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Unitarizable Highest Weight Modules of the $N = 2$ Super Virasoro Algebras: Untwisted Sectors

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Abstract. In this article, we prove the classification theorem of the unitarizable highest weight modules over the $N = 2$ super Virasoro algebras for the untwisted sector, stated by Boucher et al. (Phys Lett B 172:316–322, 1986).

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1. Introduction

In 1984, Friedan et al. [7] classified the unitarizable highest weight modules over the Virasoro algebra. This work was extended to the all sectors, Neveu-Schwarz and Ramond, of the $N = 1$ super Virasoro algebras by the same authors [8]. In 1986, Boucher et al. [1] classified the unitarizable highest weight modules over the all sectors, Neveu-Schwarz (NS), Ramond (R) and twisted (T), of the $N = 2$ super Virasoro algebras. In these references, the proofs are not given.

In 1988, Langlands [12] provided a mathematically rigorous proof of the classification theorem of [7]. A similar argument is applicable to the $N = 1$ super Virasoro algebras and was worked out by Sauvageot [13].

It was shown by Schwimmer and Seiberg [14] that NS sector and the R sector of the $N = 2$ super Virasoro algebras are isomorphic; hence, the $N = 2$ super Virasoro algebras have essentially two different isomorphism classes: the twisted sector and the untwisted sector. A relevant superconformal algebra obtained by the so-called topological twist is introduced by Dijkgraff et al. [5] in 1991. There are several works around the $N = 2$ superconformal algebras both by physicists and mathematicians. Here, we only mention that the work by Feigin et al. [6] was the starting point of further mathematical works.

The classification theorem of the unitarizable highest weight modules over the NS, R and T sectors of the $N = 2$ super Virasoro algebra were announced in [1], and was proved for the T sector by the author [10]. The aim of this article is to prove the classification theorem of [1] for the untwisted sector. Besides some technical details, the proof consists of analysis of the determinant formulas as was done by Langlands [12] for the Virasoro algebra. This paper is organized as follows:

In Section 2, after recalling some basic objects, we show that the unitarizability of highest weight modules over the NS sector of the $N=2$ super Virasoro algebra is equivalent to that of the corresponding modules over the R sector of the $N=2$ super Virasoro algebra via the spectral flow. In Section 3, the determinant formulas together with a technical lemma used in the proof of the main theorem are recalled. Section 4 is devoted to the proof of the main theorems (Theorems 4.1 and 4.2).

Throughout this article, we use the following conventions: $\sigma \in \{\pm\}$ is sometimes identified with $\sigma \in \{\pm 1\}$. In particular, for $\lambda \in \mathbb{C}$, $\lambda\sigma, \sigma\lambda$ signify $\pm\lambda$.

2. Preliminary

In this section, we recall basic objects related to this article such as $N=2$ super Virasoro algebras, Verma-type modules and contravariant forms on them.

2.1. $N=2$ SUPER VIRASORO ALGEBRAS

Here, we define the untwisted sector of the $N=2$ super Virasoro algebras.

DEFINITION 2.1. The $N=2$ **super Virasoro algebra** $\mathfrak{g}_\varepsilon$ ($\varepsilon \in \{\frac{1}{2}, 0\}$) (untwisted sector) are the Lie superalgebras

$$\mathfrak{g}_\varepsilon := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{C}I_m \oplus \bigoplus_{k\in\varepsilon+\mathbb{Z}} \{\mathbb{C}G_k^+ \oplus \mathbb{C}G_k^-\} \oplus \mathbb{C}c,$$

with the parity

$$\deg L_n = \deg I_m = \deg c = \bar{0}, \quad \deg G_k^\pm = \bar{1},$$

which satisfies the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

$$[I_m, L_n] = mI_{m+n}, \quad [I_m, I_n] = \frac{1}{3}m\delta_{m+n,0}c,$$

$$[L_m, G_n^\pm] = \left(\frac{m}{2} - n\right) G_{m+n}^\pm, \qquad [I_m, G_n^\pm] = \pm G_{m+n}^\pm,$$

$$[G_m^\pm, G_n^\pm] = 0, \quad [G_m^+, G_n^-] = 2L_{m+n} + (m-n)I_{m+n} + \frac{1}{3}\left(m^2 - \frac{1}{4}\right)\delta_{m+n,0}c.$$

The cases $\varepsilon = \frac{1}{2}, 0$ are referred to as the Neveu-Schwarz (NS) sector and the Ramond (R) sector respectively.

The Lie superalgebras $\mathfrak{g}_\varepsilon$ possess triangular decomposition

$$\mathfrak{g}_\varepsilon = (\mathfrak{g}_\varepsilon)_+ \oplus (\mathfrak{g}_\varepsilon)_0 \oplus (\mathfrak{g}_\varepsilon)_-,$$

where we set

$$(\mathfrak{g}_\varepsilon)_\pm := \bigoplus_{\pm n \in \mathbb{Z}_{>0}} \mathbb{C}L_n \oplus \bigoplus_{\pm m \in \mathbb{Z}_{>0}} \mathbb{C}I_m \oplus \bigoplus_{k \in 1-\varepsilon+\mathbb{Z}_{\geq 0}} \{\mathbb{C}G_k^+ \oplus \mathbb{C}G_k^-\},$$

$$(\mathfrak{g}_\varepsilon)_0 := \mathbb{C}L_0 \oplus \mathbb{C}I_0 \oplus \mathbb{C}c \oplus \begin{cases} \{0\} & \varepsilon = \frac{1}{2}, \\ \mathbb{C}G_0^+ \oplus \mathbb{C}G_0^- & \varepsilon = 0. \end{cases}$$

For the definition of the T sector and the topological version, see, e.g., [5,14].

2.2. REAL FORMS

Here, we classify the real forms of $\mathfrak{g}_\varepsilon$. Indeed, we classify the conjugate-linear anti-involution θ of $\mathfrak{g}_\varepsilon$. In addition, we will determine the real forms that admit a non-trivial unitarizable module. Arguments given in [2] can be easily generalized to our case; hence, we only state the results.

We denote the decomposition of $(\mathfrak{g}_\varepsilon)_0$ with respect to the $\mathbb{Z}/2\mathbb{Z}$-gradation by

$$(\mathfrak{g}_\varepsilon)_0 = (\mathfrak{g}_\varepsilon)_0^{\bar{0}} \oplus (\mathfrak{g}_\varepsilon)_0^{\bar{1}}.$$

The next proposition can be easily proved.

PROPOSITION 2.1. *The subalgebra $(\mathfrak{g}_\varepsilon)_0^{\bar{0}}$ is the unique maximal abelian* ad-*semi-simple even subalgebra of $\mathfrak{g}_\varepsilon$.*

Let us set

$$S^1 := \left\{ z \;\middle|\; |z| = 1 \right\} \subset \mathbb{C}.$$

The next results classifies all of the conjugate-linear anti-involution of $\mathfrak{g}_\varepsilon$.

