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Superopers on Supercurves

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Abstract. In this note, we introduce the generalization of opers (superopers) for a certain class of superalgebras with a root system, which admits a basis of odd roots. We study in detail SPL_2-superopers and in particular derive the corresponding Bethe ansatz equations, which describe the spectrum of the $\mathfrak{osp}(1|2)$ Gaudin model.

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1. Introduction

Opers are necessary ingredients in the study of the geometric Langlands correspondence (see e.g. [13]). They also play important role in many aspects of mathematical physics. For example, opers are very important in the theory of integrable systems, and recently, became a necessary component even in the modern quantum field theory approaches to the knot theory (see e.g. [28]).

Originally, opers were studied locally in the seminal paper of Drinfeld and Sokolov [9] as gauge equivalence classes of certain differential operators with values in some simple Lie algebra, which are the L-operators of the generalized Korteweg-de Vries (KdV) integrable models. Later, Beilinson and Drinfeld generalized this local object making it coordinate independent [2]. Namely, a G-oper on a smooth curve Σ, where G is a simple algebraic group of the adjoint type with the Lie algebra $\mathfrak{g}$, is a triple $(\mathcal{F}, \mathcal{F}_B, \nabla)$, where $\mathcal{F}$ is a G-bundle over Σ, $\mathcal{F}_B$ is its B-reduction with respect to Borel subgroup B, and ∇ is a flat connection, which behaves in a certain way with respect to $\mathcal{F}_B$. For example, in the case of PGL_2-oper, this condition just means that the reduction $\mathcal{F}_B$ is nowhere preserved by this connection. Moreover, it appears, following the results of Drinfeld and Sokolov, that the space of G-opers is equivalent to a certain space of scalar pseudodifferential operators. In the PGL_2 case, the resulting space of scalar operators is just a family of Sturm–Liouville operators and the connection transformation properties allow to consider them on all Σ as projective connections.

A really interesting story starts when we allow opers to have regular singularities. It turns out that the opers on the projective line can be described via the Bethe ansatz equations for the Gaudin model corresponding to the Langlands dual Lie algebra [12,14]. An important object on the way to understand this relation is the so-called Miura oper, which was introduced by E. Frenkel [14]. A Miura oper is an oper with one extra constraint: the connection preserves another B-reduction of $\mathcal{F}$, which we call $\mathcal{F}'_B$. The space of the Miura opers, associated to a given G-oper with trivial monodromy, is isomorphic to the flag manifold G/B. If the reduction $\mathcal{F}'_B$ corresponds to the point in a big cell of G/B, then such a Miura oper is called generic. It was shown by E. Frenkel that any Miura oper is generic on the punctured disc and that there is an isomorphism between the space of generic Miura opers on the open neighbourhood with certain H-bundle connections ($H=B/[B,B]$) [14]. The map from H-connections to G-opers is just a generalization of the standard Miura transformation in the theory of KdV integrable models.

By means of the above relation with the H-connections, it was proved for PGL_2-oper in [12] and then generalized to the higher rank in [10,14] that the eigenvalues of the Gaudin model for a Langlands dual Lie algebra $\mathfrak{g}^L$ can be described by the G-opers on $\mathbb{C}P^1$ with given regular singularities and trivial monodromy. Namely, the consistency conditions for the H-connections underlying such opers coincide with the Bethe ansatz equations for the Gaudin Model.

In this article, we are trying to generalize some of the above notions and results to the case of superalgebras. We define an analogue of the oper in the case of supergroups which allow the pure fermionic family of simple roots on a super Riemann surface, following some local considerations of [8,16] and [18]. We call such objects superopers, and in some sense they turn out to be "square roots" of standard opers. Unfortunately for all other superalgebras, the resulting formalism allows only locally defined objects (on a formal superdisc). We study in detail the simplest nontrivial case of superoper, related to the group SPL_2 (see e.g. [6]), related to superprojective transformations, and explicitly establish the relation between the $\mathfrak{osp}(1|2)$ Gaudin model studied in [17] and the SPL_2-oper on super Riemann sphere with given regular singularities.

In Section 2 we explain the relation between super projective structures on super Riemann surface and the supersymmetric version of the Sturm–Liouville operator. Then, we relate it to the flat connection on SPL_2-bundle which will give us the first example of superoper.

In Section 3 we use this experience to generalize the notion of superoper to the case of higher rank simple supergroups. However, only the supergroups which permit a pure fermionic system of simple roots allow us to construct a globally defined object on a super Riemann surface. We define Miura superopers and superopers with regular singularities in Section 4. There we study the consistency conditions for the superopers on the superconformal sphere and derive the corresponding Bethe equations. We compare the results with the $\mathfrak{osp}(1|2)$ Gaudin model and find that the Bethe ansatz equations coincide with the "body" part of the consistency condition for corresponding SPL_2 Miura superopers.

Some remarks and open questions are given in Section 5.

2. Superprojective Structures, Super Sturm–Liouville Operator and $\mathfrak{Osp}(1|2)$ Superoper

2.1. SUPER RIEMANN SURFACES AND SUPERCONFORMAL TRANSFORMATIONS

For the general information about supermanifolds and superschemes one should consult [21] and [22]. The definition of supercurve we are using in this article follows [5].

We remind that a supercurve of dimension (1|1) over some Grassmann algebra S (which is fixed throughout this paper) is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of supercommutative S-algebras over X such that $(X, \mathcal{O}_X^{\text{red}})$ is an algebraic curve (where $\mathcal{O}_X^{\text{red}}$ is obtained from $\mathcal{O}_X$ by quoting out nilpotents) and for some open sets $U_\alpha \subset X$ and some linearly independent elements $\{\theta_\alpha\}$ we have $\mathcal{O}_{U_\alpha} = \mathcal{O}_{U_\alpha}^{\text{red}} \otimes S[\theta_\alpha]$. These open sets U_α serve as coordinate neighbourhoods for supercurves with coordinates $(z_\alpha, \theta_\alpha)$. The coordinate transformations on the overlaps $U_\alpha \cup U_\beta$ are given by the following formulas $z_\alpha = F_{\alpha\beta}(z_\beta, \theta_\beta)$, $\theta_\alpha = \Psi_{\alpha\beta}(z_\beta, \theta_\beta)$, where $F_{\alpha\beta}$ and $\Psi_{\alpha\beta}$ are even and odd functions, correspondingly. A super Riemann surface Σ over some Grassmann algebra S (see [3,20,29] for good review) is a supercurve of dimension 1|1 over S, with one more extra structure: there is a subbundle $\mathcal{D}$ of $T\Sigma$ of dimension 0|1, such that for any nonzero section D of $\mathcal{D}$ on an open subset U of Σ, D^2 is nowhere proportional to $\mathcal{D}$, i.e. we have the exact sequence:

$$0 \to \mathcal{D} \to T\Sigma \to \mathcal{D}^2 \to 0. \tag{1}$$

One can pick the holomorphic local coordinates in such a way that this odd vector field will have the form $f(z,\theta)D_\theta$, where $f(z,\theta)$ is a nonvanishing function and

$$D_\theta = \partial_\theta + \theta\partial_z, \quad D_\theta^2 = \partial_z. \tag{2}$$

