to determine all quasifree vacuum states on \mathfrak{A} which have property L . Of particular interest is the case where the current is proportional to the intrinsic vector potential in the underlying GNS representation. By arguments similar to those given above one finds that the potential then is a free massive vector field with two-point function

$$W_m(g, g) = -(2\pi)^{-3} \int dp \theta(p_0) \delta(p^2 - m^2) \widehat{g}_\mu(-p) \widehat{g}^\mu(p), \quad g \in \mathcal{C}_1(\mathbb{R}^4).$$

The mass square m^2 is determined by the constant of proportionality between the current and the potential. Plugging this two-point function into the above formula yields the generating function of the corresponding vacuum state.

3.2. CLASSICAL CURRENTS

Next, we discuss the cases where the electromagnetic field is coupled to classical currents. Such currents are simultaneously measurable with all other observables and are therefore described by representations of \mathfrak{A} where the current operators $j(g) = A(\delta dg)$, $g \in \mathcal{D}_1(\mathbb{R}^4)$, are affiliated with the center. It is clear from the outset that such representations cannot be induced by vacuum states on \mathfrak{A} since the appearance of classical currents breaks the Poincaré symmetry of these states spontaneously. We are therefore led to consider a more general class of pure states ω on \mathfrak{A} which have property L . The corresponding GNS representations $(\pi, \mathcal{H}, \Omega)$ are irreducible, so their center consists of multiples of the identity and one has $\pi(V(1, \delta dg)) = e^{ij_\pi(g)} 1_{\mathcal{H}}$, $g \in \mathcal{D}_1(\mathbb{R}^4)$, where the current j_π is a conserved real-valued distribution fixed by the representation.

Whenever the current j_π is sufficiently regular it can be extended to the space $G_0 \mathcal{D}_1(\mathbb{R}^4) \supset \mathcal{D}_1(\mathbb{R}^4)$ obtained by convolution of the test functions with the time symmetric Green’s function G_0 of the wave equation (i.e., half the sum of the retarded and advanced Green’s functions). One can then define an automorphism γ of \mathfrak{A} , putting on its generating unitaries

$$\gamma(V(1, g)) \doteq e^{-ij_\pi(G_0 g)} V(1, g), \quad g \in \mathcal{C}_1(\mathbb{R}^4);$$

note that relations (2.1) to (2.3) are preserved by this map. Composing the given representation with this automorphism yields the representation $\pi_0 \doteq \pi \circ \gamma$ of \mathfrak{A}

on \mathcal{H} . Now for $g \in \mathcal{D}_1(\mathbb{R}^4)$ one has $\delta dg = \square g - d\delta g$, hence $j_\pi(\delta dg) = j_\pi(\square g)$ since j_π is conserved. As $j_\pi(G_0 \square g) = j_\pi(g)$ this implies $\pi_0(V(1, \delta dg)) = 1_{\mathcal{H}}$, $g \in \mathcal{D}_1(\mathbb{R}^4)$. Thus, the current vanishes in the representation π_0 . One may therefore consistently assume that this representation coincides with the vacuum representation of the free electromagnetic field. With this input, one obtains for the representation $\pi = \pi_0 \circ \gamma^{-1}$ the generating function

$$\omega(V(1, g)) = e^{ij_\pi(G_0 g)} e^{-W(g, g)/2}, \quad g \in \mathcal{C}_1(\mathbb{R}^4),$$

where W is the above two-point function of the free electromagnetic field.

3.3. QUANTUM CURRENTS

The rigorous construction of vacuum states on \mathfrak{A} that describe the coupling of the electromagnetic field to charged quantum fields and their currents is a long-standing open problem. In most approaches, one proceeds from time-ordered products of the underlying generating functions, denoted by $\omega(V_T(1, g))$. Relying on the relativistic spectrum condition, the vacuum states ω on \mathfrak{A} can likewise be reconstructed from these functions by methods of analytic continuation. Heuristic candidates for the time-ordered functions are Feynman path integrals of the form

$$\omega(V_T(1, g)) \doteq Z^{-1} \int dA d\psi d\bar{\psi} e^{iS(A, \psi, \bar{\psi})} e^{iA(g)},$$

where all charged fields $\psi, \bar{\psi}$ appearing in the underlying classical action S are integrated out. In spite of important progress in the rigorous construction of such integrals [1, 13], all presently available methods of determining these expectations rely on renormalized perturbation theory, cf. [10, 11, 23] and references quoted there. A perturbative approach to the computation of the unordered generating functions $\omega(V(1, g))$, based on field equations, has been developed by Steinmann [26].