PROPOSITION 2.2. *Any conjugate-linear anti-involution of $\mathfrak{g}_\varepsilon$ is of one of the following types*:

1. *For some $\alpha \in \mathbb{R}^\times$,*
   (i) *$\eta \in \mathbb{Z}$ and $\Lambda \in S^1$,*

$$\theta^+_{\alpha,\eta,\Lambda}(L_n) = \alpha^n \left(L_{-n} - \eta I_{-n} + \frac{1}{6}\eta^2 \delta_{n,0} c \right),$$

$$\theta^+_{\alpha,\eta,\Lambda}(I_n) = \alpha^n \left(-I_{-n} + \frac{1}{3}\eta \delta_{n,0} c \right),$$

$$\theta^+_{\alpha,\eta,\Lambda}(c) = c,$$

$$\theta^+_{\alpha,\eta,\Lambda}(G_n^\pm) = \Lambda^{\pm 1} \alpha^{n \pm \frac{1}{2}\eta} G^\pm_{-n \mp \eta}.$$

(ii) $\Lambda \in \mathbb{R}^\times$,

$$\theta^+_{\alpha,\Lambda}(L_n)=\alpha^n L_{-n}, \quad \theta^+_{\alpha,\Lambda}(I_n)=\alpha^n I_{-n}, \quad \theta^+_{\alpha,\Lambda}(c)=c,$$
$$\theta^+_{\alpha,\Lambda}(G^\pm_n)=\Lambda^{\pm 1}\alpha^n G^\mp_{-n}.$$

2. *For some* $\alpha \in S^1$,
   (i) $\eta \in \mathbb{Z}$ *and* $\Lambda \in \sqrt{-1}\mathbb{R}^\times$,

$$\theta^-_{\alpha,\eta,\Lambda}(L_n)=-\alpha^n\left(L_n+\eta I_n+\frac{1}{6}\eta^2\delta_{n,0}c\right),$$
$$\theta^-_{\alpha,\eta,\Lambda}(I_n)=\alpha^n\left(I_n+\frac{1}{3}\eta\delta_{n,0}c\right),$$
$$\theta^-_{\alpha,\eta,\Lambda}(c)=-c,$$
$$\theta^-_{\alpha,\eta,\Lambda}(G^\pm_n)=\mp\Lambda^{\mp 1}\alpha^{n\mp\frac{1}{2}\eta}G^\mp_{n\mp\eta}.$$

   (ii) $\Lambda \in S^1$,

$$\theta^-_{\alpha,\Lambda}(L_n)=-\alpha^n L_n, \quad \theta^-_{\alpha,\Lambda}(I_n)=-\alpha^n I_n, \quad \theta^-_{\alpha,\Lambda}(c)=-c,$$
$$\theta^-_{\alpha,\Lambda}(G^\pm_n)=\pm\Lambda^{\pm 1}\alpha^n G^\pm_n.$$

By Proposition 2.1, it is natural to consider the $(\mathfrak{g}_\varepsilon, (\mathfrak{g}_\varepsilon)^{\bar{0}}_0)$-modules. The next proposition shows for which conjugate-linear anti-involutions a non-trivial unitary $(\mathfrak{g}_\varepsilon, (\mathfrak{g}_\varepsilon)^{\bar{0}}_0)$-module exists.

PROPOSITION 2.3. *Let* V *be a non-trivial irreducible* $(\mathfrak{g}_\varepsilon, (\mathfrak{g}_\varepsilon)^{\bar{0}}_0)$*-module.*

1. *If* V *is unitary for some conjugate-linear anti-involution* θ, *then* $\theta=\theta^+_{\alpha,\Lambda}$ *for some* $\alpha, \Lambda \in \mathbb{R}_{>0}$.
2. *If* V *is unitary for some* $\alpha, \Lambda \in \mathbb{R}_{>0}$, *then* V *is unitary for* $\theta=\theta^+_{1,1}$.

Thus, in the remaining part of the paper, we shall only consider $(\mathfrak{g}_\varepsilon, (\mathfrak{g}_\varepsilon)^{\bar{0}}_0)$-module which is unitary for $\theta^+_{1,1}$. In particular, we denote θ instead of $\theta^+_{1,1}$ for simplicity.

2.3. SPECTRAL FLOWS

As one can easily see, $\mathfrak{g}_\varepsilon$ is a $\mathbb{Z}$-graded Lie superalgebra. Here, we recall a 'gauge shift' introduced by [14] which does not preserve $\mathbb{Z}$-graded structure.

Let us fix $\varepsilon \in \{\frac{1}{2}, 0\}$. For $\eta \in \frac{1}{2}\mathbb{Z}$, we define the morphism

$$\Phi_\eta : \mathfrak{g}_\varepsilon \longrightarrow \mathfrak{g}_{\varepsilon+(-1)^{2\varepsilon}(\eta-[\eta])}$$

as follows:

$$L_n \mapsto L_n+\eta I_n+\frac{1}{6}\eta^2\delta_{n,0}c, \quad I_n \mapsto I_n+\frac{1}{3}\eta\delta_{n,0}c, \quad G^\pm_n \mapsto G^\pm_{n\pm\eta}, \quad c \mapsto c.$$

Then, it follows that Φ_η is an isomorphism of Lie superalgebras, which is not an isomorphism as $\mathbb{Z}$-graded Lie superalgebras. Thus, we have

PROPOSITION 2.4.

$$\mathfrak{g}_0 \cong \mathfrak{g}_{\frac{1}{2}}.$$

2.4. VERMA-TYPE MODULES

In this subsection, we define Verma-type modules over $\mathfrak{g}_\varepsilon$ and introduce a contravariant bilinear form on them.

We set

$$(\mathfrak{g}_\varepsilon)_\geq := (\mathfrak{g}_\varepsilon)_+ \oplus (\mathfrak{g}_\varepsilon)_0.$$

For each $\varepsilon \in \{\frac{1}{2}, 0\}$, we define Verma-type modules as follows.

First, we define Verma-type modules for the NS sector, i.e., $\varepsilon = \frac{1}{2}$.

For $(z, h, q) \in \mathbb{C}^3 \cong ((\mathfrak{g}_{\frac{1}{2}})_0^{\bar{0}})^*$, let $\mathbb{C}_{z,h,q} := \mathbb{C}\mathbf{1}_{z,h,q}$ be the $(\mathfrak{g}_{\frac{1}{2}})_0 = (\mathfrak{g}_{\frac{1}{2}})_0^{\bar{0}}$-module defined by

$$L_0.\mathbf{1}_{z,h,q} = h\mathbf{1}_{z,h,q}, \quad I_0.\mathbf{1}_{z,h,q} = q\mathbf{1}_{z,h,q}, \quad c.\mathbf{1}_{z,h,q} = z\mathbf{1}_{z,h,q}, \quad \deg \mathbf{1}_{z,h,q} := \bar{0}.$$

As usual, one can define $(\mathfrak{g}_{\frac{1}{2}})_\geq$-module structure via

$$(\mathfrak{g}_{\frac{1}{2}})_+.\mathbf{1}_{z,h,q} := \{0\}.$$

The Verma-type module $M_{\frac{1}{2}}(z, h, q)$ is defined as the induced module

$$M_{\frac{1}{2}}(z, h, q) := \mathrm{Ind}_{(\mathfrak{g}_{\frac{1}{2}})_\geq}^{\mathfrak{g}_{\frac{1}{2}}} \mathbb{C}_{z,h,q}.$$

Second, we define Verma-type modules for the R sector, i.e., $\varepsilon = 0$.

