Review of AdS/CFT Integrability, Chapter IV.3:
$\mathcal{N} = 6$ Chern–Simons and Strings on $\text{AdS}_4 \times \text{CP}^3$

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Abstract. We review the duality and integrability of $\mathcal{N} = 6$ superconformal Chern–Simons theory in three dimensions and IIA superstring theory on the background $\text{AdS}_4 \times \text{CP}^3$. We introduce both of these models and describe how their degrees of freedom are mapped to excitations of a long-range integrable spin-chain. Finally, we discuss the properties of the Bethe equations, the S-matrix and the algebraic curve that are special to this correspondence and differ from the case of $\mathcal{N} = 4$ SYM theory and strings on $\text{AdS}_5 \times S^5$.

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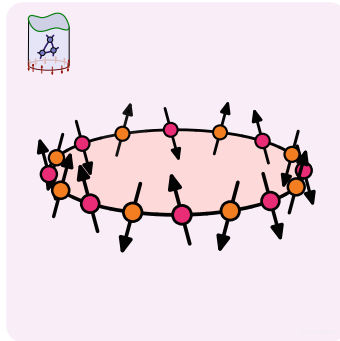
1. Introduction

Almost all statements that have been made in the other chapters of this review [1] about the duality and integrability of string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ Yang–Mills theory in four dimensions, also hold in an appropriately adopted form

for a second example of the AdS/CFT correspondence. This example has been known since June 2008 [2], and it is as concrete as the "old" one. Because the involved space–times are of one less dimension, this correspondence is often referred to as $\mathrm{Ads_4/CFT_3}$ to distinguish it from the more established $\mathrm{Ads_5/CFT_4}$.[1]

In the $\mathrm{Ads_5/CFT_4}$ case, we had IIB superstring theory on $\mathrm{AdS_5} \times S^5$ with self-dual RR five-form flux $F^{(5)} \sim N$ through $\mathrm{AdS_5}$ and S^5. This is now replaced by

> IIA superstring theory on $\mathrm{AdS_4} \times \mathrm{CP}^3$
> with RR four-form flux $F^{(4)} \sim N$ through $\mathrm{AdS_4}$ $\qquad$ (1.1)
> and RR two-form flux $F^{(2)} \sim k$ through a $\mathrm{CP}^1 \subset \mathrm{CP}^3$.

On the gauge theory side, we had $\mathcal{N} = 4$ superconformal Yang–Mills theory with coupling g_{YM} and gauge group $\mathrm{U}(N)$ on $\mathbb{R}^{1,3}$. Now this is replaced by ABJM theory:

> $\mathcal{N} = 6$ superconformal Chern–Simons-matter theory
> with gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$ on $\mathbb{R}^{1,2}$ $\qquad$ (1.2)
> and Chern–Simons levels k and $-k$.

Both theories are controlled by two and only two parameters, k and N, which take integer values. These parameters determine all other quantities like coupling constants and the effective string tension. In ABJM theory, the Chern–Simons level k acts like a coupling constant. The fields can be rescaled in such a way that all interactions are suppressed by powers of $\frac{1}{k}$, i.e. large k is the weak coupling regime. One can take a planar, or 't Hooft, limit which is given by

$$k, N \to \infty, \quad \lambda \equiv \frac{N}{k} = \text{fixed}. \qquad (1.3)$$

It is in this limit where integrability shows up and which is therefore the focus of this review. On the gravity side, the string coupling constant and effective tension are given by[2]

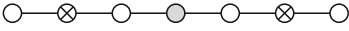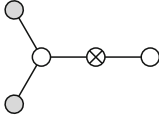$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N}, \quad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda}, \qquad (1.4)$$

where R is the radius of CP^3 and *twice* the radius of $\mathrm{AdS_4}$. These relations are qualitatively the same as in the $\mathrm{Ads_5/CFT_4}$ context. In the planar limit g_s goes to zero and thus the strings do not split or join. At small 't Hooft coupling, the background is highly curved and the string is subject to large quantum fluctuations. At large 't Hooft coupling, the background is weakly curved which renders the sigma-model weakly coupled and the string behaves classically.

The first equation in (1.4) contains a hint that the duality is about more than the relationship between (1.1) and (1.2). If we are not in the 't Hooft limit but if

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[1]Since December 2009, also an $\mathrm{Ads_3/CFT_2}$ correspondence has been discussed in the context of integrability [3].
[2]There are corrections to the second relation at two loops in the sigma model [4].

Table I. Comparison of symmetries

	Ads$_5$/CFT$_4$	Ads$_4$/CFT$_3$
Global symmetry	PSU(2, 2|4)	OSp(6|4)
Dynkin diagram	○—⊗—○—●—○—⊗—○	(Dynkin diagram of OSp(6|4))
Residual symmetry	SU(2|2)$_L$ × SU(2|2)$_R$	SU(2|2)
Representations	$(2\|2)_L \otimes (2\|2)_R = 16$ d.o.f	$(2\|2)_A \oplus (2\|2)_B = 8$ d.o.f

The Dynkin diagram of $\mathrm{PSU}(2, 2|4)$ contains two SU(2\|2) branches which represent the residual symmetries and exactly one momentum carrying root which we marked by shading it gray. This indicates that all 16 elementary excitations transform in a single irreducible representation with one fundamental index in each SU(2\|2). The Dynkin diagram of OSp(6\|4) contains only one SU(2\|2) branch, but two momentum carrying roots. Correspondingly, the eight elementary excitations transform in two copies of the fundamental representation of SU(2\|2)

we let $N \gg k^5$, then the string coupling g_s becomes large. However, strongly coupled IIA string theory is M-theory. Indeed, ABJM theory (1.2) at arbitrary value of k and N is dual to [2]

$$\text{M-theory on AdS}_4 \times S^7/\mathbb{Z}_k \qquad (1.5)$$
$$\text{with four-form flux } F^{(4)} \sim N \text{ through AdS}_4.$$

In other words, ABJM theory is the world-volume theory of a stack of N M2 branes moving on $\mathbb{C}^4/\mathbb{Z}_k$ [2]. The duality of (1.1) and (1.2) is really only a corollary of this more general M/ABJM duality in the limit where $k^5 \gg N$ and where therefore M-theory is well approximated by weakly coupled IIA string theory on a AdS$_4 \times$ CP^3 background[3]. The lecture notes [5] discuss the general M/ABJM correspondence. However, in the planar limit (1.3), where k and N grow large with equal powers, we are always in the IIA regime. Thus, by concentrating on the question of integrability we are only concerned with IIA/ABJM. An extended and largely self-contained review of the Ads$_4$/CFT$_3$ correspondence is forthcoming [6].