Such coordinates are called superconformal. The transformation between two superconformal coordinate systems (z,θ) and (z',θ') is determined by the condition that $\mathcal{D}$ should be preserved, i.e.:

$$D_\theta = (D_\theta\theta')D_{\theta'}, \tag{3}$$

so that the constraint on the transformation coming from the local change of coordinates is $D_\theta z' - \theta' D_\theta\theta' = 0$. An important nontrivial example of a super Riemann surface is the Riemann super sphere SC^*: there are two charts (z,θ), (z,θ') so that

$$z' = -\frac{1}{z}, \quad \theta' = \frac{\theta}{z}. \tag{4}$$

There is a group of superconformal transformations, usually denoted as SPL_2 which acts transitively on SC^* as follows:

$$\begin{aligned} z &\to \frac{az+b}{cz+d} + \theta \frac{\gamma z + \delta}{(cz+d)^2}, \\ \theta &\to \frac{\gamma z + \delta}{cz+d} + \theta \frac{1 + \frac{1}{2}\delta\gamma}{cz+d}, \end{aligned} \tag{5}$$

where a, b, c, d are even, $ad - bc = 1$, and γ, δ are odd. The Lie algebra of this group is isomorphic to $\mathfrak{osp}(2|1)$.

Let us introduce two more notions which we will use in the following. From now on let us call the sections of $\mathcal{D}^n$ the *superconformal fields* of dimension $-n/2$. In particular, taking the dual of the exact sequence (1), we find that a bundle of superconformal fields of dimension 1 (i.e. $\mathcal{D}^{-2}$) is a subbundle in $T^*\Sigma$. Considering the superconformal coordinate system, a nonzero section of this bundle is generated by $\eta = \mathrm{d}z - \theta \mathrm{d}\theta$, which is orthogonal to D_θ under the standard pairing.

At last, we introduce one more notation. For any element A which belongs to some free module over $S[\theta]$, where θ is a local odd coordinate, we denote the body of this element (i.e. A is stripped of the dependence on the odd variables) as $\bar{A}$.

2.2. SUPERPROJECTIVE STRUCTURES AND SUPERPROJECTIVE CONNECTIONS

Let us at first define what a superprojective connection is. We consider the following differential operator, defined locally with coordinates (z, θ):

$$D_\theta^3 - \omega(z, \theta). \tag{6}$$

The following proposition holds.

PROPOSITION 2.1. [25] *Formula* (6) *defines the operator* L, *such that*

$$L : \mathcal{D}^{-1} \to \mathcal{D}^2 \tag{7}$$

iff the transformation of ω *on the overlap of two coordinate charts* (z, θ) *and* (z', θ') *is given by the following expression:*

$$\omega(z, \theta) = \omega(z', \theta')(D_\theta \theta')^3 + \{\theta'; z, \theta\} \tag{8}$$

where

$$\{\theta'; z, \theta\} = \frac{\partial_z^2 \theta'}{D_\theta \theta} - 2 \frac{\partial_z \theta \, D_\theta^3 \theta'}{(D_\theta \theta')^2} \tag{9}$$

is a supersymmetric generalization of Schwarzian derivative.

One can show that the only coordinate transformations for which the super Schwarzian derivative vanishes are the fractional linear transformations (5).

Let us consider the covering of Σ by open subsets, so that the transition functions are given by (5). Two such coverings are considered equivalent if their union has the same property of transition functions. The corresponding equivalence classes are called superprojective structures.

It appears that like in the purely even case, there is a bijection between super projective connections and super projective structures. For a given super projective structure one can define a superprojective connection by assigning operator D_θ^3 in every coordinate chart. From Proposition 2.1 we find that the resulting object is defined globally on Σ. On the other hand, given a super projective connection on Σ, one can consider the following linear problem:

$$\left(D_\theta^3 - \omega(z,\theta)\right)\psi\left(z,\theta\right) = 0. \tag{10}$$

From the results of [1] we know that this equation has three independent solutions: two even $x(z,\theta)$, $y(z,\theta)$ and one odd $\xi(z,\theta)$. Defining $C = y/x$, $\alpha = \xi/x$, we find that $\omega(z,\theta)$ is expressed via super Schwarzian derivative, i.e. $w(z,\theta) = \{\alpha;\theta,z\}$ and the consistency conditions on C and α are such that C can be represented in terms of α in the following way:

$$C = cA + \gamma A\alpha + \delta\alpha, \tag{11}$$

where A is such a function that $(z,\theta) \to (A,\alpha)$ is a superconformal transformation. In a different basis (A,α) will be transformed via SPL_2 (5) and hence (A,α) form natural coordinates for a projective structure on Σ. Therefore, we have the following proposition (see also [15], Theorem 3.14).

PROPOSITION 2.2. *There is a bijection between the set of superprojective structures and the set of superprojective connections on* Σ.

2.3. CONNECTIONS FOR VECTOR BUNDLES OVER SUPER rIEMANN SURFACES

Let us consider a vector bundle V over the super Riemann surface with the fiber $\mathbb{C}_S^{m|n}$. Let $\mathcal{E}^0(\Sigma,V)$ be the space of sections on V over Σ and let $\mathcal{E}^1(\Sigma,V)$ be the space of 1-form valued sections. As usual, the connection is a differential operator

$$\mathrm{d}_A(fs) = df \otimes s + (-1)^{|f|} f \mathrm{d}_A s, \tag{12}$$

where f is a smooth even/odd function on Σ and $s \in \mathcal{E}^0(\Sigma,V)$. Locally, in the chart (z,θ) the connection has the following form:

$$\begin{aligned}
\mathrm{d}_A &= \mathrm{d} + A = \mathrm{d} + (\eta A_z + \mathrm{d}\theta A_\theta) + (\bar\eta A_{\bar z} + \mathrm{d}\bar\theta A_{\bar\theta}) \\
&= (\partial + \eta A_z + \mathrm{d}\theta A_\theta) + (\bar\partial + \bar\eta A_{\bar z} + \mathrm{d}\bar\theta A_{\bar\theta}) \\
&= (\eta D_z^A + \mathrm{d}\theta D_\theta^A) + (\bar\eta D_{\bar z}^A + \mathrm{d}\bar\theta D_{\bar\theta}^A).
\end{aligned} \tag{13}$$

We note that we used here the fact that $\mathrm{d}=\partial+\bar\partial$ and $\partial=\eta\partial_z+\mathrm{d}\theta D_\theta$. The expression for the curvature is

$$F=\mathrm{d}_A^2=\mathrm{d}\theta\mathrm{d}\theta F_{\theta\theta}+\eta\mathrm{d}\theta F_{z\theta}+\mathrm{d}\bar\theta\mathrm{d}\bar\theta F_{\bar\theta\bar\theta}+\bar\eta\mathrm{d}\bar\theta F_{\bar z\bar\theta}$$
$$+\eta\bar\eta F_{z\bar z}+\eta\mathrm{d}\bar\theta F_{z\bar\theta}+\bar\eta\mathrm{d}\theta F_{\bar z\theta}+\mathrm{d}\theta\mathrm{d}\bar\theta F_{\theta\bar\theta}, \tag{14}$$

where $F_{\theta\theta}=-{D_\theta^A}^2+D_z^A$, $F_{z\theta}=[D_z^A, D_\theta^A]$, $F_{z,\bar z}=[D_z^A, D_{\bar z}^A]$, $F_{z\bar\theta}=[D_z^A, D_{\bar\theta}^A]$, $F_{\theta\bar\theta}=-[D_\theta^A, D_{\bar\theta}^A]$, etc.