For $(z, h, q) \in \mathbb{C}^3 \cong ((\mathfrak{g}_0)_0^{\bar{0}})^*$ and $\sigma \in \{\pm\}$, let

$$V_{z,h,q}^\sigma := \mathbb{C}\mathbf{1}_{z,h,q}^{\sigma;\bar{0}} \oplus \mathbb{C}\mathbf{1}_{z,h,q}^{\sigma;\bar{1}}, \quad \deg \mathbf{1}_{z,h,q}^{\sigma;\tau} := \tau \in \mathbb{Z}_2$$

be the $(\mathfrak{g}_0)_\geq$-module defined by

$$\begin{aligned} L_0.v := hv, \quad c.v := zv, \quad (\mathfrak{g}_0)_+.v := \{0\} \quad (v \in V_{z,h,q}^\sigma), \\ I_0.\mathbf{1}_{z,h,q}^{\sigma;\bar{0}} := \left(q + \frac{1}{2}\sigma\right)\mathbf{1}_{z,h,q}^{\sigma;\bar{0}}, \\ G_0^\tau.\mathbf{1}_{z,h,q}^{\sigma;\bar{0}} := \delta_{\tau,-\sigma}\mathbf{1}_{z,h,q}^{\sigma;\bar{1}}, \quad G_0^\tau.\mathbf{1}_{z,h,q}^{\sigma;\bar{1}} := 2\left(h - \frac{1}{24}z\right)\delta_{\tau,\sigma}\mathbf{1}_{z,h,q}^{\sigma;\bar{0}}. \end{aligned}$$

The Verma-type module $M_0(z, h, q; \sigma)$ is defined by

$$M_0(z, h, q; \sigma) := \mathrm{Ind}_{(\mathfrak{g}_0)_\geq}^{\mathfrak{g}_0} V_{z,h,q}^\sigma.$$

Remark 2.1. Our definition of the Verma-type modules for the R sector is motivated by the following observation. For $(z,h,q)\in\mathbb{C}^3$, let $\mathbb{C}_{z,h,q}:=\mathbb{C}\mathbf{1}_{z,h,q}$ be the $(\mathfrak{g}_0)_0^{\bar{0}}$-module defined by

$$L_0.\mathbf{1}_{z,h,q}=h\mathbf{1}_{z,h,q},\quad I_0.\mathbf{1}_{z,h,q}=q\mathbf{1}_{z,h,q},\quad c.\mathbf{1}_{z,h,q}=z\mathbf{1}_{z,h,q},\quad \deg\mathbf{1}_{z,h,q}:=\bar{1}.$$

Set

$$W_{z,h,q}:=\mathrm{Ind}_{(\mathfrak{g}_0)_0^{\bar{0}}}^{(\mathfrak{g}_0)_0}\mathbb{C}_{z,h,q},$$
$$W^{\sigma}_{z,h,q}:=\mathbb{C}G_0^{\sigma}\mathbf{1}_{z,h,q}\oplus\mathbb{C}G_0^{-\sigma}G_0^{\sigma}\mathbf{1}_{z,h,q}\quad(\sigma\in\{\pm\}).$$

The $(\mathfrak{g}_0)_0$-module structure of $W_{z,h,q}$ can be summarized as follows:

1. If $h\neq\frac{1}{24}z$, then we have

$$W_{z,h,q}\cong W^{+}_{z,h,q}\oplus W^{-}_{z,h,q}.$$

2. If $h=\frac{1}{24}z$, then we have the following Jordan – Hölder series: $(\sigma\in\{\pm\})$,

$$W_{z,h,q}\supsetneq W^{+}_{z,h,q}+W^{-}_{z,h,q}\supsetneq W^{\sigma}_{z,h,q}\supsetneq\mathbb{C}G_0^{+}G_0^{-}\mathbf{1}_{z,h,q}\supsetneq\{0\}.$$

Several versions of Verma-type modules were considered in literatures (see, e.g., [9] for, what they call, chiral Verma modules and [6] for some other Verma-type modules over the topological $N=2$ superconformal algebra).

Now, we define a contravariant bilinear form on Verma-type modules as follows. By the Poincaré – Birkhoff – Witt Theorem, we have

$$U(\mathfrak{g}_\varepsilon)=U((\mathfrak{g}_\varepsilon)_0)\oplus\{U(\mathfrak{g}_\varepsilon)(\mathfrak{g}_\varepsilon)_+ +(\mathfrak{g}_\varepsilon)_- U(\mathfrak{g}_\varepsilon)\},$$

and let

$$\pi:U(\mathfrak{g}_\varepsilon)\twoheadrightarrow U((\mathfrak{g}_\varepsilon)_0)$$

be the canonical projection with respect to the above decomposition.
In the case $\varepsilon=0$, let

$$U((\mathfrak{g}_0)_0)=U((\mathfrak{g}_0)_0)^{\bar{0}}\oplus U((\mathfrak{g}_0)_0)^{\bar{1}}$$

be the decomposition with respect to the $\mathbb{Z}/2\mathbb{Z}$ gradation, and let

$$\pi^{\bar{0}}:U((\mathfrak{g}_0)_0)\twoheadrightarrow U((\mathfrak{g}_0)_0)^{\bar{0}}$$

be the canonical projection with respect to the above decomposition. For $\sigma\in\{\pm\}$, let

$$\pi_\sigma:U((\mathfrak{g}_0)_0)^{\bar{0}}\twoheadrightarrow S((\mathfrak{g}_0)_0^{\bar{0}})=\mathbb{C}[((\mathfrak{g}_0)_0^{\bar{0}})^*]$$

be the canonical projection with respect to the decomposition

$$U((\mathfrak{g}_0)_0)^{\bar{0}} = S((\mathfrak{g}_0)_0^{\bar{0}}) \oplus S((\mathfrak{g}_0)_0^{\bar{0}}) G_0^{-\sigma} G_0^{\sigma}.$$

A contravariant form on a Verma-type module is defined as follows:

NS sector: we define the contravariant form $\langle \cdot, \cdot \rangle_{z,h,q}$ on $M_{\frac{1}{2}}(z, h, q)$ by

$$\langle X.(1 \otimes \mathbf{1}_{z,h,q}), Y.(1 \otimes \mathbf{1}_{z,h,q}) \rangle_{z,h,q} := \pi(\theta(X)Y)(z, h, q), \quad (X, Y \in U(\mathfrak{g}_{\frac{1}{2}})).$$

R sector: we define the contravariant form $\langle \cdot, \cdot \rangle^{\sigma}_{z,h,q}$ on $M_0(z, h, q; \sigma)$ by

$$\langle X.(1 \otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q}), Y.(1 \otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q}) \rangle^{\sigma}_{z,h,q} := \left((\pi_\sigma \circ \pi^{\bar{0}} \circ \pi)(\theta(X)Y) \right) \left(z, h, q + \frac{1}{2}\sigma \right), \quad (X, Y \in U(\mathfrak{g}_0)).$$

Note that these forms are Hermitian. The above contravariant forms can be defined iff the weights are real, i.e., we have the next lemma:

LEMMA 2.1. *The above contravariant forms are well defined if and only if* $z, h, q \in \mathbb{R}$.

