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Return to Thermal Equilibrium

DOMINIQUE FELLAH
59 chemin de la Fare. Le Boisset, 83560 Saint-Julien, France.
e-mail: dominique.fellah@ac-aix-marseille.fr

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Abstract. We study a class of quantum dynamical semigroups on $\mathscr{B}(\mathfrak{H})$ with Lindbladian generators. We give new conditions in order to easily verify that a quantum dynamical system returns to thermal equilibrium. In the classical picture of the interacting -System+Reservoir-, our result can physically be interpreted as follows : the transition may be sufficient so that each eigenvalue energy state of the system communicates with another.

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1. Introduction

There exists an enormous amount of literature on the study of the approach to equilibrium of an open quantum system. In general, we call an open quantum system a small system $\mathscr{S}$ described by a Hilbert space interacting with a system $\mathscr{R}$, called reservoir or environment, with an infinite number of degrees of freedom. This environment is described by a C^* or W^*-algebra. If the environment admits a stationary state which is β-KMS with respect to the free dynamics then we said that the open quantum system is thermal. The basic concepts of the algebraic theory of quantum dynamical systems can be found in Bratteli and Robinson [1] as well as in Pillet [13]. Mathematical tools and their physical interpretations can also be found in Haag [10].

Let $\mathfrak{H}$ be a Hilbert space and we denote by $\mathscr{B}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$. A quantum dynamical system is a triple $(\mathscr{B}(\mathfrak{H}), \tau^t, \omega)$ where

- τ^t is a σ-weakly continuous group of $\star$-automorphisms of $\mathscr{B}(\mathfrak{H})$;
- ω is a normal stationary (τ^t-invariant for all $t \in \mathbb{R}$) state on $\mathscr{B}(\mathfrak{H})$.

We will say this system returns to thermal equilibrium at the inverse temperature β if there exists a (τ, β)-KMS state ω_e such that :

$$\forall A \in \mathscr{B}(\mathfrak{H}), \quad \lim_{t\to\infty} \mathbf{tr}\left(\rho\tau^t(A)\right) = \omega_e(A),$$

and for all density matrix ρ, i.e., a non-negative trace class operator ρ on $\mathfrak{H}$ with $\mathbf{tr}(\rho) = 1$.

In this note, we give a simple proof of the approach to equilibrium for a quantum dynamical semigroup $(\tau^t)_{t\geq 0}$. This report is a part of Fellah's thesis studying the weak coupling limit.

The particular form of the time evolution $\tau^t = \exp tK$ plays an important role in our result.

A quantum dynamical semigroup is a one parameter family $(\tau^t)_{t\geq 0}$ of linear maps of $\mathscr{B}(\mathfrak{H})$ into itself satisfying the following properties.

(1) $\tau^t(I) = I$.
(2) $\tau^{t+s} = \tau^t \tau^s$.
(3) τ^t is ultraweakly continuous.
(4) $\tau^t(A) \to A$ ultraweakly as $t \searrow 0$.
(5) τ^t is completely positive, i.e. $\sum_{1\leq i,j\leq N} B_i^* \tau^t(A_i^* A_j) B_j \geq 0$, for all integers N, all A_i and B_j of $\mathscr{B}(\mathfrak{H})$.

Then we known that there exists a (generally unbounded) map K defined on an ultraweakly dense domain $D(K)$ such that

$$\lim_{t\searrow 0} \| K - (\tau^t - \mathbb{1})/t \| = 0 ,$$

for all $A \in D(K)$.

When the semigroup is norm continuous ($\lim\limits_{t\searrow 0} \|\tau^t - \mathbb{1}\| = 0$) then K is bounded and $\tau^t = \exp(tK)$. Lindblad [12] proved that the general form of K is given by

$$K(A) = i[H, A] + \sum_{i\in I} \Big([V_i^*, A]V_i + V_i^*[A, V_i] \Big) , \tag{1}$$

where H is a bounded self-adjoint operator, $V_i \in \mathscr{B}(\mathfrak{H})$ such that the series converge ultraweakly.

We will use Frigerio's theorem (see THEOREM 12 in Appendix 3) established in Frigerio [8] after the studies made in Frigerio [9], Spohn [14, 15].

