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Review of AdS/CFT Integrability, Chapter IV.1: Aspects of Non-Planarity

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Abstract. We review the role of integrability in certain aspects of $\mathcal{N}=4$ SYM which go beyond the planar spectrum. In particular, we discuss integrability in relation to non-planar anomalous dimensions, multi-point functions and Maldacena–Wilson loops.

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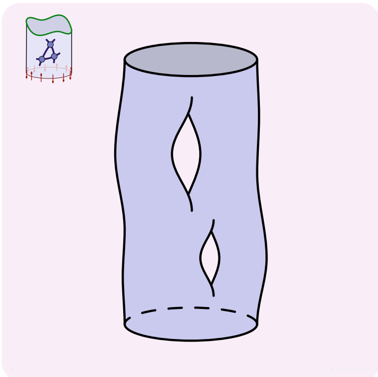

1. Introduction

The discovery of the integrability of the planar spectral problem of AdS/CFT [1–4] has provided us with a wealth of new results and tools for the study of gauge and string theory. Given this success, it is natural to investigate whether the integrability extends to other aspects of the AdS/CFT correspondence. Here, we shall discuss this possibility mainly from the gauge theory perspective and staying entirely within the maximally supersymmetric gauge theory in four dimensions, $\mathcal{N}=4$ SYM. The fate of the integrability of the planar spectral problem when reducing or completely removing the supersymmetry is discussed in the chapters [5] and [6]. A natural direction in which to search for integrability is in the non-planar version of the spectral problem. As we will review below, while the non-planar version of the dilatation generator can easily be written down (at least in some sub-sectors and to a certain loop order), attempts to diagonalize it have so far not revealed any traces of integrability. For a conformal field theory like $\mathcal{N}=4$ SYM, natural observables apart from anomalous dimensions are the structure constants which appear in the three-point functions of the theory and govern the theory's operator product expansion. Three-point functions are of course not unrelated to non-planar anomalous dimensions, as correlators of three traces can be seen as building blocks for higher genus two-point functions. As we shall see, the calculation of structure constants of $\mathcal{N}=4$ SYM is impeded by extensive operator mixing. For a certain subset of operators, this mixing can be handled via the diagonalization of the planar dilatation operator, and the structure constants can be calculated using tools pertaining to planar integrability. An integrable structure allowing to treat all types of three-point functions has not been identified.

Anomalous dimensions and structure constants are observables, which are associated with local gauge invariant operators, but in a gauge theory one of course also has at hand numerous types of non-local observables such as Wilson loops, 't Hooft loops, surface operators and domain walls. Here, we will limit our discussion to Wilson loops, more precisely to locally supersymmetric Maldacena–Wilson loops. Another type of Wilson loops, Alday–Maldacena–Wilson loops and their relation to scattering amplitudes of $\mathcal{N}=4$ SYM will be discussed in the chapters [7–9]. As was known before the discovery of the spin-chain related integrability of the AdS/CFT system, expectation values of Maldacena–Wilson loops can in certain cases be expressed in terms of expectation values of a zero-dimensional integrable matrix model and this connection has provided us with the most successful test of the AdS/CFT correspondence beyond the planar limit to date. The connection of Maldacena–Wilson loops to integrability in the form of spin-chain integrability is so far very limited.

We start by discussing the role of integrability in connection with non-planar anomalous dimensions in Section 2 and subsequently treat multi-point functions and Maldacena–Wilson loops in Sections 3 and 4.

2. Non-Planar Anomalous Dimensions

In a CFT, conformal operators, $\{\mathcal{O}_\alpha\}$, and their associated conformal dimensions, Δ_α, are characterized by being eigenstates and eigenvalues of the dilatation generator, $\hat{D}$. As a consequence of this, two-point functions of conformal operators, upon appropriate normalization take the form

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(y)\rangle = \frac{\delta_{\alpha\beta}}{(x-y)^{2\Delta_\alpha}}. \tag{2.1}$$

2.1. THE NON-PLANAR DILATATION GENERATOR

The dilatation generator, $\hat{D}$, of $\mathcal{N}=4$ SYM has a double expansion in λ and $\frac{1}{N}$ where λ is the 't Hooft coupling, which we until further notice take to be

$$\lambda = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}, \tag{2.2}$$

and where N is the order of the gauge group, $SU(N)$. By the planar limit, we mean the limit $N \to \infty$, λ fixed. At a finite order in λ, the $\frac{1}{N}$-expansion of the dilatation generator starts at order N^0 and terminates after finitely many terms, the number of which increases with the loop order. The planar dilatation generator and its loop expansion are discussed in the chapter [10]. The non-planar part of the dilatation generator was first derived at one loop order in the $SO(6)$ sector [11,12], see also [13]. The derivation was based on evaluation of Feynman diagrams and was extended to two-loop order in the $SU(2)$ sector in [2]. Later, a derivation based entirely on algebraic arguments gave the dilatation generator including non-planar parts for all fields at one-loop order [14] and for the fields in the $SU(1,1|2)$ sector at two-loop order [15]. Recently, the non-planar part of the dilatation generator was written down at order $\lambda^{3/2}$ in the $SU(2|3)$ sector [16]. In addition, the non-planar part of the dilatation generator is known in the scalar sector in a certain $\mathcal{N}=2$ superconformal gauge theory [17]. In ABJM theory [18] and ABJ theory [19], the non-planar part of the two-loop dilatation generator has been derived in a $SU(2) \times SU(2)$ sector [20,21].[1]

The diagonalization problem for the full dilatation generator of $\mathcal{N}=4$ SYM has mainly been studied in the $SU(2)$-sector, which consists of multi-trace operators built from two complex scalar fields, say X and Z. For simplicity, we shall likewise focus our discussion on this sector. The one-loop dilatation generator including the non-planar parts reads for the $SU(2)$ sector

$$\hat{D} = -\frac{\lambda}{N} :\mathrm{Tr}[X, Z][\check{X}, \check{Z}]:, \quad \text{where} \quad \check{Z}_{\alpha\beta} = \frac{\delta}{\delta Z_{\beta\alpha}}, \tag{2.3}$$

and similarly for $\check{X}$. The normal ordering symbol signifies that the derivatives should not act on the X and Z field belonging to the dilatation generator itself. Below, we illustrate how the full dilatation generator acts on a double trace operator. Notice that we only consider one out of four terms contributing to the dilatation generator and that we only represent one possible way of applying the derivatives