This lemma can be shown by applying the contravariance to L_0, c and I_0. Hereafter, we restrict ourselves to the case where the above contravariant form is well defined.

A Verma-type module $M_0(z, h, q; \sigma)$ over $\mathfrak{g}_0$ can be regarded as a $\mathfrak{g}_{\frac{1}{2}}$-module by Proposition 2.4. In the remaining section, we will study their relation.

The following proposition is a simple corollary of Proposition 2.4:

PROPOSITION 2.5. *As* $\mathfrak{g}_{\frac{1}{2}}$*-module, we have*

$$\Phi^*_{-\frac{1}{2}\sigma} M_0(z, h, q; \sigma) \cong M_{\frac{1}{2}} \left(z, h - \frac{1}{2}\sigma q + \frac{1}{24} z, q + \frac{1}{2}\sigma - \frac{1}{6}\sigma z \right),$$

where $\sigma \in \{\pm\}$ *and the morphism* $\Phi_{-\frac{1}{2}\sigma}$ *is regarded as a map*

$$\mathfrak{g}_{\frac{1}{2}} \longrightarrow \mathfrak{g}_0.$$

Proof. We have only to check that the map $M_{\frac{1}{2}}(z, h - \frac{1}{2}\sigma q + \frac{1}{24} z, q + \frac{1}{2}\sigma - \frac{1}{6}\sigma z) \longrightarrow M_0(z, h, q; \sigma)$ defined by

$$x \left(1 \otimes \mathbf{1}_{z, h - \frac{1}{2}\sigma q + \frac{1}{24} z, q + \frac{1}{2}\sigma - \frac{1}{6}\sigma z} \right) \longmapsto \Phi_{-\frac{1}{2}\sigma}(x)(1 \otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q}) \quad (x \in U((\mathfrak{g}_{\frac{1}{2}})_-)),$$

is an isomorphism. This can be done by direct computations. □

Moreover, the above isomorphism is an isometry, i.e., we have the next formula:

LEMMA 2.2. *For* $x, y \in U((\mathfrak{g}_{\frac{1}{2}})_{-})$, *we have*

$$\langle x.v, y.v\rangle_{z,h-\frac{1}{2}\sigma q+\frac{1}{24}z,q+\frac{1}{2}\sigma-\frac{1}{6}\sigma z} = \\ = \langle \Phi_{-\frac{1}{2}\sigma}(x).(1\otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q}), \Phi_{-\frac{1}{2}\sigma}(y).(1\otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q})\rangle^{\sigma}_{z,h,q}.$$

where we set $v := 1\otimes \mathbf{1}_{z,h-\frac{1}{2}\sigma q+\frac{1}{24}z,q+\frac{1}{2}\sigma-\frac{1}{6}\sigma z}$.

Proof. We first remark the following commutativity:

$$\Phi_{-\frac{1}{2}\sigma}\circ\theta = \theta\circ\Phi_{-\frac{1}{2}\sigma}, \quad (\pi_\sigma\circ\pi^{\bar{0}}\circ\pi)\circ\Phi_{-\frac{1}{2}\sigma} = \Phi_{-\frac{1}{2}\sigma}\circ\pi.$$

Hence, by definition, we have

$$\begin{aligned}
&\left\langle \Phi_{-\frac{1}{2}\sigma}(x).(1\otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q}), \Phi_{-\frac{1}{2}\sigma}(y).(1\otimes \mathbf{1}^{\sigma;\bar{0}}_{z,h,q})\right\rangle^{\sigma}_{z,h,q} = \\
&= \left\{(\pi_\sigma\circ\pi^{\bar{0}}\circ\pi)\theta(\Phi_{-\frac{1}{2}\sigma}(x))\Phi_{-\frac{1}{2}\sigma}(y)\right\}\left(z, h, q+\frac{1}{2}\sigma\right) = \\
&= \left\{(\pi_\sigma\circ\pi^{\bar{0}}\circ\pi)\Phi_{-\frac{1}{2}\sigma}(\theta(x)y)\right\}\left(z, h, q+\frac{1}{2}\sigma\right) = \\
&= \Phi_{-\frac{1}{2}\sigma}(\pi(\theta(x)y))\left(z, h, q+\frac{1}{2}\sigma\right) = \\
&= \pi(\theta(x)y)\left(z, h-\frac{1}{2}\sigma q+\frac{1}{24}z, q+\frac{1}{2}\sigma-\frac{1}{6}\sigma z\right) = \\
&= \langle x.v, y.v\rangle_{z,h-\frac{1}{2}\sigma q+\frac{1}{24}z,q+\frac{1}{2}\sigma-\frac{1}{6}\sigma z}.
\end{aligned}$$

□

Therefore, we have only to classify the unitarizable modules for one sector. Below, we restrict ourselves to the NS sector.

3. Determinant Formulae

In this section, we will recall the determinant formulae of Verma-type modules for the NS sector and a technical lemma which is used to prove the determinant formulae and which will be used later.

3.1. DETERMINANT FORMULAE

In this subsection, we recall the determinant formulae of Verma-type modules for the NS sector.

Verma-type modules admit the weight space decomposition, i.e., we have

$$M_{\frac{1}{2}}(z,h,q)=\bigoplus_{\substack{m\in\frac{1}{2}\mathbb{Z}_{\geq 0}\\ n\in\mathbb{Z}}} M_{\frac{1}{2}}(z,h,q)_{m,n},$$

$$M_{\frac{1}{2}}(z,h,q)_{m,n}:=\{u|L_0.u=(h+m)u,\ \ I_0.u=(q+n)u\}.$$

By definition and the contravariance of the bilinear form introduced in Section 2.4, different weight subspaces are perpendicular. Thus, it is enough to restrict the bilinear form to each weight subspace. Set

$$\langle\cdot,\cdot\rangle_{z,h,q;m,n}:=\langle\cdot,\cdot\rangle_{z,h,q}\big|_{M_{\frac{1}{2}}(z,h,q)_{m,n}\times M_{\frac{1}{2}}(z,h,q)_{m,n}},\qquad \left(m\in\frac{1}{2}\mathbb{Z}_{\geq 0},\,n\in\mathbb{Z}\right).$$

By the Poincaré–Birkhoff–Witt theorem, each weight subspace is of finite dimension and hence we may speak of the determinant of the matrix of pairings of a basis. These determinants are determined up to a scalar depending on the choice of a basis. We denote the determinant of $\langle\cdot,\cdot\rangle_{z,h,q;m,n}$ by $\det(z,h,q)_{m,n}$. We will state these determinants, below.

