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# Homogeneous Star Products

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**Abstract.** We give short proofs of results concerning homogeneous star products, of which S. Gutt's star product on the dual of a Lie algebra is a particular case.

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This is partly a review article; we give easy definitions and results concerning homogeneous star products of type $k > 0$. Homogeneous star products of type 1 were studied in [3], under the name "graded star products" (with a somewhat narrow symmetry condition that we will not use here); a typical example is the star product associated to the canonical Poisson bracket on the dual of a Lie algebra, and Section 3 concerning them should be considered as a review with easy proofs of some results of [1,3,4,6,9]. Homogeneous star products of type $>2$ are easily seen to be equivalent to trivial star products, and homogeneous star products of type 2 are equivalent to Moyal star products (Section 2). We do not study here homogeneous star products of type $k \leq 0$, which are not much less complicated than the general non-homogeneous case (neither is the study of homogeneous star products on a conic set not containing the origin).

## 1. Definitions

We first recall (cf. [2]) that a star product $B$ on a manifold $V$ is a formal series

$$B = \sum_{0}^{\infty} \hbar^n B_n \tag{1}$$

where $\hbar$ is the formal parameter and, for each integer $n$, $B_n$ is a bidifferential operator with smooth coefficients, i.e. in any system of smooth local coordinates we have $B_n(f, g) = \sum b_{n,\alpha,\beta}(\xi)\ \partial^\alpha f\, \partial^\beta g$.

This paper is dedicated to the memory of Moshe Flato.
I thank the referee for several judicious remarks improving the text.

It is required that $B$ defines an associative product with unit 1, i.e.

$$(f*g)*h=f*(g*h), \quad 1*f=f*1=f.$$

where we have used the notation, for $f=\sum \hbar^n f_n, g=\sum \hbar^n g_n$ formal series with smooth coefficients:

$$f*g=B(f,g)=\sum \hbar^n B_n(f,g)=B(\xi,\partial\otimes\partial)(f,g) \tag{2}$$

In operator notation:

$$B(B\otimes 1-1\otimes B)=\frac{1}{2}[B,B]=0 \tag{3}$$

where $[B,B]$ denotes the Gerstenhaber bracket [8] (see below).

In the sequel we will suppose that $V$ is a vector space.

DEFINITION 1. *A star product $B=\sum \hbar^n B_n$ on a vector space $V$ is homogeneous of type k, or k-homogeneous, if for each $n\geq 0$, $B_n$ is homogeneous of degree $-kn$ with respect to dilations $\xi\to\lambda\xi$ on $V$.*

Equivalently: $B$ is invariant by the dilations $f(\hbar,\xi)\mapsto H_\lambda f=f(\lambda^k\hbar,\lambda\xi)$ $(H_\lambda(B)(f,g)=H_\lambda(B(H_{\lambda^{-1}}f,H_{\lambda^{-1}}g))$. Equivalently again: it is invariant by the Lie derivation $L_{r\partial_r+k\hbar\partial_\hbar}$ ($r\partial_r=\sum \xi_j\partial_{\xi_j}$ denotes the radial derivation).

The domain of definition of $B$ imperatively includes the origin $0\in V$. Let us recall that a smooth homogeneous function is a homogeneous polynomial (of integral degree). Likewise a homogeneous multidifferential operator $P(f_1,\ldots,f_m)=\sum a_{\alpha_1,\ldots,\alpha_m}\partial^{\alpha_1}f_1\cdots\partial^{\alpha_m}f_m$ has polynomial coefficients. So we can as well suppose $V$ real or complex. In what follows we have also omitted the customary factor $\frac{1}{i}$ which is useless here. One might as well declare that the star product laws $B$ or Poisson brackets $c$ have complex coefficients, or that we are ultimately restricting to pure imaginary vectors of $V$.

If $B$ is a star product, the commutator law

$$C(f,g)=B(f,g)-B(g,f)$$

is a Lie bracket, and its leading coefficient

$$c(f,g)=B_1(f,g)-B_1(g,f) \tag{4}$$

is a Poisson bracket on $V$, homogeneous of degree $-k$ if $B$ is of type $k$.

