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Representations of the Witt Algebra and Gl(n)-Opers

FRANCISCO JOSE PLAZA MARTÍN
Departamento de Matemáticas y IUFFYM, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain. e-mail: fplaza@usal.es

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Abstract. We exhibit a new link between certain representations of the Witt algebra and some Gl(n)-opers on the punctured disc. As applications, we discuss the connection with the KdV hierarchy and Virasoro constraints and how the Virasoro constraints of the so-called topological recursion fit in our approach.

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1. Introduction

During the last decades, representation theory of Virasoro (and Witt) algebra has been studied in depth; in particular, its study has led to significant results in the theory of vertex operator algebras. Indeed, opers have emerged as a fundamental object in the approach to the geometric Langlands program which makes use of VOA too. More recently, representations of Virasoro algebras are playing a significant role within the framework of intersection theory and the so-called topological recursion.

In this paper we exhibit a simple and direct procedure to go back and forth between representations of the Witt algebra and opers. On the other hand, the structure of the Virasoro operators of [19,22] is unveiled. Let us state the precise claims.

THEOREM 1.1. *To every n-cyclic action of $\mathcal{W}^+$ on $\mathcal{D}^1$ one associates a* Gl(n)*-oper on the punctured disc* $D^\times := \operatorname{Spec}\mathbb{C}((z^n))$.

Furthermore, every Gl(n)*-oper structure of* $\mathbb{C}((z))$*, as a rank n vector bundle on* $D^\times := \operatorname{Spec}\mathbb{C}((z^n))$*, arises in this way* (*up to conjugation*).

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Let us explain the notions used in the statement. $\mathcal{W}$ is the Witt algebra; that is, the Lie algebra that is freely generated by $\{L_k | k \in \mathbb{Z}\}$ as a $\mathbb{C}$-vector space and endowed with the following Lie bracket $[L_i, L_j] = (i-j)L_{i+j}$. Let us denote by $\mathcal{W}^+ \subset \mathcal{W}$ the Lie subalgebra generated by $\{L_k | k \geq -1\}$. Let $\mathcal{D}^1 := \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(\mathbb{C}((z)), \mathbb{C}((z)))$ be the Lie algebra generated by first-order differential operators. An *action* of $\mathcal{W}^+$ is a Lie algebra homomorphism $\rho : \mathcal{W}^+ \to \mathcal{D}^1$. The notions of *n-cyclic* and *conjugation* will be introduced in Section 2.

Here, it suffices to recall that a $\mathrm{Gl}(n)$-oper on the punctured disc, $D^\times$, is a rank n vector bundle $\mathcal{E}$ on $D^\times$ equipped with a flag $\mathcal{E}_0 := (0) \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$ of subbundles and a flat connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_{D^\times}$ such that the induced maps $\mathcal{E}_i/\mathcal{E}_{i-1} \to (\mathcal{E}_{i+1}/\mathcal{E}_i) \otimes \Omega_{D^\times}$ are a isomorphisms of $\mathcal{O}_{D^\times}$-bundles for all i (*transversality*).

Section 3 shows how additional hypothesis satisfied by the action are reflected in the corresponding oper. Indeed, the $\mathrm{Gl}(2)$-opers associated to certain 2-cyclic actions are actually defined on $\mathrm{Spec}(\mathbb{C}[z^{-2}])$ (see Theorem 3.9). In this situation, there is a third character in this play, namely, a point in the Sato Grassmannian of $\mathbb{C}((z))$. Furthermore, its τ-function satisfies the KdV hierarchy and Virasoro-like constraints simultaneously. Strickingly, the τ-functions arising in 2D gravity [14,17] fall into this situation.

The paper ends with a further application of our results; namely, we show that the families of Virasoro algebras used in the study of the Topological Recursion [19,21,22] also fit into our framework and have a natural geometrical interpretation (see Section 4). In particular, it is shown that a family of τ-functions satisfying KdV and Virasoro is equivalent to a family of actions of the Witt algebra (Theorem 4.1). It is worth noticing that some instances of such 1-parameter families (e.g. the sine curve in [22]) are indeed the spectral curve in Eynard–Orantin theory [6]. Thus, we hope that our techniques might shed some light to the underlying geometry of Eynard–Orantin approach.

2. From Representations of $\mathcal{W}^+$ to Opers

2.1. ACTIONS OF WITT ALGEBRAS

Let V denote a 1-dimensional $\mathbb{C}((z))$-vector space and let $\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V, V)$ be the Lie algebra generated by first-order differential operators. The symbol map is

$$\sigma : \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V, V) \longrightarrow \mathrm{Der}_{\mathbb{C}}(\mathbb{C}((z))) = \mathbb{C}((z))\partial_z \xrightarrow{\sim} \mathbb{C}((z))$$

where the last map sends $f(z)\partial_z$ to $f(z)$.

We are interested in pairs (V, ρ) consisting of a 1-dimensional $\mathbb{C}((z))$-vector space, V, and a Lie algebra homomorphism

$$\mathcal{W}^+ \xrightarrow{\rho} \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V, V)$$

The pair (V, ρ) will be called an *action* of $\mathcal{W}^+$.

This subsection is concerned with explicit descriptions of the actions of $\mathcal{W}^+$ on $\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(\mathbb{C}((z)),\mathbb{C}((z)))$ that, albeit many properties of $\mathcal{W}$-modules are known, seems to be brand new. Nevertheless, similar results can be proved for V arbitrary by fixing an isomorphism $V\simeq\mathbb{C}((z))$ (see Section 2.2 for the dependence on the choice of the isomorphism).

There are two reasons for restricting ourselves to the case of first-order differential operators of V. First, this is the relevant situation when dealing with 2D gravity. Second, the equivalence of categories between Atiyah algebras and differential operator algebras [2] means that $\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V,V)$ is a natural object to study.

THEOREM 2.1. *Let V be $\mathbb{C}((z))$ and $\rho:\mathcal{W}^+\to\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V,V)$ be a $\mathbb{C}$-linear map such that $\rho\neq 0$. Then, the map ρ is a Lie algebra homomorphism if and only if there exist functions $h(z),b(z)\in\mathbb{C}((z))$ and a constant $c\in\mathbb{C}$ such that $h'(z)\neq 0$ and*

$$\rho(L_i)=\frac{-h(z)^{i+1}}{h'(z)}\partial_z-(i+1)c\cdot h(z)^i+\frac{h(z)^{i+1}}{h'(z)}b(z) \tag{1}$$

Proof. The converse is straightforward. Let us prove the direct one.

Let us write $\rho(L_k)=a_k(z)\partial_z+b_k(z)$. Since ρ is a map of Lie algebras, the expression of the bracket $[L_i,L_j]=(i-j)L_{i+j}$ yields

$$a_i(z)a_j'(z)-a_j(z)a_i'(z)=(i-j)a_{i+j}(z) \tag{2}$$
$$a_i(z)b_j'(z)-a_j(z)b_i'(z)=(i-j)b_{i+j}(z) \tag{3}$$

Observe that if $a_{-1}(z)=\sigma(\rho(L_{-1}))=0$, then we let i be equal to -1 in Equation (2) and have that $a_j(z)=0$ for all $j\geq -1$. Substituting in Equation (3), it follows that $\rho\equiv 0$.

Hence, we now assume that $a_{-1}(z)=\sigma(\rho(L_{-1}))\neq 0$. Let us fix L_{-1} and solve this system in terms of its coefficients.

Letting $i=-1$ in Equation (2), dividing by $a_{-1}(z)^2$ and integrating, it follows that $a_j(z)=-(1+j)a_{-1}(z)\int^z a_{j-1}(t)a_{-1}(t)^{-2}\,\mathrm{d}t$. Hence, $a_j(z)$ can be determined recursively from $a_{-1}(z)$. Indeed, the case $j=0$ yields $a_0(z)=a_{-1}(z)(\alpha-\int^z\frac{\mathrm{d}t}{a_{-1}(t)})$ for $\alpha\in\mathbb{C}$. Since $a_{-1}(z),a_0(z)\in\mathbb{C}((z))$, it follows that $(\alpha-\int^z\frac{\mathrm{d}t}{a_{-1}(t)})$ must lie in $\mathbb{C}((z))$; i.e., there exists $h(z)\in\mathbb{C}((z))$ such that

$$a_{-1}(z)=\frac{-1}{h'(z)}$$

Thus, setting the free term of $h(z)$ to be equal to that constant, it follows that

$$a_0(z)=\frac{-h(z)}{h'(z)}$$

Now, induction procedure proves straightforwardly that

$$a_i(z)=\frac{-h(z)^{i+1}}{h'(z)}\qquad i\geq -1$$

Let us now focus on b_i's. Firstly, let us deal with the case $h(z)=z^n$; hence, $a_i(z)=-\frac{1}{n}z^{ni+1}$ and Equation (3) acquires the following shape

$$-\frac{1}{n}z^{ni+1}b'_j(z)+\frac{1}{n}z^{nj+1}b'_i(z)=(i-j)b_{i+j}(z) \tag{4}$$

