

Orderings and Non-formal Deformation Quantization

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Abstract. We propose suitable ideas for non-formal deformation quantization of Fréchet Poisson algebras. To deal with the convergence problem of deformation quantization, we employ Fréchet algebras originally given by Gel'fand–Shilov. Ideas from deformation quantization are applied to expressions of elements of abstract algebras, which leads to a notion of “independence of ordering principle”. This principle is useful for the understanding of the star exponential functions and for the transcendental calculus in non-formal deformation quantization.

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1. Introduction

This paper summarizes various works, some still in progress, on non-formal deformation quantization (cf. [26]–[33]).

Bayen et al. [3] originally proposed a notion of deformation quantization as a formal deformation of a Poisson algebra, and this definition has been greatly developed to deepen our understanding of quantum mechanics from an algebraic point of view.

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Initial interest typically focused on the existence and equivalence of deformation quantizations of a given Poisson algebra. This problem has now been solved, first by De Wilde and Lecomte [8], Fedosov [9] and Omori et al. [25] for symplectic manifolds, and then by Kontsevich [14] for general Poisson manifolds. The settling of the existence and uniqueness problem for deformation quantization has produced many fruitful ideas for further investigation.

In analogy to asymptotic expansions for quantum observables, deformation quantization originally meant only a formal deformation, the construction of a star product as a formal power series with coefficients in the Poisson algebra. The natural next step is the study of the convergence of deformation quantization, which [3] has suggested implicitly. In the framework of C^* -algebras this subject is called strict deformation quantization. It has been studied by Rieffel [34], and e.g. Natsume [18] and Natsume et al. [19]. Among related works we can quote Weinstein [35] in the symplectic area and [5] around quantum groups and universal deformation formulas.

In this paper, we focus on non-formal deformation quantization of Poisson algebras, and in particular on suitable settings for non-formal deformation quantization. In [26] we proposed a notion of deformation quantization of a Fréchet Poisson algebra involving a convergent star product for Poisson algebras in the Fréchet category, as originally given by Gel'fand and Shilov [13] and applied to infinite dimensional Lie groups in e.g. [21]. In this paper we show that this class of Fréchet algebras successfully handles non-formal deformation quantization in the Fréchet category.

We then discuss ordering problems in this context and propose a naive concept we label the “*independence of ordering principle*” (the principle of independence of which order we choose). This concept seems to play an important role in non-formal deformation quantization. We show that this principle implies that star exponential functions of quadratic forms should be viewed as double valued functions. Finally, as an example of the strength of the “independence of ordering principle,” we show that it leads to the appearance of transcendental calculus in non-formal deformation quantization.

2. Deformation Quantization of Fréchet Poisson Algebra

To deal with non-formal deformation quantization, we have introduced a notion of deformation quantization of Poisson algebras in the Fréchet categories in [26]. We first recall it.

Let \mathcal{F} be a commutative, associative Fréchet algebra over \mathbb{C} , i.e., \mathcal{F} has a metrizable complete topology defined by a family of semi-norms, and a smooth product operation denoted by dot \cdot . \mathcal{F} is called a *Fréchet Poisson algebra* if \mathcal{F} has a continuous bilinear operation $\{ , \} : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ (called a *Poisson bracket* on \mathcal{F}) such that for any $f, g, h \in \mathcal{F}$,

- P1** $\{f, g\} = -\{g, f\}$ (skew-symmetry)
P2 $\sum_{\text{cyclic}} \{f, \{g, h\}\} = 0$ (Jacobi identity)
P3 $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (bi-derivation).

More generally, if \mathcal{F} is a Fréchet space with an associative product $*$ such that the operation $*$: $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ is continuous, we call $(\mathcal{F}, *)$ a *Fréchet algebra*. We now give a notion of *deformation quantization of a Fréchet Poisson algebra* as a family of associative products $*_{\hbar}$ on \mathcal{F} parametrized by $\hbar \in \mathbb{R}$.

For a Fréchet Poisson algebra \mathcal{F} , we define a notion of non-formal deformation quantization (cf. [26]). This is similar to the notion of strict deformation quantization defined by Rieffel [34], which was considered in the C^* -algebra categories.

DEFINITION 2.1. Let $\hbar \in \mathbb{R}$. Let \mathcal{F} be a Fréchet Poisson algebra. $(\mathcal{F}, *_{\hbar})$ is called a *deformation quantization of the Fréchet Poisson algebra \mathcal{F}* if the following conditions hold:

- (FD1)** For any \hbar , there exists an associative product $*_{\hbar}$ on \mathcal{F} so that $(\mathcal{F}, *_{\hbar})$ is a Fréchet algebra.
(FD2) $f *_{\hbar} g \rightarrow f \cdot g$ as $\hbar \rightarrow 0$ for every $f, g \in \mathcal{F}$.
(FD3) $\frac{1}{\hbar}(\{f *_{\hbar} g - f \cdot g\}) \rightarrow \frac{1}{2}\{f, g\}$ as $\hbar \rightarrow 0$ for every $f, g \in \mathcal{F}$.

As convenient spaces to set up non-formal deformation quantization, we introduce a Fréchet algebra of entire functions on the complex n -space \mathbb{C}^n with the coordinates $z = (z_1, \dots, z_n)$. Let $\mathcal{E}(\mathbb{C}^n)$ be the set of entire functions on \mathbb{C}^n . We consider the following subspace of $\mathcal{E}(\mathbb{C}^n)$: For every positive $p > 0$, set

$$\mathcal{E}_p(\mathbb{C}^n) = \{f \in \mathcal{E}(\mathbb{C}^n) \mid \|f\|_{p,s} = \sup |f| e^{-s|z|^p} < \infty, \forall s > 0\} \quad (2.1)$$

where $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$. This class of functions has been introduced by Gel'fand and Shilov [13]. The family of semi-norms $\{\|\cdot\|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^n)$ and $(\mathcal{E}_p(\mathbb{C}^n), \cdot)$ is an associative commutative Fréchet algebra, where the dot \cdot is the ordinary multiplication for functions in $\mathcal{E}_p(\mathbb{C}^n)$. It is easily seen that for $0 < p < p'$, we have a continuous embedding

$$\mathcal{E}_p(\mathbb{C}^n) \subset \mathcal{E}_{p'}(\mathbb{C}^n) \quad (2.2)$$

of commutative Fréchet algebras. It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^n)$ and that $\mathcal{P}(\mathbb{C}^n)$ is dense in $\mathcal{E}_p(\mathbb{C}^n)$ for any $p > 0$.

3. Non-formal Deformation Quantizations

We now give examples of non-formal deformation quantizations of Fréchet Poisson algebras based on the class $\mathcal{E}_p(\mathbb{C}^n)$. We note that the examples in this section have invariance or covariance properties, which will be preferable examples to treat non-formal deformation quantizations.(see also [2])

3.1. CANONICAL POISSON ALGEBRA

We set $n = 2m$. Throughout of this paper, $\mathfrak{S}(2m)$ will denote the space of *symmetric complex $2m \times 2m$ -matrices*.

We give non-formal deformation quantizations of this Poisson algebra (3.1) parametrized by $\mathfrak{S}(2m)$. Consider the canonical Poisson structures on $\mathcal{E}_p(\mathbb{C}^{2m})$:

$$\{f, g\} = f \left(\sum_{i,j=1}^{2m} \overleftarrow{\partial}_{z_i} J^{ij} \overrightarrow{\partial}_{z_j} \right) g \tag{3.1}$$

for functions $f = f(z)$ and $g = g(z)$, where $J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$ and the arrow for the differentials indicates the side of the action for functions. Then, $\mathcal{E}_p(\mathbb{C}^{2m})$ is a Fréchet Poisson algebra with the canonical Poisson bracket (3.1).

For every $K = (K^{ij}) \in \mathfrak{S}(2m)$, we define the product $*_K$ by the following formula:

$$f *_K g = f \exp \left\{ \frac{i\hbar}{2} \left(\sum_{i,j=1}^{2m} \overleftarrow{\partial}_{z_i} \Lambda^{ij} \overrightarrow{\partial}_{z_j} \right) \right\} g, \tag{3.2}$$

where $\Lambda = (\Lambda^{ij}) = (K^{ij} + J^{ij})$. It will be helpful to rewrite the formula (3.2) in the following form:

$$f *_K g = \sum_k \frac{(i\hbar)^k}{k! 2^k} \sum_{i_1 \dots i_k, j_1 \dots j_k} \Lambda^{i_1 j_1} \dots \Lambda^{i_k j_k} \partial_{z_{i_1}} \dots \partial_{z_{i_k}} f \partial_{z_{j_1}} \dots \partial_{z_{j_k}} g. \tag{3.3}$$

The product (3.3) is well-defined at least for the polynomials f, g on \mathbb{C}^{2m} and the associativity holds. This product formula gives the commutation relation:

$$z^i *_K z^j - z^j *_K z^i (= [z^i, z^j]_{*_K}) = i\hbar J^{ij}, \tag{3.4}$$

which give the same commutation relations as the Weyl algebra W_{\hbar} . Let $\mathcal{P}(\mathbb{C}^{2m})$ be the set of polynomials of (z_1, \dots, z_{2m}) . Summarizing above, we have realizations of the Weyl algebra.