Overview. In a nutshell, the differences between Ads$_5$/CFT$_4$ and Ads$_4$/CFT$_3$, see Table I, are as follows: the first duality involves theories that are invariant under the supergroup PSU(2, 2\|4) and therefore are maximally supersymmetric (32 supercharges), while the theories in the second duality are OSp(6\|4)-symmetric, a group which contains "only" 24 supercharges. After gauge fixing, the symmetry groups reduce to two and one copy of SU(2\|2), respectively. The 16 elementary excitations in the 5/4d case transform in the representation $(2|2)_L \otimes (2|2)_R$ of the residual

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[3]CP^3 arises from writing S^7 as S^1 fibered over CP^3 and by identifying the circle as the M-theory direction which shrinks to zero size by the orbifold action of $\mathbb{Z}_k$ in the large k limit.

symmetry group, while there are only *eight* elementary excitations in the 4/3d case which transform in the representation

$$(2|2)_{A-\text{particles}} \oplus (2|2)_{B-\text{particles}}. \tag{1.6}$$

In Sections 3 and 5 we will show how these two types of particles arise from the gauge and string theory degrees of freedom, respectively.

Another difference between the two dualities is that the interpolation between weak and strong coupling in $\text{Ads}_4/\text{CFT}_3$ is much more intricate. Take, e.g. the magnon dispersion relation, which due to the underlying $\text{SU}(2|2)$ symmetry is fixed in either duality to be of the form [7] (see also [8])

$$E(p) = \sqrt{Q^2 + 4h^2(\lambda)\sin^2\frac{p}{2}}, \tag{1.7}$$

where Q is the magnon R-charge and where the function $h(\lambda)$ is *not* fixed by symmetry. The fundamental magnon in $\text{Ads}_5/\text{CFT}_4$ has charge $Q = 1$, while in $\text{Ads}_4/\text{CFT}_3$ it has $Q = \frac{1}{2}$. In the $\text{Ads}_5/\text{CFT}_4$ case the function $h(\lambda)$ turned out to be simply $\sqrt{\lambda}/4\pi$, which can be argued to arise from S-duality [9]. In the present case there is no such argument and indeed the function h happens to be quite non-trivial. The weak and strong coupling asymptotics are given by

$$h(\lambda) = \begin{cases} \lambda\left[1 + c_1\lambda^2 + c_2\lambda^4 + \cdots\right] & \text{for } \lambda \ll 1, \\ \sqrt{\frac{\lambda}{2}} + a_1 + \frac{a_2}{\sqrt{\lambda}} + \cdots & \text{for } \lambda \gg 1, \end{cases} \tag{1.8}$$

where the leading terms were deduced in [10,11] and [11,12], respectively. In fact, the λ-dependence of many other quantities like the S-matrix, the Bethe ansatz, the Zhukowsky map, the universal scaling function, etc., are also related between the $\text{Ads}_5/\text{CFT}_4$ and the $\text{Ads}_4/\text{CFT}_3$ correspondence by appropriately replacing λ by $h(\lambda)$. Despite this fact, the subleading terms seem to be scheme dependent. For example, a worldsheet computation yields a non-zero a_1 [13] while the algebraic curve computation produces $a_1 = 0$ [14] which is also what is used in the Bethe ansatz proposal [15]. In order to unambiguously compare different approaches, one should therefore express all results in terms of a physical reference observable, and neither in terms of λ nor $h(\lambda)$.

2. $\mathcal{N} = 6$ Chern–Simons Matter Theory

Field content. ABJM theory is a three-dimensional superconformal Chern–Simons theory with product gauge group $\text{U}(N) \times \hat{\text{U}}(N)$ at levels $\pm k$ and specific matter content. The quiver diagram visualizing the fields of the theory and their gauge representations is drawn in Figure 1. The entire field content is given by two gauge fields A_μ and $\hat{A}_\mu$, four complex scalar fields Y^A, and four Weyl-spinors ψ_A. The matter fields are $N \times N$ matrices transforming in the bi-fundamental representation of the gauge group.
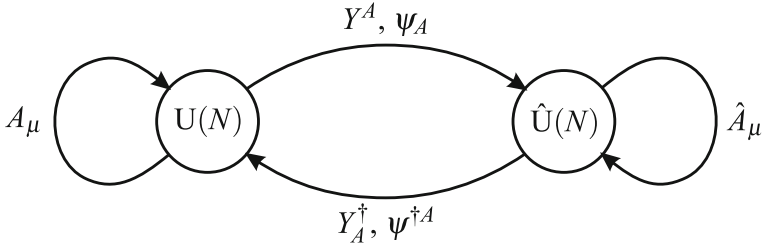
Figure 1. Quiver diagram of ABJM theory. The *arrows* indicate the representations of the fields under the gauge groups. The *arrows* are drawn from a fundamental to an anti-fundamental representation.

Table II. Representations of ABJM fields

	U(N)	Û(N)	SU(4)$_R$	SU(2)$_r$	U(1)$_\Delta$	U(1)$_b$
A_μ	**N**^{2}	**1**	**1**	**3**	1	0
$\hat{A}_\mu$	**1**	**N**^{2}	**1**	**3**	1	0
Y^A	**N**	**N̄**	**4**	**1**	$\frac{1}{2}$	1
ψ_A	**N**	**N̄**	**4̄**	**2**	1	1

Global symmetries. The global symmetry group of ABJM theory, for Chern–Simons level[4] $k > 2$, is given by the orthosymplectic supergroup OSp(6|4) [2,16] and the "baryonic" U(1)$_b$ [2]. The bosonic components of OSp(6|4) are the R-symmetry group SO(6)$_R \cong$ SU(4)$_R$ and the 3d conformal group Sp(4) $\cong$ SO(2, 3). The conformal group contains the spacetime rotations SO(3)$_r \cong$ SU(2)$_r$ and dilatations SO(2)$_\Delta \cong$ U(1)$_\Delta$. The fermionic part of OSp(6|4) generates the $\mathcal{N} = 6$ supersymmetry transformations. The baryonic charge U(1)$_b$ is +1 for bi-fundamental fields, −1 for anti-bi-fundamental fields, and 0 for adjoint fields. The representations in which the fields transform under these symmetries are listed in Table II. For more details about the OSp(6|4) group theory in this context see [17]. Finally, the model also possesses a discrete, parity-like symmetry. This might be surprising since the Chern–Simons action is not invariant but changes sign under a canonical parity transformation. The trick to make the model parity invariant is to accompany the "naive" parity transformation by the exchange of the two gauge group factors. The total transformation is a symmetry because the Chern–Simons terms for the two gauge group factors have opposite signs.