Let us write $b_j(z)$ as $\sum_k b_{j,k}z^k$, where $b_{j,k}=0$ for $k\ll 0$. Computing the coefficients of z^k, in Equation (4) one has the relation

$$-\frac{1}{n}(k-ni)b_{j,k-ni}+\frac{1}{n}(k-nj)b_{i,k-nj}=(i-j)b_{i+j,k}$$

The case $j=0$ implies that $(k-ni)b_{i,k}=(k-ni)b_{0,k-ni}$ and, therefore $b_{i,k}=b_{0,k-ni}$ for $k\neq ni$; that is, the difference between $z^{-ni}b_i(z)$ and $b_0(z)$ is a constant. Expressing this condition in terms of $b_{-1}(z)$, the following formula for $b_i(z)$ holds

$$b_i(z)=(c_i+b_{-1}(z)z^n)z^{ni} \tag{5}$$

for some $c_i\in\mathbb{C}$ and $c_{-1}=0$. Plugging this into Equation (3) and setting i equal to -1, we find a constraint for the c_i

$$jc_j+c_{-1}-(j+1)c_{j-1}=0 \qquad c_{-1}=0$$

whose general solution is

$$c_j=-c\cdot(j+1) \tag{6}$$

for a complex number $c=-c_0\in\mathbb{C}$. Bearing in mind that $h'(z)$ is invertible, there is no harm in assuming that $b_{-1}(z)$ is of the form $\frac{1}{h'(z)}b(z)$. Thus, from equations (5) and (6), the general solution for the case $h(z)=z^n$ is

$$b_i(z)=\left(-(i+1)c+z^n\frac{b(z)}{nz^{n-1}}\right)z^{ni} \tag{7}$$

The general case, i.e. for $h(z)$ arbitrary, follows from the fact that there is a $\mathbb{C}$-algebra automorphism of $\mathbb{C}[[z]]$, ϕ, such that $\phi(h(z))=z^n$ where $h(z)=a_nz^n+a_{n+1}z^{n+1}+\dots$ and $a_n\neq 0$. That is, in order to solve Equation (3), we consider ϕ, such that $\phi(h(z))=z^n$. We transform Equation (3) by ϕ, which is Equation (4), and consider its solutions (7). Thus, transforming the solutions by the inverse automorphism, ϕ^{-1}, we have that the general solution for Equation (3) is as follows

$$b_i(z)=\left(-(i+1)c+h(z)\frac{b(z)}{h'(z)}\right)h(z)^i$$

□

Remark 2.2. It is worth to observe, from the proof above, that the action ρ is determined by its restriction to the subalgebra

$$\mathfrak{sl}_2(\mathbb{C})\simeq\langle L_{-1},L_0,L_1\rangle\subset\mathcal{W}^+$$

Let us point out some consequences of Theorem 2.1. First, note that

$$[\rho(L_k), h(z)^j] = -j \cdot h(z)^{k+j} \tag{8}$$

as $\mathbb{C}$-linear operators on V. Second, if $\sigma(\rho(L_{-1}))\neq 0$, then ρ and $\sigma\circ\rho$ are injective and

$$\mathrm{Im}(\sigma\circ\rho)=\frac{1}{h'(z)}\mathbb{C}[h(z)] \tag{9}$$

EXAMPLE 2.3. Let $\widehat{\mathcal{D}}$ (a.k.a. $\mathcal{W}_{1+\infty}$) denote the unique non-trivial central extension of the Lie algebra of differential operators on the circle. The authors of [7] carried out an in-depth study of its representations with the help of the theory of vertex operator algebras. In this context, they consider two 1-parameter families of Virasoro algebras (see [7, Equation (1.7)])

$$\{L_k^+(\beta)=-z^{k+1}\partial_z-\beta(k+1)z^k \,|\, k\geq -1\}$$
$$\{L_k^-(\beta)=-z^{k+1}\partial_z-(k+\beta(-k+1))z^k \,|\, k\geq -1\}$$

Observe that these families correspond to the data $h(z)=z$, $c=\beta$ and $b(z)=0$ and $h(z)=z$, $c=1-\beta$ and $b(z)=(1-2\beta)z^{-1}$, respectively.

EXAMPLE 2.4. From the point of view of mathematical physics, recall the so-called Virasoro constraints arising in 2D quantum gravity [5,11,14,15,17]. Indeed, let us show that such differential equations are an instance of our previous Theorem. First, let us recall that a differential operator of $V=\mathbb{C}((z))$ also acts on the Sato Grassmannian $\mathrm{Gr}(V)$ [25]. Since it preserves the determinant bundle, it yields a transformation of $H^0(\mathrm{Gr}(V),\mathrm{Det}^*)=\Lambda^{\frac{\infty}{2}}V$. Having in mind that $\mathcal{W}^+$ has no non-trivial central extensions and the bosonization isomorphism, we conclude that an action $\rho:\mathcal{W}^+\to\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V,V)$ can be lifted to

$$\tilde{\rho}:\mathcal{W}^+\to \mathrm{End}(\mathbb{C}[[t_1,t_2,\ldots]])$$

We address the reader to [16] and references therein for this standard construction. An alternative approach to this construction is based on a *quantization* procedure [11]. Let us review some instances of actions appearing in this setup (see [24] for the details).

If we look for ρ such that $\tilde{\rho}(L_k)$ coincide with the operators of [15, Section 2.2], we find out that it corresponds to the data $h(z)=z^{-1}, c=\frac{1}{2}, b(z)=-z^{-1}$. The choice $h(z)=z^{-2}, c=\frac{1}{2}, b(z)=-\frac{3}{2}z^{-1}$ produces, up to rescaling, the operators consider by Dijkgraff–Verlinde–Verlinde [5, Equation 3.5] and Givental [11, Section 3]. The latter set of data can be obtained from the former by considering the subalgebra $\{\frac{1}{2}\tilde{\rho}(L_{2k})\}$. Analogously, the operators considered in [14] and [17] come from the triple $h(z)=z^{-2}, c=\frac{1}{2}, b(z)=z^{-4}-\frac{3}{2}z^{-1}$.

Sometimes, it is not relevant the precise expression of each operator $\rho(L_k)$ but what is important is the Lie algebra generated by them; that is, $\mathrm{Im}(\rho)$. It is, therefore, natural to wonder whether ρ is determined by the Lie algebra $\mathrm{Im}(\rho)$.

For studying this problem, let $\mathfrak{v}: \mathbb{C}((z)) \to \mathbb{Z} \cup \{\infty\}$ be the valuation associated with z; that is, $\mathfrak{v}(0)=\infty$ and, for $h(z)\neq 0$, $\mathfrak{v}(h(z))=a$ iff a is the largest integer number such that $h(z)\in z^a\mathbb{C}[[z]]$ and, in this situation, a will be called the *order* of h.

THEOREM 2.5. *Let* $(V=\mathbb{C}((z)), \rho_i)$ *(for* $i=1,2$*) be the action of* $\mathcal{W}^+$ *associated with elements* $h_i(z), c_i, b_i(z)$ *as in Theorem 2.1. Assume that* $h_i'(z)\neq 0$.

If $\mathrm{Im}\,\rho_1 = \mathrm{Im}\,\rho_2$ *and the signs of* $\mathfrak{v}(h_1(z))$ *and* $\mathfrak{v}(h_2(z))$ *are equal, then* $b_1(z)=b_2(z)$, $c_1=c_2$ *and* $h_1(z)=\alpha h_2(z)+\beta$ *for some* $\alpha\in\mathbb{C}^*, \beta\in\mathbb{C}$.

Conversely, if $b_1(z)=b_2(z)$, $c_1=c_2$ *and* $h_1(z)=\alpha h_2(z)+\beta$ *for some* $\alpha\in\mathbb{C}^*, \beta\in\mathbb{C}$*, then* $\mathrm{Im}\,\rho_1=\mathrm{Im}\,\rho_2$*. Moreover, there exists a Lie algebra automorphism* ϕ *of* $\mathcal{W}^+$ *such that* $\rho_2=\rho_1\circ\phi$.