A generalization of this theorem to unbounded H and V_i has been given by Fagnola and Rebolledo [6]. However their conditions are more difficult to verify. In concrete models our method is applied with more simplicity.

This concrete physical model (see for example [3, 4, 11]) can be described as follows : a small system $\mathcal{S}$ interacts with an environment $\mathcal{R}$, the Hamiltonian has the form

$$H_\lambda = H_0 + \lambda Q \otimes \phi, \quad H_0 = H_s \otimes \mathbb{1}_r + \mathbb{1}_s \otimes H_r . \tag{2}$$

We look at the time evolution of observables of the small system. Let P be the partial trace over the reservoir such that $P[H_0, \,]P = 0$. Then, under appropriate assumptions (e.g. [4]), one shows that there exists an operator K acting on $\mathscr{B}(\mathfrak{H})$ such that

$$\lim_{\lambda\to 0} P e^{-itL_0/\lambda^2} e^{itL_\lambda/\lambda^2} P = e^{tK} .$$

where $L_\lambda = [H_\lambda,]$. The weak coupling limit gives a quantum dynamical system τ^t which is Markovian with

$$\tau^t(A) = e^{tK}(A),$$

for any observable A of $\mathcal{S}$.

K has the above form (1). The theory shows that V_i are functions of Q with parameters dependent on the two-point correlation functions of $\mathcal{R}$.

The reader can study the abstract structure of the weak coupling limit in Dereziński and Jakšič [5]. In the next section, we give a general structure for a class of semigroup τ^t which the generator K has a Lindbladian form and where the terms V_i are precised with respect to a β-KMS property.

So, one obtains that the system $(\mathscr{B}(\mathfrak{H}), \tau^t)$ returns to thermal equilibrium at the inverse temperature β.

2. A class of Lindbladian Operators

Let $\mathscr{B}(\mathfrak{H})$ be the algebra of all bounded operators on a separable Hilbert space $\mathfrak{H}$. We consider H_s a Hamiltonian operator which has a purely discrete non-degenerate spectruum

$$\sigma(H_s) = \left\{ E_0 < E_1 < \cdots \right\},$$

and an orthonormal basis $(|n>)_n$ of $\mathfrak{H}$ such that

$$H_s|n> = E_n|n>,$$

for all $n \in \mathbb{N}$.

We denote by P_n the spectral projector associated to E_n. It is well-known that the spectrum of $[H_s, \cdot]$ is the set

$$\left\{ \mu \;/\; \exists (i, f),\, \mu = E_i - E_f \right\}.$$

And

$$[H_s, \cdot] = \sum_\mu \mu \overline{P}_\mu,$$

where the $\overline{P}_\mu$ are the spectral projections of $[H_s, \cdot]$, given by

$$\overline{P}_\mu(M) = \sum_{\substack{i,f \\ E_i - E_f = \mu}} P_i M P_f,$$

for all $M \in \mathscr{B}(\mathfrak{H})$.

For $Q = Q^* \in \mathscr{B}(\mathfrak{H})$ and $\mu \in \sigma([H_s, \cdot])$, we introduce the following notation

$$Q_\mu = \overline{P}_\mu(Q) = \sum_{\substack{i,f \\ E_i - E_f = \mu}} P_i Q P_f.$$

Let q be a map of $\mathbb{R}$ into $\mathbb{C}$ such that :

$$\forall \mu \in \sigma([H_s, \cdot]) \setminus \{0\}, \quad q(\mu) \neq 0 \tag{3}$$

and

$$\forall \mu \in \sigma([H_s, \cdot]), \quad |q(-\mu)|^2 = e^{-\beta\mu} |q(\mu)|^2 \tag{4}$$

where $\beta > 0$.

We will call this last property the KMS-condition for q (see below).

