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Extended Holomorphic Anomaly in Gauge Theory

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Abstract. The partition function of an $\mathcal{N}=2$ gauge theory in the Ω-background satisfies, for generic value of the parameter $\beta=-\epsilon_1/\epsilon_2$, the, in general extended, but otherwise β-independent, holomorphic anomaly equation of special geometry. Modularity together with the (β-dependent) gap structure at the various singular loci in the moduli space completely fixes the holomorphic ambiguity, also when the extension is non-trivial. In some cases, the theory at the orbifold radius, corresponding to $\beta=2$, can be identified with an "orientifold" of the theory at $\beta=1$. The various connections give hints for embedding the structure into the topological string.

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1. Introduction

Four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory was solved following Seiberg and Witten [1,2] by exploiting constraints from special geometry and modular invariance on the moduli space of vacua. Subsequently, the theory and its solution were related to string theory in several ways, engendering a variety of developments that have revolutionized our understanding of the dynamics of theories with eight supercharges.

A comparably early line of investigation was the regularization of the integral over the moduli space of instantons proposed in [3–6]. It culminated in the verification of the Seiberg–Witten prepotential directly from the instanton counting [7]. As a result, the central object today is the partition function of the $\mathcal{N}=2$ supersymmetric gauge theory in the so-called Ω-background,

$$Z(a,\epsilon_1,\epsilon_2;q). \tag{1.1}$$

Here, a are the vectormultiplet moduli, ϵ_1,ϵ_2 are the equivariant parameters for the localization with respect to the two-dimensional torus acting on $\mathbb{R}^4\cong\mathbb{C}^2$, q is the instanton counting parameter (related to the dynamical scale of the gauge theory, if any), and we have left other parameters such as masses of any matter fields implicit.

It was shown in [7,8], see also [9], that

$$\lim_{\epsilon_1,\epsilon_2\to 0}(\epsilon_1\epsilon_2 \log Z(a,\epsilon_1,\epsilon_2;q)) = \mathcal{F}^{(0)}(a;q) \tag{1.2}$$

reproduces the prepotential computed by Seiberg and Witten from the periods of a family of complex curves. Following [3,7,10], the terms of higher order in ϵ_1,ϵ_2 in (1.2) have been expected to capture some gravitational couplings of the gauge theory arising from the embedding into string theory. This has been made precise for the terms of second order in ϵ_1,ϵ_2 in [3,10], but is straightforward in higher order only for $\epsilon_1=-\epsilon_2$ [7]. In that case, one talks about terms of the form $\int d^4\theta \mathcal{F}^{(g)}\mathcal{W}^{2g}$, where $\mathcal{W}$ is the self-dual gravi-photon chiral field, and $\mathcal{F}^{(g)}$ can be obtained as the field theory limit of the genus-g topological string amplitude [11,12] on the appropriate Calabi–Yau background constructed in geometric engineering [13,14].

The computation of the higher order corrections to (1.2) from the topological string perspective has been pursued for example in [15–17]. One of the advantages is that while the field theory localization technique is applicable only in the weak-coupling regime, the topological string machinery can yield results that are valid also in expanding around other interesting points in the moduli space. This is achieved in a familiar way by the method of the holomorphic anomaly that trades holomorphicity for modular invariance [12].

To be fair, of course, the field theory limit commutes with the holomorphic anomaly, and one can study the $\mathcal{F}^{(g)}$ using just the input from special geometry provided by the Seiberg–Witten curve. This reasoning was followed in [18,19]. In fact, it was found in these works that modularity together with the gap structure in the expansion of the amplitudes around the monopole/dyon singularities provides enough boundary conditions to completely fix the ambiguity that plagues the method of the holomorphic anomaly [20].

These investigations have been restricted to the special self-dual background $\epsilon_1 = -\epsilon_2$. The purpose of the present note is to shed light on $Z(a,\epsilon_1,\epsilon_2;q)$ from the perspective of the topological B-model, Seiberg–Witten geometry, special geometry and the holomorphic anomaly, but for *general values of* ϵ_1,ϵ_2. A priori, it is not clear that the relation will persist, or what form it will take. In particular, the topological string is not known to admit a two-parameter expansion corresponding to ϵ_1,ϵ_2 (but see [21] for a recent proposal). Our results are surprisingly simple: the holomorphic anomaly equation merely experiences a *slight extension* [22] by some data contained in (1.2) to the *first order* in ϵ_1,ϵ_2, and the essential modification of the formalism is at the level of *fixing the boundary conditions*, which we are also able to determine completely. We will also find a surprising relation between the theory at the special value $\epsilon_1=-2\epsilon_2$ and a certain "orientifold" (descending from the real topological string [23–25]) of the theory at $\epsilon_2=-\epsilon_1$. We view our results mostly as an encouraging proof of principle that mirror symmetry continues to make sense for general ϵ_1, ϵ_2.

We will study in this paper $SU(2) \subset U(2)$ gauge theory coupled to $N_f = 0, 1, 2, 3$ massless hypermultiplets in the fundamental representation. We are confident that the structures we find carry over to other cases. We will proceed in the next section via special geometry, the holomorphic anomaly, the holomorphic limits and singularity structure at interesting points in moduli space. We then describe the four examples, each of which has some special illuminating features. We present the interpretation of the extension of the holomorphic anomaly equation from the point of view of Seiberg–Witten geometry in Section 4. The orientifold relation can be found in Section 5. Many readers will be familiar with most of the formulas, so we have relegated a lot of technical baggage to the appendix and the references.

2. The Expansion

Motivated in part by recent developments relating $\mathcal{N}=2$ gauge theory with two-dimensional conformal field theory and matrix models [26,27], we begin by reparameterizing the Ω-background according to

$$\epsilon_1 = \lambda\beta^{1/2}, \qquad \epsilon_2 = -\lambda\beta^{-1/2}. \tag{2.1}$$

For small q and fixed β, we then expand (1.1) in λ, which one might think of as the topological string coupling constant g_s (or the Planck constant $\hbar$),[1]

$$\log Z(a, \epsilon_1, \epsilon_2; q) \sim \sum_{n=-2}^{\infty} \lambda^n \mathcal{G}^{(n)}(a, \beta; q). \tag{2.2}$$

According to (1.2) we have

$$\mathcal{G}^{(-2)} = \mathcal{F}^{(0)}, \tag{2.3}$$

the Seiberg–Witten prepotential. In particular, it is β-independent. The term at order λ^{-1} will play a central role in our story. It takes the form

$$\mathcal{G}^{(-1)} = \left(\beta^{1/2} - \beta^{-1/2}\right)\mathcal{T}, \tag{2.4}$$

with $\mathcal{T}$ independent of β. Notice that $\mathcal{G}^{(-1)}$ vanishes in the standard topological string limit $\beta = 1$.

To proceed, we review the role of $\mathcal{F}^{(0)}$ in special geometry. We denote by u a global coordinate on the moduli space $\mathcal{M}$ of vacua, which is identified with the base space of an appropriate family of complex curves, $\mathcal{C}_u$.[2] The family of curves

[1]The index n is best thought of as running over the possible Euler numbers of Klein surfaces, a point to which we shall return.

[2]Many formulas we write will be restricted to a one-dimensional moduli space, or $SU(2)$ gauge theory with fundamental flavors. Generalizations are mostly obvious.

is equipped with a meromorphic one-form λ_{SW}, such that for appropriate choice of one-cycles A and A_D,

$$a=\oint_A \lambda_{\mathrm{SW}}, \qquad a_D=\oint_{A_D} \lambda_{\mathrm{SW}}, \tag{2.5}$$

and

$$a_D=\frac{\partial \mathcal{F}^{(0)}}{\partial a}, \tag{2.6}$$

after eliminating u from (2.5). We will not need to be explicit about the auxiliary geometric data until later. For expansion in different regions of moduli space, it is anyways more convenient to base the developments on the Picard–Fuchs differential equation, a third-order system of linear differential equations,

$$\mathcal{L}\varpi(u)=0, \tag{2.7}$$

satisfied by all periods of λ_{SW}. Using a as a local coordinate around $u\to\infty$, the Picard–Fuchs operator takes the form[3]

$$\mathcal{L}=\partial_a \frac{1}{C_{aaa}}\partial_a^2, \tag{2.8}$$

where

$$C_{aaa}=\partial_a^3\mathcal{F}^{(0)}=\partial_a^2 a_D(a)=\partial_a\tau(a) \tag{2.9}$$

is a (meromorphic) rank three symmetric tensor over $\mathcal{M}$, referred to as the Yukawa coupling, which plays a central role in special geometry. One feature of special geometry is the existence of canonical (flat) coordinates [12], providing a meaningful expansion parameter around any interesting point $u=u_*$ in $\mathcal{M}$. In such a flat coordinate $t=t(u)$, vanishing at $u=u_*$, the Picard–Fuchs operator takes again the form (2.8) with $a\to t$, i.e.,

$$\mathcal{L}=\partial_t \frac{1}{C_{ttt}}\partial_t^2, \qquad C_{ttt}=\left(\frac{\partial u}{\partial t}\right)^3 C_{uuu}. \tag{2.10}$$

