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The Serpentine Representation of the Infinite Symmetric Group and the Basic Representation of the Affine Lie Algebra $\widehat{\mathfrak{sl}_2}$

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Abstract. We introduce and study the so-called serpentine representations of the infinite symmetric group $\mathfrak{S}_\mathbb{N}$, which turn out to be closely related to the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ and representations of the Virasoro algebra.

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1. Introduction: Infinite-Dimensional Schur–Weyl Duality and the Serpentine Representation of $\mathfrak{S}_\mathbb{N}$

The serpentine representation is a remarkable representation of the infinite symmetric group $\mathfrak{S}_\mathbb{N}$, which has not yet been studied. Its importance is due to the fact that it is very closely related to the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$ and representations of the Virasoro algebra. This representation belongs to the class of so-called Schur–Weyl representations. Recall that in [12] we suggested an infinite-dimensional generalization of the classical Schur–Weyl duality for the symmetric group $\mathfrak{S}_N$ and the special linear group $SL(2,\mathbb{C})$ using a "dynamical" approach. Namely, we started from the classical Schur–Weyl duality (for definiteness, assume that $N=2n$)

$$(\mathbb{C}^2)^{\otimes N}=\sum_{k=0}^{n} M_{2k+1}\otimes H_{\pi_k}, \tag{1}$$

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where H_{π_k} is the space of the irreducible representation π_k of the symmetric group $\mathfrak{S}_N$ corresponding to the two-row Young diagram $(n+k, n-k)$ and M_{2k+1} is the $(2k+1)$-dimensional irreducible $SL(2,\mathbb{C})$-module, and considered *isometric embeddings* $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$ *that are equivariant with respect to both the actions of* $SL(2,\mathbb{C})$ *and* $\mathfrak{S}_N$, which we called *Schur–Weyl embeddings.* Given an infinite chain

$$(\mathbb{C}^2)^{\otimes 0} \overset{\alpha_0}{\hookrightarrow} (\mathbb{C}^2)^{\otimes 2} \overset{\alpha_2}{\hookrightarrow} (\mathbb{C}^2)^{\otimes 4} \overset{\alpha_4}{\hookrightarrow} \dots \tag{2}$$

of Schur–Weyl embeddings, we can consider the corresponding inductive limit. The class of all representations (called *Schur–Weyl representations*) that can be obtained in this way is described in [12, Theorem 1]: Let $\Pi^{\{\alpha_N\}}$ be the representation of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ obtained as the inductive limit of the standard representations of $\mathfrak{S}_N$ in $(\mathbb{C}^2)^{\otimes N}$ with respect to an infinite chain of Schur–Weyl embeddings (2). Then it decomposes into a countable direct sum of primary representations

$$\Pi^{\{\alpha_N\}} = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k^{\{\alpha_N\}}, \tag{3}$$

where $\Pi_k^{\{\alpha_N\}}$ is the inductive limit of the irreducible representations of $\mathfrak{S}_{2k}$, $\mathfrak{S}_{2k+2}, \dots$ corresponding to the Young diagrams $(2k), (2k+1,1), (2k+2,2), \dots$.

As an important example of such a representation, in [12] we considered the unique infinite Schur–Weyl scheme that satisfies a natural additional condition, namely, preserves the tensor structure of $(\mathbb{C}^2)^{\otimes N}$. *The main goal of this paper is to study another example of Schur–Weyl duality, namely, the* **unique** *Schur–Weyl scheme that satisfies the following additional condition: it preserves the so-called stable major index of a Young tableau.* We show that this particular representation of the infinite symmetric group, which we call the serpentine representation, can be naturally equipped with the structure of the basic representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$, with the irreducible $\mathfrak{S}_N$-modules corresponding to the irreducible Virasoro modules. This reveals new interrelations between the representation theory of the infinite symmetric group and that of the affine Lie and Virasoro algebras. The precise form of the underlying natural grading-preserving isomorphism of $\mathfrak{sl}_2$-modules is still unknown in the general case, and perhaps it is not a simple task to find it, but we present several properties of this isomorphism which are corollaries of the main theorem.

Our approach uses the result of [2] that the level 1 irreducible highest weight representations of $\widehat{\mathfrak{sl}_2}$ can be realized as certain inductive limits of tensor powers $(\mathbb{C}^2)^{\otimes N}$ of the two-dimensional irreducible representation of $\mathfrak{sl}_2$. The construction of [2] is based on the notion of the fusion product of representations, whose main ingredient is, in turn, a special grading in the space $(\mathbb{C}^2)^{\otimes N}$. A key observation underlying the results of this paper, which relies on the computation presented in [7] of the q-characters of the multiplicity spaces of irreducible $\mathfrak{sl}_2$-modules with respect to this grading, is that the fusion product under consideration can be realized in an $\mathfrak{S}_N$-module so that this special grading essentially coincides with a well-known combinatorial characteristic of Young tableaux called the major index (see Proposition 1). Thus our results provide, in particular, a kind of combinatorial description of the fusion product and show that the combinatorial notion of the major index of a Young tableau has a new representation-theoretic meaning. For instance, Corollary 2 in Section 4 shows that *the so-called stable major indices of infinite Young tableaux are the eigenvalues of the Virasoro L_0 operator, the Gelfand–Tsetlin basis of the Schur–Weyl module being its eigenbasis.*

The paper is organized as follows. In Section 2, we introduce our main object, the so-called serpentine representation of the infinite symmetric group, as well as the notion of the stable major index of an infinite Young tableau, and formulate our main Theorem 1, which states that there is a grading-preserving isomorphism of $\mathfrak{sl}_2$-modules between the basic $\widehat{\mathfrak{sl}_2}$-module $L_{0,1}$ and the space H_Π of the serpentine representation. The theorem is proved in Section 3. In Section 4, we study the above isomorphism in more detail, describing some of its properties and giving examples.