[1] We remark that our $\hat{D}$ is the dilatation generator describing the asymptotic spectrum. Hence we ignore the wrapping contributions discussed in the chapters [10,22–24]. In particular, the splitting of the dilatation operator into planar and non-planar parts that we discuss here pertains to the asymptotic regime. What is here referred to as non-planar parts of the dilatation generator might for short operators give rise to planar wrapping contributions [25].

$$\mathrm{Tr}(ZX\check{Z}\check{X})\cdot\mathrm{Tr}(XZXXZ)\,\mathrm{Tr}(XZ)=\mathrm{Tr}(ZX\check{Z}ZXXZ)\,\mathrm{Tr}(XZ)$$

$$=N\mathrm{Tr}(ZXXXZ)\,\mathrm{Tr}(XZ)+\mathrm{Tr}(ZX)\,\mathrm{Tr}(ZXX)\,\mathrm{Tr}(XZ)+\mathrm{Tr}(ZXZZZXXZ).$$

As is evident from this example, the full one-loop dilatation generator can be written as follows

$$\hat{D}=\lambda\left(\hat{D}_0+\frac{1}{N}\,\hat{D}_+ +\frac{1}{N}\,\hat{D}_-\right), \tag{2.4}$$

where $\hat{D}_+$ and $\hat{D}_-$, respectively, increases and decreases the trace number by one and where $\hat{D}_0$ conserves the number of traces. Suggestions for how to write $\hat{D}_+$ and $\hat{D}_-$ in a more explicit form can be found in [26,27]. We notice that for gauge group $SO(N)$ or $Sp(N)$, the one-loop dilatation operator will have a term which is of order $\frac{1}{N}$, but still conserves the number of traces [28]. At l-loop order, the dilatation operator can change the number of traces by at most l. Notice that since the anomalous dimensions are the *eigenvalues* of the dilatation generator, these do not necessarily have a $\frac{1}{N}$-expansion that truncates. What is more, some anomalous dimensions do not even have a well-defined double expansion in λ and $\frac{1}{N}$. An example of an operator with this property can be found in [2]. Speaking about a one-loop anomalous dimension, however, always makes sense. To calculate the leading $\frac{1}{N}$-corrections to one-loop anomalous dimensions, one can make use of standard quantum mechanical perturbation theory. Let us assume that we have found an eigenstate of the planar dilatation generator $\hat{D}_0$, i.e.

$$\hat{D}_0|\mathcal{O}\rangle=\gamma_{\mathcal{O}}|\mathcal{O}\rangle, \tag{2.5}$$

and let us treat the terms sub-leading in $\frac{1}{N}$ as a perturbation. First, let us assume that there are no degeneracies between n-trace states and $(n+1)$-trace states in the spectrum. If that is the case, we can proceed by using non-degenerate quantum mechanical perturbation theory. Clearly, the $\frac{1}{N}$ terms in equation (2.4) do not have any diagonal components, so the correction to the anomalous dimension for the state $|\mathcal{O}\rangle$ reads

$$\delta\gamma_{\mathcal{O}}=\frac{1}{N^2}\sum_{\mathcal{K}\neq\mathcal{O}}\frac{\langle\mathcal{O}|\hat{D}_+ +\hat{D}_-|\mathcal{K}\rangle\cdot\langle\mathcal{K}|\hat{D}_+ +\hat{D}_-|\mathcal{O}\rangle}{\gamma_{\mathcal{O}}-\gamma_{\mathcal{K}}}, \tag{2.6}$$

and is of order $\frac{1}{N^2}$. If there are degeneracies between n-trace states and $(n+1)$-trace states, we have to diagonalize the perturbation in the subset of degenerate states and the corrections will typically be of order $\frac{1}{N}$. We remark that the dilatation generator is *not* a Hermitian operator, but it is related to its Hermitian conjugate by a similarity transformation and therefore its eigenvalues are always real [12,29,30].

2.2. THE NON-PLANAR SPECTRUM AND INTEGRABILITY

Planar $\mathcal{N}=4$ SYM is described in terms of only one parameter, λ, and planar anomalous dimensions have a perturbative expansion in terms of this single parameter. This fact made it possible initially to search for integrability in the planar spectrum order by order in λ. In particular, the concept of perturbative integrability was introduced, meaning that at l loops the planar spectrum could be described as an integrable system when disregarding terms of order λ^{l+1} [2]. Studying this perturbative form of integrability eventually led to the all loop Bethe equations conjectured to be true perturbatively to any loop order and non-perturbatively as well [31–33]. When going beyond the planar limit, it is natural to follow a similar perturbative approach. The question of integrability beyond the planar limit has so far been addressed only perturbatively in $\frac{1}{N}$ at the one-loop order. The fact that the non-planar part of the dilatation generator introduces splitting and joining of traces enormously enlarges the Hilbert space of states of the system. This complicates the direct search for integrability via the identification of conserved charges or the construction of an asymptotic S-matrix with the appropriate properties. As a simple way of getting an indication of whether integrability persists at the non-planar level, one can test for degenerate parity pairs [2]. Parity pairs are operators with the same anomalous dimension, but opposite parity where the parity operation on a single trace operator is defined by [34]

$$\hat{P}\cdot \mathrm{Tr}\left(X_{i_1} X_{i_2} \dots X_{i_n}\right) = \mathrm{Tr}\left(X_{i_n} \dots X_{i_2} X_{i_1}\right). \tag{2.7}$$

(For a multi-trace operator, $\hat{P}$ must act on each of its single trace components.) At the planar one-loop level, one observes a lot of such parity pairs. The presence of these degeneracies has its origin in the integrability of the model. $\mathcal{N}=4$ SYM is parity invariant and its dilatation generator commutes with the parity operation, i.e.

$$\left[\hat{D}, \hat{P}\right] = 0. \tag{2.8}$$

Notice that this only tells us that eigenstates of the dilatation generator can be organized into eigenstates of the parity operator and nothing about degeneracies in the spectrum. The degeneracies can be explained by the existence of an extra conserved charge, $\hat{Q}_3$, which commutes with the dilatation generator but anti-commutes with parity, i.e.

$$\left[\hat{D}, \hat{Q}_3\right] = 0, \qquad \left\{\hat{P}, \hat{Q}_3\right\} = 0. \tag{2.9}$$