A useful property of the canonical coordinates is that in the holomorphic limit $\bar{t}\to 0$ (or $\bar{a}\to\infty$ for $t=a$), the connection of the Weil–Petersson metric $g\sim \mathrm{Im}\tau$ on $\mathcal{M}$ takes the form

$$\lim_{\bar{t}\to 0}\Gamma^u_{uu}=\partial_u \log\frac{\partial t(u)}{\partial u}. \tag{2.11}$$

[3]As is now evident, the constant is a third solution of the differential equation. This solution decouples in special cases, such as $SU(2)$ gauge theory with massless hypermultiplets.

At this stage, we are ready to write down the holomorphic anomaly equations of [12]. According to [12], the amplitudes $\mathcal{F}^{(g)}(a)$ extracted from (2.2) via

$$\mathcal{F}^{(g)}(a;q) = \mathcal{G}^{(2g-2)}(a,\beta=1;q), \tag{2.12}$$

while holomorphic in a, are not well behaved globally over $\mathcal{M}$. Instead, one should view the $\mathcal{F}^{(g)}(a)$ (for $g \geq 1$) as the holomorphic limit $\bar{a} \to \infty$ of *non-holomorphic, but globally defined* objects $\mathcal{F}^{(g)}(u,\bar{u})$. (These are denoted by the same letter, as confusion cannot arise.)

For $g>1$, the amplitudes $\mathcal{F}^{(g)}(u,\bar{u})$ satisfy the holomorphic anomaly equation

$$\bar{\partial}_{\bar{u}}\mathcal{F}^{(g)} = \frac{1}{2}\sum_{\substack{g_1+g_2=g \\ g_i>0}} \bar{C}_{\bar{u}}^{uu}\mathcal{F}_u^{(g_1)}\mathcal{F}_u^{(g_2)} + \frac{1}{2}\bar{C}_{\bar{u}}^{uu}\mathcal{F}_{uu}^{(g-1)}, \tag{2.13}$$

where $\mathcal{F}_{uu}^{(g)} = D_u\mathcal{F}_u^{(g)} = D_u^2\mathcal{F}^{(g)}$, D_u is the covariant derivative over $\mathcal{M}$, and indices are raised and lowered using the Weil–Petersson metric. The one-loop amplitude satisfies the special equation

$$\bar{\partial}_{\bar{u}}\partial_u\mathcal{F}^{(1)} = \frac{1}{2}\bar{C}_{\bar{u}}^{uu}C_{uuu}. \tag{2.14}$$

It is a natural question to ask how (2.13) should be modified away from $\beta=1$. The structure of the expansion of (1.1) (see Appendix), and in particular the generic non-vanishing of the terms of odd order in λ, suggests a possible role for the extended holomorphic anomaly equation of [22]. In the orientifold context of [24], the amplitudes $\mathcal{G}^{(n)}$ are the sums of all contributions at fixed order in string perturbation theory. When n is odd, these arise only from open and unoriented diagrams, while when n is even, we have $\mathcal{G}^{(n)} \sim \mathcal{F}^{(n/2+1)} + \cdots$. (Note that in the context of topological string orientifolds, the $\mathcal{G}^{(n)}$ do not depend on any β.) Our first main result is that the β-dependent $\mathcal{G}^{(n)}$ appearing in the gauge theory context (2.2) satisfy exactly the same extended holomorphic anomaly equation. The second result is the relation between the orientifolded theory at $\beta=1$, and the ordinary theory at $\beta=2$ (see Section 5).

To write down the extended holomorphic anomaly equation satisfied by the $\mathcal{G}^{(n)}$ with full β-dependence, we need to introduce, next to the Yukawa coupling C_{uuu}, the so-called Griffiths infinitesimal invariant,

$$\Delta_{uu} = \mathcal{G}_{uu}^{(-1)} - C_{uu}^{\bar{u}}\,\bar{\mathcal{G}}_{\bar{u}}^{(-1)}. \tag{2.15}$$

This is a rank two tensor over $\mathcal{M}$, the non-holomorphicity of which is controlled by

$$\bar{\partial}_{\bar{u}}\Delta_{uu} = -C_{uu}^{\bar{u}}\,\bar{\Delta}_{\bar{u}\bar{u}}. \tag{2.16}$$

One can show (for example using canonical coordinates) that (2.16) is equivalent to the statement that $\mathcal{G}^{(-1)}$ can be computed from an inhomogeneous Picard–Fuchs equation

$$\mathcal{L}\mathcal{G}^{(-1)} = g(u) \tag{2.17}$$

for some inhomogeneity $g(u)$, a meromorphic function over (some cover of) $\mathcal{M}$. In turn, (2.17) means that one can represent $\mathcal{G}^{(-1)}$ as a chain integral (or open period) of the Seiberg–Witten differential λ_{SW} over an appropriate divisor on the family of curves. This is the basic geometric idea behind the extension that we will discuss in more detail in Section 4.

The extended holomorphic anomaly equation [22,24], specialized to the rigid case [25], then reads, for $n > 0$

$$\bar\partial_{\bar u}\mathcal{G}^{(n)} = \frac{1}{2}\sum_{\substack{n_1+n_2=n-2\\ n_i\geq 0}} \bar C_{\bar u}^{uu}\mathcal{G}_u^{(n_1)}\mathcal{G}_u^{(n_2)} + \frac{1}{2}\bar C_{\bar u}^{uu}\mathcal{G}_{uu}^{(n-2)} - \bar\Delta_{\bar u}^{u}\mathcal{G}_u^{(n-1)}. \tag{2.18}$$

For $n = 0$, we have

$$\bar\partial_{\bar u}\partial_u\mathcal{G}^{(0)} = \frac{1}{2}\bar C_{\bar u}^{uu}C_{uuu} - \bar\Delta_{\bar u}^{u}\Delta_{uu}. \tag{2.19}$$

The Equations (2.18) and (2.19) determine the $\mathcal{G}^{(n)}$ up to certain holomorphic functions on $\mathcal{M}$. To compute the complete amplitudes, one needs to first supply an efficient algorithm for solving the holomorphic anomaly, and then find a sufficient number of boundary conditions at the various special points in $\mathcal{M}$.

To establish our main claim, we will follow the route of solving (2.18) order by order in n, and then showing that one may fix the holomorphic ambiguity so as to (i) reproduce the known results in the limit $a\to\infty$ and (ii) satisfy the expected boundary conditions at the other special points in the moduli space.

There are various ways to solve the holomorphic anomaly equation. A convenient one is the so-called polynomial algorithm of [28], described in its extended form in [29,30]. (A related approach is the "direct integration" of [31].) One starts by noticing that special geometry relates the propagator of [12], defined by the condition $\bar\partial_{\bar u}S^{uu} = \bar C_{\bar u}^{uu}$ with the Weil–Petersson curvature on $\mathcal{M}$, $\bar\partial_{\bar u}\Gamma_{uu}^{u} = -\bar C_{\bar u}^{uu}C_{uuu}$. So we may choose

$$S^{uu} = -\frac{\Gamma_{uu}^{u}}{C_{uuu}}. \tag{2.20}$$

Similarly, the terminator of [22], characterized by $\bar\partial_{\bar u}T^u = \bar\Delta_{\bar u}^{u}$ can be written as

$$T^u = -\frac{\Delta_{uu}}{C_{uuu}}. \tag{2.21}$$

The main point of [28] is then that covariant derivatives of S^{uu} and T^u are known up to some holomorphic functions. In particular,

$$D_uS^{uu} = -C_{uuu}S^{uu}S^{uu} + f^u, \qquad D_uT^u = g, \tag{2.22}$$

where in fact $g(u)$ is nothing but the inhomogeneity in (2.17), while $f^u(u)$ is a priori unknown. As a consequence of (2.22), the non-holomorphicity in the amplitudes is entirely through the S^{uu} and T^u. In fact, $\mathcal{G}^{(n)}$ may be written as polynomials in those non-holomorphic generators with coefficients that are rational functions in u (with singularities on the discriminant locus; we count this as holomorphic).

