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Two Loop Superstring Amplitudes and S_6 Representations

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Abstract. In this paper we describe how representation theory of groups can be used to shorten the derivation of two loop partition functions in string theory, giving an intrinsic description of modular forms appearing in the results of D'Hoker and Phong (Nucl Phys B639:129–181, 2002). Our method has the advantage of using only algebraic properties of modular functions and it can be extended to any genus g.

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1. Introduction

It was conjectured by Belavin and Knizhnik [2] that "any multiloop amplitude in any conformal invariant string theory may be deduced from purely algebraic objects on moduli spaces M_p of Riemann surfaces". This was a known fact for zero and for one loop amplitudes. For bosonic strings, two, three and four loop amplitudes was computed (in the same year) in [3–5] in terms of modular forms.

For superstring theories the story is much longer because of some technical difficulties. In particular, the presence of fermionic interactions makes the splitting between chiral and antichiral modes hard. Moreover, grassmanian variables arise from worldsheet supersymmetry and one needs a covariant way to integrate them out. Both problems were solved by D'Hoker and Phong, who in a series of articles [1,6–8] showed that the computation of g-loop string amplitudes in perturbation theory is strictly connected with the construction of a suitable measure on the moduli space of genus g Riemann surfaces. They claim [9,10] that the genus g

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vacuum to vacuum amplitude must take the form

$$\mathcal{A}=\int_{\mathcal{M}_g}(\det\operatorname{Im}\tau)^{-5}\sum_{\delta,\bar{\delta}}c_{\delta,\bar{\delta}}\mathrm{d}\mu[\delta](\tau)\wedge\overline{\mathrm{d}\mu[\bar{\delta}](\tau)}, \tag{1.1}$$

where δ and $\bar{\delta}$ denote two spin structures or theta characteristics, $c_{\delta,\bar{\delta}}$ are suitable constant phases depending on the details of the model and $\mathrm{d}\mu[\delta](\tau)$ is a holomorphic form of maximal rank $(3g-3,0)$ on the moduli space of genus g Riemann surfaces. The Riemann surface is represented by its period matrix τ, after a choice of canonical homology basis. Since the integrand should be independent from the choice of homology basis, it follows that the measure $\mathrm{d}\mu[\delta](\tau)$ must transform covariantly under the modular group $\mathrm{Sp}(2g,\mathbb{Z})$.

In [1] D'Hoker and Phong have given an explicit expression for the two loop measure in terms of theta constants, i.e. theta functions evaluated at the origin, $z=0$. The amplitude (1.1) is written in terms of modular forms and is manifestly modular invariant:

$$\mathrm{d}\mu[\delta](\tau)=\frac{\theta^4[\delta](\tau,0)\,\Xi_6[\delta](\tau,0)}{16\pi^6\Psi_{10}(\tau)}\prod_{I\le J}\mathrm{d}\tau_{IJ}. \tag{1.2}$$

Here $\Psi_{10}(\tau)$ is a modular form of weight ten:

$$\Psi_{10}=\prod_{\delta}\theta^2[\delta](\tau,0),$$

where δ varies on the whole set of even spin structures (consisting of ten elements). The ten $\Xi_6[\delta]$ are defined[1] by

$$\Xi_6[\delta](\tau,0):=\sum_{1\le i<j\le 3}\langle\nu_i|\nu_j\rangle\prod_{k=4,5,6}\theta^4[\nu_i+\nu_j+\nu_k](\tau,0),$$

where each even spin structure is written as a sum of three distinct odd spin structures $\delta=\nu_1+\nu_2+\nu_3$ and ν_4,ν_5,ν_6 denote the remaining three distinct odd spin structures, see Appendix A. The signature of a pair of spin structures, even or odd, is defined by:

$$\langle\kappa|\lambda\rangle:=\mathrm{e}^{\pi i(a_\kappa\cdot b_\lambda-b_\kappa\cdot a_\lambda)},\qquad \kappa=\begin{bmatrix}a_\kappa\\ b_\kappa\end{bmatrix},\ \lambda=\begin{bmatrix}a_\lambda\\ b_\lambda\end{bmatrix}.$$

In what follows we will refer to the theta constants as $\theta[\delta]:=\theta[\delta](\tau,0)$ and similar for $\Xi_6[\delta]$.