Let us first introduce some partition functions.

We define $\{P(m,n),\,\tilde{P}(m,n;k)\}_{m\in\frac{1}{2}\mathbb{Z}_{\geq 0},n\in\mathbb{Z},k\in\frac{1}{2}+\mathbb{Z}}\subset\mathbb{Z}_{\geq 0}$ by

$$\sum_{\substack{m\in\frac{1}{2}\mathbb{Z}_{\geq 0}\\ n\in\mathbb{Z}}} P(m,n)x^m y^n=\prod_{k=1}^{\infty}\frac{(1+x^{k-\frac{1}{2}}y)(1+x^{k-\frac{1}{2}}y^{-1})}{(1-x^k)^2},$$

$$\sum_{m,n}\tilde{P}(m,n;k)x^m y^n=\frac{1}{1+x^{|k|}y^{\mathrm{sgn}(k)}}\sum_{m,n}P(m,n)x^m y^n.$$

The determinant formulae, which has been presented in [1] and has been proved in [11], can be stated as follows:

THEOREM 3.1. *For* $m\in\frac{1}{2}\mathbb{Z}_{>0},\,n\in\mathbb{Z}$,

$$\det(z,h,q)_{m,n}=C_{m,n}\prod_{\substack{r\in\mathbb{Z}_{>0},s\in 2\mathbb{Z}_{>0}\\ 1\leq rs\leq 2m}}\left\{f_{r,s}(z,h,q)\right\}^{P(m-\frac{1}{2}rs,n)}\times$$
$$\times\prod_{k\in\frac{1}{2}+\mathbb{Z}}\{g_k(z,h,q)\}^{\tilde{P}(m-|k|,n-\mathrm{sgn}(k);k)},$$

where we set

$$f_{r,s}(z,h,q):=2\left(\frac{1}{3}z-1\right)h+\frac{1}{4}\left[r(\frac{1}{3}z-1)+s\right]^2-q^2-\frac{1}{4}\left(\frac{1}{3}z-1\right)^2,$$
$$g_k(z,h,q):=2h-2kq+\left(\frac{1}{3}z-1\right)\left(k^2-\frac{1}{4}\right),$$

and $\{C_{m,n}\}\subset\mathbb{R}_{>0}$ *are constants depending on the choice of a basis.*

Remark 3.1. 1. For $(m,n)\in\frac{1}{2}\mathbb{Z}_{\geq 0}\times\mathbb{Z}$, it follows that $M_{\frac{1}{2}}(z,h,q)_{m,n}\neq\{0\}$ if and only if $m\geq\frac{1}{2}n^2$.

2. In particular, for $n\in\mathbb{Z}\backslash\{0\}$, one has

$$\det(z,h,q)_{\frac{1}{2}n^2,n}=C_{\frac{1}{2}n^2,n}\prod_{\substack{k\in\frac{1}{2}+\mathbb{Z}\\ 0<\mathrm{sgn}(n)k<|n|}} g_k(z,h,q),$$

for some $C_{\frac{1}{2}n^2,n}\in\mathbb{R}_{>0}$.

We remark that the determinant formulas for some Verma-type modules over the topological $N=2$ superconformal algebra are obtained by Dörrzapf and Gota-Rivera [4].

3.2. TECHNICAL LEMMA

Here, we state a technical lemma which is used to estimate the degree of the determinant as a polynomial in h.

We define the set parametrizing a PBW basis of $U((\mathfrak{g}_{\frac{1}{2}})_-)$ as follows. Set

$$\mathcal{I}:=\left\{(i,\varepsilon)\,\middle|\, i\in\frac{1}{2}\mathbb{Z}_{>0},\quad \begin{array}{cc}\varepsilon\in\{\pm\} & i\in\frac{1}{2}+\mathbb{Z},\\ \varepsilon\in\{1,2\} & i\in\mathbb{Z}.\end{array}\right\}.$$

and define the order $\prec$ on $\mathcal{I}$ by

$$(i,\varepsilon)\prec(i',\varepsilon')\quad\Longleftrightarrow\quad\begin{cases} i<i' \\ \text{or} \\ i=i' \text{ and } (\varepsilon,\varepsilon')\in\{(-,+),(1,2)\}.\end{cases}$$

For $m\in\frac{1}{2}\mathbb{Z}_{>0}$ and $n\in\mathbb{Z}$ satisfying $m-\frac{1}{2}n^2\in\mathbb{Z}_{\geq 0}$, we set

$$\mathcal{P}_{m,n}:=\left\{((i_k,\varepsilon_k),\cdots,(i_1,\varepsilon_1))\,\middle|\,\begin{array}{c} k\in\mathbb{Z}_{>0},\quad (i_j,\varepsilon_j)\in\mathcal{I},\\ \sum_{j=1}^k i_j=m,\quad \sharp\{j|\varepsilon_l=+\}-\sharp\{j|\varepsilon_j=-\}=n,\\ (i_j,\varepsilon_j)\preceq(i_{j+1},\varepsilon_{j+1}),\\ (i_j,\varepsilon_j)\neq(i_{j+1},\varepsilon_{j+1}) \text{ if } i_j\in\frac{1}{2}+\mathbb{Z}.\end{array}\right\}.$$

The set $\mathcal{P}_{m,n}$ parametrizes a PBW basis of

$$U\left((\mathfrak{g}_{\frac{1}{2}})_-\right)_{m,n}:=\left\{x\in U((\mathfrak{g}_{\frac{1}{2}})_-)|[L_0,x]=mx,\ [I_0,x]=nx\right\}.$$

Indeed, for $(i,\varepsilon)\in\mathcal{I}$ and $\mathbb{I}=((i_k,\varepsilon_k),\ldots,(i_1,\varepsilon_1))\in\mathcal{P}_{m,n}$, set

$$x_{(i,\varepsilon)}:=\begin{cases} I_{-i} & i\in\mathbb{Z}_{>0},\ \varepsilon=1,\\ L_{-i} & i\in\mathbb{Z}_{>0},\ \varepsilon=2,\\ G^{\varepsilon}_{-i} & i\in\frac{1}{2}+\mathbb{Z}_{\geq 0},\end{cases}$$

$$e_{\mathbb{I}}:=x_{(i_k,\varepsilon_k)}\cdots x_{(i_1,\varepsilon_1)}.$$

The family $\{e_{\mathbb{I}}\}_{\mathbb{I}\in\mathcal{P}_{m,n}} \subset U((\mathfrak{g}_{\frac{1}{2}})_-)_{m,n}$ forms a PBW basis. For simplicity, we set $v_{\mathbb{I}} := e_{\mathbb{I}}.(1\otimes \mathbf{1}_{z,h,q}) \in M_{\frac{1}{2}}(z,h,q)$.

For $\mathbb{I} = ((i_k,\varepsilon_k),\ldots,(i_1,\varepsilon_1)) \in \mathcal{P}_{m,n}$, we also set

$$\rho(\mathbb{I}) := k - \sharp\{j\,|\,\varepsilon_j = 1\}, \quad \rho^c(\mathbb{I}) := \sharp\{j\,|\,\varepsilon_j = 1\}.$$

The next lemma can be proved by direct calculations.