It appears that if the connection d_A offers partial flatness, which implies $F_{\theta\theta}=F_{z\theta}=F_{\bar\theta\bar\theta}=F_{\bar z\bar\theta}=0$, then there is a superholomorphic structure on V (i.e. transition functions of the bundle can be made superholomorphic) [26]. We are interested in the flat superholomorphic connections. In this case, since $F_{\theta\theta}=0$, the connection is fully determined by the D_θ^A locally. In other words, it is determined by the following odd differential operator, which from now on will denote ∇ and call long superderivative:

$$\nabla=D_\theta+A_\theta(z,\theta), \tag{15}$$

which gives a map: $\mathcal{D}\to EndV$ so that the transformation properties for A_θ are as follows: $A_\theta\to gA_\theta g^{-1}-D_\theta g g^{-1}$, where g is a superholomorphic function providing change of trivialization.

2.4. SPL_2-OPERS

In this subsection, we give a description of the first nontrivial superoper. Suppose we have a superprojective structure on Σ. Naturally we have a structure of a flat SPL_2-bundle $\mathcal{F}$ over Σ, since on the overlaps there is a constant map to SPL_2. Let us study the corresponding flat connection on Σ. Since SPL_2 is a group of superconformal automorphisms of SC^*, one can form an associated bundle $SC^*_{\mathcal{F}}=\mathcal{F}\times_{SPL_2}SC^*$. This bundle has a global section which is just given by the superprojective coordinate functions (z,θ) on Σ. We note that it has nonvanishing (super)derivative at all points.

One can view SC^* as a flag supermanifold. Namely, consider the group SPL_2 acting in $\mathbb{C}^{2|1}=span(e_1,\xi,e_2)$, where we put the odd vector in the middle. Then, e_1 is stabilized by the Borel subgroup of upper triangular matrices. Therefore, one can identify SC^* with SPL_2/B. Since we have a nonzero section of $SC^*_{\mathcal{F}}$, we have a B-subbundle $\mathcal{F}_B$ of a G-bundle, where G stands for SPL_2. Hence, a superprojective structure gives the flat SPL_2-bundle $\mathcal{F}$ with a reduction $\mathcal{F}_B$. However, there is one more piece of data we can use: it is the condition that the (super)derivative of the section of $SC^*_{\mathcal{F}}$ is nowhere vanishing. It means that the flat connection on $\mathcal{F}$ does not preserve the B-reduction anywhere. Let us figure out which conditions does it put on the connection if we choose a trivialization of $\mathcal{F}$ induced from the $\mathcal{F}_B$ trivialization. As we discussed above, the connection is determined by the following odd differential operator:

$$\nabla = D_\theta + \begin{pmatrix} \alpha(z,\theta) & b(z,\theta) & \beta(z,\theta) \\ -a(z,\theta) & 0 & b(z,\theta) \\ \gamma(z,\theta) & a(z,\theta) & -\alpha(z,\theta), \end{pmatrix}, \tag{16}$$

so that the matrix is in the defining representation of the Lie superalgebra of SPL_2, namely $\mathfrak{osp}(1|2)$. This operator and its square describe even and odd directions for the tangent vector to SC^*. Since we have the condition that both of them are nonvanishing, and identifying tangent space with $\mathfrak{osp}(1|2)/\mathfrak{b}$ (where $\mathfrak{b}$ is the Borel subalgebra), we obtain that a is nonvanishing. It is possible to make $\gamma = 0$, by redefining ∇ by adding $\mu(\nabla)^2$ with appropriate odd function μ, which just corresponds to the choice of superconformal coordinates on SC^*. We call such a triple $(\mathcal{F}, \mathcal{F}_B, \nabla)$ a superoper. We notice that taking the square of the odd operator ∇, reducing such even operator from Σ to the underlying curve Σ^0 and getting rid of all the odd variables, we obtain the oper connection for the PGL_2-bundle. Thus superopers can be thought about as "square roots" of opers.

Using B-valued gauge transformations one can bring ∇_θ to the canonical form:

$$\nabla = D_\theta + \begin{pmatrix} 0 & 0 & \omega(z,\theta) \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{17}$$

Therefore, on a superdisc with coordinate (z, θ) the space of SPL_2 superopers can be identified with the space of differential operators $D_\theta^3 - \omega(z,\theta)$. We will see in the next section that the coordinate transformations of ω are the same as in Proposition 2.1.

Therefore, we see that there is a full analogy with the bosonic case, where the space of PGL_2-superopers was identified with the set of projective connections or equivalently with the set of projective structures.

Let us summarize the results of this section in the following theorem.

THEOREM 2.3. *There are bijections between the following three sets on a super Riemann surface* Σ*:*

(i) *Superprojective structures*
(ii) *Superprojective connections*
(iii) SPL_2*-opers.*

3. Superopers for Higher Rank Superalgebras

3.1. THE DEFINITION OF SUPEROPERS

In this section we generalize the results of the previous section to higher rank. Suppose G is a simple algebraic supergroup [4] of adjoint type over some Grassmann algebra S (which, as we remind, is fixed throughout this paper), B is its Borel subgroup, $N = [B, B]$, so that for the corresponding Lie superalgebras we have $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$. Note that $\mathfrak{g} = S \otimes \mathfrak{g}^{\text{red}}$, where $\mathfrak{g}^{\text{red}}$ is a simple Lie superalgebra over $\mathbb{C}$. As usual, $H=B/N$ with the Lie algebra $\mathfrak{h}$ and there is a decomposition: $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. The corresponding generators of simple roots will be denoted as usual: $e_1,\ldots,e_l$; $f_1,\ldots,f_l$. We are interested in the superalgebras, which have a pure fermionic system of simple roots, namely $\mathfrak{psl}(n|n)$, $\mathfrak{sl}(n+1|n)$, $\mathfrak{sl}(n|n+1)$, $\mathfrak{osp}(2n\pm 1|2n)$, $\mathfrak{osp}(2n|2n)$, $\mathfrak{osp}(2n+2|2n)$ with $n\geq 0$ and $D(2,1;\alpha)$ with $\alpha\neq 0,\pm 1$. Moreover, a necessary ingredient for our construction is the presence of the embedding of superprincipal $\mathfrak{osp}(1|2)$ subalgebra [7,11], namely that for $\chi_{-1}=\sum_i f_i$ and $\check{\rho}=\sum_i\check{\omega}_i$, where $\check{\omega}_i$ are fundamental coweights, there is such χ_1 that makes a triple $(\chi_1,\chi_{-1},\check{\rho})$ an $\mathfrak{osp}(1|2)$ superalgebra. Almost all series of superalgebras from the list above allow such an embedding; however, $\mathfrak{psl}(n|n)$ does not and we do not consider this series in this article.

As in the standard purely even case we define an open orbit $\mathbf{O}\subset[\mathfrak{n},\mathfrak{n}]^{\perp}/\mathfrak{b}$ consisting of vectors, stabilized by N and such that all the negative root components of these vectors with respect to the adjoint action of H are nonzero.