Our aim in this letter is to give an intrinsic description of the kind of modular forms appearing in two loop amplitudes, and to show how to give explicit expressions of them in terms of theta constants employing group representation

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[1] Comparing our conventions with the ones of D'Hoker and Phong one should note that our spin matrices are transposed, according with our conventions on theta functions, signatures, etc.

techniques. Our method has the advantage of using only algebraic properties of modular functions (in the spirit of [2]) and it can be extended to any genus g. In particular it can be used to overcome the difficulties encountered in [9,10] for the computation of three loop amplitudes, as will be shown in a forthcoming paper.

2. The Igusa Quartic and the Forms $\Xi_6[\delta]$

At genus two, there are ten even spin structures which correspond to ten theta functions with even characteristics. To study even powers of these functions we define:

$$\Theta[\varepsilon](\tau)=\theta\left[\begin{smallmatrix}\varepsilon\\0\end{smallmatrix}\right](2\tau,0),$$

with $[\varepsilon]=[\varepsilon_1\,\varepsilon_2]$ and we use the formula [11]:

$$\theta\begin{bmatrix}\alpha\\\beta+\gamma\end{bmatrix}(\tau,z_1+z_2)\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(\tau,z_1-z_2)=$$
$$=\sum_{\delta\in(\mathbb{Z}/2\mathbb{Z})^g}(-1)^{\beta\cdot\delta}\theta\begin{bmatrix}\delta\\\gamma\end{bmatrix}(2\tau,2z_1)\theta\begin{bmatrix}\alpha+\delta\\\gamma\end{bmatrix}(2\tau,2z_2),$$

with $z_1=z_2=0$, $\gamma=0$ and $g=2$. It follows that the fourth powers of the theta functions $\theta[\delta](\tau,z)$, evaluated at the origin, $z=0$, form a five dimensional vector space, that we call V_θ. We can choose a basis for this space of holomorphic functions on the Siegel space for $g=2$ and, for our purpose, a convenient one is:

$$P_0=\Theta^4[0\,0]+\Theta^4[0\,1]+\Theta^4[1\,0]+\Theta^4[1\,1]$$
$$P_1=2(\Theta^2[0\,0]\Theta^2[0\,1]+\Theta^2[1\,0]\Theta^2[1\,1])$$
$$P_2=2(\Theta^2[0\,0]\Theta^2[1\,0]+\Theta^2[0\,1]\Theta^2[1\,1])$$
$$P_3=2(\Theta^2[0\,0]\Theta^2[1\,1]+\Theta^2[0\,1]\Theta^2[1\,0])$$
$$P_4=4\Theta[0\,0]\Theta[0\,1]\Theta[1\,0]\Theta[1\,1],$$

The expansions of the theta constants on this basis are summarized in Table I.

The period matrix τ, that defines the Riemann surface, at genus two belongs to the complex variety $\mathbb{H}_2=\{\tau\in \mathrm{M}_2(\mathbb{C})\text{ t.c.: }{}^t\tau=\tau,\ \mathrm{Im}(\tau)>0\}$. The selected basis defines the map:

$$\varphi_4:\mathbb{H}_2\longrightarrow\mathbb{P}^4$$
$$\tau\longmapsto(P_0(\tau):P_1(\tau):P_2(\tau):P_3(\tau):P_4(\tau)).$$