Action. The ABJM action was first spelled out in all detail in [18] in $\mathcal{N} = 2$ superspace and in component form. An $\mathcal{N} = 3$ [19], an $\mathcal{N} = 1$ [20], and an $\mathcal{N} = 6$ [21] superspace version is also known. The component action using the conventions of [18] reads

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[4]We are ignoring the symmetry enhancement to OSp(8|4) at $k = 1$ and $k = 2$, because for the purpose of discussing integrability we have to work in the 't Hooft limit where k is large.

$$\mathscr{S} = \frac{k}{4\pi} \int d^3x \left[\varepsilon^{\mu\nu\lambda} \operatorname{tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right.$$
$$\left. - \operatorname{tr}(D_\mu Y)^\dagger D^\mu Y - i \operatorname{tr} \psi^\dagger \not{D} \psi - V_{\text{ferm}} - V_{\text{bos}} \right], \tag{2.1}$$

where the sextic bosonic and quartic mixed potentials are

$$V^{\text{bos}} = -\frac{1}{12} \operatorname{tr} \left[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \right.$$
$$\left. + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger \right]. \tag{2.2}$$

$$V^{\text{ferm}} = \frac{i}{2} \operatorname{tr} \left[Y_A^\dagger Y^A \psi^{\dagger B} \psi_B - Y^A Y_A^\dagger \psi_B \psi^{\dagger B} + 2 Y^A Y_B^\dagger \psi_A \psi^{\dagger B} - 2 Y_A^\dagger Y^B \psi^{\dagger A} \psi_B \right.$$
$$\left. - \varepsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \varepsilon_{ABCD} Y^A \psi^{\dagger B} Y^C \psi^{\dagger D} \right]. \tag{2.3}$$

The covariant derivative acts on bi-fundamental fields as

$$D_\mu Y = \partial_\mu Y + i A_\mu Y - i Y \hat{A}_\mu, \tag{2.4}$$

while on anti-bi-fundamental fields it acts with A_μ and $\hat{A}_\mu$ interchanged. According to the M-theory interpretation, this theory describes the low-energy limit of N M2 branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. The three-dimensional spacetime of ABJM theory is the world-volume of those M2 branes. For conventions and further details we refer to [18].

Perturbation theory and 't Hooft limit. Note that the Chern–Simons level occurs in (2.1) as an overall factor of the entire action. Alternatively, one can rescale the fields in such a way that all quadratic terms come without any factors of k and interactions of order n come with $\frac{1}{k^{n/2-1}}$. Either way, this shows that $g_{\text{CS}}^2 \equiv \frac{1}{k}$ acts like a coupling constant of ABJM theory, quite similar to g_{YM}^2 in $\mathcal{N} = 4$ SYM, though of course k has to be an integer to preserve non-abelian gauge symmetry. As announced in the introduction, the theory can be restricted to the planar sector by taking the 't Hooft limit (1.3) which introduces the effective coupling

$$\lambda \equiv g_{\text{CS}}^2 N = \frac{N}{k}. \tag{2.5}$$

In this limit the theory becomes integrable [10] (see also [11,22]) in the same sense as we are used to in planar $\mathcal{N} = 4$ SYM theory and as we will discuss below.

Gauge group. The model can be generalized to have gauge group $U(M)_k \times U(N)_{-k}$ [23]. This generalization goes by the name ABJ theory. The M-theory interpretation is given by $\min(M, N)$ M2 branes allowed to move freely on $\mathbb{C}^4/\mathbb{Z}_k$ and $|M - N|$ fractional M2 branes stuck to the singularity. The gauge theory action is formally the same as in (2.1), except that the matter fields are now given by rectangular matrices. Thus two 't Hooft couplings can be defined by

$$\lambda = \frac{M}{k}, \quad \hat{\lambda} = \frac{N}{k}, \tag{2.6}$$

and it becomes possible to take different planar limits depending on the ratio of λ and $\hat{\lambda}$. On the other hand, the generalized parity invariance of the ABJM theory is explicitly broken, because now the two gauge group factors cannot be exchanged anymore.

Deformation. It is possible to introduce independent Chern–Simons levels k and $\hat{k}$ for the two gauge groups U(N) and Û(N) that do not sum to zero. This generalized theory possesses less supersymmetry and less global symmetry. It is proposed to be dual to a type IIA background with the Romans mass $F_0 = k + \hat{k}$ turned on [24]. This modification, however, seems to break integrability [25].

3. From ABJM Theory to the Integrable Model

Spin-chain picture. The integrability of the planar ABJM theory is best described in terms of an integrable OSp(6|4) spin-chain which represents single-trace operators [10]. A qualitative difference between the case at hand and the case of $\mathcal{N}=4$ SYM is that the ABJM spin-chain is an "alternating spin-chain." Because the matter fields are in bi-fundamental representations of the product gauge group U(N) × Û(N), gauge invariant operators need to be built from products of fields that transform alternatingly in the representations $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$, e.g.

$$\mathrm{tr}(Y^1 Y_4^\dagger Y^1 Y_4^\dagger \cdots). \tag{3.1}$$

Thus, the spin-chain has even length and the fields on the odd sites are distinct from the ones on the even sites. On the odd sites, we can have any of the $4_B + 8_F$ fields $Y^A, \psi_{A\alpha}$, and on the even sites, we can have any of the $4_B + 8_F$ fields $Y_A^\dagger, \psi_\alpha^{\dagger A}$. We can also act with an arbitrary number of derivatives $D_\mu = D_{\alpha\beta}$ onto the fields, but derivatives do not introduce extra sites. Also field strength insertions do not count as extra sites as they can be written as anti-symmetrized derivatives.

Spin-chain excitations. In the spin-chain description, the ABJM fields are distinguished according to whether they represent the vacuum (or "down spin"), or elementary or multiple excitations. A convenient and common choice for the vacuum spin-chain is the BPS operator (3.1), i.e. Y^1 is the vacuum on the odd sites, and $Y_4^\dagger$ is the vacuum on the even sites.

Selecting a vacuum breaks the OSp(6|4) symmetry group of ABJM theory down to SU(2|2) × U(1)$_{\text{extra}}$ which becomes the symmetry group of the spin-chain model [10,11]. The bosonic components of this SU(2|2) are SU(2)$_G$ × SU(2)$_r$ × U(1)$_E$, where SU(2)$_G$ is the unbroken part of SU(4)$_R$, SU(2)$_r \cong$ SO(1, 2)$_r$ is the Lorentz group, and U(1)$_E$ is the spin-chain energy $E = \Delta - J$ which itself is a combination of the conformal dimension Δ and a broken SU(4)$_R$ generator J. The charges of the fields under these groups are listed and explained in Table III.