Let us consider a principal G-bundle $\mathcal{F}$ over X, which can be a super Riemann surface Σ or a formal superdisc $SD_x=\mathrm{Spec}S[\theta][[z]]$, or a punctured superdisc $D_x^{S\times}=\mathrm{Spec}S[\theta]((z))$ (see e.g. [19,27] for the definitions of the spectra of supercommutative rings), and its reduction $\mathcal{F}_B$ to the Borel subgroup B. We assume that it has a flat connection determined by a long superderivative ∇ [see (15)]. According to the example, considered in Section 2 we do not want ∇ to preserve $\mathcal{F}_B$. However, in the higher rank case this is not enough, so we have to specify extra conditions. Namely, suppose ∇' is another long superderivative, which preserves $\mathcal{F}_B$. Then we require that the difference $\nabla'-\nabla$ has a structure of superconformal field of dimension 1/2 with values in the associated bundle $\mathfrak{g}_{\mathcal{F}_B}$. We can project it onto $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}\otimes\mathcal{D}^{-1}$. Let us denote the resulting $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$-valued superconformal field as $\nabla/\mathcal{F}_B$. Now we are ready to define the superoper, which is a natural generalization of the oper.

DEFINITION 3.1. A $G-$*superoper* on X (where X is stands for super Riemann surface or formal superdisc or punctured superdisc) is the triple $(\mathcal{F},\mathcal{F}_B,\nabla)$, where $\mathcal{F}$ is a principle G-bundle, $\mathcal{F}_B$ is its B-reduction and ∇ is a long superderivative on $\mathcal{F}$, such that $\nabla/\mathcal{F}_B$ takes values in $\mathbf{O}_{\mathcal{F}_B}$.

Therefore, locally on the open subset U, with coordinates (z,θ), with respect to the trivialization of $\mathcal{F}_B$, the structure of the long superderivative is

$$D_{z,\theta}+\sum_{i=1}^{l}a_i(z,\theta)f_i+\mu(z,\theta), \tag{18}$$

where each $a_i(z,\theta)$ is an even nonzero function (meaning that these functions have nonzero body and are invertible) and $\mu(z,\theta)$ is an odd $\mathfrak{b}$-valued function. Under the change of trivialization this operator is transformed under the action of gauge transformation from the group $B(R)$, where R are either analytic or algebraic functions on U. Hence, the open subset U, the space of G-superopers on U, which will be denoted as $s\mathrm{Op}_G(U)$, can be characterized as the space of all odd operators of type (18) modulo gauge transformations from the group $B(R)$.

3.2. COORDINATE TRANSFORMATIONS AND OTHER PROPERTIES

Let us notice that one can use the H-action to make the operator (18) look as follows:

$$D_\theta + \sum_{i=1}^{l} f_i + \mu(z,\theta), \tag{19}$$

where $\mu \in \mathfrak{b}(R)$. Therefore, the space $s\mathrm{Op}_G(U)$ can be considered as the quotient of the space of operators of the form (19) [denoted as $\widetilde{s\mathrm{Op}_G}(U)$] by the action of $N(R)$. As in the pure bosonic case, $\check{\rho}$ gives a principal gradation (for those classes of superalgebras we consider), i.e. we have a direct sum decomposition $\mathfrak{b} = \oplus_{i\geq 0}\mathfrak{b}_i$. Moreover, let us remind that we denoted $\chi_{-1} = \sum_{i=1}^{l} f_i$ and there exists a unique element χ_1 of degree 1 in $\mathfrak{b}$, such that $\chi_{\pm 1}, \check{\rho}$ generate $\mathfrak{osp}(1|2)$ superalgebra. Let $\tilde{\chi}_k$ $(k = 1,\ldots,l)$ (which can be either odd or even), so that $\tilde{\chi}_2 = \chi_1^2$ be the basis of the space of the $ad(\chi_1)$ invariants. We note that the decompositions of $\mathfrak{g}$ with respect to the adjoint action of such $\mathfrak{osp}(1|2)$ triple were studied in [11]. Based on that, we have the following Lemma which is proved in a similar way as in [9] (see also Lemma 4.2.2 of [13]).

LEMMA 3.1. *The gauge action of $N(R)$ on $\widetilde{s\mathrm{Op}_G}(U)$ is free and each gauge equivalence class contains a unique operator of the form* (19) *with*

$$\mu(\theta,z) = \sum_{i=1}^{l} g_i(z,\theta)\tilde{\chi}_i, \tag{20}$$

where g_i has opposite parity to χ_i.

The sketch of the proof is as follows. We use the fact that $\mathfrak{b}_i = [\chi_{-1}, \mathfrak{b}_{i+1}] \oplus V_i$ where V_i is the subspace of degree i of the space V of all χ_{-1}-invariants. Based on this fact, one can apply gauge transformation with $\exp \mathrm{adK}$, where $K \in \mathbf{n}(R)$ to (19). Then decomposing μ and K according to the principle gradation one can solve the recursion equation for K, so that μ-part of long derivative after the transformation reduces to (20).

Now let us discuss the transformation properties of operators $\widetilde{s\mathrm{Op}_G}(U)$. Assume we have a superconformal coordinate change $(z,\theta) = (f(w,\xi), \alpha(w,\xi))$. Then according to the transformations of the long derivative we have

$$\nabla = D_\xi + (D_\xi\alpha)(w,\xi)\chi_{-1} + (D_\xi\alpha)(w,\xi)(\mu(f((w,\xi),\alpha(w,\xi)). \tag{21}$$

Considering 1-parameter subgroup $\mathbb{C}_S^{\times 1|1} \to H$ which corresponds to $\check{\rho}$, applying adjoint transformation with $\check{\rho}(D_\xi\alpha)$ we obtain:

$$D_\xi + \chi_{-1} + (D_\xi\alpha)(w,\xi)Ad_{\check{\rho}(D_\xi\alpha)} \cdot \mu(f(w,\xi),\alpha(w,\xi)) - \frac{\partial_w\alpha(w,\xi)}{D_\xi\alpha(w,\xi)}\check{\rho}. \tag{22}$$

This gives us the gluing formula for superopers on any super Riemann surface Σ.

Consider the H-bundle $\mathcal{D}^{-\check{\rho}}$ on Σ, which is determined by the property that the line bundle $\mathcal{D}^{-\check{\rho}} \times \mathbb{C}_\lambda$ is $\mathcal{D}^{-\langle\check{\rho},\lambda\rangle}$, where λ is from the lattice of characters and $\mathbb{C}_\lambda$ is the corresponding 1-dimensional representation.

The coordinate transformation formulas for superoper connection immediately lead to another characterization of this bundle via $\mathcal{F}_B$-reduction. The following statement is the supersymmetric version of Lemma 4.2.1 of [13].

LEMMA 3.2. *The H-bundle* $\mathcal{F}_H = \mathcal{F}_B \times_B H = \mathcal{F}_B/N$ *is isomorphic to* $\mathcal{D}^{-\check{\rho}}$.