Acting on a state with $\hat{Q}_3$, one obtains another state with the opposite parity but with the same energy.[2] Taking into account non-planar corrections, the degeneracies are lifted. Since parity is still conserved, this is taken as an indication (but

[2]There exist states which are unpaired and annihilated by $\hat{Q}_3$.

not a proof, obviously) of the disappearance of the higher conserved charges and thus a breakdown of integrability. Notice that in accordance with this picture, the parity pairs survive the inclusion of planar higher loop corrections. The situation in ABJM theory is the same. Degenerate parity pairs are seen at the planar level, but disappear once non-planar corrections are taken into account [20]. (For $\mathcal{N}=4$ SYM with gauge group $SO(N)$ or $Sp(N)$, parity is gauged and the concept of planar parity pairs loses its meaning [28]. For ABJ theory, parity is broken at the non-planar level [21].) Hence, it seems that one cannot hope for integrability of the spectrum of AdS/CFT beyond the planar limit, at least not in a simple perturbative sense.[3]

2.3. RESULTS ON NON-PLANAR ANOMALOUS DIMENSIONS

Prior to the derivation of the dilatation generator of $\mathcal{N}=4$ SYM, anomalous dimensions were determined through a rather complicated process which involved for each set of operators considered an explicit calculation of their two-point correlation functions through Feynman diagram evaluation. Early results on non-planar anomalous dimensions for short operators obtained by this method can be found in [36–39].

With the derivation of the dilatation generator, the calculation of anomalous dimensions was enormously simplified. At the planar level, one now even has at hand the tools of integrability and all information about the (asymptotic) spectrum is encoded in a set of algebraic Bethe equations. As argued above, similar tools are not currently available at the non-planar level. Thus to obtain spectral information beyond the planar limit, one has to explicitly diagonalize the dilatation generator in each closed subset of states. For the following discussion, it is convenient to divide the set of operators into three different types, short operators, BMN type operators and operators dual to spinning strings.

By short operators we mean operators which contain a finite, small number of fields. Such operators only mix with a finite, small number of other operators and the resulting mixing matrix can be calculated and diagonalized by hand (or using Mathematica). Various results on non-planar corrections to anomalous dimensions of short operator in the $SU(2)$ sector of $\mathcal{N}=4$ SYM can be found in [2] and [26]. Reference [26] in addition contains results on the $SL(2)$-sector of $\mathcal{N}=4$ SYM. Results for the $SU(2)\times SU(2)$ sector of ABJM and ABJ theory were obtained in [20] and [21].

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[3] The paper, [35], entitled "Hints of Integrability Beyond the Planar Limit:Non-trivial Backgrounds" deals with anomalous dimensions of operators from the $SU(2)$-sector consisting of the factor $(\det(Z))^M$ multiplying a single trace operator. In the limit $N, M \to \infty$ with $\frac{N}{M} \to 0$ and $g_{\mathrm{YM}}^2 M$ fixed, the authors find a set of conserved charges commuting with the dilatation generator. We remark, however, that in the limit considered the terms $\hat{D}_+$ and $\hat{D}_-$ do not contribute to the dilatation generator.

BMN-type operators [40] are operators consisting of many fields of one type and a few excitations in the form of fields of another type (or of derivatives). Two-excitation eigenstates can easily be written down at the planar level. In the $SU(2)$ sector, they read

$$\mathscr{O}_n^{J_0,J_1,\ldots,J_k}=\frac{1}{J_0+1}\sum_{p=0}^{J_0}\cos\left(\frac{\pi n(2p+1)}{J_0+1}\right)\mathrm{Tr}(X\,Z^p\,X\,Z^{J_0-p})\mathrm{Tr}\left(Z^{J_1}\right)\ldots\mathrm{Tr}(Z^{J_k}),\tag{2.10}$$

where $0\leq n\leq\left[\frac{J_0}{2}\right]$ and the corresponding planar eigenvalues are

$$E_n=8\lambda\sin^2\left(\frac{\pi n}{J_0+1}\right).\tag{2.11}$$

Acting with the non-planar part of the dilatation generator on BMN states only requires a finite and small number of operations and the non-planar part of the mixing matrix for BMN states can easily be written down [12]. Treating $\hat{D}_+ + \hat{D}_-$ as a perturbation of $\hat{D}_0$, one should thus be able to determine the leading non-planar corrections to the anomalous dimensions of BMN operators by standard quantum mechanical perturbation theory, cf. Section 2.2. However, degeneracies between single and multiple-trace states require the use of degenerate perturbation theory and, due to the complexity of the coupling between degenerate states, the mixing problem for BMN states was never resolved. For a discussion of this problem, see [41–43]. There is one case, however, for which there is no degeneracy issue and that is for states with mode number, $n=1$. Here, it is possible to find the leading non-planar correction to the anomalous dimension in the limit $J_i\to\infty$, $i=0,1,\ldots,k$, and $\lambda\to\infty$ with $\lambda'=\lambda/J^2$ and $g_2=J^2/N$ fixed, where $J=\sum_{i=0}^{k}J_i$. The result reads [11,44]

$$\delta E_{n=1}=\lambda' g_2^2\left(\frac{1}{12}+\frac{35}{32\pi^2}\right).\tag{2.12}$$

There exist similar results for BMN operators belonging to the $SL(2)$ sector of $\mathscr{N}=4$ SYM [45] and for BMN operators in a certain $\mathscr{N}=2$ superconformal gauge theory [17]. The result in equation (2.12) was extended to two-loop order in [12].

The third class of operators, operators dual to spinning strings, consist of an infinitely large number of background fields and an infinite number of excitations. In the $SU(2)$ sector, they take the form

$$\mathscr{O}=\mathrm{Tr}\left(Z^{J-M}X^M\right)+\cdots,\tag{2.13}$$

where $\cdots$ denotes similar terms obtained by permuting the fields and where J, $M\to\infty$, but M/J is kept finite. Acting with the non-planar dilatation generator on such an operator involves an infinite number of operations and becomes unfeasible. In [46], based on a coherent state formalism, matrix elements of the non-planar dilatation generator between operators dual to particular folded spinning strings were calculated but an explicit diagonalization of the non-planar dilatation generator for the situation in question did not seem tractable.