To study the boundary conditions that fix the holomorphic ambiguity [18,19], one expands the amplitudes in the holomorphic limit (2.11) and in the canonical coordinates (2.10) around the special points in $\mathcal{M}$, and compares with the field theory expectations. The special loci are: the weak coupling regime, the monopole/dyon points (where the Yukawa coupling blows up) and the locus where $g(u)$ is singular.

It is convenient to encode the β dependence of the boundary conditions via the asymptotic expansion of certain Schwinger integrals. These represent the contribution of integrating out in the general Ω-background the states that are becoming light [7,32]. We introduce two sets of functions $\Phi^{(n)}(\beta)$ and $\Psi^{(n)}(\beta)$ of β via

$$
\begin{aligned}
\int \frac{\mathrm{d}s}{s}\frac{\mathrm{e}^{-ts}}{(\mathrm{e}^{\epsilon_1 s}-1)(\mathrm{e}^{\epsilon_2 s}-1)} &\sim \Phi^{(0)}(\beta)\log t + \sum_{n>0}\frac{\lambda^n}{t^n}\Phi^{(n)}(\beta),\\
\int \frac{\mathrm{d}s}{s}\frac{\mathrm{e}^{-ts}}{(\mathrm{e}^{\epsilon_1 s/2}-\mathrm{e}^{-\epsilon_1 s/2})(\mathrm{e}^{\epsilon_2 s/2}-\mathrm{e}^{-\epsilon_2 s/2})} &\sim \Psi^{(0)}(\beta)\log t + \sum_{n>0}\frac{\lambda^n}{t^n}\Psi^{(n)}(\beta).
\end{aligned}
\tag{2.23}
$$

Here, $t\to 0$ is the mass of the state that is being integrated out, $\epsilon_1=\lambda\beta^{1/2}$, $\epsilon_2=-\lambda\beta^{-1/2}$, see (2.1), and we have dropped the most singular terms. Explicitly, for $n>0$,

$$
\begin{aligned}
\Phi^{(n)}(\beta) &= (n-1)!\sum_{k=0}^{n+2}\frac{(-1)^k B_k B_{n+2-k}}{k!(n+2-k)!}\beta^{k-n/2-1},\\
\Psi^{(n)}(\beta) &= (n-1)!\sum_{k=0}^{n+2}\frac{(-1)^k B_k B_{n+2-k}}{k!(n+2-k)!}(2^{1-k}-1)(2^{1-n-2+k}-1)\beta^{k-n/2-1},
\end{aligned}
\tag{2.24}
$$

where B_k are the Bernoulli numbers. Notice that $B_k=0$ for $k>1$ and odd. This makes $\Psi^{(n)}$ vanish for odd n. For $n=0$,

$$
\begin{aligned}
\Phi^{(0)} &= -\frac{1}{4}+\frac{1}{12}\beta+\frac{1}{12}\beta^{-1},\\
\Psi^{(0)} &= -\frac{1}{24}\left(\beta+\beta^{-1}\right).
\end{aligned}
\tag{2.25}
$$

The functions $\Phi^{(n)}$ were introduced in this context (but under a different name) in [7,8]. The $\Psi^{(n)}$ are well known from the study of $c=1$ string at radius $R=\beta$, see e.g., [33]. The Schwinger integrals in the general Ω-background have also been studied, for instance, in [21,34,35]. Notice that at $\beta=1$, we have for even n,

$$
\Phi^{(n)}(1)=\Psi^{(n)}(1)=-\frac{B_{n+2}}{n(n+2)}. \tag{2.26}
$$

Setting $n=2g-2$, we can recover the β-independent boundary conditions known from [8,18–20]. (For odd n, $\Phi^{(n)}(1)=\Psi^{(n)}(1)=0$).

We now briefly summarize the boundary conditions on the $\mathcal{G}^{(n)}$ that we will observe in the examples below. In the weak-coupling regime, the leading behavior of the $\mathcal{G}^{(n)}$ is controlled by the perturbative contribution to the partition function (1.1), and contains the functions $\Phi^{(n)}(\beta)$ [7,8] (see Appendix). At monopole/dyon points, we will find, extending the gap structure of [18], that $\mathcal{G}^{(n)}$ have a leading singularity $\sim t^{-n}$, followed by regular terms $\mathcal{O}(t^0)$. Quite interestingly, we will find that the leading (β-dependent) coefficient is sometimes governed by the $\Psi^{(n)}$, sometimes by the $\Phi^{(n)}$, and sometimes by an as yet unidentified function. Finally, we find that the $\mathcal{G}^{(n)}$ are regular at points where $g(u)$ is singular, but which are not monopole/dyon points.

3. Examples

In this section, we study in detail the partition function $Z(a,\epsilon_1,\epsilon_2;q)$ for $SU(2)$ gauge theory coupled to $N_f\le 3$ massless fundamental hypermultiplets. These are the simplest models with a one-dimensional Coulomb branch and an extensive literature. We have summarized in the appendix the results from instanton counting [7]. In the B-model, it suffices for the moment to recall the Picard–Fuchs differential operators written in terms of the global coordinate u. The special geometry of the moduli space has a discrete symmetry $\mathbb{Z}_{1/\alpha}$ that allows us to write the differential equation in terms of the coordinate $z=u^{1/\alpha}$. Here, α is given in the following table

N_f	0	1	2	3
α	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	1

. (3.1)

Moreover, for massless hypermultiplets, the constant solution of the third-order Picard–Fuchs equation decouples from the monodromies, so that we may work with the simpler second-order operator given by[4]

$$\mathcal{L}_\alpha=\theta(\theta-\alpha)-z\left(\theta-\frac{\alpha}{2}\right)^2, \tag{3.2}$$

where $\theta=z\frac{\mathrm{d}}{\mathrm{d}z}$. We also record that the general solution of the differential equation $\mathcal{L}_\alpha\varpi(z)=0$ around $z=\infty$ can be obtained from

$$\varpi(z;H)=\sum_{n=0}^{\infty}\frac{\Gamma(n+H+\alpha/2)\Gamma(n+H-\alpha/2)}{\Gamma(n+H+1)^2}z^{\alpha/2-n-H} \tag{3.3}$$

as

$$\varpi_0=\varpi(z;H=0)\sim z^{\alpha/2}+\cdots,\qquad \varpi_1=\partial_H\varpi(z;H=0)\sim-\varpi_0\log z+\cdots. \tag{3.4}$$

[4]We work with a strong coupling scale such that the discriminant contains the locus $u^{1/\alpha}=1$. On the A-side, we work at $q=1$.

The flat coordinate at $u\to\infty$ is $a\propto\varpi_0\sim u^{1/2}$, and matching the asymptotic behavior of the periods with the perturbative computation in the gauge theory determines the proper linear combinations yielding $a_D=\partial_a\mathcal{F}^{(0)}$ and the prepotential. The precise coefficients are not important for our purposes, as only the Yukawa coupling enters the recursion relations.

A common feature of the three models is the singularity at $1-z=1-u^{1/\alpha}=0$. In the coordinate $\tilde{z}=1-z$, with $\tilde{\theta}=\tilde{z}\frac{\mathrm{d}}{\mathrm{d}\tilde{z}}$, the Picard–Fuchs operator takes the form

$$\mathcal{L}_\alpha=\left(\tilde{z}^{-1}-1\right)\tilde{\mathcal{L}}_\alpha, \tag{3.5}$$

with

$$\tilde{\mathcal{L}}_\alpha=\tilde{\theta}(\tilde{\theta}-1)-\tilde{z}\left(\tilde{\theta}-\frac{\alpha}{2}\right)^2. \tag{3.6}$$

The solutions near $\tilde{z}=0$ can be encoded similarly as in (3.3). In most other respects, each of the four models we consider is special, so we now have to split the discussion. In each case, we start with a look at the expansion of the instanton partition function to learn whether we should use the standard or extended holomorphic anomaly equation, whether discrete symmetries are broken or not, etc.. This also dictates the ansatz we make for the holomorphic ambiguity. This information, together with the perturbative and one-instanton contribution, is summarized in the Appendix (see especially, (A.9) and (A.10), (A.11), (A.12), (A.13)).