Now one can derive the transformation properties for the canonical representatives of opers from Lemma 3.1, which will provide the transformation formulas for $g_1,\dots,g_n$. To do that, one needs to apply to the operator (19) the gauge transformation of the form

$$\exp\left(\kappa\chi_1 - \frac{1}{2}(D\kappa)[\chi_1,\chi_1]\right)\check{\rho}(D_\xi\alpha), \tag{23}$$

where $\kappa = \frac{\partial_w\alpha(w,\xi)}{D_\xi\alpha}$. Then we have that

$$\begin{aligned}
\tilde{g}_1(w,\xi) &= g_1(w,\xi)(D_\xi\alpha)^2,\\
\tilde{g}_2(w,\xi) &= g_2(w,\xi)(D_\xi\alpha)^3 + \{\alpha; w,\xi\},\\
\tilde{g}_j(w,\xi) &= g_j(w,\xi)(D_\xi\alpha)^{d_j+1}, \quad j>2.
\end{aligned} \tag{24}$$

Therefore (23) are transition functions for $\mathcal{F}_B$ and $\mathcal{F}$ bundles.

Remark. Note that the g_1-term is absent in the $\mathfrak{osp}(1|2)$; however, it often appears in the higher rank. The first example is $\mathfrak{sl}(1|2) \cong \mathfrak{osp}(2|2)$.

The formulas (24) give the following description of the space of superopers:

$$s\mathrm{Op}_G(\Sigma) \cong sProj(\Sigma) \times \oplus_{j=1, j\neq 2}^{l} \Gamma(\Sigma, \mathcal{D}^{-d_j-1}), \tag{25}$$

where $sProj(\Sigma)$ stands for superprojective connections on Σ.

In the previous section we indicated that in the $\mathfrak{osp}(1|2)$ one can introduce the oper related to a superoper, by considering ∇^2, then stripping it from the θ and S dependence, we obtain that the resulting $\overline{\nabla^2}$ has all the needed properties of $sl(2)$ oper on the curve X which is a base manifold for Σ.

A similar construction is possible in the higher rank case. Let ${}^0G$ be the reductive group, which is a base manifold for G. Due to the structure of the coordinate transformations we derived above, we find out that indeed $\overline{\nabla^2} = \nabla^2$ defines an oper on X. We refer to this object as ${}^0G$-oper, associated with the G-superoper, which we will denote as triple $({}^0\mathcal{F}, \overline{\nabla^2}, {}^0\mathcal{F}_B)$, where ${}^0\mathcal{F}$ and ${}^0\mathcal{F}_B$ denote the appropriate purely even reductions of the principal bundles.

4. Superopers with Regular Singularities, Miura Superopers and Bethe Ansatz Equations

4.1. SUPEROPERS WITH REGULAR SINGULARITIES

Consider a point x on the superc Riemann surface Σ and the formal superdisc SD_x around that point with the coordinates (z, θ). We define a G-superoper with regular singularity on SD_x as an operator of the form

$$D_\theta + \sum a_i(z,\theta) f_i + \left(\mu_1(z) + \frac{\theta}{z}\mu_0(z)\right), \tag{26}$$

modulo the $B(\mathcal{K}_x)$-transformations ($\mathcal{K}_x = \mathbb{C}[\theta]((z))$), where $a_i(z,\theta) \in \mathcal{O}_x$ are nowhere vanishing and invertible, $\mu_i(z,\theta) \in \mathfrak{b}(\mathcal{K}_x)$ $(i = 0, 1)$, such that the bodies of $\mu_i(z)$, i.e. $\overline{\mu_i} \in \mathfrak{b}^{\rm red}(\mathcal{O}_x^{\rm red})$. As before, one can eliminate a_i-dependence via H-transformations; therefore, we can talk about $N(\mathcal{K}_x)$ equivalence class of operators of the type (26) with $a_i = 1$. Let us denote by $sOp_G^{RS}(SD_x)$ the space of superopers with regular singularity. Clearly, we have the embedding: $sOp_G^{RS}(SD_x) \subset sOp_G(SD_x^\times)$.

The ${}^0G$-oper, corresponding to G-superoper (26), is the oper with regular singularity. It has the following form:

$$\partial + \chi_{-1}^2 + [\chi_{-1}, \overline{\mu_1}] + (\overline{\mu_1})^2 + \frac{1}{z}\left(\overline{\mu_0}\right), \tag{27}$$

which can be transformed to the standard form via the gauge transformation by means of $\frac{\rho}{2}(z)$:

$$\partial_z + \frac{1}{z}\left(\chi_{-1}^2 - \frac{\check{\rho}}{2} + Ad_{\frac{\check{\rho}}{2}(z)} \cdot \bar{\mu}_0\right) + v(z), \tag{28}$$

where $v(z)$ is regular.

Denoting $-\check{\lambda}$ the projection of μ_0 on $\mathfrak{h}$, we find that the residue of this differential operator is equal to $\chi_{-1}^2 - \lambda - \frac{1}{2}\check{\rho}$, however since this is an oper, only the corresponding class in $\mathfrak{h}/W$ is well defined, and we denote it as $\left(-\lambda - \frac{1}{2}\check{\rho}\right)_W$, i.e. this oper belongs to $\mathrm{Op}_G^{RS}(D_x)_{\check{\lambda}}$, see e.g. [12].

Let us refer to the space of superopers with regular singularity such that $\bar{\mu_0}(0) = \check{\lambda}$, as $s\mathrm{Op}_G^{RS}(D_x)_\lambda$.

If we consider the representation V of G one can talk about a system of differential equations $\nabla \cdot \phi_V(z,\theta)$ and their monodromy like in the purely even case.

Let $\check{\lambda}$ be the dominant integral coweight and let us introduce the following class of operators:

$$\nabla = D_\theta + \left(\sum a_i(z,\theta) f_i + \mu(z,\theta)\right), \tag{29}$$

where $a_i = z^{\langle \alpha_i, \check{\lambda} \rangle}(r_i(\theta) + z(\dots))$, so that the body of r_i is nonzero, and $\mu(z,\theta) \in \mathfrak{b}(\mathcal{O}_x)$. We call the quotient of the space of operators above by the action of $B(\mathcal{O}_x)$ as $s\mathrm{Op}_G(SD_x)_\lambda$.

The following Lemma is an analogue of Lemma 2.4. of [12].

LEMMA 4.1. *There is an injective map i:* $s\mathrm{Op}_G(SD_x)_{\check{\lambda}} \to sOp(SD_x^\times)$, *so that* $\mathrm{Im}i \subset s\mathrm{Op}_G^{RS}(SD_x)_{\check{\lambda}}$. *The image of i is a subset in the set of those elements of* $s\mathrm{Op}_G^{RS}(SD_x)_{\check{\lambda}}$, *such that the resulting oper has a trivial monodromy around x.*

Remark. Notice that the superopers corresponding to $s\mathrm{Op}_G(SD_x)_{\check{\lambda}}$ belong to $\mathrm{Op}_G(D_x)_{\check{\lambda}}$. However, here $\check{\lambda}$ is the integral dominant weight for Lie superalgebra. If we consider λ to be an integral dominant weight for the underlying Lie algebra, the monodromy for the corresponding superoper would not be necessarily trivial: the expression will include the half-integer powers of z and the monodromy will correspond to the reflection: $\theta \to -\theta$.

4.2. MIURA SUPEROPERS

Miura superoper is defined in complete analogy with the purely even case. Namely, *Miura G-superoper* is a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ where the triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a G-superoper and $\mathcal{F}'_B$ is another B-reduction preserved by ∇. Let us denote the space of such superopers as $s\mathrm{MOp}_G(\Sigma)$.