Example 1. In the classical model (e.g Lebowitz–Spohn [11]), let h be the Fourier transform of the correlations functions given by the reservoir at the inverse temperature β, then one has

$$q(\mu) = \sqrt{h(\mu)/2}.$$

Let $(\tau^t)_{t \geq 0}$ be a one-parameter semigroup with

$$\tau^t = \exp(tK),$$

where the general form of K is Lindbladian

$$K(A) = i[H, A] + \sum_{\mu \in \sigma([H_s, \cdot])} [V_\mu, A]V_\mu^* + V_\mu[A, V_\mu^*], \tag{5}$$

$$V_\mu = q(\mu) Q_\mu, \tag{6}$$

for all $A \in \mathscr{B}(\mathfrak{H})$ where the series converge ultraweakly and where $H^* = H \in \mathscr{B}(\mathfrak{H})$ is an operator commuting with H_s.

Now we are in a position to state our main result.

THEOREM 2. *We assume the following hypotheses (H 1)–(H 5) hold.*

(H 1) H_s *has a non-degenerate discrete spectrum.*
(H 2) Let q *be a map of* $\mathbb{R}$ *into* $\mathbb{C}$ *satisfying* (3) *and* (4).
(H 3) $[H_s, Q] \neq 0$.
(H 4) There exists an integer N, $N \geq 1$ *such that*

$$< n | Q^N | m > \neq 0,$$

for all (n, m) *satisfying* $n \neq m$.
(H 5) $\mathbf{tr}(e^{-\beta H_s}) < \infty$.

Then there exists a unique normal faithful and stationary state ω_e *such that :*

$$\lim_{t \to \infty} \mathbf{tr}\big(\rho \tau^t(A)\big) = \omega_e(A),$$

for all $A \in \mathscr{B}(\mathfrak{H})$ *and all* ρ *density matrix of* $\mathfrak{H}$.

The state ω_e *is given by*

$$\omega_e(A) = \mathbf{tr}(\rho_e A)\,, \quad \rho_e = Z^{-1} e^{-\beta H_s}\,, \quad Z = \mathbf{tr}(e^{-\beta H_s})\,,$$

for all $A \in \mathscr{B}(\mathfrak{H})$.

This result will be proven in Sect. 3.

Remark 3. Let us make some comments about these assumptions.

- *Intuitively, the hypothesis (H 4) says in the interacting picture of the couple -system+environment- that the small system goes from one initial energy state* E_n *to another energy state* E_m *with a non-zero probability after* N *transitions run by* Q.
  We can also think about a Markov chain of type $X_{n+1} = QX_n$, $X_N = Q^N X_0$. *The condition (H 4) implies there exists an invariant measure (Châcon's theorem).*
- *The conditions* (3) *and* (4) *of* $(H2)$ *are satisfied in the classical model (e.g* [4,11]*). Their express the KMS-conditions, but our method can built others models.*
- *It is clear that the hypothesis (H 4) excludes the case* $\dim \mathfrak{H} = 1$.
- *The hypothesis (H 3) implies* $H_s \notin \mathbb{C} \cdot \mathbb{1}$*; in particular* $\dim \mathfrak{H} \neq 1$.

We consider $\mathscr{S}$ the following subset of $\mathscr{B}(\mathfrak{H})$,

$$\mathscr{S} = \left\{ Q_\mu \,/\, \mu \in \sigma([H_s, \cdot]) \setminus \{0\} \right\}.$$

As $Q_\mu^* = Q_{-\mu}$ for all μ, then we have $\mathscr{S}^* = \mathscr{S}$, thus the commutant $\mathscr{S}'$ in $\mathscr{B}(\mathfrak{H})$ is a von Neumann subalgebra of $\mathscr{B}(\mathfrak{H})$. But the equality $V_\mu = q(\mu) Q_\mu$, where $q(\mu) \neq 0$ for any $\mu \in \sigma([H_0, \cdot]) \setminus \{0\}$, shows :

$$\left\{ V_\mu \,/\, \mu \in \sigma([H_s, \cdot]) \right\}' \subset \left\{ V_\mu \,/\, \mu \in \sigma([H_s, \cdot]) \setminus \{0\} \right\}' = \mathscr{S}'.$$

So, to apply theorem 12, it is sufficient to obtain $\mathscr{S}' = \mathbb{C} \cdot \mathbb{1}$.

Remark 4.