## 2. Homogeneous Star Products of Type $k\geq 2$

DEFINITION 2 (equivalence). *A star product $B'$ is equivalent to $B$ (resp. homogeneous-equivalent, if $B, B'$ are k-homogeneous) if there exists an asymptotic operator $p=\sum_1^\infty \hbar^n p_n$ (resp. k-homogeneous, i.e. $p_n$ is homogeneous of degree $-kn$), such that $B'=e^p(B)$.*

*$B'=e^p(B)$ means $B'(f,g)=e^p(B(e^{-p}f,e^{-p}g))$ (so $e^p(1)=1$, $p(1)=0$).*

If $p = \hbar^N p_N + \cdots$ is of order $N$, and $B' = e^p(B)$, $B' - B$ is of order $N$ and its leading coefficient is

$$[p_N, \epsilon] = \delta(p_N) : (f, g) \mapsto p_N(fg) - p_N(f)g - f\, p_N(g). \tag{5}$$

Here $\epsilon \in \mathcal{D}_2$ is the multiplication operator: $\epsilon(f, g) = fg$; $[\,,\,]$ is the Gerstenhaber bracket between 0 and 1 cochains (1 and 2-differential operators), and $\delta$ is the Hochschild differential. Here we only use $[,]$ or $\delta$ for 0 or 1-cochains, as recalled below. For the general case we refer to [8], or to the very conceptual description of [10].

Note that if $p$ is a vector field, we have $[p, \epsilon] = 0$, and the leading term of $e^p B - B$ is $[p_N, B_1]$, the Lie derivative of $B_1$.

If $B' = B + \beta$, and $[B, B] = 0$, the associativity condition is

$$\frac{1}{2}[B', B'] = [B, \beta] + \frac{1}{2}[\beta, \beta] = 0 \tag{6}$$

where $[]$ is the Gerstenhaber bracket of 1-cochains:

$$[B, B'] = B(B' \otimes 1 - 1 \otimes B') + B'(B \otimes 1 - 1 \otimes B) \tag{7}$$

If $B' - B = \hbar^N \beta_N + \cdots$ is of order $N \geq 1$, we get for the leading term

$$[\epsilon, \beta_N] = \delta(\beta_N) = 0 \tag{8}$$

where $\delta$ is the Hochschild differential. It is well known (cf. [8]) that the $\delta$-cohomology of the Hochschild complex is canonically isomorphic to the space of multivectors: to an $m$-multivector $X_1 \wedge \cdots \wedge X_m$ corresponds the skew symmetric $m$-differential operator

$$(f_1, \ldots, f_m) \mapsto \det(\langle X_i, df_j \rangle).$$

and each $m$-cocycle $\gamma$ for the $\delta$-cohomology ($\delta(\gamma) = 0$) has the form

$$\gamma = \mathfrak{a}(\gamma) + \delta p = X_1 \wedge \cdots \wedge X_m + \delta p$$

where $\mathfrak{a}$ is the complete antisymetrization.

Thus we have $\beta_N = \beta' + \delta(p_N)$, where $\beta'$ is a 2-vector field, homogeneous of degree $-kN$, and we can choose choose $p_N$ of the same degree, replacing it if need be by the homogeneous part of degree $-kN$ in its Taylor expansion. Then $e^{\hbar^N p_N}$ is homogeneous of type $k$, and it follows from (5), (6) that we have $B' - e^{\hbar^N p_N}(B) = \hbar^N \beta' + \cdots$

Since the derivations $\partial_j$ are homogeneous of degree $-1$, the degree of a homogeneous 2-vector $c = \sum c_{ij}(x)\partial_i \wedge \partial_j$ is necessarily $\geq -2$: $c$ vanishes if the degree is $\leq -3$, and its coefficients are constant if the degree is $-2$; they are linear if the degree is $-1$. So if $k \geq 3$, resp. $k = 2$, $\beta_N$ will vanish for all $N \geq 1$, resp. $N \geq 2$. Then we can proceed by induction (successive approximations) and see that $B$ and $B'$ are homogeneous-equivalent if $k \geq 3$, or if $k = 2$ and they have the same Poisson bracket:

THEOREM 3. *If $k \geq 3$, any $k$-homogeneous star product is commutative, and homogeneous-equivalent to the standard product $(f, g) \mapsto fg$.*

*A homogeneous star product of type 2 is completely determined by its Poisson bracket $c$, which has constant coefficients. It is homogeneous-equivalent to the Moyal product (cf.* [11]):

$$(f, g) \mapsto \exp\left(\frac{\hbar}{2} c(\partial_\xi, \partial_\eta)\right) \quad f(\xi)\, g(\eta)\,|_{\eta=\xi}$$

(we have used the standard notation: $\exp \frac{\hbar}{2} c(\partial_\xi, \partial_\eta) = \sum \frac{\hbar^n}{2^n n!} (\sum c_{ij} \partial_{\xi_i} \partial_{\eta_j})^n$).