2.4. COMPARISON TO STRING THEORY

To generate string theory data with which to compare non-planar corrections to anomalous dimensions, one needs to take into account string loop corrections corresponding to considering string worldsheets of higher genus. For short operators, such a comparison is currently out of sight, since we do not even have any examples of a successful comparison at the planar level, except for certain BPS states which can be shown to have vanishing anomalous dimensions [47]. Recently, it was shown at one-loop order that certain 1/4 BPS states can be labeled by irreducible representations of the Brauer algebra [48], see also [49].

The situation is slightly more encouraging in the case of BMN operators. Considering the BMN limit on the gauge theory side corresponds on the string theory side to taking the Penrose limit of the $AdS_5 \times S^5$ background, and this turns the geometry into a PP-wave. On the PP-wave, one can quantize the free IIB string theory in light cone gauge and find the corresponding free spectrum. In addition, considering higher genus effects is possible by means of light cone string field theory (LCSFT). A review of the PP-wave/BMN correspondence including an introduction to LCSFT can be found in the References [50–54]. In LCSFT, string interactions are described in terms of a three-string vertex, which encodes the information about the splitting and joining of strings. There seems to be several ways of consistently defining this three-string vertex and there exist at least three proposals for its exact form. For all proposals, however, it holds that there is a freedom of choosing a certain pre-factor of the vertex. Reference [55] constitutes the most recent review of this topic describing the different possible choices of the three vertex and containing all the relevant references. Furthermore, the authors of [55] show that the one-loop gauge theory result (2.12) can be obtained from LCSFT provided one chooses one particular of the proposed vertices, and its pre-factor in a specific way.[4] It is, however, not possible to recover the two-loop gauge theory result from the LCSFT and generically LCSFT gives rise to half-integer powers of λ' appearing in the expressions for non-planar anomalous dimensions. Such half-integer powers of λ' were also found in the analysis of worldsheet one-loop corrections to the planar energies of spinning strings [56] and eventually led to the recognition that the BMN expansion breaks down not only at strong coupling, but also at weak coupling starting at four-loop order [32,57,58]. Hence, it

[4]It should be noticed, though, that the match to the one-loop gauge theory result is obtained after a truncation to the so-called impurity conserving channel, while at the same time it is proved that generically all channels would contribute to the result. In addition, it is pointed out that an undetermined supercharge could potentially also contribute to the result.

appears that to obtain complete agreement between gauge and string theory, we are forced to consider the full $\text{AdS}_5 \times S^5$ geometry.

Finally, in the case of operators dual to spinning strings no direct comparison between gauge theory and string theory has been possible. In Reference [27], the decay of a single folded spinning string into two such strings was studied in a semi-classical approximation and a certain relation between the conserved charges of the decay products was found. If the semi-classical decay channel were the dominant one, as it is known to be in flat space, one could hope that the matrix elements for string splitting and joining found in [46] could encode some similar relation. The analysis of [46], however, did not point towards the semi-classical decay channel being the dominant one.

3. Multi-Point Functions

By multi-point functions, we mean correlation functions of the following type

$$\langle \mathscr{O}_{\Delta_1}(x_1)\mathscr{O}_{\Delta_2}(x_2)\ldots\mathscr{O}_{\Delta_n}(x_n)\rangle, \tag{3.1}$$

where the operators involved are eigenstates of the dilatation generator and carry the conformal dimensions $\Delta_1, \Delta_2, \ldots, \Delta_n$. Three-point functions play a particular role since their form is fixed by conformal invariance and since they contain the information about the structure constants C_{ijk}, which appear in the theory's operator product expansion. For appropriately normalized conformal operators, the three-point functions take the form

$$\langle \mathscr{O}_{\Delta_1}(x_1)\mathscr{O}_{\Delta_2}(x_2)\mathscr{O}_{\Delta_3}(x_3)\rangle = \frac{C_{\Delta_1\Delta_2\Delta_3}}{(x_1-x_2)^{\Delta-2\Delta_3}(x_2-x_3)^{\Delta-2\Delta_1}(x_3-x_1)^{\Delta-2\Delta_2}}, \tag{3.2}$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$.

3.1. RESULTS ON MULTI-POINT FUNCTIONS

Before the advent of the BMN paper in 2002 [40], results on multi-point functions mostly had to do with protected versions of these. A good review and a complete list of references can be found in [59]. Here, we will only very briefly list the pre-BMN results. One-, two- and three-point functions of 1/2 BPS and 1/4 BPS operators do not renormalize. Secondly, a large class of multi-point functions of 1/2 BPS operators have very simple renormalization properties. These are the so-called extremal, next-to-extremal and near extremal correlators. Extremal correlators fulfil that $\Delta_1 = \Delta_2 + \cdots + \Delta_n$ and can always be expressed entirely in terms of two-point functions. Next-to-extremal correlators obey $\Delta_1 = \Delta_2 + \cdots + \Delta_n - 2$ and factorize into a product of $n-3$ two-point functions and one three-point function. Finally, near extremal multi-point functions have the property that

$\Delta_1 = \Delta_2 + \cdots + \Delta_n - 2m$, where $2 \leq m \leq n-3$ and $4 \leq \Delta_1 \leq 2n-2$. These multi-point functions can all be expressed in terms of lower point functions. The results on multi-point functions, briefly reviewed here, can also be understood from the string theory side [59].

With the advent of the BMN limit [40], the focus was shifted from BPS operators to near BPS operators or BMN operators. As mentioned above, these are operators are created from long BPS operators by the insertion of a few impurities. A much studied set of BMN operators belonging to the $SO(6)$ sector are the following ones

$$\mathcal{O}^J_{i\,j,n} = \frac{1}{\sqrt{J N^{J+2}}} \left(\sum_{p=0}^{n} e^{\frac{2\pi i n}{J}} \mathrm{Tr}\left(\Phi_i\, Z^p\, \Phi_j\, Z^{J-p}\right) - \delta_{ij} \mathrm{Tr}\left(\bar{Z}\, Z^{J+1}\right) \right), \tag{3.3}$$

where Z is one of the three complex scalars of $\mathcal{N}=4$ SYM, say $Z = \Phi_1 + i\Phi_2$ and $i, j \in \{3,4,5,6\}$. These operators are determined by the requirement that they should be eigenvectors of the one-loop planar dilatation generator [40] in the limit $J \to \infty$. (For the exact finite J version of (3.3), see [60].) They can be organized into representations of $SO(6)$ in the obvious way. The calculation of three-point functions of non-protected operators such as BMN operators necessitates a highly non-trivial resolution of operator mixing. First, in the case of extremal correlators, to calculate the classical three-point function to leading order in $1/N$, one needs to take into account mixing between single and double trace states [61]. For BMN operators, this calculation was carried out in Reference [11,44] with the following result for the space–time independent part of the three-point functions involving two BMN operators and one 1/2 BPS operator of the form $\mathcal{O}^J = \frac{1}{\sqrt{JN^J}} \mathrm{Tr}(Z^J)$.