1. *Let us remark if* H_s *and* Q *commute then* $Q_\mu = 0$ *for all* $\mu \in \sigma([H_s, \cdot]) \setminus \{0\}$ *and* $Q_0 = Q$. *Thus*

   $$\mathscr{S}' = \mathscr{B}(\mathfrak{H})\,, \quad H_s \in \left\{ Q \right\}' = \left\{ Q_\mu \,/\, \mu \in \sigma([H_s, \cdot]) \right\}',$$

   and in order for the condition (F 2) of theorem 12 holds, it is necessary that

   $$[H_s, Q] \neq 0\,.$$

2. *In example 1, we can define* $s:\mathbb{R}\ni\mu\mapsto s(\mu)$ *the real valued function which denotes the principal part*

$$s(\mu)=\frac{1}{2\pi i}PP\Big(\int_{-\infty}^{+\infty}\frac{h(t)}{t-\mu}\mathrm{d}t\Big).$$

and the weak coupling limit (see for example [4,7,11]) *gives*

$$H=\sum_{\mu\in\sigma([H_s,\cdot])}s(\mu)Q_\mu Q_\mu^*;$$
$$V_\mu=q(\mu)Q_\mu,$$

for any $\mu\in\sigma([H_s,\cdot])$ *and where the serie converges ultraweakly.*

3. Proof of Theorem 2

We begin by the next proposition which establishes a stationary state.

PROPOSITION 5. *One assumes that he hypothesis (H 1)–(H 5) hold. Let* ρ_e *be the following density matrix*

$$\rho_e=Z^{-1}e^{-\beta H_s},\quad Z=\mathbf{tr}(e^{-\beta H_s}).$$

Then the state ω_e*, defined by*

$$\omega_e(A)=\mathbf{tr}(\rho_e A),$$

for all $A\in\mathscr{B}(\mathfrak{H})$*, is normal, faithful and stationary.*

Moreover, $(\mathscr{B}(\mathfrak{H}),\tau^t,\omega_e)$ *is a quantum dynamical system.*

Proof of Proposition 5. At this end, we consider the duality between $\mathscr{B}(\mathfrak{H})$ and the Banach space $\mathscr{T}(\mathfrak{H})$ of trace-class operators on $\mathfrak{H}$ given by the map

$$\begin{array}{ccccc}\mathscr{B}(\mathfrak{H}) & \times & \mathscr{T}(\mathfrak{H}) & \longrightarrow & \mathbb{C}\\ A & \times & \rho & \longmapsto & \mathbf{tr}(A\rho).\end{array}$$

So, we can define the adjoint operator L of K for this duality and hence

$$\mathbf{tr}\left(\rho\tau^t(A)\right)=\mathbf{tr}\left(e^{tL}(\rho)A\right),$$

for all $A\in\mathscr{B}(\mathfrak{H})$, all $\rho\in\mathscr{T}(\mathfrak{H})$ and all $t\geq 0$.

More precisely, we have

$$L(\rho)=-i[H,\rho]+\sum_\mu[V_\mu^*\rho,V_\mu]+[V_\mu^*,\rho V_\mu],$$

for all $\rho\in\mathscr{T}(\mathfrak{H})$.

A simple calculation shows that

$$Q_\mu^*\rho_e Q_\mu=e^{-\beta\mu}\rho_e Q_\mu^* Q_\mu,$$

for all $\mu\in\sigma([H_s,\cdot])$.

To compute $L(\rho_e)$, one truncates the sums with the terms $\mu > 0$ and the terms $\mu < 0$ separately (the terms $\mu = 0$ cancel because $Q_0 Q_0^* = Q_0^* Q_0$). Then the equality $Q_{-\mu} = Q_\mu^*$ and the KMS-condition (4) finally give $L(\rho_e) = 0$.

And this property suffices to finish the proof.

Note this truncation is usual in the standard description of atomic radiation : the three terms correspond to the emission term, the absorption term and a term describing a shift of the free energy levels.

We start by a sequence of lemmas and propositions which will be needed to satisfy the conditions (F 1) and (F 2) of theorem 12.