Pure gauge theory. The Yukawa coupling of the model is given by

$$C_{uuu}=\frac{2}{1-u^2}, \tag{3.7}$$

and for approriate normalization of the periods a and a_D, we obtain

$$\partial_a^2 a_D=C_{aaa}=\left(\frac{\partial u}{\partial a}\right)^3 C_{uuu}=-\frac{8}{a}-\frac{12}{a^5}-\frac{105}{4a^9}-\frac{495}{8a^{13}}+\cdots, \tag{3.8}$$

thus matching the known results, which are those of the instanton counting after setting $q=1$ (see Appendix). We also note that in our integration scheme, we have quite simply $f^u=-1/8$.

To evaluate the β-dependent higher order terms, we may use the ordinary holomorphic anomaly equation (2.13), since terms of higher odd order in the expansion (2.2) vanish identically (see Appendix). But to be more systematic, we continue to use the parameterization via the $\mathcal{G}^{(n)}$. At one loop, we find [7,9,10]

$$\mathcal{G}^{(0)}(\beta)=\frac{1}{2}\log\operatorname{Im}\tau-\frac{\beta+\beta^{-1}}{24}\log|1-u^2|^2, \tag{3.9}$$

also reproducing the instanton counting results. In higher order (even n), we solve (2.13) using the direct integration algorithm, to find $\mathcal{G}^{(n)}$ as a polynomial in the non-holomorphic propagator S^{uu} with coefficients rational functions of u, up to the constant term $A^{(g)}(u;\beta)$. Constraints on the asymptotic behavior at $u\to\infty$ and $u=\pm 1$ require the ansatz

$$A^{(g)}(u;\beta)=\frac{u^{3n/2}}{(1-u^2)^n}\sum_{i=1}^{n-1}P_i^{(n)}(\beta)u^{-2i}. \tag{3.10}$$

We may fix the coefficient functions, $P_i^{(n)}(\beta)$, by imposing the gap structure at the monopole/dyon points $z=1$. The local coordinate there is $a_D\sim\tilde{z}$, and we have

$$\mathcal{G}^{(n)}=\frac{\Psi^{(n)}(\beta)}{a_D^n}+\mathcal{O}(1), \tag{3.11}$$

with $\Psi^{(n)}$ given in (2.24). Counting parameters, there are at each order n unknown functions $P_i^{(n)}(\beta)$, which are precisely determined by as many conditions from (3.11).

This way of fixing the holomorphic ambiguity is the generalization of [18] to general values of β. One may check that it is in agreement with the expectations. In particular, the leading order term in the weak-coupling expansion takes the form

$$\mathcal{G}^{(n)}=\frac{\Phi^{(n)}(\beta)}{2^{n-1}a^n}+\cdots, \tag{3.12}$$

and the one-instanton sector is matched as well. Moreover, one can verify that the amplitudes are regular around $z=0$.

$N_f=1$: The theory with one flavor is the most interesting one from the present point of view, as here the extended holomorphic anomaly equation gears up to its full power. But first we note the Yukawa coupling

$$C_{uuu}\propto\frac{u}{1-u^3}, \tag{3.13}$$

which matches the term of order λ^{-2} from instanton counting. The term at order λ^{-1} is non-zero in this model. For $q=1$, we have

$$\mathcal{T}=\frac{\mathcal{G}^{(-1)}}{\beta^{1/2}-\beta^{-1/2}}=\frac{a}{4}-\frac{1}{4a^2}+\frac{7}{384a^8}-\frac{1131}{163840a^{14}}+\frac{3705}{917504a^{20}}+\cdots. \tag{3.14}$$

Plugging in $a=a(z)$, and acting with the Picard–Fuchs operator, we find

$$\mathcal{L}_{1/3}\mathcal{T}(z)=\frac{1}{3}z^{2/3}. \tag{3.15}$$

We defer further discussion of the geometric origin of this inhomogeneity to Section 4, and proceed with the integration of the extended holomorphic anomaly equation. At one loop, we have

$$\partial_u\mathcal{G}^{(0)}=\frac{1}{2}S^{uu}C_{uuu}+\frac{1}{2}C_{uuu}(T^u)^2+\frac{\beta+\beta^{-1}}{24}\frac{3u^2}{1-u^3}+\frac{\left(\beta^{1/2}-\beta^{-1/2}\right)^2}{8}\frac{1}{u}. \tag{3.16}$$

At higher order, the holomorphic ambiguity is parameterized by

$$A^{(n)}(u;\beta)=\begin{cases}\dfrac{u^{5n/2}}{(1-u^3)^n}\displaystyle\sum_{i=0}^{n-1}P_i^{(n)}u^{-3i}+\dfrac{1}{u^{2n}}\sum_{j=0}^{n/2}Q_j^{(n)}u^{3j} & n \text{ even}\\[2ex] \dfrac{u^{(5n-9)/2}}{(1-u^3)^{n-1}}\displaystyle\sum_{i=0}^{n-2}P_i^{(n)}u^{-3i}+\dfrac{1}{u^{2n}}\sum_{j=0}^{\frac{n-1}{2}}Q_j^{(n)}u^{3j} & n \text{ odd}\end{cases}. \tag{3.17}$$

This ansatz is slightly redundant, but more intuitive than the minimal one. The prefactor $(1-u^3)^{-n}$ captures the leading singularity at the monopole point, while the prefactor u^{-2n} is explained from the leading behavior of the solution around $u=0$. Indeed, inspecting the Picard–Fuchs equation, the flat coordinate there behaves as $a_0\sim u$, and the extension as $\mathcal{T}\sim u^2\sim a_0^2$. The maximal order of a singularity is $\mathcal{T}^{-n}$. The summation ranges are dictated by the condition that the ambiguities not spoil each other's asymptotic behavior. In particular, around $u\to\infty$, we should have the behavior $\sim a^{-n}\sim u^{-n/2}$ for n even, and $\sim a^{-n-3}\sim u^{-(n+3)/2}$ for n odd.

To fix the coefficient functions $P_i(\beta)^{(n)}$, $Q_j^{(n)}(\beta)$ in (3.17), we have n conditions from the gap structure at the monopole points $u^3=1$ (the $\mathbb{Z}_3$ symmetry remains unbroken)

$$\mathcal{G}^{(n)}=\frac{\Psi^{(n)}(\beta)}{a_D^n}+\mathcal{O}(1). \tag{3.18}$$

(Note that this is regular for n odd.) We also find that the $\mathcal{G}^{(n)}$ are regular at $u=0$, accounting for $\sim 3n/2$ conditions. Regularity at $u=0$ despite the vanishing of $\mathcal{T}$ (singularity in Δ_{uu}) is reassuring, since we would not know how to explain any singularity from integrating out a massless state. This is in contrast to the application of the extended holomorphic anomaly in the context with background D-branes [22,24,25], where the vanishing of $\mathcal{T}$ signals a tensionless domain wall and leads to so far uncontrolled singularities in the higher loop amplitudes.

After fixing the holomorphic ambiguities in this way, we can check the expansion around weak coupling with the results from instanton counting, finding complete agreement.

$N_f=2$: At first sight, the two-flavored case appears somewhat uninteresting, since the special geometry is so closely related to that of the pure gauge theory. In particular, $\alpha=1/2$ in both cases, and the Yukawa coupling,

$$C_{uuu}=\frac{1}{4(1-u^2)}, \tag{3.19}$$

and $f^u=-1$ differ only in the normalization of u.

The surprise, however, appears when we look at the loop amplitudes. Already for $n=0$, we find that turning on the β-deformation away from $\beta=1$, *breaks the*

$\mathbb{Z}_2$ *symmetry* between the monopole and dyon point at $u=+1$ and $u=-1$, respectively. Indeed, we find

$$\partial_u \mathcal{G}^{(0)} = \frac{1}{2} S^{uu} C_{uuu} + \frac{\beta - 3 + \beta^{-1}}{6} \frac{1}{1+u} + \frac{\beta + \beta^{-1}}{12} \frac{1}{1-u}. \tag{3.20}$$

We recognize in these expressions the leading behavior at the two components of the discriminant locus corresponding to $u=-1$ and $u=+1$ to be $2\Phi^{(0)}(\beta)$ and $2\Psi^{(0)}(\beta)$ from Equation (2.25). At $\beta=1$, the $\mathbb{Z}_2$ symmetry is restored, and we recover the known results [19].

