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From Integrable Lattices to Non-QRT Mappings

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Abstract. Second-order mappings obtained as reductions of integrable lattice equations are generally expected to have integrals that are ratios of biquadratic polynomials, i.e., to be of QRT-type. In this paper we find reductions of integrable lattice equations that are *not* of this type. The mappings we consider are exact reductions of integrable lattice equations proposed by Adler et al. [Comm Math Phys 233: 513, 2003]. Surprisingly, we found that these mappings possess invariants that are of the type originally studied by Hirota et al. [J Phys A 34: 10377, 2001]. Moreover, we show that several mappings obtained are linearisable and we present their linearisation.

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1. Introduction

The study of discrete integrable systems, after the revival of integrability in the late twentieth century, began with the work of Hirota in the 1970s [14]. Hirota, as well as Ablowitz and collaborators [2] and, separately Capel and his school [3], proposed lattice and differential-difference versions of many integrable evolution equations. The study of (one-dimensional) integrable mappings had to wait for more than a decade till the introduction of the QRT family of mappings. In [4] Quispel, Roberts and Thompson made the observation that the autonomous difference equations obtained by an exact reduction of an integrable differential-difference equation was an integrable mapping. From this observation they went on to derive a five parameter family of mappings which is known today as the "symmetric" QRT mapping. The latter has the form:

$$x_{n+1}=\frac{f_1(x_n)-x_{n-1}f_2(x_n)}{f_2(x_n)-x_{n-1}f_3(x_n)} \tag{1.1}$$

Here f_i are specific quartic polynomials expressed in terms of 12 parameters of which 5 correspond to genuine degrees of freedom:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \alpha_0 x^2 + \beta_0 x + \gamma_0 \\ \beta_0 x^2 + \epsilon_0 x + \zeta_0 \\ \gamma_0 x^2 + \zeta_0 x + \mu_0 \end{pmatrix} \times \begin{pmatrix} \alpha_1 x^2 + \beta_1 x + \gamma_1 \\ \beta_1 x^2 + \epsilon_1 x + \zeta_1 \\ \gamma_1 x^2 + \zeta_1 x + \mu_1 \end{pmatrix} \tag{1.2}$$

Mapping (1.1) possesses an invariant

$$K(x, y) = \frac{\alpha_0 y^2 x^2 + \beta_0 yx(y+x) + \gamma_0(y^2+x^2) + \epsilon_0 yx + \zeta_0(y+x) + \mu_0}{\alpha_1 y^2 x^2 + \beta_1 yx(y+x) + \gamma_1(y^2+x^2) + \epsilon_1 yx + \zeta_1(y+x) + \mu_1} \tag{1.3}$$

What we mean by invariance here is that if we start from $K(x_n, x_{n-1})$ and compute $\overline{K} \equiv K(x_{n+1}, x_n)$ we find that $\overline{K} = K$ when x_{n-1}, x_n, x_{n+1} are related by (1.1).

An "asymmetric" version of the mapping, which contains 8 genuine parameters was presented also by the same authors [5]. The QRT mapping has been extremely useful since it provided a multiparameter family of integrable systems, on which to test all integrability conjectures. It played thus a key role in the discovery of the singularity confinement property [6,7] and the derivation of the discrete Painlevé equations [8]. The solution of the symmetric QRT mapping can be expressed in terms of elliptic functions [9]. Recently this result was extended to the asymmetric case as well [10,11].

In the same logic as the QRT approach, one can ask the question of the nature of the integrability of the reductions of integrable lattice equations. In this paper, we have selected the family of lattice equations obtained by Adler, Bobenko and Suris who classified integrable lattice equations on quad-graphs based on the "consistency around a cube" (CAC) approach [12]. The main idea of this method is the following. One starts from a two-dimensional square lattice, define the variable on the vertices $x_{n,m}$, $x_{n,m+1}$, $x_{n+1,m}$, $x_{n+1,m+1}$ and write the *multilinear* equation relating these variables. In this way, solving for $x_{n+1,m+1}$ gives a rational expression of the other x's. For the CAC trick one adjoins a third direction, say k, and imagine the mapping giving $x_{n+1,m+1,k+1}$ as being the composition of mappings on the various planes. There exist three different ways to obtain $x_{n+1,m+1,k+1}$ and the consistency requirement is that they lead to the same result. This places severe constraints on the multilinear equation but they do not suffice to determine it completely. Adler, Bobenko and Suris have introduced two additional assumptions. They considered only a certain class of symmetrical forms for the multilinear equation and also they required that $x_{n+1,m+1,k+1}$ be independent of $x_{n,m,k}$ (the so-called tetrahedron property). Under the constraints of these simplifying assumptions they were able to produce a complete classification of lattice systems. The latter are all integrable, since the procedure also furnishes their Lax pairs.