$$\left\langle \bar{\mathcal{O}}^J_{ij,n}\; \mathcal{O}^{r\cdot J}_{kl,m}\; \mathcal{O}^{(1-r)\cdot J} \right\rangle = \frac{2\, J^{3/2} \sqrt{1-r}\, \sin^2(\pi n r)}{N \sqrt{r}\, \pi^2 \left(n^2 - m^2/r^2\right)^2} \left(1 - \frac{\lambda(n^2 - m^2/r^2)}{2J^2}\right) \times \left(\delta_{i(k}\delta_{l)j} n^2 + \delta_{i[k}\delta_{l]j} \frac{nm}{r} + \frac{1}{4}\delta_{ij}\delta_{kl}\frac{m^2}{r^2}\right), \tag{3.4}$$

where it is understood that the operators appearing on the left hand side of (3.4) have been redefined to take into account the effects of the just mentioned operator mixing.[5] To determine the order λ correction to the structure constants requires a number of considerations. First, one actually has to resolve the operator mixing problem to two loop order [39], see also the discussion in [64] as well as the remarks in [11,44]. The reason is that whereas the diagonalization of the dilatation generator to one-loop order does not introduce any coupling constant-dependent mixing of the states, this is not so at two-loop order. At one-loop order, one has a

[5]Notice that in References [13,62,63] where classical three-point functions of BMN operators also appear, the contribution to the three-point function from the mixing with double trace states was *not* taken into account.

set of states $\{\mathcal{O}_\alpha\}$ which are simultaneously eigenstates at the classical and one-loop level. However, when two-loop corrections are taken into account, these eigenstates are changed to $\{\mathcal{O}_\alpha + \lambda c_{\alpha\beta}\mathcal{O}_\beta\}$. The coupling constant-dependent modification of the states which occur at two-loop level gives contributions to the structure constants of order λ. Finally, one of course has to ensure that the structure constants one reads off from the three-point functions are renormalization scheme independent. This can be achieved by normalizing the two-point functions of the operators involved to unity at order λ; see discussion in [64].

The early papers which dealt with three-point functions ignored either one or both of the two complications from operator mixing, i.e. the mixing with multi-trace states and the mixing which naively appears to be of higher order. References [65,66] dealt with the second type of mixing phenomenon and suggested to solve it using purely algebraic means, hence avoiding the explicit evaluation of higher loop two-point functions. References [64,67,68] which studied one-loop properties of structure constants, did not take into account any of the two above-mentioned mixing issues. However, these references pointed out certain connections of three-point functions to integrable spin chains, which we will review below together with some very recent progress along the same lines [69].

3.2. MULTI-POINT FUNCTIONS AND INTEGRABILITY

As explained above, calculating three-point functions involves first dealing with a subtle mixing problem and secondly executing the Wick contractions between the appropriate eigenstates. We will follow the historical development and postpone the discussion of the mixing problem to the end of this section.

For one-loop three-point functions of scalar operators, one has tried to derive a kind of effective vertex, which when applied to the three operators involved, gives the order λ contribution to the structure constant [64,67]. When evaluating three-point functions (apart from non-extremal ones), one generically encounters two types of Feynman diagrams. One type is two-point-like involving only non-trivial contractions between fields from two of the three operators appearing in the three-point function, whereas the other type involves non-trivial contractions between fields from all three operators. The generic term of the effective vertex of [64] correspondingly acts on the indices of three different operators. However, one can show that in a certain renormalization scheme, the one-loop correction to the structure constant only obtains contributions from Feynman diagrams, which are two-point-like [67] and therefore it is possible to construct an effective vertex whose terms act at most on indices from two different operators at a time [67]. Both of the resulting effective vertices have a close resemblance to the Hamiltonian of the integrable $SO(6)$ spin chain. Notice, however, that both approaches [64,67] ignore the two particular mixing issues discussed in the previous section.

An approach to the calculation of three-point functions which explicitly exploits the integrability of the planar dilatation generator was presented in Reference [68].

Here, the field theoretic three-point functions are represented as matrix elements of certain spin operators of the integrable spin chain determining the spectrum, and it is shown how these matrix elements can in principle be expressed in terms of the elements of the spin chain's monodromy matrix. However, the method does not allow one to resolve the mixing between single and multi-trace operators.

More recently, it was understood how, for a certain subclass of operators, the mixing due to one-loop corrections and the calculation of tree-level three-point functions could be efficiently dealt with using integrability tools having their origin in the planar integrability of the theory and this led to exact results for a class of tree-level structure constants [69]. Furthermore, combining these tools with the ideas of [68], a wealth of new data on one-loop three-point functions for short operators was obtained [69]. Notice again that these studies are restricted to cases without mixing between single and multi-trace operators. Reference [70] also contains extensive data on one-loop three-point functions for short operators, but here even the single trace mixing problem was not fully resolved for all cases.