Such B-reductions of $\mathcal{F}$ are completely determined by the B-reduction of the fiber $\mathcal{F}_x$ at any point x on Σ and a set of all such reductions is given by $(G/B)_{\mathcal{F}_x} = \mathcal{F}_x \times_G G/B = (G/B)_{\mathcal{F}'_x}$. If superoper ξ has the regular singularity and a trivial monodromy, then there is an isomorphism between the space of Miura opers for such ξ and $(G/B)_{\mathcal{F}'_x}$.

The structure of the flag manifold G/B is usually quite complicated [23,24]; however, we just need the structure determined by its "body", i.e. ${}^0G/{}^0B$. For the purely even flag variety ${}^0G/\,{}^0B$, we have the standard Schubert cell decomposition, where cells ${}^0S_w = {}^0Bw_0w\,{}^0B$ are labeled by the Weyl group elements $w \in W$ and w_0 is the longest element of the Weyl group (from now on when we say Weyl group, we mean only the Weyl group corresponding to purely even Weyl reflections of the ${}^0G$ root system).

Let us denote S_w the preimage of $P: G/B \to {}^0G/\,{}^0B$. We assume that the preimage of a big cell ${}^0Bw_0^0B$ allows factorization Bw_0B. The B-reduction $\mathcal{F}'_B$ defines a point in G/B. We say that B-reductions $\mathcal{F}_{B,x}$ and $\mathcal{F}'_{B,x}$ are in relative position w if $\mathcal{F}_{B,x}$ belongs to $\mathcal{F}'_x \times_B S_w$. When w=1, we say that $\mathcal{F}_x$, $\mathcal{F}'_x$ are in generic position. A Miura superoper is called generic at a given point $x \in \Sigma$ if the B-reductions $\mathcal{F}_{B,x}$, $\mathcal{F}'_{B,x}$ are generic. Notice that if a Miura superoper is generic at x, it is generic in the neighbourhood of x. We denote the space of Miura superopers on U as $s\mathrm{MOp}_G(U)_{\mathrm{gen}}$. It is clear that the reduction of Miura superoper to $({}^0\mathcal{F}, \overline{\nabla^2}, {}^0\mathcal{F}_B, {}^0\mathcal{F}'_B)$ gives a Miura oper.

Therefore the following Proposition holds, which follows directly from the reduction to the purely even case, although one can also go along the lines of the proof of Lemma 2.6. and Lemma 2.7 of [12].

PROPOSITION 4.2. (i) *The restriction of the Miura superoper to the punctured disk is generic.*

(ii) *For a generic Miura superoper* $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ *the* H*-bundle* $\mathcal{F}'_H$ *is isomorphic to* $w_0^*(\mathcal{F}_h)$

As in the even case we can define an H-connection associated to Miura superoper on $\mathcal{F}_H \cong \mathcal{D}^{-\check{\rho}}$, which is determined by $\tilde{\nabla} = D_\theta + u(z,\theta)$, where u is $\mathfrak{h}$-valued function. Under the change of coordinates $(z,\theta) = (f(w,\xi), \alpha(w,\xi))$, the long superderivative transforms are as follows:

$$D_\xi + D_\xi\alpha \cdot u\Big(f(w,\xi), \theta(w,\xi)\Big) - \frac{\partial_w \alpha(w,\xi)}{D_\xi\alpha)}\check{\rho}. \tag{30}$$

Let us call the resulting morphism $s\mathrm{MOp}_G(U)_{\mathrm{gen}}$ to the space $Conn_U$ of the described above flat H-connections on U as $\mathbf{a}$. Now suppose we are given a long superderivative $\tilde{\nabla}$ on H-bundle $\mathcal{D}^{-\check{\rho}}$, one can construct a generic superoper as follows. Let us set $\mathcal{F} = \mathcal{D}^{-\check{\rho}} \times_H G$, $\mathcal{F}_B = \mathcal{D}^{-\check{\rho}} \times_H B$. Then, defining $\mathcal{F}'_B$ as $\mathcal{D}^{-\check{\rho}} \times_H w_0 B$ and the long superderivative on $\mathcal{F}$ as $\nabla = \chi_{-1} + \tilde{\nabla}$, we see that the constructed quadruple $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$ is a generic Miura oper.

Therefore, we obtained the following statement which is analogue of Proposition 2.8 of [12].

PROPOSITION 4.3. *The morphism* $\mathbf{a}: s\mathrm{MOp}_G(U)_{gen} \to Conn_U$ *is an isomorphism of algebraic supervarieties.*

Similarly one can define the space of Miura G-superopers of coweight $\check{\lambda}$ on SD_x via the same definition applied to $s\mathrm{Op}_G(SD_x)_{\check{\lambda}}$. Again, we have isomorphism $s\mathrm{MOp}_G(SD_x)_\lambda \cong s\mathrm{Op}_G(SD_x)_\lambda \times (G/B)_{\mathcal{F}'_x}$. We define relative positions as in the case of standard Miura superopers ($\check{\lambda} = 0$) and let $s\mathrm{MOp}_G(SD_x)_{\check{\lambda},gen}$ denote the variety of generic Miura opers of weight λ.

Finally, there is an analogue of Proposition 4.3 in this case. Let $Conn^{RS}_{SD_x,\check{\lambda}}$ denote the set of long derivatives on the H-bundle $\mathcal{D}^{-\check{\rho}}$ with regular singularity and residue $-\check{\lambda}$, namely the long derivatives of the form:

$$\tilde{\nabla} = D_\theta + \frac{\theta}{z}\check{\lambda} + u(z,\theta), \tag{31}$$

where $u(z,\theta) \in \mathfrak{h}[[z,\theta]]$. Then as before, one can construct a connection $\nabla = \tilde{\nabla} + \chi_{-1}$ and making the gauge transformation with $\check{\lambda}(z)$ we obtain the connection from $s\mathrm{Op}_G(SD_x)_{\check{\lambda}}$. Therefore, there is an isomorphism between $Conn^{RS}_{SD_x,\lambda}$ and $s\mathrm{MOp}_G(SD_x)_{\mathrm{gen},\check{\lambda}}$.

4.3. MIURA SUPEROPERS WITH REGULAR SINGULARITIES ON SC^*

First, let us consider a Miura superoper of coweight $\hat{\lambda}$ on the disc SD_x. Assume, it is not generic, but $\mathcal{F}'_{B,x}$ has the relative position w with $\mathcal{F}_{B,x}$ at x. Let us denote the space of all such Miura superopers by $s\mathrm{MOp}_G(SD_x)_{\check{\lambda},w}$.

From previous subsection we know that each such Miura superoper corresponds to some H-connection on $\mathcal{D}^{-\check{\rho}}$ over $SD_x^{\times}$. Using the results from the purely even case, one can show that the corresponding H-connection has the form

$$D_\theta + \frac{\theta}{z}\check{\nu} + f(z,\theta), \tag{32}$$

where $\check{\nu} - \frac{1}{2}\check{\rho} = w(-\check{\lambda} - \frac{1}{2}\check{\rho})$, w defines the relative position at x, $f(u,\theta)$ is such that the body of its superderivative is regular in z, i.e. $\overline{D_\theta f(z,\theta)} \in \mathfrak{h}^{\mathrm{red}}[[z]]$. Let us call the space of such connections by $Conn^{RS}_{SD_x,\check{\lambda},w}$.