## 3. Homogeneous Star Products of Type 1

Exactly as above we see that if $B, B'$ are homogeneous star products of type 1 which coincide up to order 2 ($B' - B = \hbar^3 \beta_3 + \cdots$, where $\beta_3$ is homogeneous of degree $-3$), they are homogeneous-equivalent.

Also if $B$ is 1-homogeneous, it is homogeneous-equivalent to a star product of the form $B = \epsilon + \frac{1}{2}\hbar c + \hbar^2 b_2 + \cdots$ where $c$ is a Poisson bracket homogeneous of degree $-1$, and the antisymmetric part of $b_2$ is of the form $\frac{1}{2}\hbar^2 \gamma$ with $\gamma$ a bivector homogeneous of degree $-2$, i.e. a Poisson bracket with constant coefficients.

LEMMA 4. *Notations being as above, $\gamma$ commutes with $c$, i.e. $[c, \gamma]_N = 0$ where $[\,,\,]_N$ is the Nijenhuis–Schouten bracket of multi-vectors (cf.* [12,13]).

*Proof.* the commutator law is $C(f, g) = B(f, g) - B(g, f) = \hbar c(f, g) + \hbar^2 \gamma(f, g) + O(\hbar^3)$. It satisfies the Jacobi identity, so for the coefficient of $\hbar^2$ we get:

$$\sum_{\circlearrowleft} c(\gamma(f_1, f_2), f_3) + \gamma(c(f_1, f_2), f_3) = 0$$

where $\sum_{\circlearrowleft}$ means the sum over cyclic permutations of $f_1, f_2, f_3$. This exactly means that $c$ and $\gamma$ commute ($[c, \gamma]_N = 0$); equivalently, since $\gamma$ has constant coefficients and also is a Poisson bracket, that $\hbar c + \hbar^2 \gamma$ is again a Poisson bracket.

Note further that if $B = \epsilon + \hbar c + \cdots$ and $B' = e^p(B)$ is of the same form with $p = \hbar p_1 + \cdots$ 1-homogeneous, we have $[p_1, \epsilon] = 0$, i.e. $p_1(fg) - p_1(f)g - fp_1(g) = 0$, which means that $p_1$ is a vector field with constant coefficients $p_1 = v \in V$, since it is homogeneous of degree $-1$. Then $e^{\hbar p_1}(B)$ is of the form $B(\xi + \hbar v, \partial \otimes \partial)$ (deduced from $B$ by translation). The leading antisymmetric term of $e^{\hbar p_1} B - B$ is $\frac{\hbar}{2}\gamma$, with $\gamma = c(v, \partial \wedge \partial)$.

It is known (proof recalled below) that any homogeneous Poisson bracket of degree $-1$ is the Poisson bracket of some homogeneous star product, so we get: □

THEOREM 5. *A* 1*-homogeneous star product is completely determined* (*up to equivalence*) *by its terms of order $\leq 2$. Equivalence classes of* 1*-homogeneous star products* *are in one to one correspondence with translation orbits of* 1*-homogeneous Poisson brackets of the form* $\hbar c + \hbar^2 \gamma$*, under the translation group* $\xi \mapsto \xi + \hbar v$.

A similar result was proved [3] (see also in [7]). The uniqueness assertion there (i.e. the homogeneous star product is determined up to equivalence by its Poisson bracket) is not the same as above: this is due to the fact that the authors impose a stronger symmetry condition on their star products ($B_n(g, f) = (-1)^n B_n(f, g)$), which holds for the S. Gutt and M. Kontsevich star products, but which we did not impose here (as in most of the literature).

## 4. Examples and Remarks

### 4.1. S. GUTT'S STAR PRODUCT

If $c$ is a Poisson bracket, homogeneous of degree $-1$ on $V$, it defines a Lie algebra structure on the dual $\mathfrak{g}$ of $V$: $[u, v](\xi) = \{u(\xi), v(\xi)\}$ if $u, v \in \mathfrak{g}$ are viewed as linear functions on $V$.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$: the star algebra of pseudodifferential operators on $G$ (mod smoothing operators) is well defined, so as the sub star algebra of left invariant operators (those that commute with right translations). This is a star algebra, living on the dual $V$ of $\mathfrak{g}$, in the sense that its product law can be defined by a star product on $V$. However to define a star product one must choose a total symbol, which can be done in many manners (as for pseudo-differential operators on a general manifold).

A canonical choice consists in choosing exponential coordinates near the origin of $G$, in which the product is given by the Campbell–Hausdorff formula: this is the choice made by S. Gutt in [9].