We recall the form of P_n,

$$P_n = \sum_{i=1}^{d(n)} |n^i >< n^i| \,, \quad H_s|n^i >= E_n|n^i > \,, \tag{7}$$

where $d(n) = \dim Ker(H_s - E_n \cdot \mathbb{1})$.

We denote by τ_0 the free dynamics

$$\tau_0^t(A) = e^{itH_s} A e^{-itH_s} = \sum_{\mu \in \sigma([H_s, \cdot])} e^{it\mu} \overline{P}_\mu(A) \,, \tag{8}$$

for all $A \in \mathscr{B}(\mathfrak{H})$.

Using the identities

$$\forall \mu \in \sigma([H_s, \cdot]) \,, \quad Q_\mu = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt \, e^{-it\mu} \tau_0^t(Q) \,;$$

$$\forall t \in \mathbb{R} \,, \quad \tau_0^t(Q) = \sum_{\mu \in \sigma([H_0, \cdot])} e^{it\mu} Q_\mu \,,$$

one easily shows the following proposition.

PROPOSITION 6. *The next assertions are equivalent:*

(i) $A \in \mathscr{S}'$;
(ii) $\forall t \in \mathbb{R}: \quad [A, \tau_0^t(Q)] = [A, Q]$.

Let us give some results, independent of the degeneration or not of the spectrum of H_s, which are used in the following.

COROLLARY 7. *The family* $(T_n)_{n \geq 1}$ *of self-adjoint operators, defined by the induction relations*

$$T_1 = i[H_s, Q] \,, \; T_{n+1} = i[H_s, T_n] \,,$$

satisfies, for all $A \in \mathscr{S}'$*:*

(iii) $\forall t \in \mathbb{R}, \forall n \geq 1: \quad [A, \tau_0^t(T_n)] = 0;$
(iv) $\forall n \geq 1: \quad [A, T_n] = 0.$

And

COROLLARY 8. *The restriction of* τ_0 *to* $\mathscr{S}'$ *is a* C^**-automorphism of* $\mathscr{S}'$.
In particular, the propositions (v) *and* (vi) *are equivalent :*

(v) $\forall A \in \mathscr{S}', \forall t \in \mathbb{R}, \forall n \geq 1: \quad [A, \tau_0^t(T_n)] = 0;$
and
(vi) $\forall A \in \mathscr{S}', \forall n \geq 1: \quad [A, T_n] = 0.$

The two following lemmas are taken into consideration for the end of the proof.

LEMMA 9. *Let us define the hypothesis (H) : for all* $n \geq 1$*, there exists a real number* λ_n *such that*

$$P_n Q P_n = \lambda_n P_n .$$

Then, we have

$$\{H_s\}' \cap \{Q\}' = \{H_s\}' \cap \{T_1\}',$$

where each communtant is taken in $\mathscr{B}(\mathfrak{H})$.

Proof of Lemma 9. It is clear that if $[A, H_s] = [A, Q] = 0$ then A commutes with any function of two variables H_s and Q, in particular with T_1.

Inversely, let A be an element commuting with H_s and T_1. Jacobi's identity

$$[A, [H_s, Q]] + [H_s, [Q, A]] + [Q, [A, H_s]] = 0$$

gives $[H_s, [Q, A]] = 0]$. But any observable M which commutes with H_s has the form as

$$M = \sum_n P_n M P_n ,$$

because

$$0 = P_m [M, H_s] P_n = (E_n - E_m) P_m M P_n$$

and $n \neq m$ implies $P_m M P_n = 0$.

Then, we have:

$$A = \sum_n P_n A P_n , \quad [A, Q] = \sum_n P_n [A, Q] P_n .$$

The hypothesis (H) shows

$$P_n[A, Q]P_n = [A, P_n Q P_n] = \lambda_n [A, P_n] = 0,$$

thus $[A, Q] = 0$.

LEMMA 10. *The hypotheses* $(H1)$ *and (H 4) imply*

$$\{H_s\}' \cap \{Q^N\}' = \mathbb{C} \cdot \mathbb{1}.$$

Thus, in particular

$$\{H_s\}' \cap \{Q\}' = \mathbb{C} \cdot \mathbb{1}.$$

Proof of Lemma 10 (H 1) is written as

$$d(n) = 1, \quad P_n = |n><n|$$

for all n.