By construction, the ground state spin-chain (3.1) has energy $E = \Delta - J = 0$. This spin-chain can be excited by replacing one of the vacuum fields by a different field

Table III. Charges of fields

	$SU(4)_R$ $[p_1, q, p_2]$	$SU(2)_{G'}$ J	$SU(2)_G$	$U(1)_{\text{extra}}$	$U(1)_\Delta$ Δ	$SU(2)_r$ s	$U(1)_E$ $E = \Delta - J$
Y^1	$[1, 0, 0]$	$+1/2$	0	$+1$	$1/2$	0	0
Y^2	$[-1, 1, 0]$	0	$+1/2$	-1	$1/2$	0	$1/2$
Y^3	$[0, -1, 1]$	0	$-1/2$	-1	$1/2$	0	$1/2$
Y^4	$[0, 0, -1]$	$-1/2$	0	$+1$	$1/2$	0	1
$\psi_{1\pm}$	$[-1, 0, 0]$	$-1/2$	0	-1	1	$\pm1/2$	$3/2$
$\psi_{2\pm}$	$[1, -1, 0]$	0	$-1/2$	$+1$	1	$\pm1/2$	1
$\psi_{3\pm}$	$[0, 1, -1]$	0	$+1/2$	$+1$	1	$\pm1/2$	1
$\psi_{4\pm}$	$[0, 0, 1]$	$+1/2$	0	-1	1	$\pm1/2$	$1/2$
D_0	$[0, 0, 0]$	0	0	0	1	0	1
$D_\pm$	$[0, 0, 0]$	0	0	0	1	±1	1
$Y^\dagger_1$	$[-1, 0, 0]$	$-1/2$	0	-1	$1/2$	0	1
$Y^\dagger_2$	$[1, -1, 0]$	0	$-1/2$	$+1$	$1/2$	0	$1/2$
$Y^\dagger_3$	$[0, 1, -1]$	0	$+1/2$	$+1$	$1/2$	0	$1/2$
$Y^\dagger_4$	$[0, 0, 1]$	$+1/2$	0	-1	$1/2$	0	0
$\psi^{\dagger 1\pm}$	$[1, 0, 0]$	$+1/2$	0	$+1$	1	$\pm1/2$	$1/2$
$\psi^{\dagger 2\pm}$	$[-1, 1, 0]$	0	$+1/2$	-1	1	$\pm1/2$	1
$\psi^{\dagger 3\pm}$	$[0, -1, 1]$	0	$-1/2$	-1	1	$\pm1/2$	1
$\psi^{\dagger 4\pm}$	$[0, 0, -1]$	$-1/2$	0	$+1$	1	$\pm1/2$	$3/2$

The R-symmetry group $SO(6)_R \cong SU(4)_R$ splits up into $SU(2)_{G'} \times SU(2)_G \times U(1)_{\text{extra}}$, and the conformal group $Sp(2, 2) \cong SO(2, 3)$ splits up into $U(1)_\Delta \times SU(2)_r$. The symmetry group of the spin-chain is $SU(2|2) \times U(1)_{\text{extra}} \supset SU(2)_G \times SU(2)_r \times U(1)_E \times U(1)_{\text{extra}}$. The $U(1)_J$ generator $J = \frac{p_1 + q + p_2}{2}$ is the Cartan generator of $SU(2)_{G'}$, and the $U(1)_E$ generator E is given by the difference $\Delta - J$

or by acting with a covariant derivative. This procedure increases the energy in quanta of $\delta E = 1/2$ by a total amount that can be read off from the last column in Table III. If the energy increases by $1/2$, then the excitation is an elementary one. We find that the elementary excitations on the odd and even sites are given by

$$\text{"}A\text{"-particles:} \quad (Y^2, Y^3 | \psi_{4+}, \psi_{4-}), \tag{3.2a}$$

$$\text{"}B\text{"-particles:} \quad (Y^\dagger_3, Y^\dagger_2 | \psi^{\dagger 1}_+, \psi^{\dagger 1}_-), \tag{3.2b}$$

respectively [11]. These are the two multiplets anticipated in (1.6). All other fields correspond to composite excitations and are listed in Table IV.

Subsectors. A subsector is a set of fields which is closed under the action of the spin-chain Hamiltonian, i.e. there is no overlap between spin-chains from within a subsector with spin-chains from outside. The subsectors of ABJM theory above the vacuum (3.1) are listed in Table V. To prove that these sectors are closed to all orders in perturbation theory, one defines a positive semi-definite charge

Table IV. Multi-excitations

	Multi-excitation	Made of
Double excitations	$Y_1^\dagger Y^1, Y^4 Y_4^\dagger$	$Y^2 Y_2^\dagger \pm Y^3 Y_3^\dagger$
	$\psi_2 Y_4^\dagger, \psi^{\dagger 3} Y^1$	$\psi_4 Y_2^\dagger \pm Y^3 \psi^{\dagger 1}$
	$\psi_3 Y_4^\dagger, \psi^{\dagger 2} Y^1$	$\psi_4 Y_3^\dagger \pm Y^2 \psi^{\dagger 1}$
Triple excitations	$\psi_1 Y_4^\dagger Y^1$	$Y^2 \psi^{\dagger 1} Y^3$
	$\psi^{\dagger 4} Y^1 Y_4^\dagger$	$Y_2^\dagger \psi_4 Y_3^\dagger$
	$D_\mu Y^1 Y_4^\dagger$	$\psi_4 \gamma_\mu \psi^{\dagger 1}$

In order to determine which elementary excitations a composite is made out of, one needs to compare their SU(2|2) × U(1)$_{\text{extra}}$ charges. For example, for the triple excitation ψ_1 one can check that the charges of ψ_1 together with the two background fields $Y^1 Y_4^\dagger$ coincide with the charges of the three elementary excitations $Y^2 \psi^{\dagger 1} Y^3$

Table V. Subsectors

Subsector	Vacuum	Single	Double
Vacuum	$Y^1 Y_4^\dagger$		
SU(2) × SU(2)	$Y^1 Y_4^\dagger$	$Y^2 Y_3^\dagger$	
OSp(2|2)	$Y^1 Y_4^\dagger$	$\psi_{4+} \psi_+^{\dagger 1}$	D_+
OSp(4|2)	$Y^1 Y_4^\dagger$	$Y^2 \psi_{4+} Y_3^\dagger \psi_+^{\dagger 1}$	$D_+ \psi_{3+} \psi_+^{\dagger 2}$
SU(2)	$Y^1 Y_4^\dagger$	Y^2	
SU(1|1)	$Y^1 Y_4^\dagger$	ψ_{4+}	
SU(2|1)	$Y^1 Y_4^\dagger$	$Y^2 \psi_{4+}$	
SU(3|2)	$Y^1 Y_4^\dagger$	$Y^2 Y^3 \psi_{4+} \psi_{4-}$	

This list of closed subsectors above the vacuum $\text{tr}(Y^1 Y_4^\dagger Y^1 Y_4^\dagger \cdots)$ is complete, although a specific subsector can be realized also by other fields. That would correspond to a different embedding of the sector into the full theory. Note that there is no closed SL(2) sector that is made only out of derivatives as we had in $\mathcal{N}=4$ SYM. This is because derivatives are double excitations of fermions with the above choice of vacuum. However, it is also possible to consider closed subsectors based on a different vacuum. There is, for instance, an SL(2) sector built from derivatives onto the vacuum $\text{tr}(Y^1 \psi^{\dagger 1})^L$ [26], which was studied, e.g. in [27,28]

$P = n_1 p_1 + n_2 q + n_3 p_2 + n_4 \Delta + n_5 s + n_6 b \geq 0$ from the eigenvalues of all operators that commute with the spin-chain Hamiltonian $E = \Delta - J$. These are the five Cartan generators of OSp(6|4) and the baryonic charge U(1)$_b$. The set of fields with $P = 0$ constitute a closed subsector. Different subsectors are obtained by different choices for the numbers n_i.