The closure of the image of φ_4 is the “Igusa quartic”, the vanishing locus of

$$I_4=P_4^4+P_4^2P_0^2-P_4^2P_1^2-P_4^2P_2^2-P_4^2P_3^2+$$
$$+P_1^2P_2^2+P_1^2P_3^2+P_2^2P_3^2-2P_0P_1P_2P_3$$

Table I. Expansion of $\theta^4[\delta]$ on the basis of P_i

δ	$\theta^4[\delta]$	$P_0 4$	P_1	P_2	P_3	P_4
δ_1	$\theta^4\begin{bmatrix}0&0\\0&0\end{bmatrix}$	1	1	1	1	0
δ_2	$\theta^4\begin{bmatrix}0&0\\0&1\end{bmatrix}$	1	−1	1	−1	0
δ_3	$\theta^4\begin{bmatrix}0&0\\1&0\end{bmatrix}$	1	1	−1	−1	0
δ_4	$\theta^4\begin{bmatrix}0&0\\1&1\end{bmatrix}$	1	−1	−1	1	0
δ_5	$\theta^4\begin{bmatrix}0&1\\0&0\end{bmatrix}$	0	2	0	0	2
δ_6	$\theta^4\begin{bmatrix}0&1\\1&0\end{bmatrix}$	0	2	0	0	−2
δ_7	$\theta^4\begin{bmatrix}1&0\\0&0\end{bmatrix}$	0	0	2	0	2
δ_8	$\theta^4\begin{bmatrix}1&0\\0&1\end{bmatrix}$	0	0	2	0	−2
δ_9	$\theta^4\begin{bmatrix}1&1\\0&0\end{bmatrix}$	0	0	0	2	2
δ_{10}	$\theta^4\begin{bmatrix}1&1\\1&1\end{bmatrix}$	0	0	0	2	−2

in $\mathbb{P}^4$. It is indeed immediate to verify, expressing the P_i in terms of the four theta constants $\Theta[\varepsilon]$, that this polynomial is identically zero. We can also write I_4 as:

$$I_4 = \frac{1}{192}\left[\left(\sum_\delta \theta^8[\delta]\right)^2 - 4\sum_\delta \theta^{16}[\delta]\right]. \tag{2.3}$$

We want to find a connection between the forms $\Xi_6[\delta]$ appearing in the works of D'Hoker and Phong and the Igusa quartic whose mathematical structure is well known. For this purpose, we start considering two vector spaces which we call V_Ξ and $V_{\partial_p I}$. The first one is the space generated by the ten forms $\Xi_6[\delta]$:

$$V_\Xi = \langle \ldots, \Xi_6[\delta], \ldots \rangle.$$

We will see that it is a five dimensional space. The second vector space we are interested in is the space of the derivatives of the Igusa quartic with respect to P_i:

$$V_{\partial_P I} = \left\langle \ldots, \frac{\partial I_4}{\partial P_i}, \ldots \right\rangle_{i=0,\ldots,4},$$

which is again a five dimensional space. Both spaces are generated by homogeneous polynomials of degree twelve in the theta constants $\Theta[\varepsilon]$ or, equivalently, of degree three in the P_i. We find:

THEOREM 1. *We have* $V_\Xi = V_{\partial_{P} I}$*, in particular* $\dim V_\Xi = 5$ *and Table II gives the expansion of each* $\Xi_6[\delta]$ *as linear combination of the derivative of Igusa quartic with respect to* P_i.

Table II. Expansion of the functions $\Xi_6[\delta](\tau)$ on the $\frac{\partial I_4}{\partial P_i}$

δ	$\partial_{P_0} I_4$	$\partial_{P_1} I_4$	$\partial_{P_2} I_4$	$\partial_{P_3} I_4$	$\partial_{P_4} I_4$
$\Xi_6[\delta_1]$	6	2	2	2	0
$\Xi_6[\delta_2]$	6	−2	2	−2	0
$\Xi_6[\delta_3]$	6	2	−2	−2	0
$\Xi_6[\delta_4]$	6	−2	−2	2	0
$\Xi_6[\delta_5]$	0	4	0	0	2
$\Xi_6[\delta_6]$	0	4	0	0	−2
$\Xi_6[\delta_7]$	0	0	4	0	2
$\Xi_6[\delta_8]$	0	0	4	0	−2
$\Xi_6[\delta_9]$	0	0	0	4	2
$\Xi_6[\delta_{10}]$	0	0	0	4	−2

We intend $\partial_{P_0} I_4 \equiv \frac{\partial I_4}{\partial P_i}$

Another interesting vector space is the one generated by the derivatives of the Igusa quartic with respect to the ten theta constants $\theta[\delta]$ at the fourth power:

$$V_{\partial_\theta I} := \left\langle \ldots, \frac{\partial I_4}{\partial \theta^4[\delta]}, \ldots \right\rangle.$$