Therefore, we can construct a map $\mathbf{b}^{RS}_{\lambda,w} : Conn^{RS}_{SD_x,\check{\lambda},w} \to s\mathrm{Op}^{RS}_G(SD_x)$ similarly to the previous subsection, by constructing the triple $(\mathcal{F},\nabla,\mathcal{F}_B)$ via identification $\mathcal{F} = \mathcal{D}^{-\check{\rho}} \times_H G$, $\mathcal{F}_B = \mathcal{D}^{-\check{\rho}} \times_H B$ and $\nabla = \tilde{\nabla} + \chi_{-1}$, where $\tilde{\nabla} \in Conn^{RS}_{SD_x,\check{\lambda},w}$. We denote by $Conn^{\mathrm{reg}}_{SD_x,\check{\lambda},w}$ the preimage of $s\mathrm{Op}_G(SD_x)_{\check{\lambda},w}$ under this morphism; therefore, we have the map: $\mathbf{b}_{\lambda,w} : Conn^{\mathrm{reg}}_{SD_x,\check{\lambda},w} \to s\mathrm{MOp}^{RS}_G(SD_x)_{\check{\lambda}}$, so that in the quadruple $(\mathcal{F},\nabla,\mathcal{F}_B,\mathcal{F}'_B)$ the first three terms are as above and $\mathcal{F}'_B = \mathcal{D}^{-\check{\rho}} \times_H w_0 B$. If we denote $s\mathrm{MOp}^{RS}_G(SD_x)_{\check{\lambda},w}$ those Miura superopers of coweight $\check{\lambda}$ which have the relative position w at x, then the following Proposition is true, based on the results from the purely even case (see Proposition 2.9 of [12]).

PROPOSITION 4.4. *For each $w \in W$, $\mathbf{b}_{\check{\lambda},w}$ is an isomorphism of supervarieties $Conn^{\mathrm{reg}}_{SD_x,\check{\lambda},w}$ and $s\mathrm{MOp}_G(SD_x)_{\check{\lambda},w}$.*

Let us now consider the case of $\check{\lambda} = 0$ and assume that the relative position is given by $s_{2\alpha_i}$, where α_i is a simple black root. In local coordinates, the corresponding H-connection will be given by the differential operator:

$$\tilde{\nabla} = D_\theta + \frac{\theta}{2z}\check{\alpha}_i + u(z,\theta), \tag{33}$$

where $u(z,\theta) \in \mathfrak{h}[\theta]((z))$ and $u(z,\theta) = u_1(z) + \theta u_0(z)$ and $\overline{u_0}(z) \in \mathfrak{h}^{\mathrm{red}}[[z]]$. Then applying the gauge transformation

$$\exp\left(-\frac{\theta}{2z}e_i + \frac{1}{4z}e_i^2\right) \tag{34}$$

to the Miura superoper $\tilde{\nabla} + \chi_{-1}$, we obtain that the resulting element of $s\mathrm{Op}_G(SD_x)_{\check{\lambda},s_{2\alpha_i}}$ gives the element $\mathrm{Op}_G D_{x\,\check{\lambda},s_{2\alpha_i}}$ if $\langle\check{\alpha}_i, \overline{u}_0(0)\rangle = 0$. If we consider the associate bundle corresponding to the 3-dimensional representation of the $\mathfrak{osp}(1|2)$ triple $\{e_i, f_i, \check{\alpha}_i\}$, writing explicitly all the solutions we find that this condition is also a necessary one. Namely, the following Proposition holds.

PROPOSITION 4.5. *A superoper corresponding to the H-connection given by* (33) *corresponds to* $\mathrm{Op}_G(D_x)_{\check{\lambda},s_{2\alpha_i}}$ *if and only if* $\langle\check{\alpha}_i,\overline{u_0}(0)\rangle=0$.

Now we are ready to study superopers with regular singularities over the super Riemann surface SC^*. Let us consider $\mathcal{Z}_1=(z_1,\theta_1),\dots,\mathcal{Z}_N=(z_N,\theta_N)$ on SC^*. Also, let $\check{\lambda}_1,\dots,\check{\lambda}_N,\check{\lambda}_\infty$ be the set of dominant coweights of $\mathfrak{g}$. Let us consider the H-connections on SC^* with regular singularities at the points $\mathcal{Z}_1,\dots,\mathcal{Z}_N,(\infty,0)$ and a finite number of other points $\mathcal{W}_1=(w_1,\xi_1),\dots,\mathcal{W}_m=(w_m,\xi_m)$ such that the residues of the corresponding even H-connection at z_i, w_j, ∞ are equal to $-y_i(\check{\lambda}+\frac{1}{2}\check{\rho})+\frac{1}{2}\check{\rho}$, $-y'_j(\check{\rho})+\frac{1}{2}\check{\rho}$, $-y_i(\check{\lambda}_\infty+\frac{1}{2}\check{\rho})+\frac{1}{2}\check{\rho}$, where $y_i,y'_j,y_\infty\in W$. In other words, we are considering the H-connections determined by the differential operator of the following type:

$$D_\theta-\left(\sum_{i=1}^{N}\frac{\theta-\theta_i}{z-z_i+\theta\theta_i}\left(y_i\left(\check{\lambda}+\frac{\check{\rho}}{2}\right)-\frac{\check{\rho}}{2}\right)+\sum_{j=1}^{m}\frac{\theta-\xi_j}{z-w_j+\theta\theta_j}\left(y'_j\left(\frac{\check{\rho}}{2}\right)-\frac{\check{\rho}}{2}\right)\right)+\text{nilp} \tag{35}$$

on $SC^*\backslash\infty$, where nilp stands for elements $f(z,\theta)$ from $\mathfrak{h}[\theta]((z))$ such that $\overline{f(z,\theta)}=\overline{D_\theta f(z,\theta)}=0$. Let us study its behaviour at infinity. Any connection $D_\theta+\alpha(\theta,u)$ on $\mathcal{D}^{-\check{\rho}}$ has the following expansion with respect to the coordinates $(u,\eta)=(\frac{-1}{z},\frac{\theta}{z})$:

$$D_\eta+u^{-1}\alpha(-u^{-1},-\eta u^{-1})+u^{-1}\eta\check{\rho}. \tag{36}$$

Therefore, considering $\frac{\eta}{u}$-coefficient in the expansion, we obtain the following constraint:

$$\sum_{i=1}^{N}\left(y_i(\check{\lambda}+\frac{\check{\rho}}{2})-\frac{\check{\rho}}{2}\right)+\sum_{i=1}^{m}\left(y'_i(\frac{\check{\rho}}{2})-\frac{\check{\rho}}{2}\right)=y'_\infty\left(-w_0(\check{\lambda}_\infty)+\frac{\check{\rho}}{2}\right)-\frac{\check{\rho}}{2}, \tag{37}$$

where $y'_\infty w_0=y_\infty$. This expression is expected from the consideration of the purely even case [12].

Let us denote the set of the considered above H-connections by $Conn(SC^*)^{RS}_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$.