For definiteness, in what follows we consider only the even case $N=2n$. The odd case can be treated in exactly the same way; instead of the basic representation $L_{0,1}$, it leads to the other level 1 *highest weight representation $L_{1,1}$ of $\widehat{\mathfrak{sl}_2}$.*

2. The Main Theorem

Let T_N be the set of all standard Young tableaux with N cells and at most two rows. Consider the following natural embedding $i_N : T_N \to T_{N+2}$: given a standard Young tableau τ with N cells, its image $i_N(\tau)$ is the standard Young tableau with $N+2$ cells obtained from τ by adding the element $N+1$ to the first row and the element $N+2$ to the second row. As shown in [12], it determines a Schur–Weyl embedding $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$, which, by abuse of notation, we denote by the same symbol i_N.

DEFINITION 1. The Schur–Weyl representation $\Pi := \Pi^{(i_N)}$ of the infinite symmetric group $\mathfrak{S}_\mathbb{N}$ in the space $H_\Pi = \lim((\mathbb{C}^2)^{\otimes N}, i_N)$ will be called the serpentine representation.

According to the theorem on Schur–Weyl representations (see the introduction), we have

$$\Pi = \sum_{k=0}^{\infty} M_{2k+1} \otimes \Pi_k, \tag{4}$$

where the irreducible component Π_k, which will be called the *k-serpentine* representation, is the representation of $\mathfrak{S}_\mathbb{N}$ associated with the infinite tableau

$\tau_k =$

1	2	$\cdots$	$2k$	$2k+1$	$2k+3$	$\cdots$
$2k+2$	$2k+4$	$\cdots$				

(in particular, τ_0 is the tableau with $1,3,5,\dots$ in the first row and $2,4,6,\dots$ in the second row), which can be realized in the space H_{Π_k} spanned by the set $\mathcal{T}_k$ of infinite two-row Young tableaux tail-equivalent to τ_k. It has a discrete spectrum with respect to the Gelfand–Tsetlin algebra.

In what follows, the tableaux τ_k, $k=0,1,\dots$, will be called *principal*, and a tableau tail-equivalent to τ_k for some k will be called a *serpentine tableau*; denote by $\mathcal{T}=\cup\mathcal{T}_k$ the set of all serpentine tableaux.

Now consider the well-known statistic on Young tableaux called the *major index*. It is defined as follows (see [11, Sec. 7.19]):

$$\mathrm{maj}(\tau)=\sum_{i\in\mathrm{des}(\tau)} i,$$

where, for $\tau\in T_N$,

$$\mathrm{des}(\tau)=\{i\le N-1:\ \text{the element } i+1 \text{ in } \tau \text{ lies lower than } i\}$$

is the descent set of τ.

Obviously,

$$\mathrm{maj}(i_N(\tau))=\mathrm{maj}(\tau)+(N+1). \tag{5}$$

This suggests the following important step. Given $N=2n$ and $\tau\in T_N$, denote $r_N(\tau)=n^2-\mathrm{maj}(\tau)$. Then $r_{N+2}(i_N(\tau))=r_N(\tau)$, so that we have a well-defined index on all serpentine tableaux $\tau\in\mathcal{T}$:

$$r(\tau)=\lim_{n\to\infty} r_{2n}([\tau]_{2n})=\lim_{n\to\infty}(n^2-\mathrm{maj}([\tau]_{2n})), \tag{6}$$

where $[\tau]_l$ is the tableau with l cells obtained from τ by removing all the cells with entries $k>l$. Obviously, for the principal tableaux, we have $r(\tau_k)=k^2$.

DEFINITION 2. We call $r(\tau)$ the stable major index of an infinite tableau $\tau\in\mathcal{T}$.

The stable major index determines a grading on all the spaces H_{Π_k} and hence on the whole space H_Π: for $w=u\otimes v\in M_{2k+1}\otimes H_{\Pi_k}$ we just set $\deg_r(w)=r(v)$.

Now consider the affine Lie algebra $\widehat{\mathfrak{sl}_2}=\mathfrak{sl}_2\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, its basic module $L_{0,1}$ with the homogeneous grading $\deg_H$, and the natural embedding $\mathfrak{sl}_2\subset\widehat{\mathfrak{sl}_2}$ given by $\mathfrak{sl}_2\ni x\mapsto x\otimes 1\in\widehat{\mathfrak{sl}_2}$. Our main theorem is the following.

THEOREM 1. *There is a grading-preserving unitary isomorphism of* $\mathfrak{sl}_2$*-modules between* $(L_{0,1},\deg_H)$ *and* $(H_\Pi,\deg_r)$. *The serpentine representation is the unique Schur–Weyl representation satisfying this condition.*

Remarks. 1. As mentioned in the introduction, we consider only the even case just for simplicity of notation. Considering instead of (2) the chain $(\mathbb{C}^2)^{\otimes 1} \hookrightarrow (\mathbb{C}^2)^{\otimes 3} \hookrightarrow (\mathbb{C}^2)^{\otimes 5} \hookrightarrow \dots$ and reproducing exactly the same arguments, we will obtain a grading-preserving isomorphism of the corresponding Schur–Weyl module with the other level 1 highest weight module $L_{1,1}$ of $\widehat{\mathfrak{sl}_2}$.