This structure persists at higher order as well. Amplitudes at odd n are zero. For even n, we work with the ansatz

$$A^{(n)}(u;\beta) = \frac{u^{n/2}}{(1+u)^n} \sum_{i=0}^{n-1} P_i^{(n)} u^{-i} + \frac{u^{n/2}}{(1-u)^n} \sum_{j=0}^{n-1} Q_j^{(n)} u^{-j} \tag{3.21}$$

for the holomorphic ambiguity. Then the gaps at $u=\pm 1$,

$$\mathcal{G}^{(n)} = \begin{cases} \dfrac{2\Phi^{(n)}}{a_{D,+}^n} + \mathcal{O}(1) & \text{around } u=+1 \\ \dfrac{2\Psi^{(n)}}{a_{D,-}^n} + \mathcal{O}(1) & \text{around } u=-1 \end{cases} \tag{3.22}$$

are sufficient to completely fix the $P_i^{(n)}$, $Q_j^{(n)}$, also in this case.

$N_f=3$: Our last example is an interesting mix of all the previous ones. The u-plane has no discrete symmetry, and the second component of the discriminant locus moves to $u=0$. We have

$$C_{uuu} = \frac{1}{64u(1-u)} \tag{3.23}$$

and $f^u=-16$. As for $N_f=1$, the amplitudes at odd order are generally non-zero. The term at $n=-1$, however, is just $\mathcal{G}^{(-1)} = (\beta - \beta^{-1/2})(-\frac{a}{4} + \frac{1}{4})$, and as a consequence, $\Delta_{uu}=0$. This simplifies the integration scheme, but we must still use the extended holomorphic anomaly equation. The one-loop amplitude comes out to be

$$\partial_u \mathcal{G}^{(0)} = \frac{1}{2} S^{uu} C_{uuu} + \frac{\beta + \beta^{-1}}{24} \frac{1}{1-u} + \frac{5\beta - 18 + 5\beta^{-1}}{24} \frac{1}{u}. \tag{3.24}$$

The term at $n=1$ is purely holomorphic, and we find

$$\mathcal{G}^{(1)} = \left(\beta^{1/2} - \beta^{-1/2}\right)^3 \frac{1}{u}. \tag{3.25}$$

More generally, we have a holomorphic ambiguity

$$A^{(n)}(u;\beta)=\begin{cases}\dfrac{u^{n/2}}{(1-u)^n}\displaystyle\sum_{i=0}^{n-1}P_i^{(n)}u^{-i}+\frac{1}{u^n}\sum_{j=0}^{n/2}Q_j^{(n)}u^j & n \text{ even}\\ \dfrac{u^{(n-1)/2}}{(1-u)^{n-1}}\displaystyle\sum_{i=0}^{n-2}P_i^{(n)}u^{-i}+\frac{1}{u^n}\sum_{j=0}^{(n-1)/2}Q_j^{(n)}u^j & n \text{ odd}\end{cases}. \tag{3.26}$$

As for the boundary conditions, we find at $u=+1$ a familiar gap structure

$$\mathcal{G}^{(n)}=\frac{\Psi^{(n)}}{a_D^n}+\mathcal{O}(1). \tag{3.27}$$

In contrast to $N_f=1$, $u=0$ is not a regular point (already at $\beta=1$). We also find a gap,

$$\mathcal{G}^{(n)}=\frac{X^{(n)}}{a_0^n}+\mathcal{O}(1), \tag{3.28}$$

with leading coefficients $X^{(n)}(\beta)$ that curiously are non-zero also for odd n (but vanish there at $\beta=1$). The first few are

$$\begin{aligned}X^{(0)}&=\frac{5\beta-18+5\beta^{-1}}{24}, & X^{(2)}&=\frac{-67\beta^2+540\beta-1330+540\beta^{-1}-67\beta^{-2}}{5760},\\ X^{(1)}&=\frac{(\beta^{1/2}-\beta^{-1/2})^3}{8}, & X^{(3)}&=\frac{(\beta^{1/2}-\beta^{-1/2})^3(\beta+6+\beta^{-1})}{96}.\end{aligned} \tag{3.29}$$

4. More on the Extension

Our development of the B-model formalism for general β in Section 2 involved in a central fashion the "extension" of special geometry by the term at order λ^{-1} in the expansion of $\log Z$,

$$\mathcal{G}^{(-1)}=\left(\beta^{1/2}-\beta^{-1/2}\right)\mathcal{T}. \tag{4.1}$$

In this section, we show how $\mathcal{T}$ can be recovered from the Seiberg–Witten geometry.

As a motivation, we recall that in the context of topological strings with D-branes and orientifolds [22,24], $\mathcal{T}$ is the topological disk (or disk+crosscap) amplitude. It can be written in terms of the holomorphic Chern–Simons functional, or as an integral $\sim\int^C\Omega$ of the holomorphic three-form of the Calabi–Yau over a three-chain ending on a holomorphic curve C that represents the background D-brane. See Ref. [36] for the relevant Hodge theoretic notions. The reduction of the holomorphic three-form to the present context is the Seiberg–Witten differential λ_{SW} on the curve $\mathcal{C}_u$, and the holomorphic curve C becomes a pair of points p_-, p_+, varying holomorphically with $\mathcal{C}_u$ as a function of u. Hence, we expect that for appropriate choice of $p_\pm$, we have the representation

$$\mathcal{T}=\int_{p_-}^{p_+}\lambda_{\mathrm{SW}}. \tag{4.2}$$

An important caveat is in order. In contrast to the holomorphic three-form of a Calabi–Yau threefold, the Seiberg–Witten differential is not unique. It is merely characterized by the condition that $\partial_u\lambda_{\mathrm{SW}}=\omega_u$ be the holomorphic one-form of the elliptic curve, *up to exact terms*. Modifying λ_{SW} by an exact form will change integrals such as (4.2). (Similar ambiguities play a role in recent studies of surface operators in $\mathcal{N}=2$ gauge theory, see, e.g., [37–39].) The invariant Seiberg–Witten geometry capturing the refinement should thus involve the curve, the differential and the points $p_\pm$.

From our examples, the only case where we can ask the question in an invariant way is $N_f=1$. The Seiberg–Witten curve may be written as [2],

$$\mathcal{C}_u:\ y^2=x^2(x-u)+\frac{4}{27} \tag{4.3}$$

(we chose Λ such that the discriminant is at $u^3=1$), and the differential as

$$\lambda_{\mathrm{SW}}=\sqrt{3}\frac{\mathrm{d}y}{x}=\sqrt{3}\frac{2u-3x}{y}\mathrm{d}x. \tag{4.4}$$

We claim that with respect to these choices, the correct combination of points can be obtained by intersecting the curve with the plane $x-u=0$, namely

$$p_\pm:(x,y)=\left(u,\pm\sqrt{\frac{4}{27}}\right). \tag{4.5}$$

There are various ways to check this claim. The most straightforward is to directly integrate. Indeed, the expansion around $u\to\infty$,

$$\int_{p_-}^{p_+}\lambda_{\mathrm{SW}}=\frac{4}{3u}+\frac{32}{243u^4}+\frac{512}{10935u^7}+\cdots \tag{4.6}$$

matches (up to a rational period) the solution of the inhomogeneous Picard–Fuchs equation (3.15).

Note that this discussion did not explain whether the choice of points (4.5) (with respect to the choice of differential) was distinguished in any sense. The formalism of Section 2 would go through for any other reasonable choice as well and only (4.5) lead to the correct answer. Lacking a deeper understanding, we can point out one way in which one might attempt to understand the invariant physical meaning of $\mathcal{T}$.[5]

[5]The following observations arose in a conversation with Samson Shatashvili.

In the work [40], the four-dimensional gauge theory in the Ω-background with $\epsilon_2 = 0$ was studied, and it was shown that this theory provides the quantization of the classical integrable system underlying the original four-dimensional theory. In particular, it was shown that the twisted superpotential $\mathcal{W}(a, \epsilon_1; q) = \lim_{\epsilon_2 \to 0} \epsilon_2 \log Z(a, \epsilon_1, \epsilon_2; q)$ could be identified with the Yang–Yang function of the integrable system. Moreover, the remaining parameter ϵ_1 is identified with the Planck constant of the quantization procedure. We may relate the expansion in ϵ_1 to our parametrization (2.2) via

$$\mathcal{W}(a, \epsilon_1; q) = \frac{1}{\epsilon_1}\mathcal{F}^{(0)} + \mathcal{T} + \mathcal{O}(\epsilon_1) \tag{4.7}$$

Viewed from this angle, $\mathcal{T}$ is nothing but a one-loop term, which shows its special role. Moreover, one should be able to compute it directly in the semi-classical expansion of the relevant integrable system. We leave this line of investigation for the future.