Proof. From the hypothesis $\mathrm{Im}\,\rho_1=\mathrm{Im}\,\rho_2$, it holds that there exist $\{\lambda_{kl} | k, l\geq -1\}$ such that

$$\rho_1(L_k) = \sum_{l\geq -1} \lambda_{kl}\rho_2(L_l)$$

By the explicit expression obtained in Theorem 2.1, this identity is equivalent to the equations

$$\frac{h_1(z)^{k+1}}{h_1'(z)} = \sum_{l\geq -1} \lambda_{kl}\frac{h_2(z)^{l+1}}{h_2'(z)} \tag{10}$$

and

$$\begin{aligned}&\frac{h_1(z)^{k+1}}{h_1'(z)}b_1(z) - (k+1)c_1h_1(z)^k\\ &\quad= \sum_{l\geq -1}\lambda_{kl}\left(\frac{h_2(z)^{l+1}}{h_2'(z)}b_2(z) - (l+1)c_2h_2(z)^l\right)\end{aligned} \tag{11}$$

Observe that the derivative of Equation (10) w.r.t. z yields

$$\begin{aligned}(k+1)h_1(z)^k - \frac{h_1(z)^{k+1}h_1''(z)}{h_1'(z)^2} &= \sum_{l\geq -1}\lambda_{kl}\left((l+1)h_2(z)^l - \frac{h_2(z)^{l+1}h_2''(z)}{h_2'(z)^2}\right)\\ &= \sum_{l\geq -1}\lambda_{kl}(l+1)h_2(z)^l - \frac{h_1(z)^{k+1}}{h_1'(z)}\cdot\frac{h_2''(z)}{h_2'(z)}\end{aligned} \tag{12}$$

One computes Equation (11) plus Equation (10) times $(-b_2(z))$ plus Equation (12) multiplied by c_2, and one obtains

$$\frac{h_1(z)^{k+1}}{h_1'(z)}\left((b_1(z)-b_2(z))-(k+1)(c_1-c_2)\frac{h_1'(z)}{h_1(z)}-\left(\frac{h_1''(z)}{h_1'(z)}-\frac{h_2''(z)}{h_2'(z)}\right)c_2\right)=0$$

Since this holds for all $k\geq -1$, it follows that

$$c_1-c_2=0 \tag{13}$$
$$(b_1(z)-b_2(z))-\left(\frac{h_1''(z)}{h_1'(z)}-\frac{h_2''(z)}{h_2'(z)}\right)c_2=0 \tag{14}$$

Hence $c_1=c_2$.

Further, identity (9) shows that $\frac{1}{h_1'(z)}\mathbb{C}[h_1(z)]=\frac{1}{h_2'(z)}\mathbb{C}[h_2(z)]$. Hence, there are polynomials p_i such that $\frac{1}{h_1'(z)}=\frac{p_2(h_2(z))}{h_2'(z)}$ and $\frac{1}{h_2'(z)}=\frac{p_1(h_1(z))}{h_1'(z)}$. These identities imply that

$$p_1(h_1(z))\cdot p_2(h_2(z))=1$$

The assumption about the signs of $\mathfrak{v}(h_i(z))$ implies that p_i is constant for $i=1,2$, say $p_1(x)=\alpha\in\mathbb{C}^*$. And, therefore, $h_1'(z)=\alpha h_2'(z)$, so that there exists $\beta\in\mathbb{C}$ with $h_1(z)=\alpha h_2(z)+\beta$.

Finally, substituting in Equation (14), one has that $b_1(z)=b_2(z)$.

Let us now prove the converse. Using the formula (1), a long although straightforward computation shows

$$\begin{aligned}\rho_2(L_{-1})&=\frac{1}{\alpha}\rho_1(L_{-1})\\ \rho_2(L_i)&=h_2(z)^i\left(\frac{h_2(z)}{\alpha}\rho_1(L_{-1})-(i+1)c\right)\\ &=\left(\frac{\beta}{\alpha}\right)^{i+1}\rho_1(L_{-1})+\sum_{j=0}^{i-1}\left(\binom{i}{j}+\binom{i}{j+1}\right)\left(\frac{\beta}{\alpha}\right)^{i-j}\rho_1(L_j)+\rho_1(L_i)\qquad \forall i\geq -1\end{aligned}$$

This explicit expression shows at once that $\operatorname{Im}\rho_2=\operatorname{Im}\rho_1$.

Finally, consider

$$\phi(L_i):=\left(\frac{\beta}{\alpha}\right)^{i+1}L_{-1}+\sum_{j=0}^{i-1}\binom{i+1}{j+1}\left(\frac{\beta}{\alpha}\right)^{i-j}L_j+L_i \tag{15}$$

The fact that $\phi=(\rho_1|_{\operatorname{Im}\rho_1})^{-1}\circ\rho_2$ implies that ϕ is a Lie algebra automorphism of $\mathcal{W}^+$ and that $\rho_2=\rho_1\circ\phi$. □

2.2. CONJUGATION OF ACTIONS

Let us begin with an example that will illustrate us how the notion of conjugated action (*gauge action*) should be generalized. Assume that a differential equation, $P\psi(z)=0$, is to be solved it by virtue of a replacement $\psi(z)=v(z)\phi(z)$, for a given function $v(z)$. That is, we must solve $P(v(z)\phi(z))=0$ for an unknown function $\phi(z)$. This is equivalent to solving $(v(z)^{-1}\circ P\circ v(z))\phi(z)=0$, where $v(z)$ is regarded as an operator; namely, the homothety of ratio $v(z)$. For instance, if P is a first-order differential operator with symbol $\sigma(P)$, it holds that

$$v(z)^{-1}\,(P\psi(z)) = (v(z)^{-1}\circ P\circ v(z))\phi(z) = \left(P+\sigma(P)\frac{v'(z)}{v(z)}\right)\phi(z)$$

Therefore, solving the differential equation $P\psi(z)=0$ is equivalent to solving $\left(P+\sigma(P)\frac{v'(z)}{v(z)}\right)\phi(z)=0.$

Let us recall from [12] the definition of the group of semilinear transformations and some of its properties. The group of semilinear transformations of a finite dimensional $\mathbb{C}((z))$-vector space V, denoted by $\mathrm{SGl}_{\mathbb{C}((z))}(V)$, consists of $\mathbb{C}$-linear automorphisms $\gamma: V\to V$ such that there exists a $\mathbb{C}$-algebra automorphism of $\mathbb{C}((z))$, g, satisfying

$$\gamma(f(z)\cdot v) = g(f(z))\cdot\gamma(v) \qquad \forall f(z)\in\mathbb{C}((z)),\, v\in V \tag{16}$$

and, therefore, $\mathrm{SGl}(\mathbb{C}((z)))=\mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}\,\mathbb{C}((z))\ltimes\mathbb{C}((z))^*$.

The Lie algebra of $\mathrm{SGl}_{\mathbb{C}((z))}(V)$ consists of first-order differential operators on V with scalar symbol, $\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V,V)$, and the symbol coincides with the map induced by the group homomorphism that sends γ to g (related by Equation (16)) between their Lie algebras.

THEOREM 2.6. *The space* $\mathrm{Hom}_{\textit{Lie-alg}}(\mathcal{W}^+,\mathcal{D}^1)\setminus\{0\}$ *carries an action of the group* $\mathrm{SGl}(\mathbb{C}((z)))$ *by conjugation and the quotient space is*

$$\mathbb{Z}\times\mathbb{C}\times\left(\mathbb{C}((z))/\mathbb{Z}z^{-1}+\mathbb{C}[[z]]\right)$$

More explicitly, an action ρ*, corresponding to a triple* $(h(z),c,b(z))$ *is mapped to* $(\mathfrak{v}(h(z)),c,\bar{b}(z))$ *(*$\bar{b}(z)$ *being the equivalence class of* $b(z)$*).*

Proof. Let us begin studying the action of the automorphism group $G:=\mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}$ $\mathbb{C}((z))$ (for a study and applications of this group, see [23]). Let us denote elements of G with big Greek letters $(\Phi,\Psi,\ldots)$ and, for each of them, let the corresponding small Greek letter denote the image of z; that is

$$\Phi(f(z)) = f(\phi(z))$$

and observe that $\mathfrak{v}(\phi(z))=1$ in order for Φ to be an isomorphism.

Consider the action of G on the space of actions by conjugation; i.e.

$$(\Phi, \rho) \mapsto \rho^{\Phi} \qquad \text{where } \rho^{\Phi}(L_k) := \Phi \circ \rho(L_k) \circ \Phi^{-1} \quad \forall k$$

Let us check that this definition makes sense. Let ρ be given by a triple $(h(z), c, b(z))$. It is straightforward that

$$\rho^{\Phi}(L_{-1})f(z) = (\Phi \circ \rho(L_{-1}) \circ \Phi^{-1})f(z) = \Phi\left(\left(-\frac{1}{h'(z)}\partial_z + \frac{b(z)}{h'(z)}\right) f(\phi(z))\right)$$
$$= \left(-\frac{\phi'(\phi^{-1}(z))}{h'(\phi^{-1}(z))}\partial_z + \frac{b(\phi^{-1}(z))}{h'(\phi^{-1}(z))}\right) f(z)$$

Note that expanding and derivating the identity $(\Phi \circ \Phi^{-1})(z) = z$, one gets that $\phi'(\phi^{-1}(z)) \cdot \phi(z)^{-2}\phi'(z) = 1$ and, thus

$$\frac{\phi'(\phi^{-1}(z))}{h'(\phi^{-1}(z))} = \frac{1}{\partial_z h(\phi^{-1}(z))}$$

Summing up, the transformation Φ acts on triples as follows

$$(\Phi, (h(z), c, b(z))) \mapsto (h(\phi^{-1}(z)), c, b(\phi^{-1}(z)))$$

Second, we study the action of $\mathbb{C}((z))^*$. Bearing in mind the discussion of the beginning of this subsection, we consider the action

$$(s(z), \rho) \mapsto \left(s(z) \circ \rho \circ s(z)^{-1}\right)$$

so that, in terms of triples, it holds that

$$(s(z), (h(z), c, b(z))) \mapsto \left(h(z), c, b(z) - \frac{s'(z)}{s(z)}\right)$$

One checks easily that the first defined action intertwines the second one. Hence, they yield an action of the $\mathrm{SGl}(\mathbb{C}((z)))$. □

Remarkably, the conjugation also makes sense if $v(z)$ is replaced by any linear operator on the space of functions such that $\frac{v'(z)}{v(z)}$ can be identified with an element in $\mathbb{C}((z))$. This is the case of the example at the beginning of this subsection; of functions $v(z)$ admitting an asymptotic expansion at 0; and of formal expressions $v(z) := \exp(\int s(z)\, \mathrm{d}z)$ where $s(z) \in \mathbb{C}((z))$. In the latter case, the quotient $\frac{v'(z)}{v(z)}$ will be identified with $s(z)$. The conjugation by $\exp(-\frac{2}{3}z^{-3})$ was used in [14] when solving a differential equation. For another example, let us consider $v(z)$ to be a solution of the second-order differential equation $v''(z) + \frac{1}{2}S(h)v(z) = 0$, where $S(h)$ denotes the Schwarzian derivative of h, such that $\frac{v'(z)}{v(z)} \in \mathbb{C}((z))$, which holds true in many cases (e.g. whenever $S(h) \in \mathbb{C}((z))$).