PROPOSITION 3.1. *For every $K \in \mathfrak{S}(2m)$, $(\mathcal{P}(\mathbb{C}^{2m}), *_K)$ forms an associative algebra isomorphic to the Weyl algebra W_{\hbar} .*

Proposition 3.1 gives a representation of the Weyl algebra W_{\hbar} , and it contains no further relation other than Weyl algebra. The product formula (3.3) gives also the unique expression of elements of the Weyl algebra W_{\hbar} by the usual polynomials.

According to the choice of $K = 0, K_0$, where $(0, K_0) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right)$, the product formulas are given respectively by the following formula:

$$\begin{aligned} f(z)*_0g(z) &= f \exp \frac{\hbar i}{2} \{ \overleftarrow{\partial}_v \wedge \overrightarrow{\partial}_u \} g, & (\text{Moyal product formula}) \\ f(z)*_{K_0}g(z) &= f \exp \hbar i \{ \overleftarrow{\partial}_v \overrightarrow{\partial}_u \} g, & (\Psi\text{DO product formula}) \end{aligned} \tag{3.5}$$

where $z = (z^1, \dots, z^{2m}) = (u^1, \dots, u^m, v^1, \dots, v^m)$, $\overleftarrow{\partial}_v \wedge \overrightarrow{\partial}_u = \sum_i (\overleftarrow{\partial}_{v^i} \overrightarrow{\partial}_{u^i} - \overleftarrow{\partial}_{u^i} \overrightarrow{\partial}_{v^i})$ and $\overleftarrow{\partial}_v \overrightarrow{\partial}_u = \sum_i \overleftarrow{\partial}_{v^i} \overrightarrow{\partial}_{u^i}$. The pseudo-differential operator (ΨDO) ordering is usually called ‘‘standard’’ ordering in physics.

The product formula is well-defined on $\mathcal{P}(\mathbb{C}^{2m})$ which is a dense in the space $\mathcal{E}_p(\mathbb{C}^{2m})$. We note that every exponential function $e^{ax+\beta y}$ is contained in $\mathcal{E}_p(\mathbb{C}^{2m})$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^{2m})$, and functions such as $e^{ax^2+by^2+2cxy}$ are contained in $\mathcal{E}_p(\mathbb{C}^{2m})$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^{2m})$.

THEOREM 3.1. *The product formula (3.3) gives the following:*

- (i) For $0 < p \leq 2$, the space $(\mathcal{E}_p(\mathbb{C}^{2m}), *_K)$ is a deformation quantization of $(\mathcal{E}_p(\mathbb{C}^{2m}), \cdot, \{, \})$ for every $K \in \mathfrak{S}(2m)$.
- (ii) For $p > 2$ and a fixed $\hbar \in \mathbb{R}$, the product formula (3.3) gives a continuous bilinear mapping

$$\begin{aligned} \mathcal{E}_p(\mathbb{C}^{2m}) \times \mathcal{E}_{p'}(\mathbb{C}^{2m}) &\rightarrow \mathcal{E}_p(\mathbb{C}^{2m}), \\ \mathcal{E}_{p'}(\mathbb{C}^{2m}) \times \mathcal{E}_p(\mathbb{C}^{2m}) &\rightarrow \mathcal{E}_p(\mathbb{C}^{2m}), \end{aligned} \tag{3.6}$$

for every p' such that $\frac{1}{p} + \frac{1}{p'} \geq 1$.

(cf. [26]).

Since $p > 2$, we must have $p' \leq 2$, hence by statement (i) $(\mathcal{E}_{p'}(\mathbb{C}^{2m}) : *_\hbar)$ is a Fréchet algebra. The statement (ii) means that every $\mathcal{E}_p(\mathbb{C}^{2m})$, $p > 2$, is a topological 2-sided $\mathcal{E}_{p'}(\mathbb{C}^{2m})$ -module.

Theorem 3.1 shows that the space $\mathcal{E}_p(\mathbb{C}^{2m})$ ($p \leq 2$) gives an example of non-formal deformation quantization.

3.2. $ax + b$ -GROUP

As a deformation quantization invariant under non-abelian Lie group actions, the universal formulae in the case of $ax + b$ have been studied within the context of the Wigner formalism and signal analysis (cf. [7]). We show that this universal deformation formula gives us a non-formal deformation quantization for a certain class of holomorphic functions defined by [13].

Let \mathcal{G} denote the Lie algebra of the group of affine transformations of the real line. Formally, one has $\mathcal{G} = \text{span}_{\mathbb{R}}\{A, E\}$ with table $[A, E] = 2E$. Consider the linear map $\lambda : \mathcal{G} \rightarrow C^\infty(\mathbb{R}^2) : X \mapsto \lambda_X$ defined by $\lambda_A(a, l) = 2l; \quad \lambda_E(a, l) = e^{-2a}$,

where $\mathbb{R}^2 = \{(a, l)\}$. One then checks that the map λ is a homomorphism of Lie algebras when $C^\infty(\mathbb{R}^2)$ is endowed with the symplectic Poisson bracket $\{, \} := \partial_a \wedge \partial_l$. Moreover, if $*_v^M$ denotes the formal Moyal star product on $C^\infty(\mathbb{R}^2)[[v]]$ (i.e. $u *_v^M v = u \exp(v \overleftarrow{\partial}_a \wedge \overrightarrow{\partial}_l) v$, $u, v \in C^\infty(\mathbb{R}^2)[[v]]$), one has $[\lambda_A, \lambda_E]_v = 2v\{\lambda_A, \lambda_E\}$ (where $[u, v]_v := u *_v^M v - v *_v^M u$). In particular, the formula

$$\rho_v(X)u := \frac{1}{2v} [\lambda_X, u]_v \quad X \in \mathcal{G}, \quad u \in C^\infty(\mathbb{R}^2)[[v]]$$

defines a homomorphism of Lie algebras

$$\rho_v : \mathcal{G} \rightarrow \text{Der}(C^\infty(\mathbb{R}^2)[[v]], *_v^M).$$

Explicitly, one has $\rho_v(A)u = -\partial_a u$; $\rho_v(E)u = -\frac{e^{-2a}}{v} \sinh(v\partial_l)u$. Intertwining the representation ρ_v by a transformation of the type

$$\mathcal{L}(u)(a, z) := \int_{\mathbb{R}} e^{-zl} u(a, l) dl,$$

one gets

$$\begin{aligned} \hat{\rho}_v(A)\mathcal{L}(u) &:= \mathcal{L}(\rho_v(A)u) = -\partial_a \mathcal{L}(u); \\ \hat{\rho}_v(E)\mathcal{L}(u) &:= \mathcal{L}(\rho_v(E)u) = -\frac{e^{-2a}}{v} \sinh(vz)\mathcal{L}(u), \end{aligned}$$

where we assumed $u(a, \pm\infty) = 0$. Now, set formally

$$\mathcal{Z}_v(u)(a, z) := \int_{\mathbb{R}} \exp\left(\gamma \frac{1}{v} \sinh(vz)l\right) u(a, l) dl,$$

and

$$f \bullet_v g := \mathcal{Z}_v(\mathcal{Z}_v^{-1} f \mathcal{Z}_v^{-1} g) \quad (\gamma \in \mathbb{C}_\times).$$

PROPOSITION 3.2. *For all $X \in \mathcal{G}$, $\hat{\rho}_v(X)$ is a derivation of the commutative product \bullet_v .*

DEFINITION 3.1. Let $\alpha, \beta \in \mathbb{R}^n$. The fundamental space $S_\alpha^\beta(n)$ is defined as the space of holomorphic functions $\varphi \in \mathcal{E}(\mathbb{C}^n)$ such that there exists $a, b \in (\mathbb{R}^+)^n$ and $C > 0$ with

$$|\varphi(x + iy)| \leq C \exp\left(-a|x|^{\frac{1}{\alpha}} + b|y|^{\frac{1}{1-\beta}}\right), \quad x, y \in \mathbb{R}^n,$$

where we adopt the usual notations : $a|x|^e = \sum_j a_j |x_j|^{e_j}$ ($a, x, e \in \mathbb{R}^n$); $\frac{1}{\alpha} = (\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n})$; $1 - \beta = (1 - \beta_1, \dots, 1 - \beta_n)$ (cf. [13]).