Since the algebra \mathfrak{A} embodies all basic features of the electromagnetic field, a rigorous proof that vacuum states describing interaction exist in its dual space (possibly based on other constructive schemes) would be of great physical interest. On the other hand, the unlikely possibility that no fully consistent theory of interacting electromagnetic fields can be accommodated in the general framework of local quantum field theory would likewise manifest itself in the structure of the dual space of \mathfrak{A} . Thus, this algebra provides a solid basis for further study of this existence problem.

3.4. TOPOLOGICAL CHARGES

Finally, we discuss representations of the algebra \mathfrak{A} where the properties of the intrinsic vector potential in non-contractible regions matter. The general geometrical features of nets based on such regions were studied in [8] with applications to

the electromagnetic field and potential in [9]. Here, we focus on irreducible representations of \mathfrak{V} where the (central) group theoretic commutator $[V(1, g_1), V(1, g_2)]$ has values different from 1 for certain pairs of test functions g_1, g_2 with spacelike separated, linked supports. Since the specific topology of the supports is of importance here, we interpret these values as topological charges.

At the algebraic level, let us first note that there exist functions $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ with spacelike separated supports such that $V(1, g_1)V(1, g_2) \neq V(1, g_2)V(1, g_1)$. For the equality of such spacelike separated unitaries is not required in the defining relations (2.1) to (2.3). Hence if equality holds nonetheless for a given pair of functions, this must be a consequence of the condition of locality of the electromagnetic field which is encoded in relation (2.2). In other words, there must exist functions $f_1, f_2 \in \mathcal{D}_2(\mathbb{R}^4)$ with $\text{supp} f_1 \perp \text{supp} f_2$ such that $g_1 = \delta f_1$ and $g_2 = \delta f_2$. Yet, as can be inferred from the work of Roberts [21, §1], there exist pairs $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$, having their supports in spacelike separated linked loop-shaped regions, for which this condition cannot be satisfied. Hence, the corresponding group-theoretic commutators $[V(1, g_1), V(1, g_2)]$ are non-trivial unitary operators. We denote by $\mathfrak{J} \subset \mathfrak{V}$ the C*-algebra generated by $[V(1, g_1), V(1, g_2)]$ for all pairs of functions $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ with spacelike separated supports. According to the preceding remarks, this algebra is non-trivial and, by relation (2.3), contained in the center of \mathfrak{V} . In particular, \mathfrak{J} is an abelian algebra.

Given any pure state (character) ζ on \mathfrak{J} , this state can be extended to some pure state ω_ζ on \mathfrak{V} by the Hahn–Banach theorem [2, II.6.3.2]. In the corresponding GNS representation $(\pi_\zeta, \mathcal{H}_\zeta, \Omega_\zeta)$, all elements of the algebra \mathfrak{J} are represented by multiples of the identity (the topological charges) whose values depend on the choice of the state ζ . It is an open problem whether for some suitable non-trivial choice of ζ there exist pure extensions ω_ζ which are Poincaré invariant. However, states with non-trivial topological charges are in general not regular.

4. Summary

In the present investigation, we have constructed a universal C*-algebra \mathfrak{V} of the electromagnetic field whose basic features are encoded in the defining relations (2.1) to (2.3). Even though this algebra does not contain any dynamical information, it has a sufficiently rich structure to identify in its dual space all possible vacuum states of the field which depend on its particular coupling to charged matter. The GNS representations π resulting from these states allow to construct selfadjoint generators of the spacetime transformations, comprising the desired dynamical information, and the corresponding quotient algebras $\mathfrak{V}/\ker \pi$ of observables satisfy all Haag–Kastler axioms. In the simple cases of trivial or classical currents, one can determine the underlying states without relying on any further input, such as a classical Lagrangian or canonical commutation relations. This fact shows that the universal algebra subsumes essential physical information. The still pending rigorous construction of interacting theories of the electromagnetic field amounts

to the identification of appropriate vacuum states in the dual space of \mathfrak{A} . It therefore calls for further study of this algebra.