5. Orientifold

In this section, we explain a relation between two special sets of values for the parameters of the Ω-background. The essential message is that, at least in certain cases, the theory at $\epsilon_1 = -2\epsilon_2$ can be viewed, in a very precise sense, as the orientifold of the theory at $\epsilon_1 = -\epsilon_2$. In a way, this result has been anticipated by the relation between the Nekrasov deformation of gauge theories and the β-ensemble of generalized matrix models [27]. Indeed, it is well known that when the value $\beta = -\epsilon_1/\epsilon_2 = 1$ corresponds to the $U(N)$ matrix models, then $\beta = 1/2, 2$ correspond to $SO(N)$, $Sp(N)$, respectively, which are just the orientifolds of $U(N)$. (The change of the string coupling, $\lambda = -\epsilon_1\epsilon_2 \sim 1/N$, is also accounted for in this relationship).

We uncover the relation between $\beta = 1$ and $\beta = 2$ purely from the instanton counting in the gauge theory. This has several virtues. First of all, we see the remarkable cancellation, as in the orientifold we sum only over a very small subset of the "real" instantons, but still recover the result of the full sum for different values of the parameters. Second, the identification of the β-parameter with the radius $R = \beta$ of the $c = 1$ string, and the comparison with the moduli space of $c = 1$ conformal field theories [41], is suggestive of the possible existence of an entire new branch of topological theories: The value $\beta = 2$ is precisely the point where the orbifold branch of $c = 1$ theories touches the circle branch. It would be interesting to see how to move in this direction. Finally, we will see that the relationship does not strictly hold in all cases. Namely, it fails us for an odd number of flavors. We attribute this to the accidental (in)completeness of the orientifold prescription as the right quotient. We also explain some parallels with orientifolds of the topological string.

The squareroot. The basic idea is quite simple, and indeed nothing else than a new manifestation of the "real topological string principle", developed in [23–25,42,43]. In the localization computation of [7], the instanton counting partition function is written as the sum of contributions from the fixed points of the certain group action on the moduli space of instantons. We show only the parameters ϵ_1, ϵ_2 related to $\mathbb{T}^2$ action on $\mathbb{R}^4 \cong \mathbb{C}^2 \ni (z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$.

$$Z^{\text{inst}}(\epsilon_1, \epsilon_2) = \sum_Y R_Y(\epsilon_1, \epsilon_2), \tag{5.1}$$

where Y some colored partitions label the fixed points, and $R_Y(\epsilon_1, \epsilon_2)$ is a rational function of the various parameters (see Appendix for the examples).

Consider now the real structure on $\mathbb{C}^2$: $\sigma : (z_1, z_2) \to (\bar{z}_2, \bar{z}_1)$. This commutes with a one-dimensional subtorus of $\mathbb{T}^2$, so localization still applies. In fact the subtorus is nothing else than $\epsilon_1 = -\epsilon_2$, i.e., the anti-diagonal $U(1)$. We also assume an appropriate lift to the rest of the data. In particular, σ acts on the set of Y's. The real topological string principle then instructs us to consider the sum over the invariant configurations,

$$Z^{\text{real inst}}(\epsilon_1) = \sum_{Y=\sigma(Y)} \sqrt{R_Y(\epsilon_1, -\epsilon_1)}, \tag{5.2}$$

exploiting the fact that $R_Y(\epsilon_1, -\epsilon_1)$ is a perfect square. We have implemented this procedure for $SU(2)$ gauge theory with $N_f \leq 4$ flavors. (Of course, (5.2) requires the specification of a sign for each invariant fixed point. Luckily, this is quite straightforward in the present case.) Our main result is not so much that the resulting expression makes sense (for example, in having a sensible limit as $\epsilon_1 \to 0$; this is true also for $N_f = 1, 3$). Rather, we find that for $N_f = 0, 2, 4$,

$$Z^{\text{real inst}}(\epsilon_1) = Z^{\text{inst}}(\epsilon_1, -2\epsilon_1), \tag{5.3}$$

which can be checked in the appropriate expansion.

Return of the topological string. It is well known that the $\mathcal{N} = 2$ gauge theory can be embedded into string theory via geometric engineering [13,14]. The gravitational corrections, $\mathcal{F}^{(g)} = \mathcal{G}^{(g)}(\beta = 1)$, computed in the Ω-background at $\epsilon_2 = -\epsilon_1$ are then identified with the field theory limit of the genus-g topological string amplitudes [7]. It is an important question whether the main results of the present paper, namely, extension of the holomorphic anomaly, and embedding of orientifold at $\beta = 2$, can be lifted to the full-fledged topological string on any Calabi–Yau threefold.

At the moment, we can offer one further piece of evidence that at least some aspect will survive. The Calabi–Yau's of geometric engineering are toric, and we can compute the partition functions in the topological vertex formalism [44], and also their orientifolds [43].[6] In Ref. [25], the real vertex was applied to the local $\mathbb{P}^2$ geometry (which incidentally is not even an engineering geometry), and the results were compared with those of the holomorphic anomaly equation. In particular, expanding around the conifold point of local $\mathbb{P}^2$, it was found that the genus-g Klein bottle amplitudes[7]

$$\mathcal{K}^{(g)} = \mathcal{G}^{(2g-2)} - \mathcal{F}^{(g)} = \frac{\psi^{(g)}}{a_D^{2g-2}} + \mathcal{O}(1) \tag{5.4}$$

show the universal gap structure. The first few coefficients were found to be (up to the model-dependent factor of 3^{g-1})

$$\psi^{(2)} = -\frac{3}{128}, \quad \psi^{(3)} = \frac{9}{512}, \quad \psi^{(4)} = -\frac{157}{4096}, \tag{5.5}$$

but not otherwise identified in [25]. Using these results, we can test the present idea that the topological orientifolds can be embedded in a putative refinement of the theory at $\beta = 2$ by comparing with the universal behavior found in the gauge theory examples. Taking the appropriate linear combination (5.4), we predict[8]

$$\psi^{(g)} = 2^g \Psi^{(2g-2)}(2) - \Psi^{(2g-2)}(1) = \frac{1}{2^{2g+1} g(g-1)} ((2^{2g} - 1) B_{2g} - g E_{2g-2}), \tag{5.6}$$

where E_g are Euler numbers. This indeed matches the coefficients (5.5) found in [25].

6. Conclusions

In this paper, we have uncovered some new properties of the partition function of $\mathcal{N} = 2$ gauge theory in the Ω-background, and found various hints that those structures can be lifted to the topological string. Many interesting physical and mathematical questions remain, which we hope to take up in the near future.

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[6] It can, for instance, be checked that the real topological string on $\mathbb{P}^1 \times \mathbb{P}^1$ computed from the real vertex coincides with the termwise square root of Nekrasov's five-dimensional $SU(2)$ partition function (similarly, for the geometries that engineer $N_f = 2, 4$).

[7] The $\mathcal{G}^{(2g-2)}$ featuring here don't depend on any β.

[8] The relative factor of 2^g is due to the redefinition of the string coupling in the orientifold.

Appendix A: Nekrasov's Formulae

In this appendix, we will briefly recall the instanton calculation of [7] and we collect some key observations regarding the structure of the resulting partition function for $SU(2)$ gauge group with up to four flavors.

Basics. Consider the (compactified) moduli space $\mathcal{M}_k$ of k-instantons of $U(N)$ gauge theory with N_f fundamentals in $\mathbb{R}^4 \cong \mathbb{C}^2$. According to [3,5,7], the corresponding instanton partition function, denoted as Z^{inst}, can be calculated via localization with respect to the $U(N)\times U(N_f)\times \mathbb{T}^2$ group action on $\mathcal{M}_k$, where $\mathbb{T}^2$ is the maximal torus of the $SO(4)$ rotation group of $\mathbb{R}^4$. Namely,

$$Z^{\text{inst}}(\vec{a},\vec{m},\epsilon_1,\epsilon_2;q)=\sum_k q^k\int_{\mathcal{M}_k}\mathbf{e}(V\otimes\mathbb{C}^{N_f}),\tag{A.1}$$

where q is a parameter, $\vec{a}=(a_1,a_2,\ldots,a_N)$ are coordinates of the Cartan subalgebra of $U(N)$, $\vec{m}=(m_1,m_2,\ldots,m_{N_f})$ the masses of the N_f fundamentals (coordinates on the Cartan subalgebra of the flavor group $U(N_f)$), ϵ_i are the coordinates on the Lie algebra of $\mathbb{T}^2$, $\mathbf{e}$ denotes the equivariant Euler class with respect to the group $U(N)\times U(N_f)\times\mathbb{T}^2$ and V is the bundle over $\mathcal{M}_k$ of solutions of the Dirac equation in the instanton background.