In computing these derivatives the theta constants $\theta^4[\delta]$ must be considered as independent functions and we use (2.3). $V_{\partial_\theta I}$ has dimension ten, so these polynomials are all independent. Next define the ten functions:

$$f_\delta := 2\Xi_6[\delta] - \frac{\partial I_4}{\partial \theta^4[\delta]}, \tag{2.4}$$

generating the vector space $V_f = \langle \ldots, f_\delta, \ldots \rangle$ of dimension five. Then:

$$\sum_\delta \frac{\partial I_4}{\partial \theta^4[\delta]} f_\delta = 0 \quad \text{and} \quad V_{\partial_\theta I} = V_f \oplus V_\Xi.$$

This connection of the Igusa quartic with the forms $\Xi_6[\delta]$ suggests studying the whole space of the polynomials of degree three in the P_i: $S^3 V_\theta = \langle \ldots, P_i P_j P_k, \ldots \rangle_{0 \le i \le j \le k \le 4}$, the triple symmetric tensor product of the space V_θ. We want to decompose this 35 dimensional space in a "natural" way and understand which parts of such a decomposition are involved in the measure (1.2).

3. Decomposition of $S^3 V_\theta$

To decompose the whole space $S^3 V_\theta$ in a "natural" way as a direct sum of vector spaces, $S^3 V_\theta = \bigoplus_i V_i$, we employ the theory of representations of finite groups. The point is that string amplitudes must be invariant under the action of the

Table III. Relationship between the generators of the modular group and S_6

M_1	M_2	M_3	S	Σ	T
(1 3)	(2 4)	(1 3)(2 4)(5 6)	(3 5)(4 6)	(1 2)(3 4)(5 6)	(1 3)(2 6)(4 5)

modular group $\mathrm{Sp}(2g,\mathbb{Z})$. In particular for genus two surfaces the modular group is $\mathrm{Sp}(4,\mathbb{Z})\equiv\Gamma_2$. This group can be surjectively mapped into the symmetric group S_6 with kernel $\Gamma_2(2)=\{M\in\Gamma_2,\ M\equiv \mathrm{Id}\ (\mathrm{mod}\ 2)\}$, so that $S_6\simeq\Gamma_2/\Gamma_2(2)$. The action of S_6 on the theta constants $\theta^4[\delta]$ together with the representation theory of finite groups provide the tools to understand how the space S^3V_θ decomposes in terms of invariant subspaces under the action of the modular group and which combinations of theta constants generate each subspace.

To study the action of the symmetric group S_6 on V_θ we have to relate the generators of the modular group, see Appendix B, to the elements of S_6. We report this relation in Table III.

Each generator induces a permutation of the six odd characteristics $\nu_1,\dots,\nu_6$ and thus defines an element of S_6. Writing the even characteristics as sum of three odd characteristics, as explained in Appendix A, we find how the even theta constants $\theta^4[\delta]$ transform under the action of $\mathrm{Sp}(4,\mathbb{Z})$.

We want to identify the representation of S_6 on V_θ. This can be obtained fixing a basis for V_θ, for example $\theta^4[\delta_1]$, $\theta^4[\delta_2]$, $\theta^4[\delta_3]$, $\theta^4[\delta_4]$, $\theta^4[\delta_5]$, to compute the representation matrices of M_i, S, Σ and T and thus of the generators of S_6. The symmetric group S_6 has eleven conjugacy classes and thus has eleven irreducible representations (irreps), as shown in Table IV. For example, the conjugacy class $C_{3,2}$ consists of the product of a two-cycle and a three-cycle and the character of the first ten dimensional representation, sw_{10}, for this class is 1. The space V_θ is five dimensional, therefore it must be one of the four representations of this dimension. Looking at the character of the matrix representing M_1 allows us to identify V_θ with st_5.

An alternative way to reach the same result is provideed by the Thomae formula [11,12]:

$$\theta^4[\delta]=c\,\epsilon_{S,T}\prod_{i,j\in S\, i<j}(u_i-u_j)\prod_{k,l\in T\, k<l}(u_k-u_l),$$

where u_i are the six branch points of the Riemann surface of genus two, S and T contain the indices of the odd characteristics in the two triads which yield the same even characteristic,[2] as explained in [1] or [13], $\epsilon_{S,T}$ is a sign depending on the triads, as indicated in Table V, and c is a constant independent from the characteristic.