Let A be an element commutes with H_s and Q^N,

$$A = <0|A|0> |0><0| + \sum_{n \geq 1} <n|A|n> |n><n|.$$

The hypothesis (H 4) gives $<n|A|n> = <m|A|m>$ for all n and m, because, for $n \neq m$,

$$0 = <n|[A, Q^N]|m> = (<n|A|n> - <m|A|m>) <n|Q^N|m>.$$

Thus : $A = <0|A|0> \cdot \mathbb{1}$.

The inclusion

$$\{H_s\}' \cap \{Q\}' \subset \{H_s\}' \cap \{Q^N\}'$$

establishes the second point.

Remark that the hypothesis (H 1) implies the hypothesis (H).

The end of the proof of theorem 2 is composed of the six following steps.

Step 1. If $A \in \mathscr{S}'$ then $\overline{A} \in \mathbb{C} \cdot \mathbb{1}$, where

$$\overline{A} = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} \mathrm{d}s\, \tau_0^s(A) = \sum_n <n|A|n> |n><n|.$$

As $\tau_0^s(A) \in \mathscr{S}'$, the corollary 7 shows $[\overline{A}, T_1] = 0$. Moreover $[\overline{A}, H_s] = 0$ then the lemma 9 and lemma 10 give the result.

Step 2. We have : $\forall A \in \mathscr{S}', \forall n \in \mathbb{N}, \quad <n|A|n> = <0|A|0>$.

Because, for any n, we have

$$<n|A|n>=<n|\overline{A}|n>=<0|\overline{A}|0>=<0|A|0> \ .$$

Step 3. For all $A\in\mathscr{S}'$ and for all density matrix $\rho=f(H_s)$, we have:

$$\omega_\rho(A)=<0|A|0>,$$

where

$$\omega_\rho(M)=tr(\rho M)=\sum_n f(E_n)<n|M|n> \ .$$

We have

$$\omega_\rho(A)=\sum_n f(E_n)<n|A|n>=\Big(\sum_n f(E_n)\Big)<0|A|0>=1\times<0|A|0> \ .$$

Step 4. Let A and B be two elements of $\mathscr{S}'$. For all $\gamma>\beta$ and all n, it comes

$$\sum_k e^{-\gamma(E_k-E_n)}<n|A|k><k|B|n>=<n|BA|n> \ . \tag{9}$$

We consider the state ω_γ, defined by

$$\omega_\gamma(M)=Z_\gamma^{-1}tr(e^{-\gamma H_s}M)\,,\ Z_\gamma=tr(e^{-\gamma H_s})\,,\ (Z_\gamma<Z_\beta<\infty)\,.$$

ω_e is a (τ_0,γ)-KMS state. Then, according to proposition 5.3.7 in [2], we know there exists a complex function $F_{A,B}$ which is analytic on the open strip

$$\mathscr{D}_\gamma=\big\{z;\,z\in\mathbb{C},\ 0<\mathrm{Im}(z)<\gamma\big\},$$

and bounded and continuous on its closure $\overline{\mathscr{D}_\gamma}$, such that

$$F_{A,B}(t)=\omega_\gamma(A\tau_0^t(B))\,,$$
$$F_{A,B}(t+i\gamma)=\omega_\gamma(\tau_0^t(B)A)\,,$$

for all $t\in\mathbb{R}$.

In particular, on the boundary we have

$$F_{A,B}(i\gamma)=\omega_\gamma(A\tau_0^{i\gamma}(B))=\omega_\gamma(BA)\,.$$

By ***Step 3*** , the result is that this last equality gives for all integer n:

$$<n|A\tau_0^{i\gamma}(B)|n>=<n|BA|n>,$$

for all integer n. Thus the result (9).

Step 5. $\forall A\in\mathscr{S}'$, $\forall k\geq 1\Longrightarrow<0|A|k>=0$.