2. The conditions from the statement of Theorem 1 do not uniquely determine the isomorphism, since there is a nontrivial group of transformations in H_Π that commute with $\mathfrak{sl}_2$ and preserve the grading. For more details, see the remark after Corollary 2 in Section 4. To find an explicit form of this isomorphism is an intriguing problem.

3. Proof of the Main Theorem

1. Fusion product. Our proof relies on the result of Feigin and Feigin [2] on a finite-dimensional approximation of the basic representation of $\widehat{\mathfrak{sl}_2}$, which, in turn, uses the notion of the fusion product of representations introduced in [4]. Since the corresponding construction is of importance for us, we describe it in some detail.

Given a representation ρ of $\mathfrak{sl}_2$ and $z \in \mathbb{C}$, let $\rho(z)$ be the evaluation representation of the polynomial current algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, defined as $(x \otimes t^i)v = z^i \cdot xv$. Now, given a collection $\rho_1, \dots, \rho_N$ of irreducible representations of $\mathfrak{sl}_2$ with lowest weight vectors $v_1, \dots, v_N$, and a collection $z_1, \dots, z_N$ of pairwise distinct complex numbers, we consider the tensor product of the corresponding evaluation representations: $\rho_1(z_1) \otimes \dots \otimes \rho_N(z_N)$. The crucial step is introducing a special grading in the space V_N of this representation. Set $V_N^{(m)} = U^{(m)}(e \otimes \mathbb{C}[t])(v_1 \otimes \dots \otimes v_N) \subset V_N$, where e is the raising operator in $\mathfrak{sl}_2$ and $U^{(m)}$ is spanned by homogeneous elements of degree m in t. In other words, $U^{(m)}$ is spanned by the monomials of the form $e_{i_1} \dots e_{i_k}$ with $i_1 + \dots + i_k = m$, where $e_j = e \otimes t^j$. Then we consider the corresponding filtration on V_N: $V_N^{(\le m)} = \sum_{k \le m} V_N^{(k)}$. The fusion product of $\rho_1, \dots, \rho_N$ is the graded representation with respect to the above filtration, which is realized in the space

$$V_N^* = \operatorname{gr} V_N = V_N^{(\le 0)} \oplus V_N^{(\le 1)}/V_N^{(\le 0)} \oplus V_N^{(\le 2)}/V_N^{(\le 1)} \oplus \dots. \tag{7}$$

The space $V_N^*[k] = V_N^{(\le k)}/V_N^{(\le k-1)}$ is the subspace of elements of degree k, and elements of the form $x \otimes t^l \in \mathfrak{sl}_2 \otimes \mathbb{C}[t]$ send $V_N^*[k]$ to $V_N^*[k+l]$. The degree of an element with respect to this grading will be denoted by $\widetilde{\deg}$. It is proved in [4] that V_N^* is an $\mathfrak{sl}_2 \otimes (\mathbb{C}[t]/t^N)$-module that does not depend on $z_1, \dots, z_N$ provided that they are pairwise distinct. Moreover, V_N^* is isomorphic to $\rho_1 \otimes \dots \otimes \rho_N$ as an $\mathfrak{sl}_2$-module.

We apply this construction to the case where $\rho_1 = \dots = \rho_N$ is the two-dimensional irreducible representation of $\mathfrak{sl}_2$ with the lowest weight vector v_0. In this case, $V_N^* \simeq (\mathbb{C}^2)^{\otimes N}$ as an $\mathfrak{sl}_2$-module. We equip V_N^* with the inner product such that the corresponding representation of $\mathfrak{sl}_2$ is unitary. It is proved in [2] that an inductive limit of V_N^* is isomorphic to the basic representation $L_{0,1}$ of $\widehat{\mathfrak{sl}_2}$, so that we first establish a grading-preserving isomorphism of the finite-dimensional $\mathfrak{sl}_2$-modules V_N^* and $\sum_{k=0}^n M_{2k+1}\otimes H_{\pi_k}$ and then show that it can be extended to the inductive limits of the corresponding spaces.

2. The q-character, major index, and the finite-dimensional result. Consider the decomposition of V_N^* into irreducible $\mathfrak{sl}_2$-modules:

$$V_N^* = \bigoplus_{k=0}^{n} M_{2k+1}\otimes \mathcal{M}_k.$$

By the classical Schur–Weyl duality (1), we know that the multiplicity space $\mathcal{M}_k$ coincides with the space H_{π_k} of the irreducible representation of $\mathfrak{S}_N$ with the Young diagram $(n+k, n-k)$. On the other hand, it inherits the grading from V_N^*:

$$\mathcal{M}_k = \bigoplus_{i\ge 0}\mathcal{M}_k[i], \tag{8}$$

where $\mathcal{M}_k[i]=\mathcal{M}_k\cap V^*[i]$. Consider the corresponding q-character

$$\operatorname{ch}_q \mathcal{M}_k = \sum_{i\ge 0} q^i \dim \mathcal{M}_k[i].$$

It was proved by Kedem [7] that

$$\operatorname{ch}_q \mathcal{M}_k = q^{\frac{N(N-1)}{2}}\cdot K_{(n+k,n-k),1^N}(1/q), \tag{9}$$

where $K_{\lambda,\mu}$ is the Kostka–Foulkes polynomial (see [9, Sec. III.6]).