It is worth noticing that once $\frac{v'(z)}{v(z)}$ is thought of as an element of $\mathbb{C}((z))$, $\frac{v''(z)}{v(z)}$ will be identified with $\left(\frac{v'(z)}{v(z)}\right)^2 + \left(\frac{v'(z)}{v(z)}\right)' \in \mathbb{C}((z))$. By abuse of notation, we define $\mathrm{d}\log v(z) := \frac{v'(z)}{v(z)}$.

Thus, for $P\in\mathcal{D}^1(\mathbb{C}((z)))$ and $v(z)$ as above, we consider another first-order differential operator $P^v\in\mathcal{D}^1(\mathbb{C}((z))\otimes_{\mathbb{C}}\mathbb{C}v(z))$ defined by

$$P^v(f(z)\otimes v(z)) := \left(\left(P+\sigma(P)\frac{v'(z)}{v(z)}\right)(f)\right)\otimes v(z)$$

The induced map from $\mathcal{D}^1(\mathbb{C}((z)))$ to $\mathcal{D}^1(\mathbb{C}((z))\otimes_{\mathbb{C}}\mathbb{C}v(z))$ is a Lie algebra homomorphism.

DEFINITION 2.7. The conjugated action of (V,ρ) by $v(z)$ is the pair (V^v,ρ^v), consisting of the 1-dimensional $\mathbb{C}((z))$-vector space $V^v:=V\otimes_{\mathbb{C}}\mathbb{C}v(z)$ together with the action defined by

$$\rho^v(L_k)(f(z)\otimes v(z)) := \left(\rho(L_k)(f(z))+\sigma(\rho(L_k))f(z)\frac{v'(z)}{v(z)}\right)\otimes v(z)) \tag{17}$$

In particular, if the data $h(z), c, b(z)$ define an action ρ, then $h(z), c, b(z)-\frac{v'(z)}{v(z)}$ define ρ^v.

2.3. OPERS ON THE PUNCTURED DISC

Regarding the definition of the opers, which were introduced by Drinfeld and Sokolov and generalized by Beilinson and Drinfeld, we refer interested readers to [8,9].

DEFINITION 2.8. An action (V,ρ) is said to be *n-cyclic* if

$$\{1, \rho(L_{-1})(1), \dots, \rho(L_{-1})^{n-1}(1)\}$$

is a basis of V as $\widehat{\mathbb{C}[h(z)]}_{(0)}$-module. Here ρ is given by the triple $(h(z), c, b(z))$, n is the absolute value of $\mathfrak{v}(h)$, the subindex (0) denotes the function field and the superscript $\widehat{\ }$ is the z-adic completion.

Now, we are ready to prove our main result.

Proof of Theorem 1.1. We deal with the case $\mathfrak{v}(h)<0$ being the opposite one similar. Consider an automorphism Φ of $\mathbb{C}((z))$ as $\mathbb{C}$-algebra such that $\Phi(z^{-n})=h(z)$ and conjugate the action by Φ (see 2.2). That is, it can be assumed that $h(z)=z^{-n}$.

Let $\mathcal{E}$ be the vector bundle on $\operatorname{Spec}\mathbb{C}((h(z)^{-1}))$ defined by $\mathbb{C}((z))$. Hence, the subbundles $\mathcal{E}_i$, associated with

$$\mathbb{C}((h(z)^{-1}))\otimes_{\mathbb{C}} <1, \rho(L_{-1})(1), \dots, \rho(L_{-1})^i(1)> \subseteq \mathbb{C}((z)) \qquad i=1,\dots,n$$

define a flag of vector bundles $0\subset\mathcal{E}_1\subset\ldots\subset\mathcal{E}_{n-1}\subset\mathcal{E}_n=\mathcal{E}$. Further, the inclusions are strict since ρ is n-cyclic.

Let us see that $\mathcal{E}$ carries a connection. Indeed, let $\mathrm{d}:\mathbb{C}((h(z)^{-1}))\to\Omega_{\mathbb{C}((h(z)^{-1}))/\mathbb{C}}$ be the differential and consider the $\mathbb{C}$-linear map $\rho(L_{-1})\otimes\mathrm{d}\,h-1\otimes\mathrm{d}$:

$$\begin{aligned}\mathbb{C}((z))\otimes_{\mathbb{C}}\mathbb{C}((h(z)^{-1})) &\longrightarrow \mathbb{C}((z))\otimes_{\mathbb{C}}\Omega_{\mathbb{C}((h(z)^{-1}))/\mathbb{C}}\\ f\otimes a &\longmapsto \rho(L_{-1})f\otimes a\,\mathrm{d}\,h-f\otimes\mathrm{d}\,a\end{aligned}\tag{18}$$

One checks that when composing it with the canonical map $\mathbb{C}((z))\otimes_{\mathbb{C}}\Omega_{\mathbb{C}((h(z)^{-1}))/\mathbb{C}}\to\mathbb{C}((z))\otimes_{\mathbb{C}((h(z)^{-1}))}\Omega_{\mathbb{C}((h(z)^{-1}))/\mathbb{C}}$, the images of $f\otimes a$ and of $af\otimes 1$ do coincide and, therefore, we obtain a $\mathbb{C}$-linear map

$$\mathbb{C}((z))\longrightarrow\mathbb{C}((z))\otimes_{\mathbb{C}((h(z)^{-1}))}\Omega_{\mathbb{C}((h(z)^{-1}))/\mathbb{C}}\tag{19}$$

which defines a connection $\nabla:\mathcal{E}\to\mathcal{E}\otimes\Omega_{\mathrm{Spec}\,\mathbb{C}((h(z)^{-1}))}$. Furthermore, from the equations (8) and (18) it follows that the map induced by ∇

$$\mathcal{E}_i/\mathcal{E}_{i-1}\longrightarrow(\mathcal{E}_{i+1}/\mathcal{E}_i)\otimes\Omega_{\mathrm{Spec}\,\mathbb{C}((h(z)^{-1}))}$$

sends $\rho(L_{-1})^i(1)$ to $\rho(L_{-1})^{i+1}(1)\otimes\mathrm{d}\,h$ and that it is an isomorphism of line bundles.

Let us now prove the converse. Let ∇ be the connection of the oper structure on $\mathbb{C}((z))$. For the sake of clarity, let us denote $h(z):=z^{-n}$. Recall that $\mathrm{Der}(\mathbb{C}((h(z)^{-1})))$ is generated by

$$\left\{h(z)^{k+1}\frac{\partial}{\partial h}=-\frac{1}{n}z^{-kn+1}\frac{\partial}{\partial z}\mid k\in\mathbb{Z}\right\}$$

Let $\langle,\rangle$ be the pairing of differentials with derivations. We claim that

$$\nabla_D(f):=\langle\nabla f,D\rangle\qquad\text{for } f\in\mathbb{C}((z))\,,\ D\in\mathrm{Der}(\mathbb{C}((h(z)^{-1})))$$

is a differential operator of $\mathbb{C}((z))$ as a $\mathbb{C}((h(z)^{-1}))$-module or, what amounts to the same, that $\nabla_{D,a}$

$$\nabla_{D,a}(f):=\nabla_D(af)-a\nabla_D(f)\qquad\text{for } f\in\mathbb{C}((z))\,,\ a\in\mathbb{C}((h(z)^{-1}))$$

is an endomorphism of $\mathbb{C}((z))$ as a $\mathbb{C}((h(z)^{-1}))$-module. Using the properties of the connection ∇, one obtains that

$$\begin{aligned}\nabla_{D,a}(f)&=\langle\nabla(af),D\rangle-a\langle\nabla(f),D\rangle\\&=\langle a\nabla(f)+f\,\mathrm{d}\,a,D\rangle-a\langle\nabla(f),D\rangle=f\,D(a)\end{aligned}$$

where d is the differential, $f\in\mathbb{C}((z))$ and $a\in\mathbb{C}((h(z)^{-1}))$. Accordingly, $\nabla_{D,a}$ is the homothety of ratio $D(a)$ and, therefore, linear. The same argument shows that

$$[\nabla_D,a]=Da\qquad\text{for } a\in\mathbb{C}((h(z)^{-1}))$$

where a and Da are regarded as operators on $\mathbb{C}((z))$ (by homotheties).