3.3. COMPARISON TO STRING THEORY

Given the success of the comparison of the anomalous dimensions of gauge theory operators with the energies of string states, it is natural to look for a representation of the structure constants entering the three-point functions of non-protected operators in terms of string theory quantities. With the discovery of the PP-wave limit of the type IIB string theory and the corresponding BMN limit of $\mathcal{N}=4$ SYM, hope was raised that in this limit the AdS/CFT dictionary could be extended to include the structure constants of the gauge theory and a first proposal for the translation of these into string theory was put forward in [13]. Here, some structure constants C_{ijk} were suggested to be related in a simple way to the matrix elements of the three-string vertex of the light cone string field theory. A lot of debate followed this initial proposal. First of all, it was debated whether the C_{ijk} were supposed to be the true CFT structure constants appearing after taking into account the two types of operator mixing discussed in Section 3.1, or if the translation to string theory would not involve this mixing. Secondly, as mentioned in Section 2.4, the exact form of the three-string vertex of LCSFT was also a subject of debate. The status of the discussion by the end of 2003 is well summarized in the review [54]. In 2004, Reference [71] provided a unifying description of the various earlier approaches. The true LCSFT vertex was argued to be a linear combination of the two earlier proposed ones, and the C_{ijk}s of relevance for the comparison between gauge and string theory were argued to be the true CFT structure constants. The precise translation of the gauge theory structure constants to the string theory language is well explained in [72]. All this should, however, be considered with some caution, as it has been understood that only for the full AdS/CFT system can one hope for a complete matching of string and gauge theory, cf. the discussion in Section 2.4.

In the past year, there has been quite some progress in the calculation of two- and three-point correlation functions of string states in the full $AdS_5 \times S^5$ geometry using semi-classical methods. First, in [73] (see also [74–76]), a semi-classical approach was shown to reproduce the characteristic conformal scaling of the two-point function with the energy for spinning strings with large quantum numbers and it was suggested that a similar approach could be applied to three-point functions. In [77], the semi-classical calculation of two-point functions was formulated in terms of vertex operators describing classical spinning strings [78,79]. Subsequently, the semi-classical approach was extended to the calculation of three-point functions involving two heavy states and one BPS state [80,81] and various cases of this type were considered [82–84]. Furthermore, using the vertex operator representation of the correlation functions, a number of three-point functions between two heavy states and one light non-BPS state was determined [85]. So far, an explicit comparison of the string theory three-point functions discussed here and gauge theory three-point functions has only been possible for protected correlators. However, very recently it has been suggested that an expansion of the string theory three-point functions in a large angular momentum of the heavy states might allow for a comparison with a gauge theory perturbative expansion of the same quantity, at least for the first few loop orders [86].

4. Maldacena–Wilson Loops

Wilson loops constitute an important class of gauge invariant non-local observables in any gauge theory. The idea that Wilson loops should have a dual string representation has a long history, see [87] and references therein. A realization of this idea in the context of the AdS/CFT correspondence was obtained by Maldacena, who introduced the following special type of locally supersymmetric Wilson loops [88]

$$W[C] = \frac{1}{\dim(\mathscr{R})} \mathrm{Tr}_{\mathscr{R}} \left(\mathrm{P} \exp\left[\oint_C \mathrm{d}\tau \left(i A_\mu(x) \dot{x}^\mu + \Phi_i(x) \theta^i |\dot{x}| \right) \right] \right). \tag{4.1}$$

Here, $\mathscr{R}$ denotes an irreducible representation of $SU(N)$, $x^\mu(\tau)$ is a parametrization of the loop C, $\Phi_i(x)$ are the six real scalar fields of $\mathcal{N}=4$ SYM and $\theta_i(\tau)$ is a curve on S^5. In the present section, we will use the following definition of the 't Hooft coupling constant

$$\lambda = g_{\mathrm{YM}}^2 N. \tag{4.2}$$

According to Maldacena [88], the expectation value of such a Wilson loop in the fundamental representation should be determined by the action of a string ending at the curve C at the boundary of AdS_5, i.e.

$$\langle W[C]\rangle = \int_{\partial X = C} \mathscr{D}X \exp\left(-\sqrt{\lambda}S[X]\right). \tag{4.3}$$

Expectation values of many supersymmetric Wilson loops have turned out to be expressible in terms of expectation values in integrable zero-dimensional matrix models. Furthermore, Wilson loops have provided us with the most promising test of the AdS/CFT correspondence beyond the planar limit to date. The relation between Maldacena–Wilson loops and spin chain integrability is so far rather sparse, cf. Section 4.4.

4.1. THE 1/2 BPS LINE AND CIRCLE

A Wilson loop in form of a single straight line, i.e. given by $x(\tau)=\tau$, $\theta^i(\tau)=const$, constitutes a 1/2 BPS object. Its expectation value does not get any quantum corrections and is exactly equal to one. The circular Wilson loop parameterized by

$$x(\tau) = (\cos\tau, \sin\tau, 0, 0), \tag{4.4}$$

and $\theta^i(\tau)=const$ can be obtained from the straight line by a conformal transformation and is likewise 1/2 BPS. However, its expectation value does get quantum corrections. The expectation value of the circular Wilson loop was calculated at the planar level in perturbation theory to two-loop order in [89] and it was found that only ladder-like diagrams (i.e. diagrams whose vertices all lie on the loop) contributed. The authors of [89] proposed that this could be true for all orders and showed that under that assumption the calculation of the expectation value could be reduced to a combinatorial problem the answer to which was given by an expectation value in a zero-dimensional Gaussian matrix model. Subsequently, it was understood that the reason why the problem was zero dimensional in nature was that the expectation value of the circular Wilson loop could be understood as an anomaly arising at the point at infinity when conformally mapping the straight line to a circle [90]. In addition, the proposal of [89] was extended to all orders in the $\frac{1}{N}$-expansion [90]. Stated precisely, the proposal says that the expectation value of the circular Wilson loop is given to all orders in λ and all orders in $\frac{1}{N}$ by the following expression[6]

$$\langle W_{\text{circle}}\rangle = \left\langle \frac{1}{N}\text{Tr}\,(\exp(M))\right\rangle = \frac{1}{Z}\int \mathscr{D}M \frac{1}{N}\text{Tr}\,(\exp(M)) \exp\left(-\frac{2N}{\lambda}\text{Tr}\, M^2\right). \tag{4.5}$$

Using matrix model techniques, the expectation value can be calculated *exactly* and yields [90]

$$\langle W_{\text{circle}}\rangle = \frac{1}{N} L^1_{N-1}(-\lambda/4N)\exp(\lambda/8N), \tag{4.6}$$

[6] Here, the integration is over Hermitian matrices, i.e. $\mathscr{D}M = \prod_i dM_{ii} \prod_{j>i} d\Re(M_{ij}) d\Im(M_{ij})$ and Z is the partition function of the model.

where L^1_{N-1} is a Laguerre polynomial. One can explicitly write down the genus expansion of (4.6), and then taking the strong coupling, $\lambda \to \infty$, limit of this one gets

$$\langle W_{\text{circle}} \rangle = \sum_{p=0}^{\infty} \frac{1}{N^{2p}} \frac{e^{\sqrt{\lambda}}}{p!} \sqrt{\frac{2}{\pi}} \frac{\lambda^{\frac{6p-3}{4}}}{96^p} \left[1 - \frac{3\left(12p^2 + 8p + 5\right)}{40\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \right]. \tag{4.7}$$

The possibility of the expectation value getting additional contributions from instantons was investigated in [91,92]. Recently, however, the proposal of [89,90] was proved to be true [93].