Now we use the well-known combinatorial description of the Kostka–Foulkes polynomial due to Lascoux and Schützenberger [8]. For a two-row diagram λ, their formula reduces to

$$K_{\lambda,1^N}(q) = \sum_{\tau\in[\lambda]} q^{c(\tau)}, \tag{10}$$

where $[\lambda]$ is the set of standard Young tableaux of shape λ and $c(\tau)$ is the charge of a tableau $\tau\in T_N$, defined as the sum of $i\le N-1$ such that in τ the element $i+1$ lies to the right of i (see [9]). But, obviously, for $\tau\in T_N$ we have $\operatorname{maj}(\tau)=\frac{N(N-1)}{2}-c(\tau)$. Then it follows from (9) and (10) that

$$\dim \mathcal{M}_k[i] = \#\{\tau\in[(n+k,n-k)] : \operatorname{maj}(\tau)=i\}. \tag{11}$$

The major index defines a grading in the space H_{π_k} (spanned by the standard Young tableaux of shape $(n+k,n-k)$), and hence in the whole space $\mathcal{X}_N=\sum_{k=0}^n M_{2k+1}\otimes H_{\pi_k}$, which we equip with the standard inner product. We obtain the following finite-dimensional analog of Theorem 1.

PROPOSITION 1. *There is a grading-preserving unitary isomorphism of* $\mathfrak{sl}_2$*-modules between* $(V_N^*, \widetilde{\deg})$ *and* $(\mathcal{X}_N, \mathrm{maj})$ *such that the multiplicity space* $\mathcal{M}_k$ *is spanned by the standard Young tableaux* τ *of shape* $(n+k, n-k)$ *(and hence* $\mathcal{M}_k[i]$ *is spanned by* τ *with* $\mathrm{maj}(\tau)=i$*).*

Proof. Follows from the fact that the fusion product V_N^* is isomorphic to $(\mathbb{C}^2)^{\otimes N}$ as an $\mathfrak{sl}_2$-module and Equation (11). □

Remarks. 1. Observe that the isomorphism from Proposition 1 is not unique.

2. The isomorphism from Proposition 1 determines an action of the symmetric group $\mathfrak{S}_N$ on the space V_N^*. It does not coincide with the original action of $\mathfrak{S}_N$ on $(\mathbb{C}^2)^{\otimes N}$.

3. Embeddings and the limit. It is proved in [2] that there is an embedding $j_N : V_N^* \to V_{N+2}^*$ equivariant with respect to the action of $\mathfrak{sl}_2 \otimes (\mathbb{C}[t^{-1}]/t^{-n})$, and the corresponding inductive limit $\mathcal{V} = \lim(V_N^*, j_N)$ is isomorphic to the basic representation $L_{0,1}$ of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. This embedding satisfies

$$\widetilde{\deg}(j_N x) = \widetilde{\deg}(x) - (N+1). \tag{12}$$

Since we are now considering $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]$ instead of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$, we should slightly modify the previous constructions to take the minus sign into account. Namely, instead of (8) we now have $\mathcal{M}_k = \oplus_{i \ge 0} \mathcal{M}_k[-i]$, and the isomorphism of Proposition 1 identifies $\mathcal{M}_k[-i]$ with the space spanned by the tableaux τ of shape $(n+k, n-k)$ such that $\mathrm{maj}(\tau) = i$. Denote this isomorphism between V_N^* and $\mathcal{X}_N$ by ρ_N. Observe that the only conditions we impose on ρ_N are as follows: (a) ρ_N is a unitary isomorphism of $\mathfrak{sl}_2$-modules and (b) $\rho_N \circ \widetilde{\deg} = -\,\mathrm{maj}$.

Now, since $L_{0,1} \simeq \lim(V_N^*, j_N)$, $H_\Pi = \lim(\mathcal{X}_N, i_N)$, and Proposition 1 holds, to prove Theorem 1 it suffices to show that we can choose a sequence of isomorphisms ρ_N such that the diagram

$$\begin{array}{ccc} V_N^* & \xrightarrow{\rho_N} & \mathcal{X}_N \\ \downarrow{\scriptstyle j_N} & & \downarrow{\scriptstyle i_N} \\ V_{N+2}^* & \xrightarrow{\rho_{N+2}} & \mathcal{X}_{N+2} \end{array}$$

is commutative for all N. We use induction on N. The base being obvious, assume that we have already constructed ρ_N, and let us construct ρ_{N+2}.

We have $V_{N+2}^* = j_N(V_N^*) \oplus (j_N(V_N^*))^\perp$. On the first subspace, we set $\rho_{N+2}(x) := i_N(\rho_N(j_{N+2}^{-1}(x)))$. On the second one, we define it in an arbitrary way to satisfy the desired conditions (a) and (b). The fact that this definition is correct and provides us with a desired isomorphism between V_{N+2}^* and $\mathcal{X}_{N+2}$ follows from (12) and (5). The theorem is proved.

4. The Key Isomorphism in More Detail

Our aim in this section is to study the isomorphism from Theorem 1 in more detail. For this, we first give necessary background on the Fock space realization of the basic $\widehat{\mathfrak{sl}_2}$-module.

4.1. THE FOCK SPACE REALIZATION OF THE BASIC $\widehat{\mathfrak{sl}_2}$-MODULE

Let $\mathcal{F}$ be the fermionic Fock space constructed as the infinite wedge space over the linear space with basis $\{u_k\}_{k\in\mathbb{Z}}\cup\{v_k\}_{k\in\mathbb{Z}}$. That is, $\mathcal{F}$ is spanned by the semi-infinite forms $u_{i_1}\wedge\cdots\wedge u_{i_k}\wedge v_{j_1}\wedge\cdots\wedge v_{j_l}\wedge u_N\wedge v_N\wedge u_{N-1}\wedge v_{N-1}\wedge\cdots$, $N\in\mathbb{Z}$, $i_1>\cdots>i_k>N$, $j_1>\cdots>j_l>N$, and is equipped with the inner product in which such monomials are orthonormal. Let ϕ_k be the exterior multiplication by u_k and ψ_k be the exterior multiplication by v_k, and denote by ϕ_k^*, ψ_k^* the corresponding adjoint operators. Then this family of operators satisfies the canonical anticommutation relations (CAR). We consider the generating functions $\phi(z)=\sum_{i\in\mathbb{Z}}\phi_i z^{-(i+1)}$, $\phi^*(z)=\sum_{i\in\mathbb{Z}}\phi_i^* z^i$, and the same for ψ and ψ^*.