The Campbel-Hausdorff series on $\mathfrak{g}$ is:

$$X \cdot Y = \mathrm{Log}\,(e^X e^Y) = X + Y + \sum_1^{\infty} CH_n(X, Y) \tag{9}$$

where $CH_n$ is an iterated bracket of order $n$: it is a homogeneous polynomial of degree $n+1$ of $(X, Y)$, and also a homogeneous polynomial of degree $n$ of the coefficients of the Lie bracket of $\mathfrak{g}$, or equivalently, of the linear Poisson bracket of $V$ ($CH_1 = \frac{1}{2}[X, Y]$, $CH_2 = \frac{1}{12}[X - Y, [X, Y]], \ldots$).

It is symmetric: since $\mathrm{Log}\,((e^X e^Y)^{-1}) = -(X \cdot Y) = (-Y) \cdot (-X)$, we have

$$CH_n(Y, X) = (-1)^n CH_n(X, Y).$$

We define a dilated product:

$$X \cdot_{\hbar} Y = X + Y + \sum_1^{\infty} \hbar^n CH_n(X, Y) \tag{10}$$

If $f$ is a polynomial, we denote by $\widehat{f}$ its inverse Fourier–Laplace transform: this is the distribution, supported by the origin, such that $f(\xi) = \langle \widehat{f}(X), e^{\langle \xi, X \rangle} \rangle$. The star product constructed by S. Gutt, arising from the exponential chart is, for $f, g$ polynomials:

$$B(f, g) = \langle \widehat{f}(X) \otimes \widehat{g}(Y), e^{\xi, X \cdot_\hbar Y>} \rangle$$

This can be rewritten

$$B(f, g) = \exp\left( \xi \cdot \sum_1^\infty \hbar^n CH_n(\partial_\eta, \partial_\zeta) \right) f(\eta)\, g(\zeta)|_{\eta=\zeta=\xi} \tag{11}$$

which makes sense for all formal series $f = \sum \hbar^n f_n, g = \sum \hbar^n g_n$. It is obviously a homogeneous star product of type 1.

### 4.2. M. KONTSEVICH'S CONSTRUCTION

The construction of M. Kontsevich assigns to a bivector $c$ on $V$ with smooth coefficients a formal series of bidifferential operators $K(c) = \sum_0^\infty K_n(c)$, where the bidifferential operator $K_n(c)$ is a homogeneous polynomial of degree $n$ of $c$ and its derivatives, $K_0 = \epsilon$, $K_1(c) = \frac{1}{2}c$.[1] It commutes with affine transformations of $V$, and the (formal) Gerstenhaber bracket $[K(c), K(c)]$ vanishes if the Nijenhuis bracket $[c, c]_N$ vanishes, i.e. $K(hc) = \sum \hbar^n K_n(c)$ is a star product with Poisson bracket $c$ if $c$ is a Poisson bracket.

Since the Kontsevich construction commutes with affine transformations, $K_n(c)$ is homogeneous of degree $-kn$ if $c$ is homogeneous of degree $-k$, so $K(\hbar c)$ is then a homogeneous star product of type $k$.

It is immediate to verify that $K_2(f, g)$ is symmetric in $(f, g)$ (cf. [10], §1.4.2—in fact one has $K_n(c)(g, f) = (-1)^n K_n(c)(f, g)$ for all $n$, i.e. $K(\hbar c)(g, f) = K(-\hbar c)(f, g)$). Since two star-products homogeneous of type 1 whose antisymmetric parts coincide up to order 2 are equivalent, we get back the result of [6] or [4]:

PROPOSITION 6. *If $c$ is a linear Poisson bracket, the Kontsevich star product $K(\hbar c)$ is equivalent to the Gutt star product $B$, constructed out of the Campbell–Hausdorff series.*

In [6], G. Dito describes more explicitly an asymptotic operator $P = 1 + \sum \hbar^n P_n$ which performs the transition (cf. also [4]).

### 4.3. FINAL REMARKS

If $c = \sum_1^\infty \hbar^n c_n$ if a formal Poisson bracket, homogeneous of type $k$, then $K(c)$ is a star product, homogeneous of type $k$. (The condition on $c$ means that for each $n$, $c_n$ is a bivector homogeneous of degree $-n$, and $\sum_{p+q=n} [c_p, c_q]_N = 0$.)

[1] For commodity we inserted the coefficient $\frac{1}{2}$ although it is not present in [10].

Such a series is empty if $k>2$, and is reduced to its first term $\hbar c_1$ if $k=2$.

It is of the form $\hbar c+\hbar^2\gamma$ if $k=1$, with $c$ is a Poisson bracket homogeneous of degree $-1$ an $\gamma$ a 2-vector with constant coefficient such that $[c,\gamma]_N=0$.