[2]For example for δ_4, $S=\{1,4,5\}$ and $T=\{2,3,6\}$.

Table IV. Characters of the conjugacy classes of the irreps of S_6

S_6	Partition	C_1	C_2	C_3	$C_{2,2}$	C_4	$C_{3,2}$	C_5	$C_{2,2,2}$	$C_{3,3}$	$C_{4,2}$	C_6
id_1	[6]	1	1	1	1	1	1	1	1	1	1	1
alt_1	$[1^6]$	1	−1	1	1	−1	−1	1	−1	1	1	−1
st_5	$[2^3]$	5	−1	−1	1	1	−1	0	3	2	−1	0
sta_5	$[3^2]$	5	1	−1	1	−1	1	0	−3	2	−1	0
rep_5	[5 1]	5	3	2	1	1	0	0	−1	−1	−1	−1
repa_5	$[2\,1^4]$	5	−3	2	1	−1	0	0	1	−1	−1	1
n_9	[4 2]	9	3	0	1	−1	0	−1	3	0	1	0
na_9	$[2^2\,1^2]$	9	−3	0	1	1	0	−1	−3	0	1	0
sw_{10}	$[3\,1^3]$	10	−2	1	−2	0	1	0	2	1	0	−1
swa_{10}	$[4\,1^2]$	10	2	1	−2	0	−1	0	−2	1	0	1
s_{16}	[3 2 1]	16	0	−2	0	0	0	1	0	−2	0	0

Table V. Relative signs between the theta constants $\theta^4[\delta]$ for the Thomae formula

146 235	126 345	125 346	145 236	124 356	156 234	123 456	134 256	136 245	135 246
δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}
−1	1	1	−1	1	−1	1	−1	−1	−1

The Thomae formula shows that S_6 acts on the theta constants by permuting the branch points. Evaluating in this way the effect of permutations, and comparing the characters we find again that the representation V_θ must be identified with st_5.

Thus the representation on the space $S^3 V_\theta$ is the $\mathrm{S}^3(\mathrm{st}_5)$ that decomposes as follows:

$$\mathrm{S}^3(\mathrm{st}_5) = \mathrm{id}_1 + \mathrm{n}_9 + \mathrm{repa}_5 + 2\mathrm{st}_5 + \mathrm{sw}_{10}. \tag{3.5}$$

The presence of id_1, the trivial representation of S_6, implies the existence of an invariant polynomial. Its expression in terms of the basis P_i, up to a scalar, is:

$$\Psi_6 = P_0^3 - 9P_0(P_1^2 + P_2^2 + P_3^2 - 4P_4^2) + 54P_1P_2P_3, \tag{3.6}$$

and essentially it is the modular form of weight six appearing in [1].

We will now identify some subspaces of $S^3 V_\theta$ in the decomposition (3.5). All these subspaces must be invariant over the action of the modular group otherwise a modular transformation of $\theta^4[\delta]$ would send an element of a subspace in another one. We summarize the results in Table VI.