Spin-chain Hamiltonian. Various works have computed the spin-chain Hamiltonian for different subsectors to different loop orders with different methods in different approximations. The first results were obtained in the SU(4) sector[5] at two[6] loops [10,22] where the spin-chain Hamiltonian reads

$$H = \frac{\lambda^2}{2} \sum_{l=1}^{2L} \left(2 - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2} \right). \tag{3.3}$$

with $P_{l,m}$ and $K_{l,m}$ being the permutation and the trace operator, respectively, and $2L$ being the length of the spin-chain. This Hamiltonian has been proven to be integrable by means of an algebraic Bethe ansatz [10,22]. In the SU(2) × SU(2) sector, independently studied in [11], the trace operators annihilate the states and the Hamiltonian reduces to the sum of two decoupled Heisenberg $XXX_{1/2}$ Hamiltonians, one acting onto the even sites and one acting onto the odd sites. The only coupling between these two sublattices comes from the cyclicity condition which says that the *total* momentum of all excitations has to be zero (mod 2π), not individually for the even and odd sites. Nevertheless, the Hamiltonians will continue to be decoupled up to six loop order [15].

The extension of the two-loop Hamiltonian to the full theory was derived in [26,29]. The integrability in the OSp(4|2) sector was proved by means of a Yangian construction [26]. The generalization to ABJ theory at two loops was studied in the scalar sector [30] and the full theory [29], which at this perturbative order amounts to replacing λ^2 in the ABJM result by $\lambda\hat{\lambda}$, cf. (2.6). That means that the absence of parity in ABJ theory is not visible at two-loop order.

Beyond two loops only the dispersion relation, i.e. the eigenvalue of the Hamiltonian on spin-chains with a single excitation, is known to date. It is of the general form (1.7). The expansion of the interpolating function h to four-loop order was computed for the ABJM and the ABJ theory in [31–33] with the result

$$h^2(\lambda, \hat{\lambda}) = \lambda\hat{\lambda} - (\lambda\hat{\lambda})^2 \left[\frac{2\pi^2}{3} + \frac{\pi^2}{6} \left(\frac{\lambda - \hat{\lambda}}{\sqrt{\lambda\hat{\lambda}}} \right)^2 \right], \tag{3.4}$$

where the ABJM expression is obtained from this by setting the two 't Hooft couplings equal to each other. We see that $h(\lambda, \lambda)$ is for the form (1.8) with $c_1 = -\pi^2/3$. Note that (3.4) is invariant under the exchange of λ and $\hat{\lambda}$, even though ABJ theory lacks manifest parity invariance. The fact that parity is not broken in the spin-chain picture is *not* a consequence of integrability, because as shown in [30] there are integrable but parity breaking spin-chain Hamiltonians already at two loops. Alternative explanations for the non-visibility of parity breaking were proposed [30]. In ABJ theory one can also study the limit $\lambda \gg \hat{\lambda}$ [31].

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[5]This sector is closed at two-loop order but not beyond.
[6]There is no contribution to the Hamiltonian at an odd number of loops as in three dimensions no such Feynman diagram is logarithmically divergent.

In this limit, the Hamiltonian of the $SU(2) \times SU(2)$ sector is, at any loop order, proportional to two decoupled Heisenberg spin-chain Hamiltonians [31]. An exact expression for the λ-dependent prefactor, which gives a prediction for the function $h(\lambda, \hat{\lambda})$ in the limit $\hat{\lambda} \ll \lambda$, has been conjectured in [34]. Very recently, even for the case when $\lambda = \hat{\lambda}$ an all-order guess for $h^2(\lambda)$ was made [33], that is in line with the weak and strong coupling data.

At six loops only a subset of Feynman diagrams have been evaluated, namely those which move the impurities along the spin-chain by the maximal amount that is possible at this loop order [35]. The contributions from this subset to the dilatation operator are consistent with the corresponding spin-chain being integrable [35].

Also non-planar contributions to the two-loop dilatation operator have been computed in the $SU(2) \times SU(2)$ sector [36]. The degeneracy of the dimensions of parity pairs at the planar level, which is a signature of integrability, is lifted by the non-planar contributions [36]. At the non-planar level, one can also observe the breaking of parity in the ABJ theory already at two loops [37].

4. Superstrings on AdS$_4 \times$ CP^3

String background. AdS$_4 \times$ CP^3 with two- and four-form fluxes turned on is a solution to IIA supergravity that preserves 24 out of 32 supersymmetries [38], i.e. unlike AdS$_5 \times S^5$ it is not maximally supersymmetric. The AdS$_4 \times$ CP^3 super-space geometry has been constructed in [39]. The fermionic coordinates $\Theta^{1..32} = \left(\vartheta^{1..24}, \upsilon^{1..8} \right)$ split into 24 coordinates ϑ, which correspond to the unbroken super-symmetries of the background, and eight coordinates υ corresponding to the broken supersymmetries.

Green–Schwarz action. Although formal expressions for the Green–Schwarz super-string action exist for any type II supergravity background [40], in practice it is generically hopeless to find exact expressions for the supervielbeins. Nevertheless, utilizing the connection to M-theory on AdS$_4 \times S^7$, all functions that are required to write down the Nambu–Goto form of the action, in particular the supervielbe-ins and the NS–NS two-form superfield, were explicitly spelled out in [39], and the simpler κ-gauge-fixed version was given in [41]. However, it is probably fair to say that working with this action is still quite cumbersome as the explicit expressions are rather involved.