Every element $\varphi \in \mathcal{S}_\alpha^\beta(n)$ is entirely determined by its restriction the “real axis” $\varphi(x) \quad x \in \mathbb{R}^n$. We will often identify the space $\mathcal{S}_\alpha^\beta(n)$ with the subspace $(\mathcal{S}_\alpha^\beta(n))_x$ of $C^\infty(\mathbb{R}^n)$ constituted by the restrictions. In order to consider only non-trivial spaces, we will assume $\alpha + \beta \geq 1; \alpha > 0; \beta > 0$. We will denote by $\mathcal{F}(u)(\xi)$ the Fourier transform of the function $u \in L^1(\mathbb{R}^n)$:

$$\mathcal{F}(u)(\xi) := \int_{\mathbb{R}^m} e^{i\xi x} u(x) dx,$$

where ξx denotes the canonical dot product on \mathbb{R}^m . For even $n = 2m$, we will denote by J the endomorphism of \mathbb{R}^{2m} defined by the matrix $J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$. We denote by ω the bilinear symplectic structure on \mathbb{R}^{2n} defined by $\omega(x, y) := x J y$.

DEFINITION 3.2. We define the symplectic Fourier transform of the function $u \in L^1(\mathbb{R}^{2m})$ as

$$S\mathcal{F}(u)(y) := \int_{\mathbb{R}^{2m}} e^{i\omega(x,y)} u(x) dx \quad (y \in \mathbb{R}^{2m}).$$

DEFINITION 3.3. The Weyl product between u and v in $L^1(\mathbb{R}^{2m})$ is defined by

$$u *^w_q v := S\mathcal{F}^{-1} [S\mathcal{F}(u) \times_q S\mathcal{F}(v)],$$

where

$$u \times_q v(x) := \int_{\mathbb{R}^{2m}} e^{iq\omega(x,y)} u(y) v(x-y) dy \quad (q \in \mathbb{R}).$$

DEFINITION 3.4. Let $q \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. We define the twisting map $\phi_{q,\theta} : \mathbb{C} \rightarrow \mathbb{C}$ by

$$\begin{cases} \phi_{q,\theta}(z) = \frac{e^{i\theta}}{q} \sinh(iqz) & \text{if } q \neq 0 \\ \phi_{0,\theta}(z) = z. \end{cases}$$

Let us consider the fundamental space $\mathcal{S}_{(\alpha_1, \alpha_2)}^{\sigma(\alpha_1, \alpha_2)}(2)$, $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. Let $\varphi \in \mathcal{S}_{(\alpha_1, \alpha_2)}^{\sigma(\alpha_1, \alpha_2)}(2)$ and consider the partial function $\varphi_{x_1} : x \mapsto \varphi(x_1, x)$. For all $x_1 \in \mathbb{R}$, the function φ_{x_1} belongs to $\mathcal{S}_{\alpha_2}^{\alpha_1}(1) =: \mathcal{S}_{\alpha_2}^{\alpha_1}$. Therefore provided some restrictions on (α_1, α_2) , the function $\mathcal{L}^{-1}(\phi_{q,k\frac{\pi}{2}}^{-1})^* \mathcal{L}(\varphi_{x_1})$ ($k=0, 1$) is well defined as an element of \mathcal{S}'_I .

DEFINITION 3.5. We define the linear map

$$\mathcal{S}_{(\alpha_1, \alpha_2)}^{\sigma(\alpha_1, \alpha_2)}(2) \xrightarrow{\tau_q^{(k)}} C^\infty(\mathbb{R}^2) \quad (k=0, 1)$$

by

$$\tau_q^{(k)} := \text{id}_{x_1} \otimes \left(\mathcal{L}^{-1} \circ (\phi_{q,k\frac{\pi}{2}}^{-1})^* \circ \mathcal{L} \right)_{x_2} \quad (x_1, x_2) \in \mathbb{R}^2.$$

We denote by $E_{(\alpha_1, \alpha_2)}^{(k)}$ its range in $C^\infty(\mathbb{R}^2)$. The inverse map

$$\text{id}_{x_1} \otimes \left(\mathcal{L}^{-1} \circ \phi_{q,k\frac{\pi}{2}}^* \circ \mathcal{L} \right)_{x_2} \Big|_{E_{(\alpha_1, \alpha_2)}^{(k)}}$$

will be denoted by $T_q^{(k)}$. It yields a linear isomorphism $T_q^{(k)} : E_{(\alpha_1, \alpha_2)}^{(k)} \rightarrow \mathcal{S}_{(\alpha_1, \alpha_2)}^{\sigma(\alpha_1, \alpha_2)}(2)$.

DEFINITION 3.6. The product $*_q^{(k)}$ on $E_{(\alpha_1, \alpha_2)}^{(k)}$ will be referred as the twisted Weyl product.

Let $A = (\alpha_1, \alpha_2) \in (0, 1)^2$ with $\alpha_1 + \alpha_2 \geq 1$. Set $\mathcal{S}_A := \mathcal{S}_{(\alpha_1, \alpha_2)}^{\sigma(\alpha_1, \alpha_2)}(2)$. The function $\hat{f}(y_1, x_2) := f(iy_1, x_2)$ determines completely f . So that we have an injection $(\mathcal{S}_A)_x \rightarrow C^\infty(\mathbb{R}^2) : f \mapsto \hat{f}$. We denote by $\widehat{*}_q^w$ the product on $\widehat{\mathcal{S}}_A$ obtained by transporting Weyl’s product on $(\mathcal{S}_A)_x$ via $f \mapsto \hat{f}$. Observe that $\widehat{\mathcal{S}}_A$ is still stable under the pointwise multiplication whose $\widehat{*}_q^w$ is a non-commutative deformation of $\widehat{\mathcal{S}}_A$. Observe also that for every $\psi \in \widehat{\mathcal{S}}_A$ and $y_1 \in \mathbb{R}$ the partial function $x_2 \mapsto \psi(y_1, x_2)$ is in $\mathcal{S}_{\alpha_2}^{\alpha_1}$. Therefore the transformations $\tau_q^{(k)}$ and $T_q^{(k)}$ are well defined on $\widehat{\mathcal{S}}_A$.

This procedure gives a non-formal deformation quantization [6].

THEOREM 3.2. Set $\widehat{E}_{(\alpha_1, \alpha_2)}^{(k)} := \tau_q^{(k)}(\widehat{\mathcal{S}}_A)$. Then for all $a, b \in \widehat{E}_{(\alpha_1, \alpha_2)}^{(k)}$, the formula

$$a \widehat{*}_q^{(k)} b := \tau_q^{(k)} \left(T_q^{(k)} a \widehat{*}_q^w T_q^{(k)} b \right)$$

defines an associative \mathbb{R} -algebra structure on $\widehat{E}_{(\alpha_1, \alpha_2)}^{(k)}$. The space $\widehat{E}_{(\alpha_1, \alpha_2)}^{(k)}$ contains elements of exponential growth.

3.3. HEISENBERG LIE ALGEBRA

We formulate the deformation quantization of a Fréchet Poisson algebra associated with the $(2m + 1)$ -dimensional Heisenberg Lie algebra in the class \mathcal{S}_b^a of generalized functions on \mathbb{C}^{2m+1} given by Gel’fand and Shilov [13] (cf. [32]). Let \mathbb{C}^{n+1} be a complex $(n + 1)$ -space with complex coordinates $z = (z_0, z_1, \dots, z_n)$ and $\mathcal{P}(\mathbb{C}^{n+1})$ the set of all polynomials on \mathbb{C}^{n+1} . The usual pointwise multiplication

$$(f \cdot g)(p) = f(p) \cdot g(p)$$

for polynomial functions f, g on \mathbb{C}^{n+1} gives a commutative associative structure on $\mathcal{P}(\mathbb{C}^{n+1})$.

To define a system of semi-norms, we use the following notations: \tilde{p} and \tilde{b} , etc. denotes $(n + 1)$ -tuples $\tilde{p} = (p_0, p_1, \dots, p_n)$ and $\tilde{b} = (b_0, b_1, \dots, b_n)$ with $p_i > 0$

and $b_i > 0$ for $0 \leq i \leq n$. Forgetting p_0 and b_0 , we let \tilde{p}_* and \tilde{b}_* denote $\tilde{p}_* = (p_1, \dots, p_n)$ and $\tilde{b}_* = (b_1, \dots, b_n)$, respectively.