LEMMA 3.1. *Let $\mathbb{I},\mathbb{I}_1,\mathbb{I}_2 \in \mathcal{P}_{m,n}$. For $h,z \in \mathbb{R}_{>0}$ such that h,z and $\frac{h}{z}$ are big enough, one has*

1. $\langle v_{\mathbb{I}}, v_{\mathbb{I}}\rangle_{z,h,q} = c_{\mathbb{I}} h^{\rho(\mathbb{I})} z^{\rho^c(\mathbb{I})}(1+o(1)) \quad \exists\ c_{\mathbb{I}} \in \mathbb{R}_{>0}$,
2. $\langle v_{\mathbb{I}_1}, v_{\mathbb{I}_2}\rangle_{z,h,q} = o(h^{\frac{\rho(\mathbb{I}_1)+\rho(\mathbb{I}_2)}{2}} z^{\frac{\rho^c(\mathbb{I}_1)+\rho^c(\mathbb{I}_2)}{2}}) \quad \mathbb{I}_1 \neq \mathbb{I}_2$.

4. Unitarity

In this section, we state and prove the classification theorem of the unitarizable highest weight modules over $\mathfrak{g}_{\frac{1}{2}}$.

4.1. MAIN RESULTS

For $m \in \mathbb{R}$ satisfying $m \geq 2$ and $r,s,t \in \mathbb{R}$, we set

$$z_m := 3\left(1-\frac{2}{m}\right), \quad h_{r,s}(m) := \frac{4rs-1}{4m}, \quad q_t(m) := \frac{t}{m}. \tag{1}$$

The main theorem we are going to prove is as follows:

THEOREM 4.1. [1] *The form $\langle\cdot,\cdot\rangle_{z,h,q}$ on $M_{\frac{1}{2}}(z,h,q)$ is positive semi-definite only if one of the following three conditions are satisfied*:

1. *$z \geq 3$ and (z,h,q) satisfies $g_n(z,h,q) \geq 0$ for all $n \in \frac{1}{2}+\mathbb{Z}$.*
2. *$z \geq 3$ and (z,h,q) satisfies $g_n(z,h,q) = 0$ and $g_{n+\mathrm{sgn}(n)}(z,h,q) < 0$ for some $n \in \frac{1}{2}+\mathbb{Z}$ and $f_{1,2}(z,h,q) \geq 0$.*
3. *$z = z_m, h = h_{j,k}(m)$ and $q = q_{j-k}(m)$ for some $m \in \mathbb{Z}_{\geq 2}$ and $j,k \in \frac{1}{2}+\mathbb{Z}$ satisfying $0 < j,k,\ j+k \leq m-1$.*

Remark 4.1. As we will see below, the first condition in Theorem 4.1 is sufficient. Moreover, it follows from the result obtained in [3] that the third condition in Theorem 4.1 is also sufficient.

For $k \in \mathbb{Z}$ and $(r,s) \in \mathbb{Z}_{>0} \times 2\mathbb{Z}_{>0}$, we set

$$f^{\mathrm{R}}_{r,s}(z,h,q) := 2\left(\frac{1}{3}z-1\right)\left(h-\frac{1}{24}z\right) - q^2 + \frac{1}{4}\left[r\left(\frac{1}{3}z-1\right)+s\right]^2,$$

$$g^{\mathrm{R}}_k(z,h,q) := 2h - 2kq + \left(\frac{1}{3}z-1\right)\left(k^2-\frac{1}{4}\right) - \frac{1}{4}.$$

The next theorem is a corollary of the Theorem 4.1.

THEOREM 4.2. *The form* $\langle\cdot,\cdot\rangle^{\sigma}_{z,h,q}$ *on* $M_0(z,h,q;\sigma)$ *is positive semi-definite only if one of the following three conditions are satisfied*:

1. $z\geq 3$ *and* $(z,h,q;\sigma)$ *satisfies* $g^{\mathrm{R}}_n(z,h,q)\geq 0$ *for all* $n\in\mathbb{Z}$.
2. $z\geq 3$ *and* $(z,h,q;\sigma)$ *satisfies* $g^{\mathrm{R}}_n(z,h,q)=0$ *and* $g^{\mathrm{R}}_{n+sgn(n-\frac{1}{2}\sigma)}(z,h,q)<0$ *for some* $n\in\mathbb{Z}$ *and* $f^{\mathrm{R}}_{1,2}(z,h,q)\geq 0$.
3. $z=z_m$ *for some* $m\in\mathbb{Z}_{\geq 2}$ *and*

$$h=\frac{jk}{m}+\frac{1}{24}z_m,\quad q=\frac{j-k}{m},$$

for some $j,k\in\mathbb{Z}$ *such that* $j-\frac{1}{2}\sigma, k+\frac{1}{2}\sigma>0$ *and* $j+k\leq m-1$.

Indeed, by Proposition 2.5 and Lemma 2.2, one sees that a unitarizable $\mathfrak{g}_{\frac{1}{2}}$-module corresponds to a unitarizable $\mathfrak{g}_0$ module. Hence, the remaining part of the article is devoted to the proof of Theorem 4.1.

For simplicity, we let $|\mathrm{vac}\rangle:=1\otimes\mathbf{1}_{z,h,q}\in M_{\frac{1}{2}}(z,h,q)$ be a highest weight vector.

LEMMA 4.1. *If the form* $\langle\cdot,\cdot\rangle_{z,h,q}$ *is positive semi-definite, then one has* $h\geq 0$ *and* $z\geq 0$. *In particular, if* $q\neq 0$, *one has* $z>0$.

Proof. One has

$$\langle I_{-1}|\mathrm{vac}\rangle, I_{-1}|\mathrm{vac}\rangle\rangle_{z,h,q}=\langle|\mathrm{vac}\rangle,[I_1,I_{-1}]|\mathrm{vac}\rangle\rangle_{z,h,q}=\frac{1}{3}z\geq 0,$$
$$\langle L_{-1}|\mathrm{vac}\rangle, L_{-1}|\mathrm{vac}\rangle\rangle_{z,h,q}=\langle|\mathrm{vac}\rangle,[L_1,L_{-1}]|\mathrm{vac}\rangle\rangle_{z,h,q}=2h\geq 0,$$

which implies $z\geq 0$ and $h\geq 0$. For $n\in\frac{1}{2}+\mathbb{Z}_{\geq 0}$, one has

$$\begin{aligned}\langle G^{\mp}_{-n}|\mathrm{vac}\rangle, G^{\mp}_{-n}|\mathrm{vac}\rangle\rangle_{z,h,q}=\langle|\mathrm{vac}\rangle,[G^{\pm}_n,G^{\mp}_{-n}]|\mathrm{vac}\rangle\rangle_{z,h,q}=\\ =2h\pm 2qn+\frac{1}{3}\left(n^2-\frac{1}{4}\right)z\geq 0,\end{aligned}$$

which implies $z>0$ if $q\neq 0$. □

4.2. PROOF I: THE CASE $z\geq 3$

In this subsection, we prove Theorem 4.1 in the case when $z\geq 3$. We remark that if the form $\langle\cdot,\cdot\rangle_{z,h,q}$ is positive semi-definite, then one has

$$(\mathbf{P}):\det(z,h,q)_{m,n}\geq 0 \text{ for any } (m,n)\in\frac{1}{2}\mathbb{Z}_{\geq 0}\times\mathbb{Z}.$$