Coset action. A more pragmatic approach to strings on AdS$_4 \times$ CP^3 has been taken in [42,43]. The observation is that AdS$_4$ is the coset $SO(2,3)/SO(1,3)$ and CP^3 is the coset $SO(6)/U(3)$, and that $SO(2,3) \times SO(6)$ is the bosonic subgroup of $OSp(6|4)$. Thus, the idea is to write the superstrings action as a sigma-model on the supercoset

$$\frac{OSp(6|4)}{SO(1,3) \times U(3)},$$

(4.1)

analogously to the $PSU(2,2|4)/SO(1,4) \times SO(5)$ coset model for superstrings on $AdS_5 \times S^5$ [44], which itself was inspired by the WZW-type action for strings in flat space [45]. Again it is possible to define a $\mathbb{Z}_4$ grading [46] of the (complexified) algebra [42,43], and when this grading is used to split up the current one-form $A = -g^{-1}dg = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)}$, constructed from a parametrization of the coset representatives g, then the coset action is given by

$$\mathscr{S} = -\frac{R^2}{4\pi\alpha'} \int d\sigma \, d\tau \, \mathrm{str} \left[\sqrt{-h} \, h^{\alpha\beta} \, A^{(2)}_\alpha A^{(2)}_\beta + \kappa \varepsilon^{\alpha\beta} \, A^{(1)}_\alpha A^{(3)}_\beta \right]. \tag{4.2}$$

The explicit form of this sigma-model action can look quite differently depending on the choice of coset representative and the choice of gauge [42,43,47,48].

Fermions, κ-symmetry and singular configurations. There is a subtle problem with the coset action (4.2). The supercoset (4.1) has only 24 fermionic directions, which is the number of supersymmetries preserved by the background. However, independent of how many supersymmetries are preserved, the Green–Schwarz superstring always requires two Majorana–Weyl fermions with a total number of 32 degrees for freedom. Thus, the coset model misses eight fermions and can therefore not be equivalent to the GS string! This problem did not exist in the case of $AdS_5 \times S^5$ because that background is maximally supersymmetric and the corresponding supercoset has 32 fermionic directions.

It has been argued that the eight missing fermions υ are part of the 16 fermionic degrees of freedom that due to κ-gauge symmetry are unphysical anyway, i.e. to think of the coset action on (4.1) as an action with κ-symmetry partially gauge-fixed. Of the remaining 24 fermions ϑ, further 8 should then be unphysical. For this interpretation to be correct, the rank of κ-symmetry of the coset action must be 8. This is in fact true for generic bosonic configurations [42,43]: unfortunately, however, not for strings that move only in the AdS part of the background, in which case the rank of κ-symmetry is 12 [42]. This means that on such a "singular configuration" the coset model is a truncation of the GS string where instead of removing eight unphysical fermions (from 32 to 24), four physical fermions have been put to zero, while four unphysical fermions have been retained.

The upshot is that the coset model is generically equivalent to the GS string, but *not* on singular backgrounds. The consequence is that these singular backgrounds cannot be quantized semi-classically within the coset description.

Near plane-wave expansion. One method for dealing with a curved RR-background at the quantum level is to take a Penrose limit of the geometry which leads to a solvable plane-wave background and then to include curvature corrections perturbatively. Penrose limits of the $AdS_4 \times CP^3$ background were studied in

[11,12,49–51]. The near plane-wave Hamiltonian was derived in a truncation[7] to the bosonic sector in [52,53], for a sector including fermions in [54], and for the full theory in [50].

Alternative approaches. The pure spinor formulation of the superstring on $AdS_4 \times CP^3$ was developed in [55–57]. This approach is suitable for the covariant quantization of the string. Another possibility to obtain an action for the $AdS_4 \times CP^3$ string is to start from the supermembrane on $AdS_4 \times S^7$ and perform a double dimensional reduction [58].

5. From $AdS_4 \times CP^3$ to the Integrable Model

Evidence for integrability. The purely bosonic sigma-model on $AdS_4 \times CP^3$ is integrable at the classical level, though quantum corrections spoil the integrability [59,60]. For the super-coset model, classical integrability is also proven [42,43]. The Lax connection found in [61] for the $AdS_5 \times S^5$ case as a means of writing the equations of motion in a manifestly integrable form is directly applicable here. Moreover, the absence of particle production in the coset sigma-model has been shown explicitly for bosonic amplitudes at tree-level [62]. However, we know that the full GS string is more than the coset model. Therefore, although there are generic arguments in favor of the integrability of the whole theory, the direct proof of the integrability of the complete $AdS_4 \times CP^3$ superstring still remains an open problem [41]. Different integrable reductions of the sigma model have also been studied [63–65].

Matching $AdS_4 \times CP^3$ to ABJM theory. The metric on $AdS_4 \times CP^3$ has the two factors:

$$ds^2 = R^2 \left[\frac{1}{4} ds^2_{AdS_4} + ds^2_{CP^3} \right], \tag{5.1}$$

where R is the radius of CP^3 which is twice the radius of AdS_4. This relative size is demanded by supersymmetry and comes out automatically when one starts from the coset action (4.2). The radius R is related to the 't Hooft coupling λ of ABJM theory by (1.4). In global coordinates the metric for AdS_4 reads

$$ds^2_{AdS_4} = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \left(d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \tag{5.2}$$

with coordinate ranges $\rho = 0 \ldots \infty$, $t = -\infty \ldots \infty$, $\theta = 0 \ldots \pi$, and $\varphi = 0 \ldots 2\pi$. The metric on CP^3 is the standard Fubini-Study metric and can be written as

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[7]This truncation is not consistent and the absence of the fermions yields divergences, which were regularized using ζ-function regularization. Up to so-called "non-analytic" terms, the result is correct.

$$ds_{CP^3}^2 = d\xi^2 + \cos^2\xi \sin^2\xi \left[d\psi + \frac{1}{2}\cos\theta_1 \, d\varphi_1 - \frac{1}{2}\cos\theta_2 \, d\varphi_2 \right]^2$$
$$+ \frac{1}{4}\cos^2\xi \left[d\theta_1^2 + \sin^2\theta_1 \, d\varphi_1^2 \right] + \frac{1}{4}\sin^2\xi \left[d\theta_2^2 + \sin^2\theta_2 \, d\varphi_2^2 \right]. \tag{5.3}$$

The coordinates (θ_1, φ_1) and (θ_2, φ_2) parameterize two two-spheres, the angle $\xi = 0 \ldots \frac{\pi}{2}$ determines their radii, and the angle $\psi = 0 \ldots 2\pi$ corresponds to the $U(1)_R$ isometry.