DEFINITION 3.7. Let r_0 and N_0 be positive real numbers and nonnegative integers, respectively. We define semi-norms $\|\cdot\|_{\tilde{p}, \tilde{b}}$, $\|\cdot\|_{\tilde{p}_*, \tilde{b}_*, r_0}$ and $\|\cdot\|_{\tilde{p}_*, \tilde{b}_*, N_0}$ on $\mathcal{P}(\mathbb{C}^{n+1})$ as follows:

$$\|f\|_{\tilde{p}, \tilde{b}} = \sup_{(z_0, \dots, z_n) \in \mathbb{C}^{n+1}} |f| \exp\left(-\sum_{i=0}^n b_i |z_i|^{p_i}\right), \quad (3.7)$$

$$\|f\|_{\tilde{p}_*, \tilde{b}_*, r_0} = \sup_{|z_0| \leq r_0} \sup_{(z_1, \dots, z_n) \in \mathbb{C}^n} |f| \exp\left(-\sum_{i=1}^n b_i |z_i|^{p_i}\right), \quad (3.8)$$

$$\|f\|_{\tilde{p}_*, \tilde{b}_*, N_0} = \sum_{k=0}^{N_0} \|f_k(z_1, \dots, z_n)\|_{\tilde{p}_*, \tilde{b}_*}, \quad (3.9)$$

where we expand a polynomial f as $f(z_0, z_1, \dots, z_n) = \sum_{k=0} f_k(z_1, \dots, z_n) z_0^k$ as a power series of z_0 variable.

We denote the completions of $\mathcal{P}(\mathbb{C}^{n+1})$ under the systems of seminorms $\{\|\cdot\|_{\tilde{p}, \tilde{b}}\}_{\tilde{b}}$, $\{\|\cdot\|_{\tilde{p}_*, \tilde{b}_*, r_0}\}_{\tilde{b}_*, r_0}$ and $\{\|\cdot\|_{\tilde{p}_*, \tilde{b}_*, N_0}\}_{\tilde{b}_*, N_0}$, respectively, by

$$(E.1): \mathcal{E}_{\tilde{p}}(\mathbb{C}^{n+1}), \quad (E.2): \mathcal{E}_{\text{Hol}, \tilde{p}_*}(\mathbb{C}^{n+1}), \quad (E.3): \mathcal{E}_{\infty, \tilde{p}_*}(\mathbb{C}^{n+1}).$$

We abbreviate the notation by using $\mathcal{E}_{\Omega}(\mathbb{C}^{n+1})$ for any one of $\mathcal{E}_{\tilde{p}}(\mathbb{C}^{n+1})$, $\mathcal{E}_{\text{Hol}, \tilde{p}_*}(\mathbb{C}^{n+1})$ and $\mathcal{E}_{\infty, \tilde{p}_*}(\mathbb{C}^{n+1})$. Then, $\mathcal{E}_{\Omega}(\mathbb{C}^{n+1})$ is a commutative associative Fréchet algebra.

We remark that $\mathcal{E}_{\tilde{p}}(\mathbb{C}^{n+1})$ and $\mathcal{E}_{\text{Hol}, \tilde{p}_*}(\mathbb{C}^{n+1})$ are subalgebras of all entire functions $\mathcal{E}(\mathbb{C}^{n+1})$ on \mathbb{C}^{n+1} , and $\mathcal{E}_{\infty, \tilde{p}_*}(\mathbb{C}^{n+1})$ is the space $\mathcal{E}_{\tilde{p}_*}(\mathbb{C}^n)[[z_0]]$ of all formal power series of z_0 with coefficients in $\mathcal{E}_{\tilde{p}_*}(\mathbb{C}^n)$ with the z_0 -adic direct product topology.

We set $n = 2m$ and let $(z_0, z_1, \dots, z_n) = (z, x_1, \dots, x_m, y_1, \dots, y_m)$. We are mainly concerned with the case $p_1 = p_i$ for $1 \leq i \leq n$, so we set $p = (p_0, p_1, \dots, p_1)$ and $p_* = (p_1, \dots, p_1)$.

We set

$$\{f, g\}_H = z \left(f \left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x \right) g \right), \quad (3.10)$$

for functions $f = \overleftarrow{f}(z, x_1, \dots, x_m, y_1, \dots, y_m)$ and $g = g(z, x_1, \dots, x_m, y_1, \dots, y_m)$, where $\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x$ stands for a bidifferential operator:

$$f \left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x \right) g = \sum_{i=1}^m (\partial_{x_i} f \cdot \partial_{y_i} g - \partial_{y_i} f \cdot \partial_{x_i} g). \quad (3.11)$$

We have

$$\{z, x_i\}_H = 0, \quad \{z, y_i\}_H = 0, \quad \{x_i, y_j\}_H = z\delta_{ij}. \tag{3.12}$$

which gives a linear Poisson structure on \mathbb{C}^{2m+1} associated to the Heisenberg Lie algebra.

LEMMA 3.1. *Let $\mathcal{E}_\Omega(\mathbb{C}^{2m+1})$ be one of (E.1)–(E.3). Then $(\mathcal{E}_\Omega(\mathbb{C}^{2m+1}), \cdot, \{, \}_H)$ is a Fréchet Poisson algebra.*

Let $\mathcal{E}_\Omega(\mathbb{C}^{2m+1})$ be a Fréchet Poisson algebra given by Lemma 3.1. We consider a noncommutative product which gives a deformation quantization of $(\mathcal{E}_\Omega(\mathbb{C}^3), \cdot, \{, \}_H)$: Setting $\hbar = 1$, we define a product $f * g$ for $f, g \in \mathcal{P}(\mathbb{C}^3)$ by the product formula, also called the Moyal product formula:

$$f * g = \sum_{p=0}^{\infty} \frac{(iz)^p}{2^p p!} \left(f \left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x \right)^p g \right). \tag{3.13}$$

As for $f, g \in \mathcal{P}(\mathbb{C}^{2m+1})$, the product (3.13) is an associative product.

Similar to Theorem 3.1, we have a non-formal deformation quantization of linear Poisson algebra of Heisenberg type [32]:

THEOREM 3.3. *Let $(\mathcal{E}_\Omega(\mathbb{C}^{2m+1}), \cdot, \{, \}_H)$ be a Fréchet Poisson algebra given by Lemma 3.1. Assume that p is given by $p = (p_0, p_1, \dots, p_1)$. Then, we have $(\mathcal{E}_\Omega(\mathbb{C}^{2m+1}), *)$ is an associative Fréchet algebra if and only if Ω satisfies one of the following:*

- (A1) For $\Omega = p = (p_0, p_1, \dots, p_1)$, $0 < p_1 \leq \frac{2p_0}{p_0+1}$,
- (A2) For $\Omega = (Hol, p_*)$, $p_* = (p_1, \dots, p_1)$, $0 < p_1 \leq 2$,
- (A3) For $\Omega = (\infty, p_*)$, $p_* = (p_1, \dots, p_1)$, $0 < p_1$.

4. Independence of Ordering Principle

In this section, we propose an idea for treating elements of an abstract algebra by taking the Weyl algebra as an example, which will be called *independence of ordering principle*. In the following sections, we go back to the case of the non-formal deformation quantization of the canonical Poisson algebra defined by Section 3.1.

4.1. INTERTWINERS

The intertwiner between the K -ordered and the K' -ordered expressions is explicitly given as follows:

PROPOSITION 4.1. *For every $K, K' \in \mathfrak{S}(2m)$, the intertwiner is defined by*

$$I_K^{K'}(f) = \exp\left(\frac{i\hbar}{4} \sum_{i,j} (K'^{ij} - K^{ij}) \partial_{u_i} \partial_{u_j}\right) f \quad (= I_0^{K'} (I_0^K)^{-1}(f)), \tag{4.1}$$

and it gives an algebra isomorphism $I_K^{K'} : (\mathcal{P}(\mathbb{C}^{2m}), *_K) \rightarrow (\mathcal{P}(\mathbb{C}^{2m}); *_{K'})$. Namely, the following identity holds for any $f, g \in \mathcal{P}(\mathbb{C}^{2m})$:

$$I_K^{K'}(f *_K g) = I_K^{K'}(f) *_{K'} I_K^{K'}(g), \tag{4.2}$$

In this manner, intertwiners does not change the product structure, but does change the expression of elements as usual complex valued functions.

Define the infinitesimal intertwiner at $K \in \mathfrak{S}(2m)$ to the direction Γ as follows:

$$dI_K(\Gamma)(f) = \left. \frac{d}{dt} \right|_{t=0} I_K^{K+t\Gamma}(f) = \frac{i\hbar}{4} \sum_{i,j} \Gamma^{ij} \partial_{u_i} \partial_{u_j} f. \tag{4.3}$$

This is viewed as a flat connection on the trivial bundle $\coprod_{K \in \mathfrak{S}(2m)} \text{Hol}(\mathbb{C}^{2m})$ where $\text{Hol}(\mathbb{C}^{2m})$ is the space of all entire functions on \mathbb{C}^{2m} .