Let a_n^ϕ and a_n^ψ be the systems of bosons constructed from the fermions $\{\phi_k\}$ and $\{\psi_k\}$, respectively: $a_n^\phi=\sum_{k\in\mathbb{Z}}\phi_k\phi_{k+n}^*$ for $n\neq 0$ and $a_0^\phi=\sum_{n=1}^\infty\phi_n\phi_n^*-\sum_{n=0}^\infty\phi_{-n}^*\phi_{-n}$, and similarly for a^ψ. They satisfy the canonical commutation relations (CCR), i.e., form a representation of the Heisenberg algebra $\mathfrak{A}$. Denote $a^\phi(z)=\sum_{n\in\mathbb{Z}}a_n^\phi z^{-(n+1)}$, and similarly for a^ψ.

Let V be the operator in $\mathcal{F}$ that shifts the indices by 1:

$$V(w_{i_1}\wedge w_{i_2}\wedge\ldots)=V_0(w_{i_1})\wedge V_0(w_{i_2})\wedge\ldots,\qquad V_0(u_i)=u_{i+1},\qquad V_0(v_i)=v_{i-1}.$$

The vacuum vector in $\mathcal{F}$ is $\Omega=u_{-1}\wedge v_{-1}\wedge u_{-2}\wedge v_{-2}\wedge\ldots$. We also consider the family of vectors

$$\Omega_0=\Omega,\quad \Omega_{2n}=V^{-n}\Omega_0,\quad n\in\mathbb{Z}.$$

In the space $\mathcal{F}$, we have a canonical representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}=\mathfrak{sl}_2\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, which is given by the following formulas. Given $x\in\mathfrak{sl}_2$, denote $X(z)=\sum_{n\in\mathbb{Z}}x_n z^{-(n+1)}$. Then

$$E(z)=\psi(z)\phi^*(z),\qquad F(z)=\phi(z)\psi^*(z),$$
$$h_n=a_{-n}^\psi-a_{-n}^\phi,\qquad d=\frac{h_0^2}{2}+\sum_{n=1}^\infty h_{-n}h_n,\qquad c=1.$$

We have $\mathcal{F}=\mathcal{H}_0\otimes\mathcal{K}_0+\mathcal{H}_1\otimes\mathcal{K}_1$, where $\mathcal{H}_0\simeq L_{0,1}$ and $\mathcal{H}_1\simeq L_{1,1}$ are the irreducible level 1 highest weight $\widehat{\mathfrak{sl}_2}$-modules and $\mathcal{K}_0$ and $\mathcal{K}_1$ are the multiplicity spaces. Observe also that $e_{-(N+1)}\Omega_{-N}=\Omega_{-(N+2)}$.

Note that the operators $a_n=\frac{1}{\sqrt{2}}h_n$ satisfy the CCR, i.e., form a system of free bosons, or generate the Heisenberg algebra $\mathfrak{A}_h$. The vectors $\{\Omega_{2n}\}_{n\in\mathbb{Z}}$ introduced above are exactly singular vectors for this Heisenberg algebra: $h_k\Omega_m=0$ for $k>0$,

$h_0\Omega_m = m\Omega_m$. The representation of $\mathfrak{A}_h$ in $\mathcal{H}_0$ breaks into a direct sum of irreducible representations:

$$\mathcal{H}_0 = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}_0[2k], \tag{13}$$

where $\mathcal{H}_0[2k]$ is the charge $2k$ subspace, i.e., the eigenspace of h_0 with eigenvalue $2k$:

$$\mathcal{H}_0[2k] = \{v\in\mathcal{H}_0 : h_0 v = 2kv\} = \mathbb{C}[h_0, h_1, \ldots]\Omega_{2k}.$$

Now, given a representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$, we can use the Sugawara construction to obtain the corresponding representation of the Virasoro algebra Vir. It can also be described in the following way. As noted above, the operators $a_n = \frac{1}{\sqrt{2}}h_n$ form a system of free bosons. Given such a system, a representation of Vir can be constructed as follows ([13]; see also [6, Ex. 9.17]):

$$L_n = \frac{1}{2}\sum_{j\in\mathbb{Z}} a_{-j}a_{j+n}, \quad n\neq 0; \qquad L_0 = \sum_{j=1}^{\infty} a_{-j}a_j. \tag{14}$$

Thus we obtain a representation of Vir in $\mathcal{F}$ and, in particular, in $\mathcal{H}_0$. In this representation, the algebras generated by the operators of Vir and $\mathfrak{sl}_2 \subset \widehat{\mathfrak{sl}_2}$ are mutual commutants, and we have the decomposition

$$\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} M_{2k+1} \otimes L(1, k^2), \tag{15}$$

where M_{2k+1} is the $(2k+1)$-dimensional irreducible $\mathfrak{sl}_2$-module and $L(1,k^2)$ is the irreducible Virasoro module with central charge 1 and conformal dimension k^2.