Let us now set $D = \frac{1}{n} z^{n+1} \frac{\partial}{\partial z} = -\frac{1}{h'(z)} \frac{\partial}{\partial z}$. Since ∇_D is a first order differential operator of $\mathbb{C}((z))$, it can be written as $-\frac{1}{\alpha'} \frac{\partial}{\partial z} + \frac{\beta}{\alpha'}$ for certain functions $\alpha, \beta \in \mathbb{C}((z))$. It follows that

$$\frac{1}{n} z^{n+1} \frac{\partial a}{\partial z} = Da = [\nabla_D, a] = \left[-\frac{1}{\alpha'} \frac{\partial}{\partial z} + \frac{\beta}{\alpha'}, a \right] = -\frac{1}{\alpha'} \frac{\partial a}{\partial z} \qquad a \in \mathbb{C}((h(z)^{-1}))$$

and, thus, $\alpha' = h'(z)$. Similarly, one has that the symbol of the differential operator ∇_D is equal to D.

Let us consider the linear map

$$\begin{aligned} &\mathcal{W}^+ \xrightarrow{\rho} \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(\mathbb{C}((z)), \mathbb{C}((z))) \\ &L_i \mapsto \rho(L_i) := \nabla_{D_i} \end{aligned}$$

where $D_i := -\frac{h(z)^{i+1}}{h'(z)} \frac{\partial}{\partial z}$. The fact that ρ is a morphism of Lie algebras is derived from the flatness of ∇ as follows

$$[\rho(L_i), \rho(L_j)] = [\nabla_{D_i}, \nabla_{D_j}] = \nabla_{[D_i, D_j]} = \nabla_{(i-j)D_{i+j}} = (i-j)\rho(L_{i+j})$$

It remains to check the compatibility with the construction given in the first half of the proof. For this goal, recall from [4, Lemma 1.3] that there exists a cyclic vector $v(z) \in \mathbb{C}((z))$ for the oper $(\mathbb{C}((z)), \nabla)$. In particular, this fact implies that $\{v(z), \rho(L_{-1})(v), \dots, \rho(L_{-1})^{n-1}(v)\}$ is a basis of $\mathbb{C}((z))$ as $\mathbb{C}((z^n))$-vector space. Conjugate ρ by $\frac{1}{v(z)}$ so that

$$\{1, \rho^{\frac{1}{v}}(L_{-1})(1), \dots, (\rho^{\frac{1}{v}}(L_{-1}))^{n-1}(1)\}$$

becomes a basis of $\mathbb{C}((z))$. Considering the action $(\mathbb{C}((z)), \rho^{\frac{1}{v}})$, the conclusion follows. □

Remark 2.9. Recalling the close relationship between vertex algebras and infinite dimensional representations of the Virasoro algebra (e.g. [13]), we expect to interpret the action of $\mathcal{W}^+$ on $\mathbb{C}[[t_1, t_2, \dots]]$ in terms of vertex operators. Further, the techniques of [8, Chap. 5] can be applied to the above results in order to associate to a general action (V, ρ) a $\mathrm{Gl}(n)$-oper on the abstract punctured disc $D^\times = \operatorname{Spec} \mathbb{C}((\bar{z}))$. Both facts will help to understand our approach within Frenkel's framework of the geometric Langlands program [9]. It is worth noticing the salient role of the punctured disc in this picture [10, Remark 1]

3. The KdV Hierarchy and $Gl(2)$-Opers

3.1. STABILIZER

Following Section 2.1, let (V, ρ) be an action of $\mathcal{W}^+$.

Let $U \subset V$ denote a $\mathbb{C}$-vector subspace and let A_U denote its stabilizer; that is,

$$A_U := \operatorname{Stab}(U) = \{f \in \mathbb{C}((z)) | fU \subseteq U\}$$

We say that U is L_{-1}-*stable* when $\rho(L_{-1})U \subseteq U$. Similarly, we say that U is $\mathcal{W}^+$-stable when $\rho(L)U \subseteq U$ for all $L \in \mathcal{W}^+$; or, what is tantamount to this, $\rho(L_k)U \subseteq U$ for all $k \geq -1$.

Let us fix the following notation. Let $V_+ \subseteq V$ denote a $\mathbb{C}[[z]]$-submodule of V and, as above, let $\mathfrak{v}$ be the valuation defined by z. Recall that the Sato Grassmannian of V, $\mathrm{Gr}(V)$, consists of those subspaces $U \subseteq V$ such that $U \cap V_+$ and $V/(U + V_+)$ are finite dimensional.

THEOREM 3.1. *Let* (V, ρ) *be an action of* $\mathcal{W}^+$ *and let* $h(z)$ *be given as in Theorem 2.1. Let* U *be a subspace of* $\mathrm{Gr}(V)$.

If U *is* L_{-1}*-stable and* $A_U \neq \mathbb{C}$*, then* U *is* $\mathcal{W}^+$*-stable and* $A_U = \mathbb{C}[h(z)]$.

Proof. First, let us show that $\mathfrak{v}(f(z)) < 0$ for each $f(z) \in A_U$ non-constant. Indeed, if $\mathfrak{v}(f(z)) > 0$ then $U \cap V_+$ cannot be finite-dimensional since $U \neq (0)$. On the other hand, if $\mathfrak{v}(f(z)) = 0$, then $\bar{f}(z) := f(z) - f(0)$ belongs to A_U and $\mathfrak{v}(\bar{f}(z)) > 0$, which again contradicts the hypotheses. Thus, it must hold that $\mathfrak{v}(f(z)) < 0$.

We shall now prove that $A_U \subseteq \mathbb{C}[h(z)]$. From the previous paragraph, let us take $f(z) \in A_U \setminus \mathbb{C}[h(z)]$ such that $\mathfrak{v}(f(z))$ attains the value $\max\{\mathfrak{v}(f(z)) | f(z) \in A_U \setminus \mathbb{C}[h(z)]\}$. Since U is stable under L_{-1} and under the multiplication by $f(z)$, it follows that $[f(z), \rho(L_{-1})] = \frac{f'(z)}{h'(z)} \in A_U$. Note that $\mathfrak{v}(\frac{f'(z)}{h'(z)}) = \mathfrak{v}(f(z)) - \mathfrak{v}(h(z)) > \mathfrak{v}(f(z))$, since $\mathfrak{v}(h(z))$ is negative and $\mathfrak{v}(f(z)) \neq 0$. Bearing in mind that $f(z)$ is such that $\mathfrak{v}(f(z))$ is maximal among elements of $A_U \setminus \mathbb{C}[h(z)]$, we have that

$$\frac{f'(z)}{h'(z)} \in \mathbb{C}[h(z)]$$

and, thus, $f(z) \in \mathbb{C}[h(z)]$. That is, $A_U \subseteq \mathbb{C}[h(z)]$.

Let us see that $A_U = \mathbb{C}[h(z)]$. Since $A_U \neq \mathbb{C}$, let $p(x)$ be a non-constant polynomial of minimal degree such that $p(h(z)) \in A_U$. Similar to the above, one has that $[p(h(z)), \rho(L_{-1})] = p'(h(z)) \in A_U$ and, thus, $p'(x)$ must be constant and $p(x)$ is of the form $ax + b$. Therefore, $\mathbb{C}[h(z)] = \mathbb{C}[p(h(z))] \subseteq A_U \subseteq \mathbb{C}[h(z)]$.

It remains to show that, in the case $A_U = \mathbb{C}[h(z)]$, U is $\mathcal{W}^+$-stable. This follows from the fact that $h(z)U \subseteq U$ and from the relation $\rho(L_i) = h(z)^i (h(z)\rho(L_{-1}) - (i+1)c)$ for all $i \geq 0$. □

Remark 3.2. Observe that, in the case $A_U = \mathbb{C}[h(z)]$, the connection of Theorem 1.1 can be introduced in an alternative way. Indeed, the map $h(z)^n \partial_h \mapsto L_{n-1}$, for $n \geq 0$, provides a section of the canonical map $\mathcal{D}^1_{A_U/\mathbb{C}}(U) \to \mathrm{Der}_{\mathbb{C}}(A_U)$ which, by [2, Section 1.1]), is an integrable connection on $\mathcal{E}$ on $\mathrm{Spec}\, \mathbb{C}((h(z)^{-1}))$.

Remark 3.3. The previous result can be generalized by dropping out the condition on the dimension of $U \cap V_+$. Under the remaining hypotheses, one can prove that if U is L_i-stable for $i=-1,0$ and $A_U \neq \mathbb{C}$, then $\mathbb{C}((h(z)^{-1})) \subseteq \widehat{(A_U)}_{(0)}$. Furthermore, $[\quad, \rho(L_i)]$ induces a derivation on $(A_U)_{(0)}$ and on $\widehat{(A_U)}_{(0)}$. Here $(A_U)_{(0)}$ denote the function field of A_U and $\widehat{(A_U)}_{(0)}$ its z-adic completion.