Although the differential equation for parallel translations may not be solved for general initial function, every global parallel section (if exists) $\{f(K); K \in \mathfrak{S}(2m)\}$ of this bundle is naturally identified with an element of $\text{Hol}(\mathbb{C}^{2m})$ via a fixed ordered expression. One may regard such a parallel section as an element of extended Weyl algebra, and the evaluating at K is its K -ordered expression.

4.2. EXTENSION OF PRODUCTS

Let $\mathcal{P}(\mathbb{C}^{2m})[[\hbar]]$ be the space of all formal power series of \hbar with coefficients of polynomials on \mathbb{C} . Obviously, $*_K$ -products and the intertwiners extend naturally to $\mathcal{P}(\mathbb{C}^{2m})[[\hbar]]$ by the same formula. $(\mathcal{P}(\mathbb{C}^{2m})[[\hbar]], *_K)$ is an associative algebra and $I_K^{K'}$ is an algebra isomorphism of $(\mathcal{P}(\mathbb{C}^{2m})[[\hbar]], *_K)$ onto $(\mathcal{P}(\mathbb{C}^{2m})[[\hbar]], *_{K'})$.

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^{2m})$ and $\mathcal{P}(\mathbb{C}^{2m})$ is dense in $\mathcal{E}_p(\mathbb{C}^{2m})$ for any $p > 0$ in the Fréchet topology defined by the family of seminorms $\{\| \cdot \|_{p,s}\}_{s>0}$.

THEOREM 4.1. *For $0 < p \leq 2$, the intertwiner $I_K^{K'}$ extends to give an isomorphism of $(\mathcal{E}_p(\mathbb{C}^{2m}), *_K)$ onto itself (cf. [29]).*

It is easily seen that the following identities hold on $\mathcal{E}_p(\mathbb{C}^{2m})$, $p \leq 2$

$$I_{K'}^K I_K^{K'} = 1, \quad I_{K'}^{K''} I_K^{K'} = I_K^{K''}. \tag{4.4}$$

Hence, for every $f \in \mathcal{E}_p(\mathbb{C}^{2m})$, the set $f_*(\mathbf{u}) = \{I_0^K(f); K \in \mathfrak{S}(2m)\}$ is a globally defined parallel section.

For every $f \in \mathcal{E}_p(\mathbb{C}^{2m})$ such that $p \leq 2$, $f(K) = I_0^K(f)$ is a globally defined parallel section. Thus, we naturally extends our object to the space of all parallel sections $\{f(K); K \in \mathfrak{S}(2m)\}$ of the trivial bundle

$$\coprod_{K \in \mathfrak{S}(2m)} (\mathcal{E}_p(\mathbb{C}^{2m}), *_{K}), \quad (0 < p \leq 2).$$

On the other hand, several anomalous phenomena occur on the space $(\mathcal{E}_{2+}(\mathbb{C}^{2m}), *_{K}) = \bigcap_{p>2} (\mathcal{E}_p(\mathbb{C}^{2m}), *_{K})$.

4.3. INDEPENDENCE OF ORDERING PRINCIPLE

We have introduced the notion of *ordered expression* to realize elements of the Weyl algebra using the K -ordered product (3.3). Obviously, the algebraic structure of $(\mathcal{P}(\mathbb{C}^{2m}), *_{K})$ depends only on the skew part of Λ . Elements of the Weyl algebra are expressed in terms of elements of $\mathcal{P}(\mathbb{C}^{2m})$ via a K -ordered expression once we choose a $*_{K}$ -product defined by (3.3). For every K and K' , the corresponding K - and K' -ordered expressions have explicit relations given by an intertwiner.

We interpret the various expressions for the Weyl algebra elements together with the intertwining relations as the *independence of ordering principle* (IOP) (exactly means “the principle of independence of which order we choose.”), parallel to the standard notion in geometry and physics that geometric and physical objects are coordinate free quantities. These suggestions in geometry and physics seem parallelly to propose a naive idea of IOP in treating abstract algebra.

As will be seen below, the IOP has nontrivial implications, since many delicate anomalous phenomena appear when considering star exponential functions. Moreover, these ordered expressions occur in the transcendental calculus of non-formal deformation quantization. The main goal of this paper is to propose a method for handling anomalous objects appearing in the construction of star exponential functions of quadratic forms in the Weyl algebras.

In fact, we show below that there may be difficulties in implementing the IOP. In spite of these difficulties, we are hopeful that the IOP provides deeper insight into abstract algebra.

5. Star Exponentials of Linear Functions

Set $f(K) = I_0^K(f)$. Then, we easily see that $I_K^{K'} f(K) = f(K')$. We think that the set $\{(K; I_0^K(f)); K \in \mathfrak{S}(2m)\}$ expresses a certain element $f_*(z)$ of the Weyl algebra W_{\hbar}^{2m} , where $z = (z_1, \dots, z_{2m})$ are the generators of the Weyl algebra which are also identified with the coordinates of the complex \mathbb{C}^{2m} . $f(K)$ is viewed as the K -ordered expression of $f_*(z)$. We denote this by

$$:f_*(z):_K = f(K), \quad (\text{precisely } :f_*(z):_{K,z} = f(K),)$$

where $: :_{K,z}$ means the K -ordered expression with respect to the generators z . If a generator system is fixed, $: :_{K,z}$ is simply denoted by $: :_K$.

Let $\lambda = (\lambda_1, \dots, \lambda_{2m}) \in \mathbb{C}^{2m}$. By a direct calculation of the intertwiner, we see that

$$I_K^{K'} (e^{\frac{1}{i\hbar} \langle \lambda, z \rangle}) = e^{\frac{1}{4i\hbar} \langle \lambda(K' - K), \lambda \rangle} e^{\frac{1}{i\hbar} \langle \lambda, z \rangle}. \tag{5.1}$$

For $\lambda \in \mathbb{C}^{2m}$ and $K \in \mathfrak{S}(2m)$, we denote by $\langle \lambda, z \rangle = \sum_{i,j} \lambda_i z_j$ and $\langle \lambda K, \lambda \rangle = \sum_{i,j} K^{ij} \lambda_i \lambda_j$. The set $\{e^{\frac{1}{4i\hbar} \langle \lambda K, \lambda \rangle} e^{\frac{1}{i\hbar} \langle \lambda, z \rangle}; K \in \mathfrak{S}(2m)\}$ is viewed as a single element in a transcendentially extended Weyl algebra.

$$: e_*^{\frac{1}{i\hbar} \langle \lambda, z \rangle} :_K = e^{\frac{1}{4i\hbar} \langle \lambda K, \lambda \rangle} e^{\frac{1}{i\hbar} \langle \lambda, z \rangle} = e^{\frac{1}{4i\hbar} \langle \lambda K, \lambda \rangle + \frac{1}{i\hbar} \langle \lambda, z \rangle}. \tag{5.2}$$

It is easy to check that the exponential law holds for every ordered expression. Hence one may write

$$e_*^{s \frac{1}{i\hbar} \langle \lambda, z \rangle} * e_*^{t \frac{1}{i\hbar} \langle \lambda, z \rangle} = e_*^{(s+t) \frac{1}{i\hbar} \langle \lambda, z \rangle}.$$

Viewing that differentiation is defined for star exponential functions, one may write

$$\frac{d}{ds} e_*^{s \frac{1}{i\hbar} \langle \lambda, z \rangle} = \frac{1}{i\hbar} \langle \lambda, z \rangle * e_*^{s \frac{1}{i\hbar} \langle \lambda, z \rangle}.$$

$e_*^{s \frac{1}{i\hbar} \langle \lambda, z \rangle}$ may be called *ordering free expression* of star exponential functions.

Suppose $\text{Re} \frac{1}{i\hbar} \langle \lambda K, \lambda \rangle < 0$. Then it is clear that for every $\alpha \in \mathbb{C}$, the integral

$$\int_{-\infty}^{\infty} : e_*^{t(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)} :_K dt$$

converges. Using standard estimates in our Fréchet context, the following definitions

$$\begin{aligned} \vartheta_* \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) &= \sum_{n \in \mathbb{Z}} e_*^{n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)} \\ \vartheta_{*+} \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) &= \sum_{n=0}^{\infty} e_*^{n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)}, \quad \vartheta_{*-} \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) = \sum_{n=0}^{\infty} e_*^{-n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)} \end{aligned} \tag{5.3}$$

make sense. These are parallel sections defined on an open (but not dense) domain \mathfrak{D}_λ , where

$$\mathfrak{D}_\lambda = \left\{ K \in \mathfrak{S}(2m); \text{Re} \frac{1}{i\hbar} \langle \lambda K, \lambda \rangle < 0 \right\}.$$

Remark 5.1. We use notations such as $e_*^{s \frac{1}{i\hbar} \langle \lambda, z \rangle}$, $\vartheta_{*+}(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)$ without $: :_K$ for parallel sections defined on some open domain of expressions. These may be called *elements written by the independence of ordering*.