The partition function (A.1) can be expressed as follows, which is convenient for explicit computation [7]

$$Z^{\text{inst}}(\vec{a},\vec{m},\epsilon_1,\epsilon_2;q)=\sum_{\vec{Y}}\frac{\prod_{i=1}^{N_f}\prod_\gamma f_\gamma^{\vec{Y}}(m_i)}{\prod_{\alpha,\beta}n_{\alpha,\beta}^{\vec{Y}}}q^{|\vec{Y}|},\tag{A.2}$$

where $\vec{Y}=(Y_1,Y_2,\ldots,Y_N)$ is an N-tuple of partitions (Young diagrams), q a parameter, $n_{\alpha,\beta}^{\vec{Y}}$ collects the contribution from the gauge sector (we use the form presented in [9,45,46])

$$\begin{aligned}n_{\alpha,\beta}^{\vec{Y}}=&\prod_{s\in Y_\alpha}(-l_{Y_\beta}(s)\epsilon_1+(a_{Y_\alpha}(s)+1)\epsilon_2+a_{\beta\alpha})\\&\times\prod_{t\in Y_\beta}((l_{Y_\alpha}(t)+1)\epsilon_1-a_{Y_\beta}(t)\epsilon_2+a_{\beta\alpha}),\end{aligned}\tag{A.3}$$

with s and t running over all boxes (i,j) in Y_α and Y_β, respectively, $a_Y(i,j):=\mu_i^Y-j$, $l_Y(i,j):=\mu_j^{Y^t}-i$ (μ_i^Y and $\mu_i^{Y^t}$ denote the i-th column of the Young diagram Y and its transpose, respectively), $f_\gamma^{\vec{Y}}(m)$ represents the contribution from a single fundamental of mass m, i.e.,

$$f_\gamma^{\vec{Y}}(m)=\prod_{i=1}^{\mu_1^{Y_\gamma^t}}\prod_{j=1}^{\mu_i^{Y_\gamma}}\big(a_\gamma+m+\epsilon_1(i-1)+\epsilon_2(j-1)\big),\tag{A.4}$$

$a_{\beta_\alpha} = a_\beta - a_\alpha$, and indices α, β, γ running over $1, \ldots, N$. Note that one can restrict to $SU(N) \subset U(N)$ by enforcing $\sum_i a_i = 0$.

The instanton partition function (A.1) must be supplemented by a perturbative part, denoted as Z^{pert}, to yield the full partition function in the Ω-background,

$$Z(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2; q) = Z^{\text{pert}}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2)\, Z^{\text{inst}}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2; q). \tag{A.5}$$

We also give some details on the perturbative part of (A.5). Namely, one takes [7,8,47],

$$\log Z^{\text{pert}}(\vec{a}, \vec{m}, \epsilon_1, \epsilon_2) = \sum_{\alpha,\beta} \gamma_{\epsilon_1,\epsilon_2}(a_{\alpha\beta}) - \sum_{\gamma} \sum_{i=1}^{N_f} \gamma_{\epsilon_1,\epsilon_2}(a_\gamma + m_i), \tag{A.6}$$

with

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \sum_{n=3}^{\infty} \frac{c_n(\epsilon_1, \epsilon_2)}{n(n-1)(n-2)} x^{2-n} + \mathcal{O}(1), \tag{A.7}$$

and c_n defined via the expansion

$$\frac{1}{(\mathrm{e}^{\epsilon_1 t} - 1)(\mathrm{e}^{\epsilon_2 t} - 1)} = \sum_{g=0}^{\infty} \frac{c_g(\epsilon_1, \epsilon_2)}{g!} t^{g-2}. \tag{A.8}$$

The lower order terms in (A.7) are not so important for our considerations and therefore we omit to display them explicitly. Especially, in the $SU(2)$ case with N_f massless flavors (we drop the $\vec{m}$ parameter), we infer

$$\log Z^{\text{pert}} \sim \sum_n \lambda^n \frac{(1 - 2^n N_f)}{2^{n-1} a^n} \Phi^{(n)}(\beta), \tag{A.9}$$

where as in the main text, $\beta = -\epsilon_1/\epsilon_2$, and $\Phi^{(n)}(\beta)$ are functions defined in (2.24).

Observations. When expanding Z as in Equation (2.2), we have to remember to first expand $\log Z$ in powers of q. We may then set $q = 1$. We consider $SU(2)$ gauge theory with $N_f \leq 4$.

Pure $SU(2)$: We set $a_1 = -a_2$ and drop the empty $\vec{m}$ parameter. Terms at odd n vanish, except for $n = -1$, where we have $\mathcal{G}^{(-1)} = (\beta^{1/2} - \beta^{-1/2})\frac{a}{2}$, see [47]. For the one-instanton sector, we obtain

$$\log Z^{\text{inst}}(a, \beta^{1/2}\lambda, -\beta^{-1/2}\lambda)|_{q^1} = -\sum_{i=0}^{\infty} \frac{(\beta - 1)^{2i}}{2^{2i+1}\beta^i} \frac{\lambda^{2i-2}}{a^{2i+2}} =: \mathcal{F}_1^{\text{even}}. \tag{A.10}$$

$N_f=1$: For a single flavor of mass m, terms at odd powers of λ are non-zero, and already $\mathcal{G}^{(-1)}$ is highly non-trivial. See (3.14) for the explicit expression when $m=0$. The one-instanton sector is

$$\log Z^{\text{inst}}(a,m,\beta^{1/2}\lambda,-\beta^{-1/2}\lambda)|_{q^1} = \sum_{i=0}^{\infty}\frac{(\beta-1)^{2i+1}}{2^{2i+2}\beta^{i+1/2}}\frac{\lambda^{2i-1}}{a^{2i+2}}+m\,\mathcal{F}_1^{\text{even}} =: \mathcal{F}_1^{\text{odd}}+m\mathcal{F}_1^{\text{even}}. \tag{A.11}$$

$N_f=2$: We have two flavors with masses $\vec{m}=(m_1,m_2)$. The 1-instanton sector reads

$$\log Z^{\text{inst}}(a,\vec{m},\beta^{1/2}\lambda,-\beta^{-1/2}\lambda)|_{q^1}=(m_1+m_2)\,\mathcal{F}_1^{\text{odd}}+(a^2+m_1m_2)\,\mathcal{F}_1^{\text{even}}. \tag{A.12}$$

As in the $N_f=1$ case, generally odd powers of λ cannot be avoided. However, for the special choice $m_1=-m_2$ all odd power terms vanish, as can be inferred by expanding as well the higher instanton sectors. So that case is very similar to the pure gauge theory. Note that as in the $N_f=1$ case, β dependence in all terms of order λ^{-1} can be factored out. The $\mathbb{Z}_2$ parity symmetry is broken for $\beta\neq 1$ as well.

$N_f=3$: The masses of the flavors are $\vec{m}=(m_1,m_2,m_3)$. The 1-instanton sector reads

$$\begin{aligned}\log Z^{\text{inst}}(a,\vec{m},\beta^{1/2}\lambda,-\beta^{-1/2}\lambda)|_{q^1} =&(a^2+m_1(m_2+m_3)+m_2m_3)\,\mathcal{F}_1^{\text{odd}}\\ &+(a^2(m_1+m_2+m_3)+m_1m_2m_3)\,\mathcal{F}_1^{\text{even}}.\end{aligned} \tag{A.13}$$

As in the $N_f=1$ case, generally odd powers of λ cannot be avoided. However, expanding as well the higher instanton sectors shows that in the massless case ($m_i=0$) almost all terms of order λ^{-1} drop out, and we have $\mathcal{T}=-\frac{a}{4}+\frac{1}{4}$.

$N_f=4$: The four flavor case with mass vector $\vec{m}=(m_1,m_2,m_3,m_4)$ is very similar to the two flavor case. Therefore, we will be brief. The main observation is that for choosing two pairs of masses with different signs, the odd powers of λ drop out, for example for the choice $m_1=-m_2$ and $m_3=-m_4$.