The background admits five Killing vectors:

$$E = -i\partial_t, \quad S = -i\partial_\varphi, \quad J_{\varphi_1} = -i\partial_{\varphi_1}, \quad J_{\varphi_2} = -i\partial_{\varphi_2}, \quad J_\psi = -i\partial_\psi \tag{5.4}$$

leading to the five conserved charges: the worldsheet energy E, the AdS-spin S and the CP^3 momenta $J_{\varphi_1}, J_{\varphi_2}$, and J_ψ. Note that this is one conserved charge less than in the $AdS_5 \times S^5$ case where there are two AdS-spins. This shows that $AdS_4 \times CP^3$ is less symmetric. The charges (5.4) are one choice of Cartan generators of $SO(3,2) \times SU(4)$. The angular momenta J_{φ_1} and J_{φ_2} correspond to the Cartan generators of two $SU(2)$ subgroups that on the gauge theory side transform (Y^1, Y^2) and (Y^3, Y^4), respectively. The angular momentum J_ψ is the $U(1)_R$ generator. Thus, the angular momenta are related to the charges in Table III according to

$$J_{\varphi_1} = \frac{1}{2}p_1, \quad J_{\varphi_2} = \frac{1}{2}p_2, \quad J_\psi = q + \frac{1}{2}(p_1 + p_2). \tag{5.5}$$

These relations are important for identifying classical strings with gauge theory operators. It also suggests a parametrization of CP^3 inside $\mathbb{C}^4$ in terms of the embedding coordinates

$$y^1 = \cos\xi \, \cos\frac{\theta_1}{2} \, e^{i(+\varphi_1+\psi)/2} \qquad\qquad y^3 = \sin\xi \, \cos\frac{\theta_2}{2} \, e^{i(+\varphi_2-\psi)/2} \tag{5.6}$$
$$y^2 = \cos\xi \, \sin\frac{\theta_1}{2} \, e^{i(-\varphi_1+\psi)/2} \qquad\qquad y^4 = \sin\xi \, \sin\frac{\theta_2}{2} \, e^{i(-\varphi_2-\psi)/2}$$

which can be identified one-to-one with the scalar fields Y^A of ABJM theory.

Worldsheet spectrum. In order to relate the string description to the spin-chain picture, we need to quantize the worldsheet theory. It is only known how to do this by semiclassical means, i.e. by expanding the string about a classical solution and quantizing the fluctuations. As can be seen from the charges, the classical string solution that corresponds to the vacuum spin-chain, or in other words to the gauge theory operator $\mathrm{tr}(Y^1 Y_4^\dagger)^L$ (with L large so that the string becomes classical), is a point-like string that moves along the geodesic parametrized by $t = \kappa\tau$, $\psi = \kappa\tau$, located at the center of AdS_4 ($\rho = 0$) and the equator of CP^3 ($\xi = \pi/4$), and furthermore sitting at the north pole of the first sphere ($\theta_1 = 0$) and at the south pole of the other sphere ($\theta_2 = \pi$). Expanding the fields in fluctuations of order $\lambda^{-1/4}$ yields the mass spectrum given in Table VI.

Table VI. Spectrum of fluctuations about the point-like string

Field	Mass	Dispersion relation
t, ψ	0	$\omega_n = n$
$x_{1,2,3}, \xi$	κ	$\omega_n = \sqrt{\kappa^2 + n^2}$
$\theta_{1,2}, \varphi_{1,2}$	$\kappa/2$	$\omega_n = \sqrt{(\kappa/2)^2 + n^2} \pm \kappa/2$

Two linear combinations of $\theta_{1,2}$ and $\varphi_{1,2}$ possess the dispersion relation with $+\kappa/2$, and two other linear combinations the one with $-\kappa/2$

The massless fluctuations $\tilde{t}$ and $\tilde{\psi}$ can be gauged away, i.e. set to zero. This is the usual light-cone gauge, $t + \psi \sim \tau$, with one light-cone direction in AdS$_4$ and one in CP^3. We are left with four light excitations ($\theta_{1,2}, \varphi_{1,2}$) from CP^3 and four heavy excitations of which one (ξ) comes from CP^3 and the other three ($x_{1,2,3}$) from AdS$_4$. For the eight physical fermions the same pattern is found: four light excitations of mass $\kappa/2$ and four heavy excitations of mass κ.

These worldsheet modes transform in definite representations of the residual symmetry group SU(2|2) $\times$ U(1)$_{\text{extra}}$ that is left after fixing the light-cone gauge [66]. The light fields form two (2|2)-dimensional supermultiplets [48]:

"A"-particles: (X^a, ψ_α),   (5.7a)

"B"-particles: $(X_a^\dagger, \psi^{\dagger\alpha})$,   (5.7b)

where $a = 1, 2$ and $\alpha = 1, 2$ are SU(2)$_G \times$ SU(2)$_r$ indices. The doublet of complex scalars X^a is a combination of $\theta_{1,2}$ and $\varphi_{1,2}$, and the fermions are written in terms of a complex spinor ψ_α. These two supermultiplets correspond precisely to the A- and B-particles (3.2) in the spin-chain picture, respectively!

The heavy fields form one (1|4|3)-dimensional supermultiplet ($\xi, \chi_\alpha^a, x_{1,2,3}$) [48]. The bosonic components are literally the coordinates used above, and the fermionic component is a doublet of Majorana spinors. These heavy fields, however, do not count as independent excitations in the spin-chain description; they are rather an artifact of the above analysis which is done at infinite coupling λ. When going to finite coupling they "dissolve" into two light particles [48]. At the technical level this is seen by looking at which particle poles appear in Green's functions at *not* strictly infinite coupling [48,54]. The first observation is that in the free theory the pole for the heavy particles with mass κ coincides with the branch point of the branch cut that accounts for the pair production of two light modes with mass $\frac{\kappa}{2}$ each. When interactions are turned on, i.e. when $1/\sqrt{\lambda}$ corrections are considered, the pole moves into the branch cut, and the statement is that the exact propagator has a branch cut only.

Giant magnons. As we have just seen, the worldsheet fluctuations match the spin-chain excitations, but only as far as their charges are concerned. The dispersion relation of the worldsheet excitations is relativistic rather than periodic as in (1.7). In order to see the periodic dispersion relation also on the string theory side,

macroscopically many quanta must be excited. The results are classical string solutions known as giant magnons [67], or dyonic giant magnons [68,69] if they have at least two non-zero angular momenta. The dispersion relation of all dyonic giant magnons are of the form (1.7) for appropriate values for Q.

The variety of giant magnons in CP^3 is somewhat larger than in S^5. The simplest types are obtained by embedding the HM giant magnon [67] into subspaces of CP^3 [11] (see also [70]). There are two essentially different choices: one may either pick a proper two-sphere inside CP^3 or a two-sphere with antipodes identified. According to these subspaces the former choice leads to what is called the CP^1 ($\cong S^2$) giant magnon [11] and the latter choice to the so-called RP^2 ($\cong S^2/\mathbb{Z}_2$) giant magnon [11,12].

The RP^2 giant magnon is in fact a threshold bound state of two HM giant magnons, one inside each of the S^2s parametrized by (θ_1, φ_1) and (θ_2, φ_2) in (5.3) [12]. Therefore, this kind of giant magnon is sometimes referred to as the $S^2 \times S^2$ magnon or as the $SU(2) \times SU(2)$ magnon. This is, however, somewhat misleading as the two constituent magnons do not move independently.