Denote the K -ordered expression of (5.3) by

$$\vartheta_K \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) =: \sum_{n \in \mathbb{Z}} e_*^{n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)} :_K.$$

Then, we see

$$\vartheta_K \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) = \sum_{n \in \mathbb{Z}} e^{n^2 \frac{1}{4i\hbar} \langle \lambda K, \lambda \rangle} e^{n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)}. \tag{5.4}$$

Setting $\tau = \frac{1}{4i\hbar} \langle \lambda K, \lambda \rangle$ and $q = e^\tau$, $w = \frac{1}{2}(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)$, we see the K -ordered expression of $\vartheta_*(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)$ is given as the ordinary theta function

$$\vartheta_K(w, q) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2nw}.$$

The independence of ordering principle suggests that for orderings, for which identity :

$$e_*^{\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle} * \sum_{n=-\infty}^{\infty} e_*^{n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)} = \sum_{n=-\infty}^{\infty} e_*^{n(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)}, \tag{5.5}$$

is defined, we have the following properties in the K -ordered expression.

PROPOSITION 5.1. (i) (*Pseudo-periodicity*)

$$\vartheta_K \left(\alpha + \frac{1}{2} \langle \lambda K, \lambda \rangle + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) = e^{-\left(\frac{1}{4i\hbar} \langle \lambda K, \lambda \rangle + \alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle\right)} \vartheta_K \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right)$$

(ii) (*Reflectivity*)

$$\vartheta_K \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) = \vartheta_K \left(- \left(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle \right) \right).$$

Note also there are two different inverses of $1 - e_*^{(\alpha + \frac{1}{i\hbar} \langle \lambda, z \rangle)}$, which apparently breaks associativity (cf. [31]). See [22] for other associativity breaking phenomena.

6. Star Exponential Functions of Quadratic Functions

In this section, we show how the star exponential functions of quadratic functions on \mathbb{C}^{2m} are understood reasonably. As for computations of star exponential functions, we refer [4, 15, 17]. Our idea is to use K -ordered expressions and to glue the star exponential functions defined locally on domains via the extended intertwiners. This procedure will propose us to view the star exponential functions as double valued elements (cf. [27]–[31]).

6.1. EXTENDED INTERTWINERS

Let A be a $2m \times 2m$ symmetric complex matrix and $\langle zA, z \rangle$ the quadratic function associated with A , where $z = (z_1, \dots, z_{2m})$. For simplicity we denote $ge^{\frac{1}{i\hbar}\langle zA, z \rangle}$ by $(g; A)$.

We call g and A the amplitude and the phase part of $ge^{\frac{1}{i\hbar}\langle zA, z \rangle}$. In this notation, we see that

$$I_0^K(g; A) = (g \det(I - AK)^{-\frac{1}{2}}; T_K(A)),$$

where

$$T_K : \mathfrak{S}(2m) \rightarrow \mathfrak{S}(2m), \quad T_K(A) = \frac{1}{I - AK}A$$

is viewed as the phase part of the intertwiner I_0^K .

Computing the inverse $I_K^0 = (I_0^K)^{-1}$, and the composition $I_0^{K'} I_K^0$, we easily see

$$I_K^{K'}(g; A) = \left(g \det(I - A(K' - K))^{-\frac{1}{2}}; \frac{1}{I - A(K' - K)}A \right). \tag{6.1}$$

This mapping has a singularity at A such that $\det(I - A(K' - K)) = 0$ and the sign ambiguity can not be removed. $T_K^{K'}(A) = \frac{1}{I - A(K' - K)}A$ is viewed as the phase part of the intertwiner.

We rewrite (6.1) by

$$I_K^{K'} \left(\frac{g}{\sqrt{\det(I - AK)}}; \frac{1}{I - AK}A \right) = \left(\frac{g}{\sqrt{\det(I - AK')}}; \frac{1}{I - AK'}A \right) \tag{6.2}$$

if $I - AK, I - AK'$ are invertible.

Let $\mathcal{D}_K = \{A \in \mathfrak{S}(2m); \det(I - AK) \neq 0\}$. For every K ,

$$\tilde{\mathcal{D}}_K = \left\{ \left(\frac{g}{\sqrt{\det(I - AK)}}; \frac{1}{I - AK}A \right); K \in \mathcal{D}_K \right\}$$

is a double covering of \mathcal{D}_K .

PROPOSITION 6.1. *The intertwiner $I_K^{K'}$ is then a 2-to-2 mapping from $\tilde{\mathcal{D}}_K$ to $\tilde{\mathcal{D}}_{K'}$.*

Note that the intertwiners do not satisfy a cocycle condition of the kind that one-to-one mappings do.

Both mappings $A \rightarrow -AJ$ and $A \rightarrow JA$ give isomorphisms of $\mathfrak{S}(2m)$ onto $sp(m, \mathbb{C})$, the Lie algebra of $Sp(m, \mathbb{C})$, where

$$sp(m, \mathbb{C}) = \{X \in M(2m, \mathbb{C}); {}^tXJ + JX = 0\}.$$

Thus, we set $\alpha = -AJ, \xi = -QJ, \kappa = JK, \kappa' = JK'$ for every $A, Q, K, K' \in \mathfrak{S}(2m)$. These are elements of $sp(m, \mathbb{C})$. We translate everything in terms of $sp(m, \mathbb{C})$. Then, we see

PROPOSITION 6.2. *The covering space $\tilde{\mathcal{D}}_K$ through $(c, 0)$ is translated into*

$$\tilde{\mathcal{D}}_K = \left\{ c\sqrt{\det(I+(I+\kappa)\xi)} e^{\frac{1}{i\hbar}\langle z(\xi^J), z \rangle}; \xi \in \mathcal{D}_\kappa \right\}, \tag{6.3}$$

which is a double covering space of \mathcal{D}_κ .

Proposition 6.2 shows in particular that one can know the amplitude whenever one knows the phase. Moreover, the covering space $\tilde{\mathcal{D}}_K$ through $(1, 0)$ is closed under the $*_\kappa$ -product, for the $*_\kappa$ -product of two elements must be the same on the covering space by the uniqueness of the real analytic solutions of linear differential equations.

The intertwiner $I_{\kappa'}^{\kappa'}$ is translated easily by using κ, κ' , and these intertwiners may be viewed as coordinate transformations:

$$\begin{array}{ccc} \tilde{\mathcal{D}}_\kappa \supset \pi^{-1}(\mathcal{D}_\kappa \cap \mathcal{D}_{\kappa'}) & \xrightarrow{I_{\kappa'}^{\kappa'}} & \pi^{-1}(\mathcal{D}_{\kappa'} \cap \mathcal{D}_\kappa) \subset \tilde{\mathcal{D}}_{\kappa'} \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{D}_\kappa \supset \mathcal{D}_\kappa \cap \mathcal{D}_{\kappa'} & = & \mathcal{D}_{\kappa'} \cap \mathcal{D}_\kappa \subset \mathcal{D}_{\kappa'} \end{array}$$

However intertwiners $I_{\kappa'}^{\kappa'}$ are 2-to-2 mappings. Thus the union $\bigcup_\kappa \tilde{\mathcal{D}}_\kappa$ is a manifold-like object glued by 2-to-2 coordinate transformations (cf. [30] for a little more general setting to obtain ℓ -to- ℓ , or \mathbb{Z} -to- \mathbb{Z} coordinate transformations).

6.2. BLURRED LIE GROUP

We show the union $\bigcup_\kappa \tilde{\mathcal{D}}_\kappa$ has a group-like structure, which we would like to call *Blurred Lie group*. A reduced version of this section to the commutative case has been seen in [32].

First of all, recall that the Cayley transform $C_0(X) = \frac{I-X}{I+X}$ has following properties:

$$\begin{aligned} X \in sp_{\mathbb{C}}(m) &\Leftrightarrow C_0(X) \in Sp(m, \mathbb{C}), \quad C_0^2(X) = X, \\ &\det(I + C_0(X)) = (\det(I + X))^{-1}. \end{aligned} \tag{6.4}$$

Let $\mathcal{D}_0 = \{X \in sp(m, \mathbb{C}); \det(I+X) \neq 0\}$. $C_0 : \mathcal{D}_0 \rightarrow Sp(m, \mathbb{C})$ is a local coordinate system of $Sp(m, \mathbb{C})$. Define the twisted Cayley transform by

$$\begin{aligned} C_\kappa(\alpha) &= (I - (I-\kappa)\alpha) \frac{1}{I + (I+\kappa)\alpha} = \frac{1}{I + \alpha(I+\kappa)} (I - \alpha(I-\kappa)), \\ (C_\kappa)^{-1}(Y) &= \frac{1}{I - \kappa + Y(I+\kappa)} (I - Y). \end{aligned} \tag{6.5}$$

$C_\kappa : \mathcal{D}_\kappa \rightarrow Sp(m, \mathbb{C})$ gives also a local coordinate system of $Sp(m, \mathbb{C})$.