Now one can associate to any such connection a G-oper on SC^* with regular singularities at the points $(\mathcal{Z}_i)$, $(\mathcal{W}_j)$, $(\infty,0)$ by setting, in familiar way, $\mathcal{F}=\mathcal{D}^{-\check{\rho}}\times_H G$, $\mathcal{F}_B=\mathcal{D}^{-\check{\rho}}\times_H B$.

Let us denote the set of superopers with regular singularities at $\mathcal{Z}_1\dots\mathcal{Z}_N,(\infty,0)$, whose restriction to the formal superdisc at any point $\mathcal{Z}_i$ or $(\infty,0)$ belongs to the space $s\mathrm{Op}_G(SD_{\mathcal{Z}_i})_{\check{\lambda}}$ or $s\mathrm{Op}_G(SD_{(\infty,0)})_{\check{\lambda}_\infty}$, by $s\mathrm{Op}_G(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$.

Then let $Conn(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}\subset Conn(SC^*)^{RS}_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$ be those H-connections with regular singularities, which are associated to $s\mathrm{Op}_G(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$ under the above correspondence. Therefore we have the map

$$Conn(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}\to s\mathrm{Op}_G(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}. \tag{38}$$

We can construct a Miura superoper associated with the image of this map, namely $\mathcal{F}'_B = \mathcal{D}^{-\check{\rho}} \times_H w_0 B$. Therefore, this map can be lifted to

$$\mathbf{b}_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty} : Conn(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty} \to s\mathrm{MOp}_G(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}. \tag{39}$$

Similarly to the purely even case, one can argue that this map is an isomorphism. Notice that for a given superoper $\tau \in s\mathrm{Op}_G(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$ (because of the absence of nontrivial monodromy), the space $s\mathrm{MOp}_G(SC^*)_\tau$ of the corresponding Miura superopers is isomorphic to G/B.

Similarly to the argument in the purely even case, we obtain the following theorem, which is an analogue of Theorem 3.1 of [12].

THEOREM 4.6. *The set of all connections* $Conn(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$, *which corresponds to a given oper* $\tau \in s\mathrm{Op}_G(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$, *is isomorphic to the set of points of the flag variety G/B.*

4.4. SPL_2-SUPEROPERS AND SUPER BETHE ANSATZ EQUATIONS

As we explained in the introduction one of the goals of this article is to establish the relation between the SPL_2-superopers with singularities on supersphere and the $\mathfrak{osp}(1|2)$ Gaudin model, namely that the Bethe equations of the corresponding Gaudin model will encode the relations between the residues of a given oper. This is expected because of the analogy with the standard purely even case where the same relation was established between PGL_2-opers with singularities on the Riemann sphere and the $\mathfrak{sl}(2)$ Gaudin model. It appeared that this relation is the simplest example of the so-called Geometric Langlands correspondence (for more details see e.g. [12]). We expect, again, based on the analogy with the purely even case that there is a higher rank generalization, which will possibly lead to the proper formulation of Geometric Langlands correspondence for simple superalgebras.

So, in this subsection we return back to the simplest nontrivial example of the superoper, related to supergroup SPL_2. In subsection 4.4 we obtained, that for a fixed superoper τ one can trivialize $\mathcal{F}$ using the fiber at $(\infty,0)$. Therefore, we have the trivialization of G/B-bundle and the map: $\phi_\tau : SC^* \to G/B$, so that $(\infty,0)$ maps into the point orbit of G/B. Also, in the case $G = SPL_2$, $G/B \cong SC^*$.

Similar to the purely even case, let us call the superoper τ $non-degenerate$ if i) $\phi_\tau(\mathcal{Z}_i)$ is in generic position with B, for any $i = 1, \dots, N$, and ii) the relative position of $\phi_\tau(x)$ and B is either generic or corresponds to a reflection for all $x \in SC^* \backslash (\infty,0)$. Since PGL_2 opers are non-degenerate for the generic choice of z_i, and those are the opers corresponding to the SPL_2-superopers, then any $\tau \in s\mathrm{Op}_{SPL(2)}(SC^*)_{(\mathcal{Z}_i),(\infty,0);\check{\lambda}_i,\check{\lambda}_\infty}$ for the generic choice of $\mathcal{Z}_i$ is non-degenerate. Also, let us consider the unique Miura superoper structure for τ, such that $\mathcal{F}_{B,(\infty,0)}$ and $\mathcal{F}'_{B,(\infty,0)}$ coincide, i.e. correspond to the point orbit in G/B.

The corresponding H-connections will have the following form:

$$\tilde{\nabla} = D_\theta - \sum_{i=1}^{N} \frac{\theta-\theta_i}{z-z_i-\theta\theta_i}\check{\lambda}_i + \sum_{j=1}^{m} \frac{\theta-\xi_j}{z-w_j-\theta\xi_j}\frac{\check{\alpha}}{2} + \text{nilp}, \tag{40}$$

where $\check{\lambda}_i = l_i\check{\omega}$, so that $l_i \in \mathbb{Z}_+$. Imposing the constraint from Proposition 4.5, we obtain that the following equations should hold for the corresponding oper to be monodromy free:

$$\sum_{i=1}^{N} \frac{2l_i}{w_j - z_i} - \sum_{s=1}^{m} \frac{2}{w_j - w_s} = 0 \tag{41}$$

Also, let us recall that the coweights $\check{\lambda}_i$ should also satisfy (37), which in our case simplifies to:

$$\sum_{i=1}^{N} l_i - m = l_\infty \tag{42}$$

Note, that the corresponding PGL_2-oper coweights, i.e. $2l_i$, are even: superopers associated with the odd weights will have a monodromy which will correspond to a reflection in θ variable, as it was explained above (see Remark after Lemma 4.1). The Equations (41) are exactly the Bethe ansatz equations for the $\mathfrak{osp}(1|2)$ Gaudin model studied in [17].

5. Some Remarks

In this article, we studied superopers for simple superalgebras with the root system, which admits the basis of odd roots. However, one can define a similar object for other types of superalgebras, just in such case it can be only locally defined (i.e. on a superdisc). The analogue of the expression (18) will be

$$\nabla = D_{z,\theta} + \sum_e a_e\Big(z,\theta\Big) f_e + \sum_o \theta a_o\Big(z,\theta\Big) f_o + \mu\Big(z,\theta\Big), \tag{43}$$

where the summation is over even and odd roots correspondingly and $a_{e,f}(z,\theta)$ are the even functions of z,θ with nonzero body. The resulting connection cannot be defined globally on the super Riemann surface; however, the operator $\nabla^2|_{\theta=0}$ can give rise to a connection for a G-bundle over a smooth curve underlying the super Riemann surface, while $\overline{\nabla^2}$ will give an oper for the underlying semisimple supergroup. This construction gives a generalization of opers in the case of any simple superalgebra.

In this paper we briefly considered an important relation between the spectrum of the Gaudin model and superopers on SC^*, which in fact could give an example of geometric Langlands correspondence in the case of superalgebras. For SPL_2-superopers and the Gaudin model for $\mathfrak{osp}(1|2)$ the spectrum was determined in fact by the underlying PGL_2-oper. Unfortunately so far the Gaudin models were not studied in the case of other superalgebras yet, so it is not clear whether such a relation holds for higher rank superalgebras.

We will address these and other important questions in the forthcoming publications.

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