Given an action (V,ρ), let us introduce the *first-order stabilizer* of a subspace $U \subset V$ as

$$A_U^1 := \{D \in \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V,V) \mid D(U) \subseteq U\}$$

For a $\mathcal{W}^+$-stable subspace $U \subset V$ such that $A_U = \mathbb{C}[h(z)]$, there is a canonical exact sequence of Lie algebras

$$0 \longrightarrow A_U \longrightarrow A_U^1 \longrightarrow \operatorname{Der}_{\mathbb{C}}(A_U) \longrightarrow 0, \tag{20}$$

and, bearing in mind that $\operatorname{Im}(\rho) \subseteq A_U^1$, one concludes that ρ induces a splitting and, accordingly,

$$A_U^1 = \mathbb{C}[h(z)] \otimes_{\mathbb{C}} < 1, \rho(L_{-1}) > = A_U \oplus \operatorname{Im}(\rho)$$

as Lie subalgebras of $\mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V,V)$.

Remark 3.4. It is worth mentioning the paper [1, Section 2.1] where the authors study subspaces of the Sato Grassmannian, which are stable by the multiplication by a power of z as well as by the action of a first-order differential operator. Then, they investigate the matrix integral representation of the corresponding τ-function.

3.2. STABLE SUBSPACES

In this section we aim to construct explicitly a subspace fulfilling our requirements; namely, invariance under the action and under the homothety z^{-2}. Because of this fact and of Theorem 3.1, we shall assume, henceforth, that $h(z) = z^{-2}$.

A naive candidate would be the $\mathbb{C}[h(z)]$-module generated by 1 under the action of $\rho(L_{-1})$. Nevertheless, we shall need to consider a conjugate of it (Section 2.2). For this, we shall choose a solution of the Airy equation and decompose $b(z)$ in a suitable way. Let us be more precise.

First, we choose $w(z)$, a formal solution of the Airy equation

$$w''(z) + \frac{1}{2} S(h(z)) w(z) = 0 \tag{21}$$

where S denotes the Schwarzian derivative; that is,

$$S(h) := \frac{h'''(z)}{h'(z)} - \frac{3}{2}\left(\frac{h''(z)}{h'(z)}\right)^2$$

It is a straightforward check that $w(z)$ satisfies the Airy equation iff $f(z) = \frac{w'(z)}{w(z)}$ satisfies the Riccati equation

$$f(z)^2 + f'(z) + \frac{1}{2} S(h(z)) = 0 \tag{22}$$

Note, in particular, that $h'(z)^{-1/2}$ satisfies the Equation (21); or, equivalently, $\mathrm{d}\log(h'(z)^{-1/2}) = -\frac{1}{2}\frac{h''(z)}{h'(z)}$ satisfies Equation (22).

Recalling from [18, Chapters 6 and 9] the basic properties of the solutions of the Airy and Riccati equations, we know that in our situation the solutions of Equation (22) are meromorphic; i.e. $\frac{w'(z)}{w(z)} \in \mathbb{C}((z))$. Thus, it makes sense to conjugate a given action by $w(z)$ (see Section 2.2).

Let the following operator be given

$$P = -\frac{1}{h'(z)}\partial_z + \frac{b(z)}{h'(z)}$$

and let us express $b(z)$ w.r.t. the decomposition

$$\begin{aligned}\mathbb{C}((z)) &\simeq \mathbb{C}[h(z)]h'(z) \oplus \left(\mathbb{C}[h(z)] + \mathbb{C}[[z]]z^{-1}\right)\\ &\simeq \mathbb{C}[z^{-2}]z^{-3} \oplus \left(\mathbb{C}[z^{-2}] + \mathbb{C}[[z]]z^{-1}\right)\end{aligned}$$

since $h(z) = z^{-2}$ and $h'(z) = -2z^{-3}$. That is, we write

$$b(z) = u(h(z))h'(z) + \left(\frac{v'(z)}{v(z)} - \frac{1}{2}h''(z)\right) \tag{23}$$

where $u(h(z)), v(z)$ are uniquely determined by: $u(x)$ is a polynomial; and, $v(z)$ is the formal expression

$$v(z) := \exp \int \left(b(z) - u(h(z))h'(z) + \frac{1}{2}h''(z)\right) \mathrm{d}z.$$

Note that $v(z)$ satisfies that $\frac{v'(z)}{v(z)} \in \mathbb{C}[h(z)] + z^{-1}\mathbb{C}[[z]]$.

LEMMA 3.5. *Given* $P = -\frac{1}{h'(z)}\partial_z + \frac{b(z)}{h'(z)}$, *let* $w(z), u(h(z)), v(z)$ *be as above. It then holds that*

$$\left(P^2 - 2u(h(z))P + (u'(h(z)) + u(h(z))^2)\right)(1 \otimes w(z) \otimes v(z)) = 0$$

Proof. Note that the l.h.s. in the statement is rewritten as

$$\begin{aligned}&(P - u(h(z)))^2 (1 \otimes w(z) \otimes v(z))\\ &= \left(\left(P - u(h(z)) - \frac{1}{h'(z)}\frac{v'(z)}{v(z)}\right)^2 (1 \otimes w(z))\right) \otimes v(z)\\ &= \left(\left(-\frac{1}{h'(z)}\partial_z - \frac{1}{2}\frac{h''(z)}{h'(z)}\right)^2 (1 \otimes w(z))\right) \otimes v(z)\\ &= \frac{1}{h'(z)^2}\left(\left(\partial_z^2 + \frac{1}{2}S(h(z))\right)(1 \otimes w(z))\right) \otimes v(z)\end{aligned}$$

In order to see that the last expression vanishes, note that

$$\begin{aligned}\partial_z^2(1\otimes w(z)) &= \partial_z\left(\partial_z(1\otimes w(z))\right)\partial_z\left(\left(\partial_z+\frac{w'(z)}{w(z)}\right)(1)\otimes w(z)\right)\\ &= \partial_z\left(\frac{w'(z)}{w(z)}\otimes w(z)\right)=\left(\left(\frac{w'(z)}{w(z)}\right)'+\left(\frac{w'(z)}{w(z)}\right)^2\right)\otimes w(z)\\ &= -\frac{1}{2}S(h(z))\end{aligned}$$

where the last equality comes from the fact that $\frac{w'(z)}{w(z)}$ solves the Riccati Equation (22). □

Remark 3.6. In [14], the authors are able to solve the second-order differential equation $\left(\frac{3}{2}\bar{z}+\frac{1}{2\bar{z}}\partial_{\bar{z}}-\frac{1}{4\bar{z}^2}\right)^2\phi(\bar{z})=\bar{z}^2\phi(\bar{z})$ (their $\bar{z}$ variable and our z variable are related by $\bar{z}=(\frac{1}{3})^{\frac{1}{3}}z^{-1})$ by the substitution $\phi(\bar{z})=\bar{z}^{1/2}\exp(\frac{2}{3}\bar{z}^{-3})\psi(\bar{z})$ where $\psi(\bar{z})$ is a solution of the Airy equation. However, this makes sense since they show that $\phi(\bar{z})$ has an asymptotic expansion in $\mathbb{C}[[\bar{z}^{-1}]]$. Observe that the previous Lemma can be thought of as an abstract formalization of this procedure.

THEOREM 3.7 (Existence). *Let $w(z)$ be a solution of the Airy Equation (21) linearly independent with $h'(z)^{-\frac{1}{2}}$. Let $(\mathbb{C}((z)),\rho)$ be defined by $(h(z)=z^{-2},c,b(z))$ and let $v(z)$ be a formal function such that Equation (23) is fulfilled. Let V^{wv} be the $\mathbb{C}((z))$-vector space $\mathbb{C}((z))\otimes w(z)\otimes v(z)$ with the conjugated action ρ^{wv}.*

It then holds that the $\mathbb{C}$-vector subspace of V^{wv}

$$\mathcal{U}(w):=\langle 1\otimes w(z)\otimes v(z),\rho^{wv}(L_{-1})(1\otimes w(z)\otimes v(z))\rangle\otimes_{\mathbb{C}}\mathbb{C}[h(z)]$$

is $\mathcal{W}^+$-stable, it is a $\mathbb{C}[h(z)]$-module of rank 2 and it belongs to $\mathrm{Gr}(V^{wv})$.

For the basics of Sato Grassmannian and τ-functions, see [25].