Set $T_{\kappa'-\kappa}(\alpha) = \frac{1}{I - \alpha(\kappa' - \kappa)}\alpha$ for $\alpha \in sp(m, \mathbb{C})$ such that $\det(I - \alpha(\kappa' - \kappa)) \neq 0$. The identity $\det(I - \alpha(\kappa' - \kappa)) = \det(I + \alpha(\kappa' - \kappa))$ gives that $T_{\kappa'-\kappa}^{-1}(\alpha) = \frac{1}{I + \alpha(\kappa' - \kappa)}\alpha$.

It is easy to see that $T_{\kappa'-\kappa}(\alpha) \in sp(m, \mathbb{C})$ if $\alpha \in sp(m, \mathbb{C})$. We have also

$$\frac{1}{I-\alpha(\kappa'-\kappa)}(I+\alpha(I+\kappa))=I+T_{\kappa'-\kappa}(\alpha)(I+\kappa').$$

LEMMA 6.1. $C_\kappa(\mathcal{D}_\kappa)$ is open dense in $Sp(m, \mathbb{C})$, and $\bigcup\{C_\kappa(\mathcal{D}_\kappa); \kappa \in sp(m, \mathbb{C})\} = Sp(m, \mathbb{C})$, $\bigcap\{C_\kappa(\mathcal{D}_\kappa); \kappa \in sp(m, \mathbb{C})\} \ni I$.

We regard $T_{\kappa'-\kappa}$ as a coordinate transformation on $Sp(m, \mathbb{C})$, and the intertwiner $I_{\kappa'}^{\kappa'}$ as a coordinate transformation

$$I_{\kappa'}^{\kappa'}: \mathbb{C} \times \mathcal{D}_{\kappa, \kappa'} \rightarrow \mathbb{C} \times \mathcal{D}_{\kappa', \kappa}, \quad (\mathcal{D}_{\kappa, \kappa'} = \mathcal{D}_\kappa \cap \mathcal{D}_{\kappa'}). \tag{6.6}$$

By Proposition 6.2, the covering space $\widetilde{\mathcal{D}}_\kappa$ is naturally translated to the double covering space of $C_\kappa(\mathcal{D}_\kappa)$. We denote this space by $\widetilde{C}_\kappa(\mathcal{D}_\kappa)$.

We have

LEMMA 6.2. The intertwiner $I_{\kappa'}^{\kappa'}: \widetilde{\mathcal{D}}_\kappa \rightarrow \widetilde{\mathcal{D}}_{\kappa'}$ is a morphism in the sense that $I_{\kappa'}^{\kappa'}$ is a homomorphism as 2-to-2 mapping where they are defined.

This is immediately translated to the following:

PROPOSITION 6.3. Intertwiners are morphisms between $\widetilde{C}_\kappa(\mathcal{D}_\kappa)$ and $\widetilde{C}_{\kappa'}(\mathcal{D}_{\kappa'})$ as 2-to-2 mappings.

Thus the union forms a Lie group-like object glued by 2-to-2 coordinate transformations. If this were a manifold, then it would have had to be a connected double cover of $Sp(m, \mathbb{C})$, which is simply connected.

We have

$$(g; \alpha) *_\kappa (g'; \beta) = \left(gg' \left(\frac{1}{P} \right)^{\frac{1}{2}}; C_\kappa^{-1}(C_\kappa(\alpha)C_\kappa(\beta)) \right), \tag{6.7}$$

where $P = I + \alpha(I - \kappa)\beta(I + \kappa)$.

Now we fix arbitrarily $\kappa_0 \in sp(m, \mathbb{C})$. For every $(g; \alpha)$, the family

$$\{I_{\kappa_0}^\kappa(g, \alpha); \kappa \in sp(m, \mathbb{C})\}$$

is a densely defined double valued parallel section. Moving κ , we have the set of all densely defined double valued parallel sections \mathfrak{G} such that $\mathfrak{G}(\kappa) = \mathbb{C}e^{\mathfrak{G}(m)}$. Singularities move when κ moves. Hence, for every $(g; a), (g'; b)$, $I_{\kappa_0}^\kappa(g; a) *_\kappa I_{\kappa_0}^\kappa(g'; b)$ is defined for every κ moving in an open dense domain. This will be denoted by $(g; a) * (g'; b)$.

THEOREM 6.1. *Under the notion of double valued sections, the product $(g; a) * (g'; b)$ is defined as a double valued section. However the associativity holds only as double valued sections. In spite of this, $(1; 0)$ is the identity with respect to $*_{\kappa}$ -product for all κ .*

By the above observation, the independence of ordering principle will give us naturally the necessity of double valued elements.

Remark 6.1. Note that the work by Olver [20] seems a similar direction to our work. Also, adding an interesting comment suggested by Daniel Sternheimer: The notion of “blurred Lie group” is related to the question of integrability to the corresponding Lie group of a skew-symmetric Lie algebra representation in Hilbert space, which is known to require e.g. a dense domain of common analytic vectors for generators [11]. When these are absent we get “local” representations that can sometimes be made into true representations of an infinite-dimensional Lie group (cf. [10]). When starting with generators of the regular representation (on $L^2(G)$) of a simply connected compact simple Lie group G and making it nonsimply connected by removing a suitable subset of Haar measure zero, then taking e.g. a double covering G_2 of it (see [12] and [16]) we get one parameter unitary groups that do not close to what would have had to be G_2 , but to an infinite subgroup of the unitary group of $L^2(G_2)$, hard to study and with sometimes strange algebraically indecomposable subrepresentations [16]. It seems that our notion of “blurred Lie group” can provide a subgroup of that huge infinite group more amenable to study.

7. Transcendental Calculus

In the previous section, we showed that anomalous phenomena occur for the star exponential functions of the quadratic forms, and proposed to view them as double valued elements. On the other hand, choosing suitable family of K -ordered expressions together with the intertwiners $I_K^{K'}$ leads us to an advantage for the transcendental calculus in non-formal deformation quantization.

We think that notion and properties derived by a broad family of K -ordered expressions will be acceptable as independence of ordering.

In this section, we consider the case $m = 1$ for simplicity. Namely, we put $z_1 = u$ and $z_2 = v$ as below.

7.1. THE STAR EXPONENTIAL FUNCTION $e_*^{i(z + \frac{1}{i\hbar} uv)}$

Let $K = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & \delta \end{bmatrix}$. The product $*_{(\varepsilon, \delta)}$ and the ordered expression $: :_{(\varepsilon, \delta)}$ stand for $*_K$ and $: :_K$, respectively.

The (ε, δ) -ordered expression of the star exponential function $e_*^{t \frac{1}{i\hbar} 2uv}_{(\varepsilon, \delta)}$ is given by

$$:e_*^{t \frac{1}{i\hbar} 2uv}_{(\varepsilon, \delta)}: = \frac{2}{\Delta} \exp\left(\left(\frac{e^t - e^{-t}}{\Delta}\right)^2 \delta \frac{1}{i\hbar} u^2 + \frac{e^t - e^{-t}}{\Delta} \frac{1}{i\hbar} 2uv\right), \quad (7.1)$$

where $\Delta = (e^t + e^{-t}) - \varepsilon(e^t - e^{-t})$. The general ordered expression is a little more complicated involving the square root in the amplitude.

Note that if the function f has the form $f = h(uv)$, then $I_{(\varepsilon, 0)}^{(\varepsilon', 0)}(h(uv))$ is also a function of uv . From here on, we mainly concern with functions of uv alone. We set $\frac{2}{i\hbar} uv = (zA, z)$, where $z = (u, v)$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The intertwiner $I_{(\varepsilon, 0)}^{(\varepsilon', 0)}$ is given as follows:

$$I_{(\varepsilon, 0)}^{(\varepsilon', 0)}(ge^{t \frac{2}{i\hbar} uv}) = g \frac{1}{1-t(\varepsilon' - \varepsilon)} e^{\frac{1}{i\hbar} \frac{1}{1-t(\varepsilon' - \varepsilon)} 2uv} \quad (7.2)$$

Solving the evolution equation for the exponential function, we see that $e_*^{t \frac{1}{i\hbar} 2uv}$ is given by

$$:e_*^{t \frac{1}{i\hbar} 2uv}:_0 = \frac{1}{\cosh t} e^{\frac{1}{i\hbar} 2uv \tanh t} \quad (7.3)$$

in the Weyl ordering, and by

$$:e_*^{t \frac{1}{i\hbar} 2uv}:_{K_0} = e^t e^{\frac{1}{i\hbar} (e^{2t} - 1)uv} \quad (7.4)$$

in the K_0 -ordered expression.