The equality (9) gives, in choosing $n=0$,

$$<0|A|0><0|B|0>+\sum_{k\geq 1}e^{-\gamma(E_k-E_0)}<0|A|k><k|B|0>=<0|BA|0> \ .$$

With $E_k - E_0 \geq E_1 - E_0 > 0$, for all $k \geq 1$ and Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}&\Big|\sum_{k\geq 1} e^{-\gamma(E_k-E_0)} <0|A|k><k|B|0>\Big|^2\\ &\qquad\leq e^{-2\gamma(E_1-E_0)}\Big(\sum_{k\geq 1}|<0|A|k>|^2\Big)\times\Big(\sum_{k\geq 1}|<k|B|0>|^2\Big)\\ &\qquad\leq e^{-2\gamma(E_1-E_0)}<0|AA^*|0>|\times <0|BB^*|0> .\end{aligned}$$

And therefore, taking $\gamma \to +\infty$, we have:

$$<0|A|0><0|B|0>=<0|BA|0> .$$

By ***Step 2*,** it follows that

$$<n|A|n><n|B|n>=<n|BA|n> ,$$

for all n. The choice $B = A^*$ implies $|<0|A|0>|^2 = <0|A^*A|0>$ and

$$\sum_{k\geq 1} e^{-\gamma E_k}|<0|A|k>|^2 = 0 .$$

Hence : $<0|A|k> = 0$, for all $k \geq 1$.
***Step 6*.** $\forall A \in \mathscr{S}'$, $\forall (k,n)$, $k \neq n \Longrightarrow <n|A|k> = 0$.

Let $B = A^*$, $|<n|A|n>|^2 = <n|A^*A|n>$. The equality (9) gives

$$\sum_{k\geq 1} e^{-\gamma(E_k-E_n)}|<n|A|k>|^2 = |<n|A|n>|^2 .$$

Thus

$$\sum_{k\geq 1,\, k\neq n} e^{-\gamma(E_k-E_n)}|<n|A|k>|^2 = 0 ,$$

i.e. $<n|A|k> = 0$, for all (k,n) such that $k \neq n$.

So

$$\begin{aligned}A &= \sum_{k,n}|<n|A|k>|n><k| = \sum_n <n|A|n>|n><n|\\ &= <0|A|0>\cdot\sum_n |n><n| = <0|A|0>\cdot\mathbb{1} .\end{aligned}$$

In other words, $\mathscr{S}' = \mathbb{C}\cdot\mathbb{1}$.

Finally, the conditions (F 1) and (F 2) of theorem 12 are satisfied.

Remark 11. In adapting these conditions, it will be strongly motivating to extend the above method when the operators are unbounded. At the start, the operators are *defined in the weak sense sesquilinear forms in $\mathfrak{H}$: there exists a domain $\mathfrak{D}$, dense in $\mathfrak{H}$ such that*

$$< x|K(A)|y >=< Gx|Ay > + < x|GA|y > + \sum_{i\in I} < V_i x|AV_i|y > ,$$

for all $(x, y) \in \mathfrak{D}^2$.

Appendix

THEOREM 12 [8]. *Let* $\tau^t = \exp(tK)$ *be a one-parameter semigroup of* $\mathscr{B}(\mathfrak{H})$ *where* K *has the Lindblad general form :*

$$K(A) = i[H, A] + \sum_{i\in I} \Big([V_i^*, A]V_i + V_i^*[A, V_i]\Big), \tag{10}$$

for any $A \in \mathscr{B}(\mathfrak{H})$ *where* $H^* = H$ *and* V_i *are in* $\mathscr{B}(\mathfrak{H})$ *and the series converge ultra-weakly.*

We assume that the following hyotheses hold :

(F 1) the subspace of $\mathscr{B}(\mathfrak{H})$ *generated by* $\{V_i; i \in I\}$ *is a self-adjoint set;*
(F 2) the commutant $\{V_i; i \in I\}'$ *in* $\mathscr{B}(\mathfrak{H})$ *is equal to* $\mathbb{C} \cdot \mathbb{1}$.

Let ω *be a normal stationary state for* τ^t *then*

(i) ω *is faithful;*
(ii) *for all* $A \in \mathscr{B}(\mathfrak{H})$, *we have :* $w^* - \lim_{t\to\infty} \tau^t(A) = \omega(A) \cdot \mathbb{1}$.

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