The dyonic generalization of the CP^1 giant magnon moves in a CP^2 subspace of CP^3 and was found for momentum $p = \pi$ in [71] and for general momenta in [72]. This giant magnon does not have an analog in $AdS_5 \times S^5$. The CP^2 dyonic giant magnons are in one-to-one correspondence with the elementary spin chain excitations (3.2): the polarizations of the giant magnons match the flavors of the excitations [73]. In [73] it has also been shown that the classical phase shifts in the scattering of these dyonic giant magnons are consistent with the S-matrix proposed by Ahn and Nepomechie [74]. The general scattering solutions of N giant magnons have also been known since very recently [75], in fact for the much wider context of giant magnons on CP^n, $SU(n)$ and S^n [76].

The dyonic generalization of the RP^2 giant magnon moves in a RP^3 subspace of CP^3 and was found in [63]. This giant magnon is the CDO dyonic giant magnon on S^3 [69] embedded into RP^3. It can be regarded as a composite of two CP^2 dyonic magnons with equal momenta [73]. Finally, by the dressing method one can also find a two-parameter one-charge solution [72,77,78].

6. Solving Ads_4/CFT_3 Using Integrability

In this section, we will briefly discuss those aspects of the methods employed to solve the Ads_4/CFT_3 model that differ from the ones in the Ads_5/CFT_4 case. For an introduction to these tools, we refer to the other chapters of this review. For the Bethe ansatz see [79], for the S-matrix see [80], for the algebraic curve see [81], and for the thermodynamic Bethe ansatz and the Y-system see [82,83].

Asymptotic Bethe equations. The Bethe equations for the two-loop $SU(4)$ sector were derived within the algebraic Bethe ansatz scheme in [10], where also the extension of the Bethe equations to the full theory, though still at one loop, were

conjectured. The form of these equations is quite canonical and the couplings between the Bethe roots is encoded in the Dynkin diagram of OSp(6|4), see Table I. The all-loop extension of the Bethe equations was conjectured in [15].

The fact that we now have two types of momentum carrying roots—call them u and v—means that the conserved charges are given by sums over all roots of both of these kinds

$$Q_n = \sum_{j=1}^{K_u} q_n(u_j) + \sum_{j=1}^{K_v} q_n(v_j), \tag{6.1}$$

where q_n is the charge carried by a single root. The spin-chain energy, or anomalous dimension, or string light-cone energy, is the second charge $E = h(\lambda)Q_2$. The other Bethe roots—call them r, s, and w—are auxiliary roots and influence the spectrum only indirectly through their presence in the Bethe equations.

The SU(2) $\times$ SU(2) sector is given by only exciting the momentum carrying roots. The SU(4) sector uses the roots u, v, r, though this sector is only closed at two loops. The four components of an A-particle, cf. (3.2) and (5.7), correspond to the states with one u root and excitation numbers $\{K_r, K_s, K_w\} = \{0, 0, 0\}$, or $\{1, 0, 0\}$, or $\{1, 1, 0\}$, or $\{1, 1, 1\}$ for the auxiliary roots. The same holds for the B-particle if the u-root is replaced by one of type v. This accounts for all light excitations. The heavy excitations are given by a stack of one of each kind of the momentum carrying roots. This is the Bethe ansatz way of seeing that the heavy excitations are compounds.

This Bethe ansatz has been put to a systematic test by comparing the predicted eigenvalues with the direct diagonalization of the spin-chain Hamiltonian for various length-4 and length-6 states at two loops [84].

S-Matrix. It has been shown that the proposed all-loop Bethe ansatz can be derived from an exact two-particle S-matrix [74]. The alternating nature of the spin-chain naturally breaks the S-matrix up into pieces: interactions between two A-particles, between two B-particles, and between one of each kind [74], where each piece is proportional to the old and famous SU(2|2) S-matrix [7,85] from Ads$_5$/CFT$_4$. Crossing symmetry relates AA- and BB- to AB-scattering and therefore does not fix the overall scalar factor for any of them uniquely. A solution that is consistent with the Bethe equations was made in [74] and uses the BES dressing phase [86].

This S-matrix does not have poles that correspond to the heavy particles, which is in line with them not being asymptotic states. The heavy particles occur, however, as intermediate states. That is seen from the fact that they appear as internal lines in the Feynman diagrams that are used to derive the worldsheet S-matrix from scattering amplitudes [48].

The S-matrix has the peculiarity that the scattering of A- and B-particles is reflectionless [87]. Though at first unexpected, this property has been confirmed

perturbatively at weak [88] and at strong coupling [48]. This reflectionlessness would follow straightforwardly if one assumes that the two terms in (6.1) were individually conserved [89].

Algebraic curve. The algebraic curve for the Ads_4/CFT_3 duality was constructed from the string coset sigma-model in [90]. It is a ten-sheeted Riemann surface $q(x)$ whose branches—or quasi-momenta—are pairwise related $q_{1,2,3,4,5} = -q_{10,9,8,7,6}$. The physical domain is defined for spectral parameter $|x| > 1$. The values of the quasi momenta within the unit circle are related to their values outside it by an inversion rule [90]. Branch cut and pole conditions are identical to the ones in the Ads_5/CFT_4 case. The Virasoro constraints demand that the quasi momenta $q_1, \ldots, q_4$ all have a pole with the same residue at $x = 1$ and another one at $x = -1$, while the quasi momentum q_5 cannot have a pole at $x = \pm 1$.

For a given algebraic curve, the charges of the corresponding string solution are encoded in the large x asymptotics. For example, the curve

$$q_1(x) = \cdots = q_4(x) = \frac{L}{2g}\frac{x}{x^2 - 1}, \quad q_5(x) = 0. \tag{6.2}$$

carries the charges $(\Delta_0, S, J_{\varphi_1}, J_{\varphi_2}, J_\psi) = (L, 0, \frac{L}{2}, \frac{L}{2}, L)$ and $\delta\Delta = 0$ of $\text{tr}(Y^1 Y_4^\dagger)^L$ and thus corresponds to the vacuum. String excitations are represented by additional poles that connect the various branches. A dictionary between the polarizations of the excitations and the different branch connections is given in [90]. The light modes can be recognized as those which connect a non-trivial sheet with a trivial sheet in (6.2), and the heavy modes are those which connect two non-trivial sheets.

Thermodynamic Bethe ansatz and Y-system. The Y-system for the $OSp(6|4)$ spin-chain was conjectured along with the corresponding equations for Ads_5/CFT_4 in [91]. A derivation of the Y-system, i.e. writing down the asymptotic Bethe ansatz at finite temperature for the mirror theory, formulating the string hypothesis, and Wick rotating back to the original theory, was performed in [92,93], and a modification of the original conjecture was found.

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