Apart from these constructive aspects, the universal algebra provides a solid basis for the structural analysis, the physical interpretation and the classification of theories incorporating electromagnetism. For example, based on the Haag–Kastler axioms, a general scattering theory for the electromagnetic field has been developed in [5], the notorious infrared problems appearing in this context were analyzed in [6, 12] and, more recently, the possible charge structures and statistics appearing in theories of the electromagnetic field were determined in [7]. Thus, the universal algebra \mathfrak{A} is a concrete and physically significant example fitting into the general algebraic framework of relativistic quantum field theory.

Furthermore, the universal algebra \mathfrak{A} seems to fully encode the geometrical features underlying gauge theories. In particular, the locality properties of the electromagnetic field, encoded in the commutation relations (2.2) and (2.3) of the intrinsic vector potential, lead to the emergence of new types of topological charges that can be described by cohomological invariants associated with linked commutators. We believe that these aspects are not confined to the electromagnetic field but should be present also in non-Abelian gauge theories. Thus, further investigations of these structures seem warranted.

Acknowledgements

DB gratefully acknowledges the hospitality and financial support extended to him by the University of Rome “Tor Vergata” which made this collaboration possible. FC and GR are partially supported by PRIN-2010/2011. FC is also supported by the ERC Advanced Grant 227458 OACFT “Operator Algebras and Conformal Field Theory”.

Appendix

We show in this appendix that the commutator of the intrinsic vector potential, smeared with test functions that have spacelike separated supports, lies in the center of the polynomial algebra \mathfrak{P} . The argument is based on refinements of Poincaré’s lemma that put emphasis on the support properties of the co-primitives. For the convenience of the reader, we outline the proofs of these basic results, noting that the subsequent facts adopted from differential geometry and algebraic

topology on \mathbb{R}^4 carry over to Minkowski space since they do not depend on the choice of a metric.

Local Poincaré Lemma: Let $g \in \mathcal{C}_1(\mathbb{R}^4)$, where $\text{supp } g$ is contained in a given open star-shaped region $\mathcal{R} \subset \mathbb{R}^4$ (e.g., a double cone). There exists $f \in \mathcal{D}_2(\mathbb{R}^4)$ with $\text{supp } f \subset \mathcal{R}$ such that $\delta f = g$.

Proof. Making use of the fact that $\star\star \upharpoonright \mathcal{D}_r(\mathbb{R}^4) = (-1)^{r+1} \upharpoonright \mathcal{D}_r(\mathbb{R}^4)$, $r = 0, \dots, 4$, one obtains $d\star f = -\star \delta f$ for $f \in \mathcal{D}_2(\mathbb{R}^4)$. It is then apparent that finding a co-primitive for g is equivalent to finding a primitive f' for $h \doteq -\star g \in \mathcal{D}_3(\mathbb{R}^4)$, i.e., $df' = h$. Note that h has the same support as g since the Hodge operator does not change supports. Since $dh = \star \delta g = 0$ it follows from the compact Poincaré lemma [18, Lem. 17.27] that there is some $f'' \in \mathcal{D}_2(\mathbb{R}^4)$ such that $df'' = h$. To modify f'' so as to obtain a co-primitive supported in \mathcal{R} , we make use of the fact that $\text{supp } h \subset \mathcal{R}$ is compact. Hence, there is an open neighborhood \mathcal{N} of $\mathbb{R}^4 \setminus \mathcal{R}$ such that $df'' = h = 0$ on \mathcal{N} . We choose $\mathcal{N} \sim \mathbb{R}^4 \setminus \mathcal{R}$ such that $\text{supp } g \subset \mathbb{R}^4 \setminus \mathcal{N}$, where \sim denotes homotopy equivalence [18, p. 614]. Since \mathcal{R} is star shaped it is homotopic to a point $o \in \mathcal{R}$ and one has $\mathcal{N} \sim \mathbb{R}^4 \setminus \mathcal{R} \sim \mathbb{R}^4 \setminus \{o\}$.