The charge k subspace $\mathcal{H}_0[k]$ contains a series of singular vectors $\xi_{k,m}$ of Vir with energy $(k+m)^2$:

$$L_n\xi_{k,m} = 0 \quad \text{for } n = 1, 2, \ldots, \qquad L_0\xi_{k,m} = (k+m)^2.$$

Let us use the so-called homogeneous vertex operator construction of the basic representation of $\widehat{\mathfrak{sl}_2}$ ([5], see also [6, Sec. 14.8]). In this realization,

$$E(z) = \Gamma_-(z)\Gamma_+(z)z^{-h_0}V^{-1}, \qquad F(z) = \Gamma_+(z)\Gamma_-(z)z^{h_0}V, \tag{16}$$

where

$$\Gamma_\pm(z) = \exp\left(\mp\sum_{j=1}^{\infty}\frac{z^{\pm j}}{j}h_{\pm j}\right)$$

and the operators $\Gamma_\pm(z)$ satisfy the commutation relation

$$\Gamma_+(z)\Gamma_-(w) = \Gamma_-(w)\Gamma_+(z)\left(1-\frac{z}{w}\right)^2. \tag{17}$$

Using the boson–fermion correspondence (see [6, Ch. 14]), we can identify $\mathcal{H}_0$ with the space $\Lambda\otimes\mathbb{C}[q,q^{-1}]$, where Λ is the algebra of symmetric functions (see [9]). In particular, consider the charge 0 subspace $\mathcal{H}[0]=\mathcal{H}_0[0]$, which is identified with Λ. We can use the following representation of the Heisenberg algebra generated by $\{h_n\}_{n\in\mathbb{Z}}$:

$$h_n \leftrightarrow 2n\frac{\partial}{\partial p_n}, \qquad h_{-n}=p_n, \qquad n>0, \tag{18}$$

where p_j are Newton's power sums. Note that the representation (18) of $\mathfrak{A}_h$, and hence the corresponding representation (14) of Vir, are not unitary with respect to the standard inner product in Λ. To make it unitary, we should consider the inner product in Λ defined by

$$\langle p_\lambda, p_\mu\rangle=\delta_{\lambda\mu}\cdot z_\lambda\cdot 2^{l(\lambda)}, \tag{19}$$

where p_λ are the power sum symmetric functions, $z_\lambda=\prod_i i^{m_i}m_i!$ for a Young diagram λ with m_i parts of length i, and $l(\lambda)$ is the number of nonzero rows in λ.

Denote the singular vectors of Vir in $\mathcal{H}[0]$ by $\xi_m:=\xi_{0,m}$. According to a result by Segal [10], in the symmetric function realization (18),

$$\xi_n \leftrightarrow c\cdot s_{(n^n)}, \tag{20}$$

where $s_{(n^n)}$ is the Schur function indexed by the $n\times n$ square Young diagram and c is a numerical coefficient.

4.2. FURTHER ANALYSIS OF THE KEY ISOMORPHISM

Comparing the structure (4) of the serpentine representation with (15), we obtain the following result.

COROLLARY 1. *The space H_{Π_k} of the k-serpentine representation of the infinite symmetric group has a natural structure of the Virasoro module $L(1,k^2)$.*

Our aim is to study this Virasoro representation in Π_k (or, which is equivalent, the corresponding representation of the infinite symmetric group in the Fock space). In particular, from the known theory of the basic module $L_{0,1}$, we immediately obtain the following result.

COROLLARY 2. *In the above realization of the Virasoro module $L(1,k^2)$, the Gelfand–Tsetlin basis of H_{Π_k} (which consists of the infinite two-row Young tableaux tail-equivalent to τ_k) is the eigenbasis of L_0, and the eigenvalues are given by the stable major index r:*

$$L_0\tau=r(\tau)\tau.$$

Now we see that the isomorphism in Theorem 1 is in fact defined up to the commutant of L_0 in each Π_k.

Let ω_{-2k} be the lowest vector in M_{2k+1}. Then a natural basis of $\mathcal{V}$ is $\{e_0^m\omega_{-2k}\otimes\tau : m=0,1,\ldots,2k,\ \tau\in\mathcal{T}_k\}$. Denoting $\mathcal{V}_k=M_{2k+1}\otimes H_{\Pi_k}$ and $\mathcal{V}_k[0]=\{v\in\mathcal{V}_k : h_0v=0\}$, we have $\mathcal{V}_k[0]=e_0^k\omega_{-2k}\otimes H_{\Pi_k}$, so that we may identify $\mathcal{V}_k[0]$ with H_{Π_k} via the correspondence

$$c(t)\cdot e_0^k\omega_{-2k}\otimes t\leftrightarrow t,$$

where $c(t)$ is a normalizing constant. On the other hand, it is shown in [2] that $V_{2n}^*\simeq\mathbb{C}[e_0,\ldots,e_{-(2n-1)}]\Omega_{-2n}\subset\mathcal{F}$ as an $\mathfrak{sl}_2\otimes(\mathbb{C}[t^{-1}]/t^{-2n})$-module, and the limit space $\mathcal{V}$ coincides with $\mathcal{H}_0$. Under this correspondence, the charge 0 subspace $\mathcal{H}[0]$ is identified with $\mathcal{V}[0]=\{v\in\mathcal{V} : h_0v=0\}$. Thus we have

$$\mathcal{H}[0]\simeq H_\Pi[0]=\bigoplus_{k=0}^{\infty}H_{\Pi_k}, \tag{21}$$

where $H_\Pi[0]$ is the space spanned by all serpentine tableaux, which is the space of the countable sum of the k-serpentine representations Π_k of $\mathfrak{S}_\mathbb{N}$ without multiplicities, and the following corollary holds.

COROLLARY 3. *The space $H_\Pi[0]$ has a structure of an irreducible representation of the Heisenberg algebra $\mathfrak{A}$.*

Now, using results of [2], one can easily prove the following lemma.