Proof. Let us denote $P:=\rho^{wv}(L_{-1})$. Lemma 3.5 implies that $\mathcal{U}(w)$ is a P-stable $\mathbb{C}[h(z)]$-module. The $\mathcal{W}^+$-stability follows from those facts and from the following relations

$$\begin{aligned}\rho^{wv}(L_i)&=h(z)^i(h(z)\rho^{wv}(L_{-1})-(i+1)c)\\ \rho^{wv}(L_{-1})(p(h(z))\otimes v(z))&=p(h(z))\rho^{wv}(L_{-1})(1\otimes v(z))-p'(h(z))\otimes v(z)\end{aligned}$$

Let us prove that $P(1\otimes w(z)\otimes v(z))\notin\mathcal{U}(w)\otimes_{\mathbb{C}[h(z)]}\mathbb{C}((h(z)^{-1}))$. Being $w(z)$ and $h'(z)^{-\frac{1}{2}}$ linearly independent solutions of the Airy equation, it follows that $S(w(z)\cdot h'(z)^{\frac{1}{2}})=S(h(z))$, and they therefore differ by a Möbius transformation

$$w(z)\cdot h'(z)^{\frac{1}{2}}=\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\qquad\text{for some }\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in\mathrm{PGL}(2,\mathbb{C})\tag{24}$$

Noting that $h(z)=z^{-2}$ and Equation (24), it holds that

$$\frac{1}{h'(z)}\frac{w'(z)}{w(z)}=\frac{1}{h'(z)}\,\mathrm{d}\log\left(w(z)h'(z)^{\frac{1}{2}}\right)-\frac{1}{2}\frac{h''(z)}{h'(z)^2}\in\mathbb{C}[[z^2]]$$

Computing how P acts, we have

$$\begin{aligned}&P(f(z)\otimes w(z)\otimes v(z))\\&=\left(-\frac{1}{h'(z)}\partial_z+u(h(z))-\frac{1}{2}\frac{h''(z)}{h'(z)}-\frac{1}{h'(z)}\frac{w'(z)}{w(z)}\right)(f(z))\otimes w(z)\otimes v(z)\end{aligned}$$

and note that the term $\frac{1}{2}\frac{h''(z)}{h'(z)}$ on the r.h.s. shifts the order by an odd integer while all the other terms shift it by an even integer. Hence, $\mathcal{U}(w)$ is a free $\mathbb{C}[h(z)]$-module of rank 2.

Finally, in order to prove that $\mathcal{U}(w)$ lies in the Sato Grassmannian, where we are considering $V_+^{wv}:=\mathbb{C}[[z]]\otimes w(z)\otimes v(z)$, one has to show the following two conditions

$$\begin{aligned}\dim_{\mathbb{C}}\left(\mathbb{C}[[z]]\otimes w(z)\otimes v(z)\cap\mathcal{U}(w)\right)<\infty\\ \dim_{\mathbb{C}}\mathbb{C}((z))\otimes w(z)\otimes v(z)/\left(\mathbb{C}[[z]]\otimes w(z)\otimes v(z)+\mathcal{U}(w)\right)<\infty\end{aligned}\tag{25}$$

Bearing in mind that $u(h(z))-\frac{1}{2}\frac{h''(z)}{h'(z)}-\frac{1}{h'(z)}\frac{w'(z)}{w(z)}$ does not belong to $\mathbb{C}((z^2))$, both conditions follow from the previous claims. □

The above constructed subspace depends clearly on the choice of a solution of the Airy equation. The following result studies what this dependence looks like.

PROPOSITION 3.8. *Let $(\mathbb{C}((z)),\rho)$ be an action of $\mathcal{W}^+$ defined by the data $\{h(z)=z^{-2},c,b(z)\}$. Let $w_1(z),w_2(z)$ be two solutions of* (21).

Then, up to $\mathbb{C}^*$, there is a unique isomorphism of $\mathbb{C}((z))$-vector spaces $V^{w_1}\xrightarrow{\sim}V^{w_2}$ which is compatible w.r.t. the actions of the conjugated actions ρ^{w_1} and ρ^{w_2}.

Proof. We begin by constructing one isomorphism; we shall then prove the uniqueness.

Let us consider (V^{w_i},ρ^{w_i}) as the conjugated action by $w_i(z)$ (for $i=1,2$); that is, the $\mathbb{C}((z))$-vector space V^{w_i} is given by $\mathbb{C}((z))\otimes_{\mathbb{C}}\mathbb{C}w_i(z)$ and ρ^{w_i} by Equation (17).

From [18, Chapter 6], we know that the fact that w_1,w_2 solve (21) yields

$$S\left(\frac{w_1(z)}{w_2(z)}\right)=S(h)$$

and, therefore,

$$\frac{w_1(z)}{w_2(z)}=\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\qquad\text{for some }\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\mathrm{PGL}(2,\mathbb{C})\tag{26}$$

Let us now check that the $\mathbb{C}((z))$-linear map

$$\begin{aligned} \mathbb{C}((z))\otimes_{\mathbb{C}}\mathbb{C}w_1(z) &\longrightarrow \mathbb{C}((z))\otimes_{\mathbb{C}}\mathbb{C}w_2(z) \\ 1\otimes w_1(z) &\mapsto \frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\otimes w_2(z) \end{aligned} \tag{27}$$

gives rise to an isomorphism that is compatible with the actions of ρ^{w_1} on the l.h.s. and of ρ^{w_2} on the r.h.s.; that is, one has to show that

$$\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right)\cdot\rho^{w_1}(L_k)(f(z)\otimes w_1(z)) = \rho^{w_2}(L_k)\left(\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right)f(z)\otimes w_2(z)\right)$$

We shall only prove the case $k=-1$, $f(z)=1$, since the general case goes along the same lines.

First, taking logarithms and derivatives in Equation (26), we obtain

$$\frac{w_1'(z)}{w_1(z)} = \frac{w_2'(z)}{w_2(z)} + \left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right)^{-1}\cdot\partial_z\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right) \tag{28}$$

On the one hand, one computes the image of

$$\rho^{w_1}(L_{-1})(1\otimes w_1(z)) = \left(-\frac{1}{h'(z)}\frac{w_1'(z)}{w_1(z)}+\frac{b(z)}{h'(z)}\right)\otimes w_1(z)$$

by the map (27) and one obtains

$$\begin{aligned} &\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\left(-\frac{1}{h'(z)}\cdot\frac{w_1'(z)}{w_1(z)}+\frac{b(z)}{h'(z)}\right)\otimes w_2(z) \\ &\quad = -\frac{1}{h'(z)}\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\cdot\frac{w_2'(z)}{w_2(z)}+\partial_z\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right)-\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\cdot b(z)\right)\otimes w_2(z) \end{aligned} \tag{29}$$

where we have used the identity (28).

On the other hand, one has

$$\begin{aligned} &\rho^{w_2}(L_{-1})\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\otimes w_2(z)\right) \\ &\quad = \left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right)\cdot\left(-\frac{1}{h'(z)}\cdot\frac{w_2'(z)}{w_2(z)}+\frac{b(z)}{h'(z)}\right)\otimes w_2(z) \\ &\qquad -\frac{1}{h'(z)}\partial_z\left(\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right)\otimes w_2(z) \end{aligned}$$

and, since this expression coincides with Equation (29), it follows that (27) is an isomorphism compatible with the actions.

Let us denote by ϕ the isomorphism (27) and let $\psi: V^{w_1}\to V^{w_2}$ be another isomorphism compatible with the actions. The statement will be proved if we can show that $\phi\circ\psi^{-1}$ belongs to $\mathbb{C}^*$.

Let $f(z)$ be defined by $\psi(1\otimes w_1(z)) = f(z)\otimes w_2(z)$. Thus

$$(\phi\circ\psi^{-1})(1\otimes w_2(z)) = f(z)^{-1}\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\otimes w_2(z)$$

is a $\mathbb{C}((z))$-linear automorphism of V^{w_2} that is compatible with the action of ρ^{w_2}; that is, $(\phi\circ\psi^{-1})\circ\rho^{w_2} = \rho^{w_2}\circ(\phi\circ\psi^{-1})$ and, bearing in mind Equation (8), it follows that

$$\frac{1}{h'(z)}\partial_z\left(f(z)^{-1}\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}\right) = 0$$

and, hence, $f(z) = \lambda\cdot\frac{\alpha h(z)+\beta}{\gamma h(z)+\delta}$ for $\lambda\in\mathbb{C}^*$ and the statement follows. □

3.3. $Gl(2)$-OPERS

THEOREM 3.9. *Let* $(\mathbb{C}((z)),\rho)$ *be defined by* $(h(z)=z^{-2}, c, b(z))$. *If* $\mathrm{Res}_{z=0}\frac{b(z)}{h'(z)} = \frac{3}{2}$, *then it defines a* Gl(2)-*oper* $\mathcal{E}$ *on* Spec $\mathbb{C}[h(z)]$. *Moreover, the* τ-*function of* $\mathcal{E}$, *which is a point of the Sato Grassmannian, satisfies the KdV hierarchy and a set of Virasoro-like constraints.*

Proof. The condition $\mathrm{Res}_{z=0}\frac{b(z)}{h'(z)} = \frac{3}{2}$ means that there exist a polynomial u and a formal function v such that Equation (23) is fulfilled. Let V^{wv} be the $\mathbb{C}((z))$-vector space $\mathbb{C}((z))\otimes w(z)\otimes v(z)$ with the conjugated action ρ^{wv} where $w(z)$ is a solution of the Airy Equation (21) linearly independent with $h'(z)^{-\frac{1}{2}}$.