Now the corresponding de Rham cohomology groups of homotopic manifolds are isomorphic, cf. [18, Thm. 17.11], so it follows from [18, Cor. 17.23] that the second de Rham cohomology group of \mathcal{N} is trivial. Thus, every closed two-form f'' on \mathcal{N} is exact and there is some smooth one-form g'' such that $dg'' = f''$ on \mathcal{N} . Picking some smooth characteristic function χ with $\chi \upharpoonright \mathbb{R}^4 \setminus \mathcal{R} = 1$ and $\chi \upharpoonright \mathbb{R}^4 \setminus \mathcal{N} = 0$ we put $f' = f'' - d\chi g''$. Clearly, $\text{supp } f' \subset \mathcal{R}$ and $df' = df'' = h$. Thus, $f \doteq -\star f'$ is the desired co-primitive of g . □

Causal Poincaré Lemma: Let $g \in \mathcal{C}_1(\mathbb{R}^4)$ and $\mathcal{O} \subset \mathbb{R}^4$ an open double cone satisfying $\mathcal{O} \perp \text{supp } g$. There exists $f \in \mathcal{D}_2(\mathbb{R}^4)$ such that $\delta f = g$ and $\text{supp } f \perp \mathcal{O}$.

Proof. Since g has compact support there are open double cones $\mathcal{O}_2 \supset \mathcal{O}_1 \supset \overline{\mathcal{O}}$ such that $\text{supp } g \subset \mathcal{O}'_1 \cap \mathcal{O}_2$; we put $\mathcal{K} \doteq \overline{\mathcal{O}'_1 \cap \mathcal{O}_2}$, where the bar denotes closure, cf. the Figure 1. Note that the collar-shaped region \mathcal{K} is simply connected in four spacetime dimensions. Since $\text{supp } g \subset \mathcal{O}_2$ and \mathcal{O}_2 is star shaped, there exists according to the preceding lemma a co-primitive $f' \in \mathcal{D}_2(\mathbb{R}^4)$, i.e., $\delta f' = g$, which has support in \mathcal{O}_2 .

To exhibit a co-primitive f of g which has support in a neighborhood of \mathcal{K} , we have to rely on methods of algebraic topology [4]. To this end, we consider the function $f'' \doteq -\star f' \in \mathcal{D}_2(\mathbb{R}^4)$ and note that $df'' = \star \delta f' = \star g = 0$ on $\mathbb{R}^4 \setminus \mathcal{K}$. Since \mathcal{K} is compact it follows from the Alexander duality theorem [4, Cor. 8.6] that the second (real) homology group of $\mathbb{R}^4 \setminus \mathcal{K}$ is isomorphic to the first cohomology group

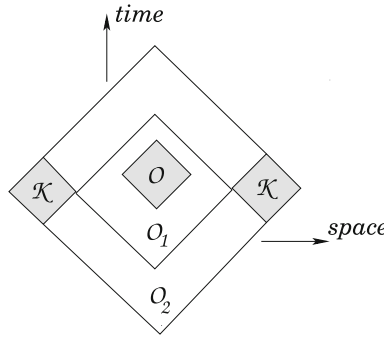


Figure 1. Collar-shaped region \mathcal{K} in the spacelike complement of \mathcal{O} .

of \mathcal{K} , $H_2(\mathbb{R}^4 \setminus \mathcal{K}) \approx H^1(\mathcal{K})$; note that these groups coincide with the corresponding reduced groups in the case at hand. Moreover, since \mathcal{K} is simply connected we have $0 = H^1(\mathcal{K}) \approx H_2(\mathbb{R}^4 \setminus \mathcal{K}) \approx H^2(\mathbb{R}^4 \setminus \mathcal{K})$, where the latter isomorphism relies on the fact that the coefficient group of interest here is \mathbb{R} , cf. [20, Thm. 6.9]. Finally, applying de Rham’s Theorem we obtain $H_{dR}^2(\mathbb{R}^4 \setminus \mathcal{K}) \approx H^2(\mathbb{R}^4 \setminus \mathcal{K}) = 0$, cf. [4, Thm. 6.9], hence every closed two-form on $\mathbb{R}^4 \setminus \mathcal{K}$ is exact.