The expectation value of the circular Wilson loop can be found from the string theory recipe (4.3) in the strong coupling limit by performing a saddle point analysis. It turns out that the string action is dominated by its bosonic part at the saddle point and the calculation becomes equivalent to determining the area of the minimal area surface ending at the loop C. The minimal surface area, however, diverges and requires a regularization, which results in the saddle point action being negative [88]. The minimal area corresponding to the circle was first determined in [94,95] and led to the first crude estimate of the expectation value of the planar circular Wilson loop from the string theory side $\langle W_{\text{circle}} \rangle^{\text{string}} \sim e^{\sqrt{\lambda}}$. Later, the string analysis was extended to include sub-leading corrections in λ coming from integration over zero-modes and to include higher genus surfaces [90]. This led to the following string theory estimate of the expectation of the circular Wilson loop

$$\langle W_{\text{circle}} \rangle^{\text{string}} \propto \sum_{p=0}^{\infty} \frac{1}{N^{2p}} \frac{e^{\sqrt{\lambda}}}{p!} \lambda^{\frac{6p-3}{4}} \left[1 + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) \right]. \tag{4.8}$$

The matching between (4.7) and (4.8) provides a piece of evidence in favour of the validity of the AdS/CFT correspondence beyond the planar level. To reproduce the additional factor $\sqrt{\frac{2}{\pi}}$ appearing in (4.7) from string theory, one needs to take into account the fluctuations about the minimal surface. The framework for performing this calculation at the planar level was laid out in [96] and recently interesting progress was achieved in the explicit evaluation of the missing sub-leading contribution in the planar case [97].

4.2. MORE SUPERSYMMETRIC WILSON LOOPS

In Reference [98], Zarembo found a series of Wilson loops of 1/4, 1/8 and 1/16 BPS type, which can be viewed as generalizations of the 1/2 BPS Wilson line in the higher dimensional subspaces $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$. These Wilson loops all have trivial expectation values. This was argued from the gauge theory side in [98,99] and an understanding from the string theory perspective was provided in [100]. Finally, it was explained by topological arguments in [101].

The first example of a 1/4 BPS Wilson loop with non-trivial expectation value was found by Drukker [102]. Later, a large family of supersymmetric Wilson loops with non-trivial expectation values was identified [103–105]. This family of loops constitute generalizations of the 1/2 BPS circular loop above. The most generic type is 1/16 BPS and exists on an S^3 sub-manifold of four-dimensional space–time. Loops further restricted to an S^2 are 1/8 BPS, and their expectation values were conjectured to be equal to the analogous expectation values in the zero instanton sector of two-dimensional Yang-Mills theory on a sphere [104,105], which implies that they can again be evaluated using a matrix model. More precisely, for such loops we should have

$$\langle W[C]\rangle = \frac{1}{N} L^1_{N-1}\left(g^2_{\mathrm{YM}}\frac{\mathscr{A}_1\mathscr{A}_2}{\mathscr{A}^2}\right)\exp\left[-\frac{g^2_{\mathrm{YM}}}{2}\frac{\mathscr{A}_1\mathscr{A}_2}{\mathscr{A}^2}\right], \tag{4.9}$$

where $\mathscr{A}_1$ and $\mathscr{A}_2$ are the two areas of the sphere bounded by the loop and $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_2 = 4\pi$. Perturbative gauge theory arguments supporting the conjecture have been presented in [104–107], and string theoretic arguments in favour of the conjecture appeared in [108]. The conjecture was further supported by studies using localization techniques in [109].

A unifying and exhaustive description of all supersymmetric Wilson loops has been given in [110] and it was found that the two classes of Wilson loops described, respectively, by Zarembo and Drukker et al. are indeed the two most natural ones.

Some aspects of the analysis outlined above have been generalized to $\mathscr{N}=6$ supersymmetric Chern–Simons matter theory. The 1/2 BPS Wilson loop has been constructed [111] and its expectation value shown to be expressible in terms of an expectation value in a zero-dimensional supermatrix model [111,112]. In addition, one has identified a 1/6 BPS Wilson loop [113–115], the expectation value of which can likewise be calculated using a matrix model [112,116].

4.3. HIGHER REPRESENTATIONS

Having obtained the result (4.5) and using the Schur polynomial formula, one has access to the expectation value of the 1/2 BPS circular Wilson loop in any given irreducible representation of $SU(N)$. When the rank of the representation, k, i.e. the number of boxes in the Young tableau, fulfils that $k \sim \mathscr{O}(N)$, the appropriate string theory description of the Wilson loop is in terms of Dp-branes rather than fundamental strings. Early ideas in this direction were presented in [117–119]. The precise dictionary between Wilson loops in higher representations and Dp-branes was found in [120]. A Wilson loop operator in a representation given by a Young diagram with M rows and K columns with n_i boxes in the i'th row and m_j boxes in the jth column has two different string realizations. One is in terms of K D3-branes carrying electric charges $n_1,\ldots,n_K$ and the other is in terms of M D5-branes carrying electric charges $m_1,\ldots,m_M$. In both cases, as long as $k \ll N^2$, one should be able to determine the expectation value of the Wilson loop by treating the Dp-brane using the probe approximation, i.e. ignoring the back reaction of the $AdS_5 \times S^5$ geometry.[7]

For the completely symmetric and the completely antisymmetric representation of rank k, the gauge theory expectation value of the 1/2 BPS circular Wilson loop has been extracted from the matrix model in the limit $N \to \infty$, $k \to \infty$ with k/N fixed using saddle point techniques. In the antisymmetric case, the result in the large λ limit reads [124]

$$\langle W_{A_k}(C)\rangle = \exp\left[\frac{2N}{3\pi}\sqrt{\lambda}\sin^3\theta_k\right], \tag{4.10}$$

where θ_k is given by

$$\pi\left(1-\frac{k}{N}\right) = (\theta_k - \sin(\theta_k)\cos(\theta_k)). \tag{4.11}$$