References

1. Seiberg, N., Witten, E.: Monopoles, duality and chiral symmetry breaking in $N=2$ supersymmetric QCD. Nucl. Phys. B **431**, 484 (1994). arXiv:hep-th/9408099
2. Seiberg, N., Witten, E.: Monopole condensation, and confinement in $N=2$ supersymmetric Yang-Mills theory. Nucl. Phys. B **426**, 19 (1994). Erratum-ibid. B **430**, 485 (1994). arXiv:hep-th/9407087
3. Losev, A., Nekrasov, N., Shatashvili, S.L.: Issues in topological gauge theory. Nucl. Phys. B **534**, 549 (1998). arXiv:hep-th/9711108

4. Moore, G.W., Nekrasov, N., Shatashvili, S.: Integrating over Higgs branches. Commun. Math. Phys. **209**, 97 (2000). arXiv:hep-th/9712241
5. Losev, A., Nekrasov, N., Shatashvili, S.L.: Testing Seiberg-Witten solution. arXiv:hep-th/9801061
6. Moore, G.W., Nekrasov, N., Shatashvili, S.: D-particle bound states and generalized instantons. Commun. Math. Phys. **209**, 77 (2000). arXiv:hep-th/9803265
7. Nekrasov, N.A.: Seiberg-Witten prepotential from instanton counting. Adv. Theor. Math. Phys. **7**, 831 (2004). arXiv:hep-th/0206161
8. Nekrasov, N., Okounkov, A.: Seiberg-Witten theory and random partitions. arXiv:hep-th/0306238
9. Nakajima, H., Yoshioka, K.: Instanton counting on blowup. Part I: 4-dimensional pure gauge theory. arXiv:math.ag/0306198
10. Moore, G.W., Witten, E.: Integration over the u-plane in Donaldson theory. Adv. Theor. Math. Phys. **1**, 298 (1998). arXiv:hep-th/9709193
11. Antoniadis, I., Gava, E., Narain, K.S., Taylor, T.R.: Topological amplitudes in string theory. Nucl. Phys. B **413**, 162 (1994). arXiv:hep-th/9307158
12. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. Commun. Math. Phys. **165**, 311 (1994). arXiv:hep-th/9309140
13. Klemm, A., Lerche, W., Mayr, P., Vafa, C., Warner, N.P.: Self-dual strings and $N=2$ supersymmetric field theory. Nucl. Phys. B **477**, 746 (1996). arXiv:hep-th/9604034
14. Katz, S.H., Klemm, A., Vafa, C.: Geometric engineering of quantum field theories. Nucl. Phys. B **497**, 173 (1997). arXiv:hep-th/9609239
15. Klemm, A., Marino, M., Theisen, S.: Gravitational corrections in supersymmetric gauge theory and matrix models. JHEP **0303**, 051 (2003). arXiv:hep-th/0211216
16. Iqbal, A., Kashani-Poor, A.K.: Instanton counting and Chern-Simons theory. Adv. Theor. Math. Phys. **7**, 457 (2004). arXiv:hep-th/0212279
17. Eguchi, T., Kanno, H.: Topological strings and Nekrasov's formulas. JHEP **0312**, 006 (2003). arXiv:hep-th/0310235
18. Huang, M.X., Klemm, A.: Holomorphic anomaly in gauge theories and matrix models. JHEP **0709**, 054 (2007). arXiv:hep-th/0605195
19. Huang, M.X., Klemm, A.: Holomorphicity and modularity in Seiberg-Witten theories with matter. arXiv:0902.1325[hep-th]
20. Ghoshal, D., Vafa, C.: $C=1$ String as the topological theory of the conifold. Nucl. Phys. B **453**, 121 (1995). arXiv:hep-th/9506122
21. Antoniadis, I., Hohenegger, S., Narain, K.S., Taylor, T.R.: Deformed topological partition function and Nekrasov backgrounds. arXiv:1003.2832[hep-th]
22. Walcher, J.: Extended holomorphic anomaly and loop amplitudes in open topological string. Nucl. Phys. B **817**, 167 (2009). arXiv:0705.4098[hep-th]
23. Bouchard, V., Florea, B., Marino, M.: Counting higher genus curves with crosscaps in Calabi-Yau orientifolds. JHEP **0412**, 035 (2004). arXiv:hep-th/0405083
24. Walcher, J.: Evidence for tadpole cancellation in the topological string. Comm. Number Theor. Phys. **3**, 111–172 (2009). arXiv:0712.2775[hep-th]
25. Krefl, D., Walcher, J.: The real topological string on a local Calabi-Yau. arXiv:0902.0616[hep-th]
26. Alday, L.F., Gaiotto, D., Tachikawa, Y.: Liouville correlation functions from four-dimensional gauge theories. Lett. Math. Phys. **91**, 167 (2010). arXiv:0906.3219 [hep-th]
27. Dijkgraaf, R., Vafa, C.: Toda theories, matrix models, topological strings, and $N=2$ gauge systems. arXiv:0909.2453[hep-th]

28. Yamaguchi, S., Yau, S.T.: Topological string partition functions as polynomials. JHEP **0407**, 047 (2004). arXiv:hep-th/0406078
29. Alim, M., Lange, J.D.: Polynomial structure of the (open) topological string partition function. JHEP **0710**, 045 (2007). arXiv:0708.2886[hep-th]
30. Konishi, Y., Minabe, S.: On solutions to Walcher's extended holomorphic anomaly equation. arXiv:0708.2898[math.AG]
31. Grimm, T.W., Klemm, A., Marino, M., Weiss, M.: Direct integration of the topological string. JHEP **0708**, 058 (2007). arXiv:hep-th/0702187
32. Gopakumar, R., Vafa, C.: M-theory and topological strings. Part I, II. arXiv:hep-th/9809187, arXiv:hep-th/9812127
33. Gross, D.J., Klebanov, I.R.: One-dimensional string theory on a circle. Nucl. Phys. B **344**, 475 (1990)
34. Hollowood, T.J., Iqbal, A., Vafa, C.: Matrix models, geometric engineering and elliptic genera. JHEP **0803**, 069 (2008). arXiv:hep-th/0310272
35. Iqbal, A., Kozcaz, C., Vafa, C.: The refined topological vertex. JHEP **0910**, 069 (2009). arXiv:hep-th/0701156
36. Morrison, D.R., Walcher, J.: D-branes and Normal Functions. Adv. Theor. Math. Phys. **13**, 553–598 (2009). arXiv:0709.4028[hep-th]
37. Alday, L.F., Gaiotto, D., Gukov, S., Tachikawa, Y., Verlinde, H.: Loop and surface operators in $N = 2$ gauge theory and Liouville modular geometry. JHEP **1001**, 113 (2010). arXiv:0909.0945[hep-th]
38. Kozcaz, C., Pasquetti, S., Wyllard, N.: A & B model approaches to surface operators and Toda theories. arXiv:1004.2025[hep-th]
39. Dimofte, T., Gukov, S., Hollands, L.: Vortex counting and Lagrangian 3-manifolds. arXiv:1006.0977[hep-th]
40. Nekrasov, N.A., Shatashvili, S.L.: Quantization of integrable systems and four dimensional gauge theories. arXiv:0908.4052[hep-th]
41. Dijkgraaf, R., Verlinde, E.P., Verlinde, H.L.: $C = 1$ Conformal field theories on Riemann surfaces. Commun. Math. Phys. **115**, 649 (1988)
42. Bouchard, V., Florea, B., Marino, M.: Topological open string amplitudes on orientifolds. JHEP **0502**, 002 (2005). arXiv:hep-th/0411227
43. Krefl, D., Pasquetti, S., Walcher, J.: The real topological vertex at work. Nucl. Phys. B **833**, 153 (2010). arXiv:0909.1324[hep-th]
44. Aganagic, M., Klemm, A., Marino, M., Vafa, C.: The topological vertex. Commun. Math. Phys. **254**, 425 (2005). arXiv:hep-th/0305132
45. Flume, R., Poghossian, R.: An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential. Int. J. Mod. Phys. A **18**, 2541 (2003). arXiv:hep-th/0208176
46. Bruzzo, U., Fucito, F., Morales, J.F., Tanzini, A.: Multi-instanton calculus and equivariant cohomology. JHEP **0305**, 054 (2003). arXiv:hep-th/0211108
47. Nakajima, H., Yoshioka, K.: Lectures on instanton counting. arXiv:math/0311058