Since $:e_*^{t \frac{2}{i\hbar} uv}:_\varepsilon = I_0^{(\varepsilon, 0)}\left(\frac{1}{\cosh t} e^{\frac{1}{i\hbar} 2uv \tanh t}\right)$, we see that

$$:e_*^{t \frac{1}{i\hbar} 2uv}:_\varepsilon = \frac{2}{(1-\varepsilon)e^t + (1+\varepsilon)e^{-t}} \exp\left(\frac{e^t - e^{-t}}{(1-\varepsilon)e^t + (1+\varepsilon)e^{-t}} \frac{1}{i\hbar} 2uv\right). \quad (7.5)$$

Note that $(1-\varepsilon)e^t + (1+\varepsilon)e^{-t} = 0$ if and only if $e^{2t} = \frac{\varepsilon+1}{\varepsilon-1}$. Hence, $:e_*^{t \frac{1}{i\hbar} 2uv}:_{(\varepsilon, \delta)}$ has a singular point at $2t = \log \frac{\varepsilon+1}{\varepsilon-1} + 2\pi i \mathbb{Z}$. However, if $\varepsilon = \pm 1$, then $:e_*^{t \frac{1}{i\hbar} 2uv}:_{(\pm 1, \delta)}$ are entire functions with respect to t .

The properties, like singularities, of star exponential functions in ordered expressions will be general properties as an independence of ordering.

7.2. INVERSES

Formula (5.2) gives in particular

$$:e_*^{t(z + \frac{1}{i\hbar} v)}:_{(\varepsilon, \delta)} = e^{\frac{1}{4i\hbar} t^2 \delta} e^{t(z + \frac{1}{i\hbar} v)}. \quad (7.6)$$

It follows that if $\text{Im} \delta < 0$, then $e^{\frac{1}{4i\hbar}t^2\delta}$ is rapidly decreasing in t . By (7.6), the integrals

$$: \int_{-\infty}^0 e_*^{t(z+\frac{1}{i\hbar}v)} dt :_{(\varepsilon, \delta)}, \quad - : \int_0^{\infty} e_*^{t(z+\frac{1}{i\hbar}v)} dt :_{(\varepsilon, \delta)}.$$

converge. Both integrals are respective inverses of $z+\frac{1}{i\hbar}v$, and are denoted by $(z+\frac{1}{i\hbar}v)_{+*}^{-1}$, $(z+\frac{1}{i\hbar}v)_{-*}^{-1}$, respectively, with the subscript (ε, δ) omitted.

PROPOSITION 7.1. *If $\text{Im} \delta < 0$, then the (ε, δ) -ordered expression of the difference of the two inverses is given by*

$$: \left(z + \frac{1}{i\hbar}v \right)_{+*}^{-1} - \left(z + \frac{1}{i\hbar}v \right)_{-*}^{-1} :_{(\varepsilon, \delta)} = \int_{-\infty}^{\infty} e^{\frac{1}{4i\hbar}t^2\delta} e^{t(z+\frac{1}{i\hbar}v)} dt.$$

This difference is holomorphic in z .

We now consider the inverses of $(z+\frac{1}{i\hbar}uv)$ which have the different features from the above. We formally set

$$\left(z + \frac{1}{i\hbar}uv \right)_{+*}^{-1} = \int_{-\infty}^0 e_*^{t(z+\frac{1}{i\hbar}uv)} dt, \quad \left(z + \frac{1}{i\hbar}uv \right)_{-*}^{-1} = \int_0^{\infty} e_*^{t(z+\frac{1}{i\hbar}uv)} dt. \tag{7.7}$$

(7.7) makes sense for the (ε, δ) -ordered expression such that $\text{Im} \delta < 0$. We have the following result (cf. [33], [23], [24]).

THEOREM 7.1. *The inverses $(z+\frac{1}{i\hbar}uv)_{+*}^{-1}$, $(z-\frac{1}{i\hbar}uv)_{-*}^{-1}$ extend to holomorphic functions in z on $\mathbb{C} - \{-(\mathbb{N}+\frac{1}{2})\}$ in (ε, δ) -ordered expression such that $\text{Im} \delta < 0$.*

COROLLARY 7.1. *Let $\text{Im} \delta < 0$. If $-\frac{1}{2} < \text{Re} z < \frac{1}{2}$, then the difference of the two inverses is given by*

$$\left(z + \frac{1}{i\hbar}uv \right)_{+*}^{-1} - \left(z + \frac{1}{i\hbar}uv \right)_{-*}^{-1} = \int_{-\infty}^{\infty} e_*^{t(z+\frac{1}{i\hbar}uv)} dt. \tag{7.8}$$

Its (ε, δ) -ordered expression is holomorphic on this strip.

Although we restrict ordered expressions, Corollary 7.1 gives us a suggestion that there are two inverses for $(z+\frac{1}{i\hbar}uv)$. Furthermore, the following discrete picture will be a general feature for uv .

7.3. STAR FUNCTIONS

By suitable choices of K -orderings, we can produce several interesting star functions. In this subsection, we briefly show star gamma functions and star sine functions.

We first recall the ordinary gamma function and beta function:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

The star gamma function and the star beta function may be defined by replacing x with $z \pm \frac{uv}{i\hbar}$:

$$\begin{aligned} \Gamma_* \left(z \pm \frac{uv}{i\hbar} \right) &= \int_{-\infty}^\infty e^{-e^\delta} e_*^{\delta(z \pm \frac{uv}{i\hbar})} d\delta, \\ B_* \left(z \pm \frac{uv}{i\hbar}, y \right) &= \int_{-\infty}^0 e_*^{\delta(z \pm \frac{uv}{i\hbar})} (1 - e^\delta)^{y-1} d\delta. \end{aligned} \tag{7.9}$$

The star gamma function is defined in the $(\varepsilon, 0)$ -ordered expression where $\varepsilon \in \mathbb{C} - \{\varepsilon \geq 1\} \cup \{\varepsilon \leq -1\}$. On the other hand, we can define the star sine function $\sin_* \pi(z + \frac{1}{i\hbar} uv)$ via the formula

$$\sin_* \pi \left(z + \frac{1}{i\hbar} uv \right) = \frac{1}{2i} \left(\exp_* \left(\pi i \left(z + \frac{1}{i\hbar} uv \right) \right) - \exp_* \left(-\pi i \left(z + \frac{1}{i\hbar} uv \right) \right) \right),$$

in the $(\varepsilon, 0)$ -ordered expression where $\text{Re } \varepsilon < 0$. Adapting the arguments of the infinite product formulas for complex analytic functions (cf. [1]) to a star product version, we can show that in a appropriate context, the star gamma function and the star sine function have infinite product formulas (cf. [33]).

$$\begin{aligned} \Gamma_* \left(z + \frac{uv}{i\hbar} \right) &= e_*^{-\gamma(z + \frac{uv}{i\hbar})} * \left(z + \frac{uv}{i\hbar} \right)_{*+}^{-1} * \prod_{k=1}^\infty * \left(\left(1 + \frac{1}{k} \left(z + \frac{1}{i\hbar} uv \right) \right)_{*+}^{-1} * e_*^{\frac{1}{k}(z + \frac{1}{i\hbar} uv)} \right), \\ \sin_* \pi \left(z + \frac{1}{i\hbar} uv \right) &= \pi \left(z + \frac{1}{i\hbar} uv \right) * \lim_{n \rightarrow \infty} \prod_{k=1}^n * \left(1 - \frac{1}{k} \left(z + \frac{1}{i\hbar} uv \right) \right) * e_*^{\frac{1}{k}(z + \frac{1}{i\hbar} uv)} \\ &\quad * \prod_{k=1}^n * \left(1 + \frac{1}{k} \left(z + \frac{1}{i\hbar} uv \right) \right) * e_*^{-\frac{1}{k}(z + \frac{1}{i\hbar} uv)}. \end{aligned}$$

These computations suggest the following theorem, which indicate a general feature on the discrete picture for the element uv as independence of ordering.

THEOREM 7.2. $\sin_* \pi(z + \frac{1}{i\hbar} uv) * \Gamma_*(z + \frac{1}{i\hbar} uv)$ is defined as an entire function of z , vanishing at $z \in \mathbb{N} + \frac{1}{2}$ in any $(\varepsilon, 0)$ -ordered expression such that $\text{Re } \varepsilon < 0$, and $\varepsilon \in \mathbb{C} - \{\varepsilon \geq 1\} \cup \{\varepsilon \leq -1\}$.

The statements mentioned in this section have been announced with an outline of proofs in [33] (see also [23] and [24]). The complete proofs with careful checks of associativity will appear elsewhere.

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