Now, Theorem 3.7 gives us an $\mathbb{C}$-vector subspace $\mathcal{U}(w)\subset V^{wv}$ which is $\mathcal{W}^+$-stable, it is a $\mathbb{C}[h(z)]$-module of rank 2 and it belongs to $\mathrm{Gr}(V^{wv})$. Applying to $\mathcal{U}(w)$ similar arguments to those used in the proof of Theorem 1.1, the first statement follows.

Finally, bearing in mind Theorem 3.7, we know that the $\mathbb{C}[z^{-2}]$-module attached to the Gl(2)-oper fulfills:

(i) $\mathcal{U}(w)$ belongs to the Sato Grassmannian,
(ii) $z^{-2}\mathcal{U}(w)\subset\mathcal{U}(w)$,
(iii) $\rho(L_k)\mathcal{U}(w)\subseteq U$ for $k\geq -1$.

One can now translate this properties into properties of the τ-function of $\mathcal{U}(w)$, $\tau_{\mathcal{U}(w)}(t)\in\mathbb{C}[[t_1,t_2,\dots]]$. Proceeding along the lines of Example 2.4 (see also[25]), one obtains that the above conditions are equivalent to:

(i') KP-hierarchy,
(ii') $\partial_{t_{2i}}\tau(t)=0$ (KdV hierarchy, provided that KP is fulfilled),
(iii') $\bar{L}_k\tau(t)=0$, for $k\geq -1$ (Virasoro constraints), for certain differential operators $\{\bar{L}_k\}_{k\geq -1}$ with $[\bar{L}_i,\bar{L}_j]=(i-j)\bar{L}_{i+j}$. □

Let us say a word on the significance of this Theorem. We have shown that there is a deep relationship among the following three sets: (a) 2-cyclic actions of

$\mathcal{W}^+$; (b) functions $\tau(t)$ satisfying the KdV hierarchy and Virasoro-like constraints; (c) Gl(2)-opers on Spec($\mathbb{C}[z^{-2}]$). Surprisingly, the τ-functions arising in 2D gravity [14,17] fall into this scheme (see [24] for the details).

4. Universal Family and Topological Recursion

Recent results on the so-called *topological recursion* involve *families* of τ-functions depending on an infinite number of parameters such that the whole family lies entirely on the space of functions satisfying KdV and Virasoro constraints (see, for instance, [16,19,21,22]). One of these families already appeared in Kontsevich's work [17, Section 3.4]. It is worth mentioning the existence of relevant 1-parameter families; for instance, the one *connecting* the Witten–Kontsevich partition function with the Hurwitz partition function ([20], see also [3]), another one *connecting* Witten–Kontsevich and Mirzakhani theories [22], and a third one the Witten–Kontsevich partition function with the generating function of linear Hodge integrals defined on the moduli space of stable curves [16].

In this section, a natural a general procedure to obtain the above-mentioned families will be provided.

Let us consider a family of independent variables $\mathbf{s} := (s_1, s_2, \ldots)$. For a sequence of non-negative integers, $\mathbf{m} := (m_1, m_2, \ldots)$, with $m_i = 0$ for all $i \gg 0$ define:

$$|\mathbf{m}| := \sum_{i\geq 1} i m_i \qquad \|\mathbf{m}\| := \sum_{i\geq 1} m_i \qquad \mathbf{m}! := \prod_{i\geq 1} m_i! \qquad \mathbf{s}^{\mathbf{m}} := \prod_{i\geq 1} s_i^{m_i}$$

Based on Mulase–Safnuk's approach [22], Liu–Xu considered the operators [19, Equation (9)]:

$$\begin{aligned}\bar{L}'_n(\mathbf{s}) := & -\frac{1}{2}\sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!}\mathbf{s}^{\mathbf{m}}\partial_{q_{|\mathbf{m}|+n+1}} + \sum_{i=0}^{\infty}(i+\frac{1}{2})q_i\partial_{q_{i+n}} \\ & + \frac{1}{2}\sum_{i=1}^{n}\partial_{q_{i-1}}\partial_{q_{n-i}} + \frac{q_0{}^2}{4}\delta_{n,-1} + \frac{1}{16}\delta_{n,0}\end{aligned}$$

for $n \geq -1$ (their exact expression corresponds to a rescaling by a double factorial). They showed that

$$[\bar{L}'_i(\mathbf{s}), \bar{L}'_j(\mathbf{s})] = (i-j)\bar{L}'_{i+j}(\mathbf{s}) \qquad \text{for } i, j \geq -1$$

and, therefore, they generate a family of Witt algebras depending on the parameters $\mathbf{s}$. We may write

$$\bar{L}'_n(\mathbf{s}) = -\frac{1}{2}\sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!}\mathbf{s}^{\mathbf{m}}\partial_{q_{|\mathbf{m}|+n+1}} + \bar{L}'_n(0)$$

where $\bar{L}'_n(0)$ denotes the value of $\bar{L}'_n(\mathbf{s})$ at $\mathbf{s}=0$. Observe that the operators $\bar{L}'_n(0)$ coincide with those of [5, Equation 3.5] and [11, Section 3] (up to rescaling of the variables q_i).

Recalling Example 2.4, the action on $\mathbb{C}((z))$ corresponding to the above operators can be written down explicitly

$$\rho'_{\mathbf{s}}(L_n) := \frac{1}{2}\sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!}\mathbf{s}^{\mathbf{m}} z^{-2(|\mathbf{m}|+n)-3} + \frac{1}{2}z^{-2n}\left(z\partial_z + \frac{1-2n}{2}\right) \forall n \geq -1$$

By Theorem 2.1, the action $\rho'_{\mathbf{s}}$ is attached to a triple $(h(z), c, b(z))$. Actually, bearing in mind that $\frac{h(z)^{n+1}}{h'(z)} = -\frac{1}{2}z^{-2n+1}$, and regarding $\mathbf{s}$ as parameters, we obtain that the action $\rho'_{\mathbf{s}}$ is attached to the data $(h(z) = z^{-2}, c = \frac{1}{2}, b_{\mathbf{s}}(z))$ where

$$b_{\mathbf{s}}(z) := -\sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!}\mathbf{s}^{\mathbf{m}} z^{-2|\mathbf{m}|-4} - \frac{3}{2}z^{-1}$$

The fact that we are concerned with the KdV case can be equivalently stated in three forms: (a) the associated triple has $h(z) = z^{-2}$; (b) the subspace U of the Sato Grassmannian satisfies that $z^{-2}U \subseteq U$; and, (c) the corresponding τ-function, $\tau_U(t)$, does not depend on t_{2i}; i.e. $\tau_U(t) \in \mathbb{C}[[t_1, t_3, \ldots]]$. Thus, if no confusion arises $\rho'_{\mathbf{s}}(L_n)$ can be thought of as operators acting on $\mathbb{C}[[t_1, t_3, \ldots]]$.

We conclude that the action induced by $\rho'_{\mathbf{s}}$ on $\mathbb{C}[[t_1, t_3, \ldots]]$ is the *universal* action for the case of KdV (i.e. $h(z) = z^{-2}$). In particular, this agrees with the idea addressed in [22] that a certain 1-parameter family, which would correspond to Eynard–Orantin's *spectral curve*, deforms the Witten–Kontsevich theory to other cases where the Virasoro also appears. Thus, we are led to the following generalization of [22, Theorem 1.2] (see also [16, Theorem 2.1] and [19, Theorem 4.4]).

THEOREM 4.1. *Let $\tau_{\mathbf{s}}(t) \in \mathbb{C}[[t_1, t_3, \ldots]]$ be the τ-function associated to $\rho'_{\mathbf{s}}$. Then, $\tau_{\mathbf{s}}(t)$ satisfies the Virasoro constraints corresponding to operators $\bar{L}'_n(\mathbf{s})$ above (as in Section 3.2) and, moreover, it holds that*

$$\tau_{\mathbf{s}}(t) = \tau_0(\tilde{t})$$

where $\tilde{t}_{2i+1}$ is equal to t_i for $i = 0, 1$ and to $t_{2i+1} - \frac{1}{(2i+1)!!}\sum_{|\mathbf{m}|=i-1} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!}\mathbf{s}^{\mathbf{m}}$ for $i > 1$.

Conversely, let $\tau(t) \in \mathbb{C}[[t_1, t_3, \ldots]]$ be a τ-function for the KdV satisfying the Virasoro constraints. Then, there exist values of $\mathbf{s}$, say $\mathbf{s}_0 := (s_1, s_2, \ldots)$ such that

$$\tau(t) = \tau_{\mathbf{s}_0}(t)$$

Proof. It is enough to observe that under that change of variables, the operators $\bar{L}'_n(\mathbf{s})$ in t_{2i+1} are transformed into the operators $\bar{L}'_n(0)$ in $\tilde{t}_{2i+1}$.

Bearing in mind that, regarding $\mathbf{s}$ as parameters, the action $\rho'_{\mathbf{s}}$ is universal, the converse follows. □

Finally, the previous Theorem can be used to strengthen Theorem 3.9 and it provides a link between Gl(2)-opers and τ-functions fulfilling the KdV hierarchy and the Virasoro constraints simultaneously. These promising facts deserve further research.

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