The next lemma treats the simplest case following the above-guiding principle:

LEMMA 4.2. *Suppose that* $z \geq 3$. *If one has* $g_n(z,h,q) \geq 0$ *for any* $n \in \frac{1}{2} + \mathbb{Z}$, *then the form* $\langle \cdot, \cdot \rangle_{z,h,q}$ *is positive semi-definite.*

Proof. We first show that under the assumption of the lemma, one has $f_{r,s}(z,h,q) > 0$ for any $(r,s) \in \mathbb{Z}_{>0} \times 2\mathbb{Z}_{>0}$.

Suppose that $z > 3$, there exists $k \in \frac{1}{2} + \mathbb{Z}$ such that

$$\left| k\left(\frac{1}{3}z - 1\right) - q \right| \leq \frac{1}{2}\left(\frac{1}{3}z - 1\right).$$

It follows that

$$\begin{aligned} & f_{r,s}(z,h,q) - \left(\frac{1}{3}z - 1\right) g_k(z,h,q) = \\ & = \frac{1}{4}\left[r\left(\frac{1}{3}z - 1\right) + s \right]^2 - \left[k\left(\frac{1}{3}z - 1\right) - q \right]^2 \geq \\ & \geq \frac{1}{4}\left\{ \left[r\left(\frac{1}{3}z - 1\right) + s \right]^2 - \left(\frac{1}{3}z - 1\right)^2 \right\} > 0. \end{aligned}$$

If $z = 3$, the assumption of the lemma implies $q = 0$. Hence, one has $f_{r,s}(z,h,q) = \frac{1}{4}s^2 > 0$.

Now, for a fixed q and sufficiently big h, z and $\frac{h}{z}$, Lemma 3.1 implies that $\langle \cdot, \cdot \rangle_{z,h,q}$ is positive definite. Hence, by the connectivity of the domain under consideration, the lemma follows. □

Now, we consider when the condition (**P**) is satisfied besides the case of Lemma 4.2.

By Remark 3.1, one sees that if the condition (**P**) is satisfied and there exists $n \in \frac{1}{2} + \mathbb{Z}$ such that $g_n(z,h,q) = 0$, then one should have

$$g_k(z,h,q) \geq 0 \qquad \begin{cases} k < n & n > 0, \\ k > n & n < 0. \end{cases}$$

In this case, it follows that the condition $g_{n+\mathrm{sgn}(n)}(z,h,q) \geq 0$ is included in the case treated in Lemma 4.2. Hence, we consider the condition $g_{n+\mathrm{sgn}(n)}(z,h,q) < 0$, below. We remark that this condition does not violate the condition (**P**) because of Theorem 3.1 and the next lemma:

LEMMA 4.3. *For* $(m,n) \in \frac{1}{2}\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ *and* $k \in \frac{1}{2} + \mathbb{Z}$, *one has*

$$\widetilde{P}(m - |k|, n - \mathrm{sgn}(k); k) \geq \widetilde{P}(m - (|k|+1), n - \mathrm{sgn}(k); k + \mathrm{sgn}(k)).$$

Proof. For $l \in \frac{1}{2}+\mathbb{Z}$, one has the next relation

$$\widetilde{P}(m,n;l)+\widetilde{P}(m-|l|,n-\mathrm{sgn}(l);l)=P(m,n),$$

by definition. Hence, it is equivalent to show

$$\widetilde{P}(m,n;k)\leq \widetilde{P}(m,n;k+\mathrm{sgn}(k)).$$

By definition, the series

$$\begin{aligned}&\sum_{m,n}\widetilde{P}(m,n;k+\mathrm{sgn}(k))x^m y^n-\sum_{m,n}\widetilde{P}(m,n;k)x^m y^n=\\&=\left(\frac{1}{1+x^{|k|+1}y^{\mathrm{sgn}(k)}}-\frac{1}{1+x^{|k|}y^{\mathrm{sgn}(k)}}\right)\prod_{n>0}\frac{(1+x^{n-\frac{1}{2}}y)(1+x^{n-\frac{1}{2}}y^{-1})}{(1-x^n)^2}=\\&=\frac{x^{|k|}(1-x)y^{\mathrm{sgn}(k)}}{(1+x^{|k|}y^{\mathrm{sgn}(k)})(1+x^{|k|+1}y^{\mathrm{sgn}(k)})}\prod_{n>0}\frac{(1+x^{n-\frac{1}{2}}y)(1+x^{n-\frac{1}{2}}y^{-1})}{(1-x^n)^2}\end{aligned}$$

has non-negative coefficients, from which the lemma follows. □

We notice that for $n\in\mathbb{Z}\setminus\{0\}$, the next formula follows from Theorem 3.1:

$$\begin{aligned}&\det(z,h,q)_{\frac{1}{2}n^2+1,n}=\\&=C_{\frac{1}{2}n^2+1,n}f_{1,2}(z,h,q)\prod_{\substack{k\in\frac{1}{2}+\mathbb{Z}\\0<\mathrm{sgn}(n)k<|n|+1}}\{g_k(z,h,q)\}^{\widetilde{P}(\frac{1}{2}n^2+1-|k|,n-\mathrm{sgn}(k);k)},\end{aligned}\tag{2}$$

for some $C_{\frac{1}{2}n^2+1,n}\in\mathbb{R}_{>0}$. Remark that this formula holds for any (z,h,q). Hence, we assume that $f_{1,2}(z,h,q)\geq 0$ holds. Under this assumption, one has

$$f_{r,s}(z,h,q)-f_{1,2}(z,h,q)=\frac{1}{4}\left\{\left[r\left(\frac{1}{3}z-1\right)+s\right]^2-\left[\left(\frac{1}{3}z-1\right)+2\right]^2\right\}\geq 0,$$

which implies the condition (**P**) by Theorem 3.1.

Thus, we have shown that, for $z\geq 3$, the first and the second conditions in Theorem 4.1 are necessary. Moreover, Lemma 4.2 implies that the first condition is also sufficient.

4.3. PROOF II: THE CASE $0\leq z<3$

Here, in this subsection, we assume that there exists $m\in\mathbb{R}_{\geq 2}$ such that $z=z_m$ (see (1) for the definition of z_m). By Lemma 4.1, we also assume $h\geq 0$.