[If $k\leq 0$, a case which we do not study here, the series can have infinitely many terms.]

It follows from Theorem 5 above that any star product homogeneous of type 1 with Poisson bracket $c$ is homogeneous-equivalent to a star product $K(\frac{1}{2}(hc+h^2\gamma))$, where $\gamma$ is a 2-vector with constant coefficients, $[c,\gamma]_N=0$; two such star products are homogeneous-equivalent if and only if there exists a constant vector $v$ such that $\gamma=L_v(c)=[v,c]$, equivalently $c+\hbar\gamma=c(x+\hbar v,\partial\wedge\partial)$.[2]

If $\gamma$ is a 2-vector, let us denote by $\tilde{\gamma}(x,y)$ the antisymmetric form it defines on the dual $\mathfrak{g}$ of $V$. We define a bracket on $\mathfrak{g}_1=\mathfrak{g}_0\oplus\mathfrak{g}$ ($\mathfrak{g}_0=\mathbb{R}$ or $\mathbb{C}$, the trivial Lie algebra of rank 1) by

$$[(t,x),(s,y)]_1 \;=\; (\tilde{\gamma}(x,y),[x,y])$$

where $[x,y]$ is the Lie bracket on $\mathfrak{g}$ defined by $c$.

The condition $[c,\gamma]_N=0$ means that $[,]_1$ is a Lie bracket on $\mathfrak{g}_1$, thus defining a central extension $\mathfrak{g}_0\to\mathfrak{g}_1\to\mathfrak{g}$. Then $\gamma$ is of the form $L_v c$ if and only if this extension splits (is trivial). We can thus slightly reformulate Theorem 5:

PROPOSITION 7. *Equivalence classes of homogeneous star products of type 1 with Poisson bracket c are (canonically) in 1–1 correspondence with elements of* $Ext^1(\mathfrak{g},\mathfrak{g}_0)$.

Thus if $\mathrm{Ext}^1(\mathfrak{g},\mathfrak{g}_0)$ vanishes, e.g. if $\mathfrak{g}$ is semi simple, all 1-homogeneous star products with Poisson bracket $c$ are equivalent (in fact all, homogeneous or not, cf. [5]). At the opposite end, if $c=0$, any 1-homogeneous star product is isomorphic to a star product $K(\hbar^2\gamma)$, with $\gamma$ an arbitrary 2-vector with constant coefficients, i.e. a Moyal star product with $\hbar$ replaced by $\hbar^2$:

$$B(f,g)=\exp\left(\frac{\hbar^2}{2}\gamma(\partial_\xi,\partial_\eta)\right)\; f(\xi)\,g(\eta)|_{\eta=\xi}.$$

## References

1. Arnal, D., Ben Amar, N., Masmoudi, M.: Cohomology of good graphs and Kontsevich linear star products. Lett. Math. Phys. **48**(4), 291–306 (1999)
2. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation theory and quantization I, II. Ann. Phys. **111**, 61–131 (1977)
3. Ben Amar, N., Chaabouni, M.: Properties of graded star products. Russ. J. Math. Phys. **12**(3), 288–300 (2005)

[2] This would not usually be true without the homogeneity assumption.

4. Ben Amar, N., Chaabouni, M.: Hochschild cohomology and equivalence of graded star products. Acta Math. Univ. Comenian (N.S.) **75**(1), 31–41 (2006)
5. Chloup-Arnould, V.: Star products on the algebra of polynomials on the dual of a semi-simple Lie algebra. Acad. Roy. Belg. Bull. Cl. Sci. (6) 8, no. 7–12, 263–269 (1997)
6. Dito, G.: Kontsevich Star Product on the dual of a Lie Algebra. Lett. Math. Phys **48**, 307–322 (1999)
7. Gammella, A.: Tangential star products. L.M.P. **51**(1), 1–15 (2000)
8. Gerstenhaber, M.: On the deformation of rings and algebras. Ann. Math. **79**, 59–103 (1964)
9. Gutt, S.: An explicit $*$-product on the cotangent bundle of a Lie group. Lett. Math. Phys. **7**(3), 249–258 (1983)
10. Kontsevich, M.: Deformation Quantization of Poisson Manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003). arXiv: q-alg/9709040
11. Moyal, J.E.: Quantum mechanics as a statistical theory. Proc. Camb. Philos. Soc. **45**, 99–124 (1949)
12. Nijenhuis, A.: Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58. Indag. Math. **17**, 390–397, 398–403 (1955)
13. Schouten, J. A.: On the differential operators of first order in tensor calculus. Rapport ZA 1953-012. Math. Centrum Amsterdam, 6 (1953)