So, we conclude that there exists a smooth one-form g'' such that $dg'' = f''$ on $\mathbb{R}^4 \setminus \mathcal{K}$. Choosing some open bounded region \mathcal{N} , $\mathcal{K} \subset \mathcal{N} \subset \mathcal{O}'$, and a smooth characteristic function χ which satisfies $\chi \upharpoonright \mathcal{K} = 1$ and $\chi \upharpoonright \mathbb{R}^4 \setminus \mathcal{N} = 0$, we define $f''' \doteq f'' - d(1 - \chi)g''$. Then, $df''' = df'' = \star g$ on \mathbb{R}^4 and $\text{supp } f''' \subset \mathcal{N}$. Hence, $f \doteq \star f'''$ is a co-primitive of g which has its support in $\mathcal{N} \subset \mathcal{O}'$, completing the proof. \square

After these preparations, we turn now to the analysis of commutators of the intrinsic vector potential, smeared with test functions having arbitrary spacelike separated supports. Let $g \in \mathcal{C}_1(\mathbb{R}^4)$ and let g_y be its translate for given $y \in \mathbb{R}^4$. As one sees by a straightforward computation, there exists a co-primitive $f_y \in \mathcal{D}_2(\mathbb{R}^4)$ of the difference $(g_y - g) \in \mathcal{C}_1(\mathbb{R}^4)$, given by

$$x \mapsto f_y^{\mu_1 \mu_2}(x) \doteq (1/2) \int_0^1 dt (y^{\mu_1} g^{\mu_2}(x - ty) - y^{\mu_2} g^{\mu_1}(x - ty)).$$

It has support in the cylindrical region $\{\text{supp } g + ty : 0 \leq t \leq 1\}$ which, for sufficiently small y , is contained in an arbitrarily small neighborhood of the support of g . Now let $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ have spacelike separated supports. Then, for sufficiently small translations $y \in \mathbb{R}^4$, there is a co-primitive $f_{2y} \in \mathcal{D}_2(\mathbb{R}^4)$ of $(g_{2y} - g)$ with $\text{supp } f_{2y} \perp \text{supp } g_1$. By a partition of unity one can decompose this co-primitive into a sum $f_{2y} = \sum_{m=1}^n f_{2y,m}$ of elements $f_{2y,m} \in \mathcal{D}_2(\mathbb{R}^4)$ which have their supports in double cones $\mathcal{O}_m \perp \text{supp } g_1$, $m = 1, \dots, n$. Thus,

$$\begin{aligned} [A(g_1), A(g_{2y} - g_2)] &= [A(g_1), A(\delta f_{2y})] \\ &= \sum_{m=1}^n [A(g_1), A(\delta f_{2y,m})] = \sum_{m=1}^n [A(\delta f_{1,m}), A(\delta f_{2y,m})], \end{aligned}$$

where in the last equality the causal Poincaré lemma has been used according to which there exist co-primitives $f_{1,m} \in \mathcal{D}_2(\mathbb{R}^4)$ of g_1 such that $\text{supp } f_{1,m} \perp \mathcal{O}_m$, $m = 1, \dots, n$. Because of locality of the electromagnetic field one has

$$[A(\delta f_{1,m}), A(\delta f_{2y,m})] = 0, \quad m = 1, \dots, n,$$

and consequently $[A(g_1), A(g_{2y})] = [A(g_1), A(g_2)]$ for small y . Applying the same argument to the translates of g_1 , one finds that $[A(g_{1y}), A(g_{2y})] = [A(g_1), A(g_2)]$ for sufficiently small y . This equality extends to arbitrary translations $y \in \mathbb{R}^4$ by iteration. It then follows from locality and the Jacobi identity that the commutator $[A(g_1), A(g_2)]$ commutes with any other operator $A(g) \in \mathfrak{P}$ if $g_1, g_2 \in \mathcal{C}_1(\mathbb{R}^4)$ and $\text{supp } g_1 \perp \text{supp } g_2$. Hence, it lies in the center of \mathfrak{P} , completing the proof.

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