The next lemma is a crucial step:

LEMMA 4.4. *Suppose that there exists* $\sigma \in \{\pm 1\}$ *such that*

$$g_k(z,h,q) \neq 0 \quad \forall\ \sigma k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}.$$

Then, the form $\langle \cdot, \cdot \rangle_{z,h,q}$ *is not positive semi-definite.*

Proof. By definition, one has

$$g_k(z_m,h,q) = -\frac{2}{m}\left(k^2 - \frac{1}{4}\right) - 2qk + 2h,$$

which implies that $g_k(z,h,q) < 0$ for sufficiently big $|k|$. Hence, by Remark 3.1, there exists $N \in \mathbb{Z}_{>0}$ such that $\det(z,h,q)_{\frac{1}{2}N^2,\sigma N} < 0$. □

This lemma implies

LEMMA 4.5. *Suppose that the form* $\langle \cdot, \cdot \rangle_{z,h,q}$ *is positive semi-definite. Then, there exists* $k \in \frac{1}{2} + \mathbb{Z}$ *such that*

$$h = \frac{1}{4m}\{4k(k+qm) - 1\}.$$

In addition, one has $qm \in \mathbb{Z}$.

Proof. By Lemma 4.4, there exists $k \in \frac{1}{2} + \mathbb{Z}$ such that $g_k(z_m,h,q) = 0$. Considering this as an equation of h and solving it, the first statement follows. To show the second statement, we remark

$$g_l(z,h,q) = 0 \quad \Longleftrightarrow \quad l^2 + qml - \left(mh + \frac{1}{4}\right) = 0.$$

Lemma 4.4 implies that this equation of l should have two solutions $l = k_\pm \in \frac{1}{2} + \mathbb{Z}$ which implies that $qm = -(k_+ + k_-) \in \mathbb{Z}$. □

By the assumption $h \geq 0$, one may assume that either $k, k+qm \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ or $k, k+qm \in -\frac{1}{2} + \mathbb{Z}_{\leq 0}$ is valid. Hence, hereafter, we assume that there exists $m \in \mathbb{R}_{\geq 2}$ and $j, k \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ such that $z = z_m, h = h_{j,k}(m)$ and $q = q_{j-k}(m)$, where $z_m, h_{j,k}(m), q_{j-k}(m)$ are defined in (1).

With this expression, one obtains

LEMMA 4.6. *If the form* $\langle \cdot, \cdot \rangle_{z,h,q}$ *is positive semi-definite, then one has* $0 < j+k \leq m-1$.

Proof. By definition, one has

$$f_{1,2}(z_m, h_{j,k}(m), q_{j-k}(m)) = \frac{(m-1)^2 - (j+k)^2}{m^2},$$

$$g_l(z_m, h_{j,k}(m), q_{j-k}(m)) = -\frac{2}{m}(l+j)(l-k).$$

Now, we assume that the form $\langle\cdot,\cdot\rangle_{z,h,q}$ is positive semi-definite and $j+k>m-1$. The latter condition implies $f_{1,2}(z_m, h_{j,k}(m), q_{j-k}(m))<0$.

By Theorem 3.1, one has, for some $C_{1,0}\in\mathbb{R}_{>0}$,

$$\det(z,h,q)_{1,0}=C_{1,0}f_{1,2}(z,h,q)g_{\frac{1}{2}}(z,h,q)g_{-\frac{1}{2}}(z,h,q),$$

which implies that if both $j\neq\frac{1}{2}$ and $k\neq\frac{1}{2}$ hold, then one has $\det(z,h,q)_{1,0}<0$ which contradicts to the assumption. Hence, we may assume that either $j=\frac{1}{2}$ or $k=\frac{1}{2}$ holds below. It follows from the condition $m\geq 2$ that $(j,k)\neq(\frac{1}{2},\frac{1}{2})$.

By (2), it follows that if $j=\frac{1}{2}$ and $k>\frac{3}{2}$, one has $\det(z,h,q)_{\frac{3}{2},1}<0$, and that if $j>\frac{3}{2}$ and $k=\frac{1}{2}$, one has $\det(z,h,q)_{\frac{3}{2},-1}<0$, which contradict to the assumption. Hence, it is enough to consider the cases $(j,k)=(\frac{1}{2},\frac{3}{2}),(\frac{3}{2},\frac{1}{2})$. In these cases, one has $m<3$ by assumption. Now, one has

$$\begin{cases}\langle G^-_{-\frac{3}{2}}|\mathrm{vac}\rangle, G^-_{-\frac{3}{2}}|\mathrm{vac}\rangle\rangle_{z,h,q}=2\left(1-\frac{3}{m}\right)<0 & \text{for } (j,k)=\left(\frac{1}{2},\frac{3}{2}\right),\\ \langle G^+_{-\frac{3}{2}}|\mathrm{vac}\rangle, G^+_{-\frac{3}{2}}|\mathrm{vac}\rangle\rangle_{z,h,q}=2\left(1-\frac{3}{m}\right)<0, & \text{for } (j,k)=\left(\frac{3}{2},\frac{1}{2}\right),\end{cases}$$

which are contradictions. □

Thus, if $m\in\mathbb{Z}_{\geq 2}$, this lemma implies that the third condition in Theorem 4.1 is necessary. Hence, to terminate the proof of Theorem 4.1, it is sufficient to prove

LEMMA 4.7. *Suppose that $m\in\mathbb{R}_{\geq 2}\backslash\mathbb{Z}$. Then, the $\langle\cdot,\cdot\rangle_{z_m,h_{j,k}(m),q_{j-k}(m)}$ is not positive semi-definite.*

Proof. By Lemma 4.6, it is sufficient to prove this lemma for $j,k\in\frac{1}{2}+\mathbb{Z}$ such that $0<j,k,\ j+k<m-1$. We write (z,h,q) in place of $(z_m,h_{j,k}(m),q_{j-k}(m))$ for simplicity.

Notice that for $n\in\mathbb{Z}_{>0}$, one has $M_{\frac{1}{2}}(z,h,q)_{\frac{1}{2}n^2,n}=\mathbb{C}G^+_{-(n-\frac{1}{2})}\cdots G^+_{-\frac{3}{2}}G^+_{-\frac{1}{2}}|\mathrm{vac}\rangle$, which together with Remark 3.1 implies that

$$w:=\begin{cases}|\mathrm{vac}\rangle & k=\frac{1}{2},\\ G^+_{-(k-1)}\cdots G^+_{-\frac{3}{2}}G^+_{-\frac{1}{2}}|\mathrm{vac}\rangle & k>\frac{1}{2},\end{cases}$$

satisfies $\langle w,w\rangle_{z,h,q}>0$. For $N\in\mathbb{Z}_{>0}$, if we set

$$w_N:=\frac{1}{N!}G^+_{-(k+N)}G^+_{-(k+N-1)}\cdots G^+_{-(k+1)}.w,$$

one can show the next formula by the induction on N:

$$\langle w_N,w_N\rangle_{z,h,q}=\left(\frac{2}{m}\right)^N\binom{m-j-k-1}{N}\langle w,w\rangle_{z,h,q}.$$

Hence, the assumption $m \notin \mathbb{Z}$ implies that there exists $N \in \mathbb{Z}_{>0}$ such that $\langle w_N, w_N \rangle_{z,h,q} < 0$. □

Therefore, we have completed the proof of Theorem 4.1.

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