LEMMA 1. *A basis in $F_{2n}=\mathbb{C}[e_0,\ldots,e_{-(2n-1)}]\Omega_{-2n}$ is*

$$\{e_0^{i_0}e_{-1}^{i_1}\ldots e_{-(2n-1)}^{i_{2n-1}} : 0\le k\le 2n-(i_0+\cdots+i_{2n-1})\}\Omega_{-2n}.$$

In particular, a basis of $F_{2n}[0]=F_{2n}\cap\mathcal{H}[0]$ is

$$\left\{\prod e_0^{i_0}e_{-1}^{i_1}\ldots e_{-n}^{i_n} : i_0+i_1+\cdots+i_n=n\right\}\Omega_{-2n}. \tag{22}$$

On the other hand, as mentioned above, $\mathcal{H}[0]$ can be identified with the algebra of symmetric functions Λ via (18). Denote by Φ the obtained isomorphism between $H_\Pi[0]$ and Λ, which thus associates with every serpentine tableau $\tau\in\mathcal{T}$ a symmetric function $\Phi(\tau)\in\Lambda$ such that $r(\tau)=\deg\Phi(\tau)$.

PROPOSITION 2. *Under the isomorphism Φ, the principal tableaux correspond to the Schur functions with square Young diagrams:*

$$\Phi(\tau_k)=\mathrm{const}\cdot s_{(k^k)}.$$

Proof. Follows from Segal's [10] result (20), since it is not difficult to see that the singular vector of Vir in $\mathcal{V}_k[0]$ is just $e_0^k\omega_{-2k}\otimes\tau_k$. □

Denote by $T^{(N)}$ the (finite) set of infinite two-row tableaux that coincide with some τ_n, $n=0,1,\ldots$, from the Nth level.

PROPOSITION 3. *Let* $H_\Pi^{(N)}$ *be the subspace in* $H_\Pi[0]$ *spanned by all* $\tau\in T^{(N)}$. *Then*

$$\Phi(H_\Pi^{(2k)})=\Lambda_{k\times k},$$

where $\Lambda_{k\times k}$ *is the subspace in* Λ *spanned by the Schur functions indexed by Young diagrams lying in the* $k\times k$ *square.*

Proof. From all the above identifications, $H_\Pi^{(2k)}\leftrightarrow F_{2k}[0]$. Now the claim follows from the result proved in [3] that in the symmetric function realization $F_{2k}[0]$ corresponds to $\Lambda_{k\times k}$. □

In the next theorem, we refine this result, giving an explicit formula for the Schur basis in $\Lambda_{k\times k}$ in terms of the basis (22) in $H_\Pi^{(2k)}\simeq F_{2k}[0]$. In fact, we would like to have an explicit formula for Φ or Φ^{-1}, expressing, say, a Schur function in terms of serpentine tableaux. At the moment we cannot provide such a general formula, but the theorem below is a step toward solving this problem, reducing it to describing the action of the operators e_{-m} in the space of serpentine tableaux. Besides, Propositions 2 and 3 can easily be derived from formula (23), the former by taking $\nu=(k^k)$ and the latter by counting the dimensions.

THEOREM 2. *In the symmetric function realization, the correspondence between the Schur function basis in* $\Lambda_{k\times k}$ *and the basis* (22) *in* $H_\Pi^{(2k)}\simeq F_{2k}[0]$ *is given by*

$$s_\nu=\sum_{\mu=(0^{r_0}1^{r_1}2^{r_2}\ldots)\subset(k^k)}\frac{K_{\nu\mu}}{\prod_{j=0}^k r_j!}e_{-(k-\mu_1)}\ldots e_{-(k-\mu_k)}\Omega_{-2k},\qquad \nu\subset(k^k),\tag{23}$$

where $K_{\lambda\mu}$ *are Kostka numbers.*

Proof. We generalize Wasserman's [14] proof of Segal's result (20) (a similar computation is also given in an earlier paper [1]).

Let $0\le i_1,\ldots,i_k\le k$. Then, obviously,

$$e_{-i_1}\ldots e_{-i_k}\Omega_{-2k}=\left[\prod_{j=1}^k z_j^{i_j-1}\right]E(z_k)\ldots E(z_1)\Omega_{-2k},$$

where by [monomial]$F(z_1,\ldots,z_m)$ we denote the coefficient of this monomial in $F(z_1,\ldots,z_m)$ (in particular, $[1]F(z_1,\ldots,z_m)$ is the constant term of F). Now, using the representation (16), the commutation relation (17), and the obvious facts that

$V^{-k}\Omega_{-2k}=\Omega_0$ and $\Gamma_+(z)\Omega_0=\Omega_0$, we obtain

$$E(z_k)\dots E(z_1)\Omega_{-2k}=\prod_{j=1}^{k} z_j^{2(k-j)} \prod_{1\le j<i\le k}\left(1-\frac{z_i}{z_j}\right)^2 \Gamma_-(z_k)\dots\Gamma_-(z_1)\Omega_0.$$