Table VI. Decomposition of the given subspaces

Space	Dimension	Representation
$\boldsymbol{\langle P_0^3+\cdots+54P_1P_2P_3\rangle \equiv V_I}$	**1**	$\mathbf{id_1}$
$\boldsymbol{\langle \partial_{P_i} I_4\rangle \equiv \langle \Xi_6[\delta]\rangle \equiv V_\Xi}$	**5**	$\mathbf{st_5}$
$\boldsymbol{\langle 2\Xi_6[\delta]-\frac{\partial I_4}{\partial\theta^4[\delta]}\rangle \equiv V_f}$	**5**	$\mathbf{repa_5}$
$\boldsymbol{\langle \theta^4[\delta_i]\sum_{\delta'}\theta^8[\delta']\rangle \equiv V_S}$	**5**	$\mathbf{st_5}$
$\langle \frac{\partial I_4}{\partial\theta^4[\delta_i]}\rangle$	10	$\mathrm{st}_5\oplus\mathrm{repa}_5$
$\langle\theta^{12}[\delta_i]\rangle$	10	$\mathrm{st}_5\oplus\mathrm{repa}_5$
$\langle\theta^{12}[\delta_i], \frac{\partial I_4}{\partial\theta^4[\delta_j]}\rangle$	15	$2\mathrm{st}_5\oplus\mathrm{repa}_5$
$\langle\theta^{12}[\delta_i], \Xi_6[\delta]\rangle$	15	$2\mathrm{st}_5\oplus\mathrm{repa}_5$
$\langle\theta^{12}[\delta_i], \theta^4[\delta_j]\sum_{\delta'}\theta^8[\delta']\rangle$	15	$2\mathrm{st}_5\oplus\mathrm{repa}_5$
$\langle\theta^{12}[\delta_i], \theta^4[\delta_j]\sum_{\delta'}\theta^8[\delta'], \partial_{\delta_k}I_4\rangle$	15	$2\mathrm{st}_5\oplus\mathrm{repa}_5$
$\boldsymbol{\langle\theta^4[\delta_i]\theta^4[\delta_j]\theta^4[\delta_k]\rangle_{\delta_i+\delta_j+\delta_k \text{ odd}}}$	**20**	$\mathbf{st_5\oplus repa_5\oplus sw_{10}}$
$\boldsymbol{\langle\theta^4[\delta_i]\theta^8[\delta_j]\rangle}$	**34**	$\mathbf{2st_5\oplus repa_5\oplus n_9\oplus sw_{10}}$
$\langle\theta^4[\delta_i]\theta^4[\delta_j]\theta^4[\delta_k]\rangle_{\delta_i,\delta_j,\delta_k \text{ even}}$	35	S^3V_θ
$\langle\theta^4[\delta_i]\theta^4[\delta_j]\theta^4[\delta_k]\rangle_{\delta_i+\delta_j+\delta_k \text{ even}}$	35	S^3V_θ

The final decomposition of the whole space S^3V_θ is then:

$$S^3V_\theta = V_I\oplus V_\Xi\oplus V_f\oplus V_S\oplus V_9\oplus V_{10},$$

where V_I is the subspace generated by the invariant polynomial Ψ_6 (3.6), V_Ξ is generated by the forms $\Xi_6[\delta]$, V_f is generated by the functions defined in (2.4) and V_9 and V_{10} are parts of the subspaces of dimension 20 or 34 given in Table VI.

Note that Ψ_6 cannot be written as a linear combination of the products $\theta^4[\delta_i]\theta^4[\delta_j]\theta^4[\delta_k]$ for $\delta_i+\delta_j+\delta_k$ an odd characteristic, in contradiction to the claim in [14], because the subspace V_I is not contained in $\langle\theta^4[\delta_i]\theta^4[\delta_j]\theta^4[\delta_k]\rangle_{\delta_i+\delta_j+\delta_k \text{ odd}}$. Instead Ψ_6 can be written as a linear combination of the products $\theta^4[\delta_i]\theta^4[\delta_j]\theta^4[\delta_k]$ for $\delta_i+\delta_j+\delta_k$ an even characteristic, as correctly said in [1]. Indeed these products of theta constants span the whole S^3V_θ.

4. Conclusions

In this letter we clarified the algebraic properties of the modular structures underlying two loop superstring amplitudes. In the papers of D'Hoker and Phong it was shown that the crucial ingredients are the modular forms $\Xi_6[\delta]$ appearing in (1.2). In Section 2 we have connected the forms $\Xi_6[\delta]$ to the mathematically well known Igusa quartic. This clarifies the origin of such forms which result to live in a given five dimensional subspace of the vector space of cubic polynomials in the fourth powers of the 10 even theta constants. We studied the whole space in Section 3 where we decomposed it in irreducible representations (irreps) of the group S_6, a quotient of the modular group. In this way we identified the irrep corresponding to the space generated by the forms $\Xi_6[\delta]$. Our analysis can be extended to any genus g and gives a direct and quick strategy for searching modular forms with certain properties. However, there are some difficulties in carrying on such a generalization. Possibly Equation (1.1) is no more true for genus $g>2$ for the following reasons (E. Witten, Private Communication): D'Hoker and Phong obtained (1.1) from a chiral splitting which works using the fact that, for a $g=2$ super Riemann surface with an even spin structure, there are two even holomorphic differentials and no odd ones. The second point necessary for the splitting is that by taking the periods of the two holomorphic differentials, one associates to the original super Riemann surface M an abelian variety J, so that one maps the given super Riemann surface M to the ordinary Riemann surface M' that has M for its Jacobian. For a $g>2$ super Riemann surface with an even spin structure there are "generically" g even holomorphic differentials and no odd ones, but it is possible to have odd ones for special complex structures on M. So, in an arbitrary genus g where we can have also odd holomorphic differentials, this procedure can not be carried on. Also, if there are no odd holomorphic differentials, taking the periods of the even holomorphic differentials will give us an abelian variety, but it won't necessarily be the Jacobian of an ordinary Riemann surface. Its period can differ from those of an arbitrary Riemann surface by terms that are bilinear in fermionic moduli. Thus equation (1.1) requires an improvement for $g>2$.