This result matches the result of a supergravity calculation on the string theory side using D5-brane probes [125]. For the completely symmetric representation, the situation is more involved since in the large N analysis one encounters two different saddle points. Which one dominates depends on the precise values of λ and k/N. If one considers the limit of large λ and N with a fixed value of κ, defined by

$$\kappa = \frac{\sqrt{\lambda}k}{4N}, \tag{4.12}$$

one finds [124,126]

$$\langle W^{(1)}_{S_k}[C]\rangle = \exp\left[2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}(\kappa)\right)\right]. \tag{4.13}$$

This result matches a supergravity calculation carried out using D3-brane probes [119]. The same saddle point dominates in the limit $\lambda \to \infty$, $k \to \infty$, $N \to \infty$ with k/N fixed. In other regions of the parameter space, the second saddle point might come into play and, in general, one has that the expectation value of the Wilson loop in the symmetric representation is a sum of two terms, i.e. $W_{S_k}[C] = W^{(1)}_{S_k}[C] + W^{(2)}_{S_k}[C]$.

When the rank of the representation reaches the size $k \sim \mathcal{O}(N^2)$, the probe approximation breaks down as the back reaction of the $AdS_5 \times S^5$ geometry can no longer be ignored. In this case, the resulting string background can be described as a bubbling geometry [127]. The determination of the bubbling geometry corresponding to 1/2 BPS Wilson loops was initiated in [128,129] and completed in [130]. The calculation of the expectation value of the Wilson loop from the gauge theory side still proceeds via the matrix model and was carried out in [131, 132].

[7] In particular, it is expected that energies of certain spinning D3- and D5-branes correspond to anomalous dimensions of local twist operators (cf. the chapter [121]) carrying higher representations of the gauge group [122,123].

Like the 1/2 BPS Wilson loop, the less supersymmetric Wilson loops can be studied in higher representations of the gauge group. This was done for a number of 1/4 BPS Wilson loops in [133]. There also exist numerous results on correlation functions involving multiple Wilson loops, as well as Wilson loops and local operators for loops in various representations.

4.4. OTHER INSTANCES OF INTEGRABILITY OF WILSON LOOPS

As explained in Section 4.1, expectation values of Wilson loops in the strong coupling, $\lambda \to \infty$ limit can be evaluated by finding a classical string solution with appropriate boundary conditions. The string sigma model describing type IIB strings on $AdS_5 \times S^5$ is known to be classically integrable [3,4] and this fact was exploited in Reference [134] to find the strong coupling expectation values of numerous Wilson loops with $x^\mu(t)$ and $\theta^i(t)$ periodic. More recently, a class of polygonal (non-supersymmetric) Wilson loops built from light like segments have attracted attention due to their relation with gluon scattering amplitudes [135]. The minimal surfaces corresponding to these loops have turned out to be described by integrable systems of the Hitchin type. For a discussion of Wilson loops related to scattering amplitudes and the relevant set of references, we refer to the chapters [7–9].

It seems difficult to relate the expectation value of supersymmetric Wilson loops to integrable spin chains, but there exists one special construction which exposes such a relation. In Reference [136], the authors studied insertion of composite operators into Wilson loops. The Wilson loop was taken to be a straight line or a circle and θ^i to describe a single point on S^5. Furthermore, the composite operator was assumed to be built from two complex scalars $Z = (\Phi_1 + i\Phi_2)/\sqrt{2}$ and $X = (\Phi_3 + i\Phi_4)/\sqrt{2}$. It is possible to assign a conformal dimension to such an inserted operator, and to determine this dimension one has to solve a certain mixing problem involving two-point functions of the type

$$\left\langle W_{\text{line}}\left[\mathcal{O}_\beta^\dagger(t)\mathcal{O}_\alpha(0)\right]\right\rangle = \left\langle \frac{1}{N}\text{Tr}\left(P\,\mathcal{O}_\alpha^\dagger(t)\mathcal{O}_\beta(0)\exp\left[i\int(A_t + i\,\Phi_6)\text{d}t\right]\right)\right\rangle. \quad (4.14)$$

An operator insertion $\mathcal{O}_\Delta$ with a well-defined conformal dimension fulfils

$$\left\langle W_{\text{line}}\left[\mathcal{O}_\Delta^\dagger(t)\mathcal{O}_\Delta(0)\right]\right\rangle \sim \frac{1}{t^{2\Delta}}. \quad (4.15)$$

The above mixing problem was studied at the planar one-loop order in [136] and mapped onto the problem of diagonalizing the Hamiltonian of an $SU(2)$ open Heisenberg spin chain with completely reflective boundary conditions. This spin chain is integrable and can be solved by Bethe ansatz. For a description of the Bethe equations associated with integrable open spin chains, we refer to the chapter [5]. The string dual of the inserted operator can be identified and a successful comparison between the gauge theory side and string theory side for inserted operators of BMN type and of the type dual to spinning strings was carried out in [136].

5. Conclusion

The search for spin chain-like integrable structures in $\mathcal{N}=4$ SYM regarding non-planar anomalous dimensions and Maldacena–Wilson loops has so far not provided us with very strong positive results. Maldacena–Wilson loops are more naturally related to zero-dimensional integrable matrix models than to spin chains, and non-planar anomalous dimensions have not yet provided us with any traces of integrability. It is possible that one can learn more about non-planar anomalous dimensions by studying the three-point functions or structure constants of the theory. Non-trivial operator mixing issues, however, make the evaluation of structure constants quite involved. For a subset of single trace operators, the mixing is an entirely planar effect and can in principle be handled using tools originating from the planar integrability of the theory. In the generic case, however, single trace operators will mix with multi-trace operators and the calculation of structure constants requires a diagonalization of the non-planar dilatation operator. The most naive approach to studying non-planar anomalous dimensions, namely doing perturbation theory in $\frac{1}{N}$ requires dealing with the splitting and joining of spin chains and leads to a Hilbert space of states for which the standard concepts of integrability, such as the asymptotic S-matrix and two-particle scattering, do not immediately apply. Going beyond the planar limit hence seems to require a rethinking of the entire framework of integrability or invoking some non-perturbative way of handling the higher topologies.

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