Observe that, in view of (18) and the well-known fact from the theory of symmetric functions, $\Gamma_-(z)$ is exactly the generating function of the complete symmetric functions. Hence, expanding the product $\Gamma_-(z_k)\dots\Gamma_-(z_1)\Omega_0$ by the Cauchy identity ([9, I.4.3]) and making simple transformations, we obtain

$$E(z_k)\dots E(z_1)\Omega_{-2k}=(-1)^{k(k-1)/2}\prod_{j=1}^{k} z_j^{k-1} a_\delta(z)a_\delta(z^{-1})\sum_{\lambda:l(\lambda)\le k} s_\lambda(z^{-1})s_\lambda,$$

where

$$a_\delta(z)=\prod_{1\le i<j\le k}(z_i-z_j)=\det[z_i^{k-j}]_{1\le i,j,\le k}$$

is the Vandermonde determinant, $a_\delta(z^{-1})$ is the similar determinant for the variables $z^{-1}=(z_1^{-1},\dots,z_k^{-1})$, $l(\lambda)$ is the length of the diagram λ (the number of nonzero rows), $s_\lambda(z^{-1})$ is the Schur function calculated at the variables z^{-1}, and s_λ is the Schur function as an element of Λ identified with $\mathcal{H}[0]$. Thus we have

$$e_{-i_1}\dots e_{-i_k}\Omega_{-2k}=(-1)^{k(k-1)/2}\cdot[1]\left(\prod_{j=1}^{k} z_j^{k-i_j} a_\delta(z)a_\delta(z^{-1})\sum_\lambda s_\lambda(z^{-1})s_\lambda\right).$$

For convenience, set $\widetilde{e}_p:=e_{-(k-p)}$, $0\le p\le k$. Given $0\le\alpha_1,\dots,\alpha_k\le k$, we have

$$\widetilde{e}_{\alpha_1}\dots\widetilde{e}_{\alpha_k}\Omega_{-2k}=[1]\left(\prod_{j=1}^{k} z_j^{\alpha_j} a_\delta(z)\sum_{l(\lambda)\le k} a_{\lambda+\delta}(z^{-1})s_\lambda\right), \tag{24}$$

where $a_{\lambda+\delta}(x)=\det[x_i^{\lambda_j+k-j}]_{1\le i,j\le k}=s_\lambda(x)a_\delta(x)$. Consider a Young diagram $\mu=(\mu_1,\dots,\mu_k)=(0^{r_0}1^{r_1}2^{r_2}\dots)$. Let us sum (24) over all different permutations $\alpha=(\alpha_1,\dots,\alpha_k)$ of the sequence $(\mu_1,\dots,\mu_k)$. Note that the operators e_j commute with each other, so that the left-hand side does not depend on the order of the factors. In the right-hand side, $\sum_\alpha\prod z_j^{\alpha_j}=m_\mu(z)$, a monomial symmetric function. Thus we have

$$\frac{k!}{\prod_{j=0}^{k} r_j!}\widetilde{e}_{\mu_1}\dots\widetilde{e}_{\mu_k}=[1]\left(m_\mu(z)a_\delta(z)\sum_{l(\lambda)\le k} a_{\lambda+\delta}(z^{-1})s_\lambda\right). \tag{25}$$

Let ν be a Young diagram with at most k rows and at most k columns, i.e., $\nu\subset(k^k)$. We have

$$s_\nu(z)=\sum_\mu K_{\nu\mu}m_\mu(z), \tag{26}$$

where $K_{\nu\mu}$ are Kostka numbers. It is well known that $K_{\nu\mu}=0$ unless $\mu\le\nu$, where $\le$ is the standard ordering on partitions: $\mu\le\nu \iff \mu_1+\dots+\mu_i\le\nu_1+\dots+\nu_i$ for every $i\ge1$. In particular, $\mu_1\le\nu_1\le k$. Besides, since we consider only k nonzero variables $z_1,\dots,z_k$, it also follows that $m_\mu(z)=0$ unless $l(\mu)\le k$. Thus the sum in (26) can be taken only over diagrams $\mu\subset(k^k)$, for which Equation (25) holds. Multiplying this equation by $K_{\nu\mu}$ and summing over μ yields

$$\sum_{\mu=(0^{r_0}1^{r_1}2^{r_2}\dots)\subset(k^k)}\frac{k!}{\prod_{j=0}^k r_j!}K_{\nu\mu}\widetilde{e}_{\mu_1}\dots\widetilde{e}_{\mu_k}=[1]\left(s_\nu(z)a_\delta(z)\sum_{l(\lambda)\le k}a_{\lambda+\delta}(z^{-1})s_\lambda\right).$$

By the orthogonality relations, the right-hand side is equal to $k!s_\nu$, and the desired formula (23) follows. □

4.3. EXAMPLES

In this section, we present the results of computing $\Phi(\tau)$ for the serpentine tableaux with $r(\tau)\le4$ (note that although the conditions of Theorem 1 do not determine the isomorphism uniquely, these relations hold for *any* isomorphism satisfying them) in terms of Newton's power sums p_k. We write down only the "nontrivial" part of a tableau, meaning that it should be continued up to an infinite tableau in the "serpentine" way. We also omit the normalizing coefficients of $\Phi(\tau)$, which are their norms in the inner product (19).

$r(\tau)$	τ	$\Phi(\tau)$ up to a constant	$r(\tau)$	τ	$\Phi(\tau)$ up to a constant
0	τ_0	$1=s_\emptyset$	4	$\tau_2=$ 1 2 3 4	$p_1^4+3p_2^2-4p_1p_3=s_{(2^2)}$
1	$\tau_1=$ 1 2	$p_1=s_{(1)}$	4	1 3 4 2 5 6	$p_1^4-3p_2^2+2p_1p_3$
2	1 2 4 3	p_2	4	1 2 4 6 3 5 7 8	$p_1^4+12p_2^2+32p_1p_3$
2	1 2 3 4	p_1^2	4	1 3 4 6 2 5	$p_1^2p_2-p_4$
3	1 2 3 4	$p_1^3-p_3$	4	1 2 4 6 8 3 5 7	$p_1^2p_2+4p_4$
3	1 2 4 6 3 5	$p_1^3+8p_3$			
3	1 2 4 3 5 6	p_1p_2			

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