Such issue and similar, together with the application of our analysis to the construction of genus three amplitudes and to open and type O string amplitudes will be the goals of future papers.

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Appendix A: Spin Structure

At genus two there are sixteen independent characteristics, six odd and ten even. The odd characteristics are:

$$\nu_1=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix},\quad \nu_2=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix},\quad \nu_3=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\quad \nu_4=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad \nu_5=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad \nu_6=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}.$$

Table VII. Combinations of odd characteristics that form the same even characteristic

Triad	146 235	126 345	125 346	145 236	124 356	156 234	123 456	134 256	136 245	135 246
$[\delta]$	$\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$
δ	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}

The even characteristics are:

$$\delta_1=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad \delta_2=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \delta_3=\begin{bmatrix}0&0\\1&0\end{bmatrix},\quad \delta_4=\begin{bmatrix}0&0\\1&1\end{bmatrix},\quad \delta_5=\begin{bmatrix}0&1\\0&0\end{bmatrix},$$

$$\delta_6=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \delta_7=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \delta_8=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \delta_9=\begin{bmatrix}1&1\\0&0\end{bmatrix},\quad \delta_{10}=\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

Each even characteristic can be obtained in two distinct way as a sum of three odd characteristics [1,13] as shown in Table VII.

In the first line are listed the indices of the two sets of three odd characteristics that summed give the same even characteristic.

Appendix B: The Modular Group Sp(4, $\mathbb{Z}$)

The modular group Sp(4, $\mathbb{Z}$) is defined by the matrices $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ satisfying:

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}0&I\\-I&0\end{pmatrix}{}^t\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}0&I\\-I&0\end{pmatrix},$$

where A, B, C, $D\in \mathrm{M}_2(\mathbb{Z})$. The group is generated by:

$$M_i=\begin{pmatrix}I&B_i\\0&I\end{pmatrix},\qquad B_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad B_2=\begin{pmatrix}0&0\\0&1\end{pmatrix},\qquad B_3=\begin{pmatrix}0&1\\1&0\end{pmatrix};$$

$$S=\begin{pmatrix}0&I\\-I&0\end{pmatrix};\qquad \Sigma=\begin{pmatrix}\sigma&0\\0&-\sigma\end{pmatrix},\qquad \sigma=\begin{pmatrix}0&1\\-1&0\end{pmatrix};$$

$$T=\begin{pmatrix}\tau_+&0\\0&\tau_-\end{pmatrix},\qquad \tau_+=\begin{pmatrix}1&1\\0&1\end{pmatrix},\qquad \tau_-=\begin{pmatrix}1&0\\-1&1\end{pmatrix}.$$

The action of the modular group on a characteristic κ (even or odd), at genus $g=2$, is given by:

$$\begin{pmatrix}{}^t\tilde{a}\\{}^t\tilde{b}\end{pmatrix}=\begin{pmatrix}D&-C\\-B&A\end{pmatrix}\begin{pmatrix}{}^ta\\{}^tb\end{pmatrix}+\mathrm{Diag}\begin{pmatrix}C\cdot{}^tD\\A\cdot{}^tB\end{pmatrix},$$

where a and b are the rows of the characteristic $\kappa=\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$. Diag($M$) for a $n\times n$ matrix M is an $1\times n$ column vector whose entries are the diagonal entries of M. The action of a modular transformation on a period matrix is

$$\tilde{\tau}=(A\tau+B)(C\tau+D)^{-1},$$

Table VIII. Transformation of the even characteristics under the action of the modular group

Triad	$[\delta]$	δ	M_1	M_2	M_3	S	Σ	T	$\epsilon^4(\delta, M_1)$	$\epsilon^4(\delta, M_2)$
146 235	$\begin{bmatrix}0&0\\0&0\end{bmatrix}$	δ_1	δ_3	δ_2	δ_1	δ_1	δ_1	δ_1	+	+
126 345	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$	δ_2	δ_4	δ_1	δ_2	δ_5	δ_3	δ_4	+	+
125 346	$\begin{bmatrix}0&0\\1&0\end{bmatrix}$	δ_3	δ_1	δ_4	δ_3	δ_7	δ_2	δ_3	+	+
145 236	$\begin{bmatrix}0&0\\1&1\end{bmatrix}$	δ_4	δ_2	δ_3	δ_4	δ_9	δ_4	δ_2	+	+
124 356	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	δ_5	δ_6	δ_5	δ_6	δ_2	δ_7	δ_5	+	−
156 234	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	δ_6	δ_5	δ_6	δ_5	δ_8	δ_8	δ_6	+	−
123 456	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$	δ_7	δ_7	δ_8	δ_8	δ_3	δ_5	δ_9	−	+
134 256	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	δ_8	δ_8	δ_7	δ_7	δ_6	δ_6	δ_{10}	−	+
136 245	$\begin{bmatrix}1&1\\0&0\end{bmatrix}$	δ_9	δ_9	δ_9	δ_{10}	δ_4	δ_9	δ_7	−	−
135 246	$\begin{bmatrix}1&1\\1&1\end{bmatrix}$	δ_{10}	δ_{10}	δ_{10}	δ_9	δ_{10}	δ_{10}	δ_8	−	−

Table IX. Transformation of the odd characteristics under the action of the modular group

$[\nu]$	ν	M_1	M_2	M_3	S	Σ	T
$\begin{bmatrix}0&1\\0&1\end{bmatrix}$	ν_1	ν_3	ν_1	ν_3	ν_1	ν_2	ν_3
$\begin{bmatrix}1&0\\1&0\end{bmatrix}$	ν_2	ν_2	ν_4	ν_4	ν_2	ν_1	ν_6
$\begin{bmatrix}0&1\\1&1\end{bmatrix}$	ν_3	ν_1	ν_3	ν_1	ν_5	ν_4	ν_1
$\begin{bmatrix}1&0\\1&1\end{bmatrix}$	ν_4	ν_4	ν_2	ν_2	ν_6	ν_3	ν_5
$\begin{bmatrix}1&1\\0&1\end{bmatrix}$	ν_5	ν_5	ν_5	ν_6	ν_3	ν_6	ν_4
$\begin{bmatrix}1&1\\1&0\end{bmatrix}$	ν_6	ν_6	ν_6	ν_5	ν_4	ν_5	ν_2

and on the theta functions:

$$\theta[\tilde{\kappa}](\tilde{\tau}, {}^t(C\tau + D)^{-1}z) = \epsilon(\kappa, M)\det(C\tau + D)^{\frac{1}{2}} e^{\pi i {}^t z(C\tau+D)^{-1}Cz}\theta[\kappa](\tau, z).$$

The phase factor $\epsilon(\kappa, M)$, satisfying $\epsilon^8(\kappa, M) = 1$, depends both on the characteristic κ and on the matrix M generating the transformation. For the even characteristics $\delta = \begin{bmatrix}a\\b\end{bmatrix}$ the fourth powers of ϵ are given by:

$$\epsilon^4(\delta, M_i) = e^{\pi i {}^t a B_i a} \quad i = 1, 2$$
$$\epsilon^4(\delta, M_3) = \epsilon^4(\delta, S) = \epsilon^4(\delta, \Sigma) = \epsilon^4(\delta, T) = 1.$$

The action of the six generators on the theta characteristics and on the triads are reported in Table VIII.

In Table IX we report the action of the generators of the modular group on the odd characteristics.

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