Four equations were listed in the ABS "Q list" on which we shall focus here.

$$\begin{aligned}
\mathrm{Q}_1:\quad & \alpha(x_{n,m}-x_{n,m+1})(x_{n+1,m}-x_{n+1,m+1})+\\
& \quad +\beta(x_{n,m}-x_{n+1,m})(x_{n,m+1}-x_{n+1,m+1})+\gamma=0\\
\mathrm{Q}_2:\quad & \alpha(x_{n,m}-x_{n,m+1})(x_{n+1,m}-x_{n+1,m+1})+\\
& \quad +\beta(x_{n,m}-x_{n+1,m})(x_{n,m+1}-x_{n+1,m+1})+\\
& \quad +\gamma(x_{n,m}+x_{n,m+1}+x_{n+1,m}+x_{n+1,m+1})+\delta=0\\
\mathrm{Q}_3:\quad & \alpha(x_{n,m}x_{n+1,m+1}+x_{n,m+1}x_{n+1,m})+\\
& \quad +\beta(x_{n,m}x_{n+1,m}+x_{n,m+1}x_{n+1,m+1})+\\
& \quad +\gamma(x_{n,m}x_{n,m+1}+x_{n+1,m}x_{n+1,m+1})+\delta=0\\
\mathrm{Q}_4:\quad & A\left((x_{n,m}-b)(x_{n,m+1}-b)-(a-b)(c-b)\right)\times\\
& \quad \times\left((x_{n+1,m}-b)(x_{n+1,m+1}-b)-(a-b)(c-b)\right)+\\
& \quad +\left(B(x_{n,m}-a)(x_{n+1,m}-a)-(b-a)(c-a)\right)\times\\
& \quad \times\left((x_{n,m+1}-b)(x_{n+1,m+1}-b)-(b-a)(c-a)\right)=ABC(a-b)
\end{aligned}$$

where α and β are lattice parameters (and γ, δ are expressed in terms of them) associated with each lattice direction n, m. Here we have taken Q_4 in the form given by Nijhoff [13]. It is obtained by considering that the lattice parameters α, β appearing in the equation take values on the elliptic curve $v^2=4u^3-g_2u-g_3$. We have $(a, A)=(\wp(\alpha), \wp'(\alpha))$, $(b, B)=(\wp(\beta), \wp'(\beta))$ and introduce $(c, C)=(\wp(\beta-\alpha), \wp'(\beta-\alpha))$, where $\wp$ is the Weierstrass elliptic function. The six quantities are related by the well-known addition formula for elliptic functions $A(c-b)-B(a-c)+C(b-a)=0$.

Concerning the Q_4 equation there exists an alternative form (due to Hietarinta, as quoted in [14,23])

$$\begin{aligned}
\mathrm{Q}_4:\quad & \operatorname{sn}\alpha(x_{n,m}x_{n+1,m+1}+x_{n,m+1}x_{n+1,m})-\\
& -\operatorname{sn}\beta(x_{n,m}x_{n+1,m}+x_{n,m+1}x_{n+1,m+1})-\\
& -\operatorname{sn}(\alpha-\beta)(x_{n,m}x_{n,m+1}+x_{n+1,m}x_{n+1,m+1})+\\
& +\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)(1+k^2x_{n,m}x_{n,m+1}x_{n+1,m}x_{n+1,m+1})=0.
\end{aligned}$$

The four equations above form a degeneration cascade. Indeed starting from Q_4 above and taking $k=0$ the elliptic sines become circular sines and Q_4 goes over to a Q_3 in the form (also given by Hietarinta)

$$\begin{aligned}
\mathrm{Q}_3:\quad & \sin\alpha(x_{n,m}x_{n+1,m+1}+x_{n,m+1}x_{n+1,m})-\\
& -\sin\beta(x_{n,m}x_{n+1,m}+x_{n,m+1}x_{n+1,m+1})-\\
& -\sin(\alpha-\beta)(x_{n,m}x_{n,m+1}+x_{n+1,m}x_{n+1,m+1})+\delta\sin\alpha\sin\beta\sin(\alpha-\beta)=0.
\end{aligned}$$

(This is of course just a renaming of the parameters of the Q_3 previously given.) Working with the latter form of Q_3 we can recover Q_2 through a more complicated limit. Using capital letters for the variables of Q_3 and lower-case for those of Q_2 we put $X=1+\epsilon x$, $A=\alpha$, $B=\beta$, $\Gamma=-\alpha-\beta+\epsilon\gamma$, $\Delta=-2\epsilon\gamma+\epsilon^2\delta$ in Q_3 and obtain Q_2 at the limit $\epsilon\to 0$. Finally putting $\gamma=0$ in Q_2 leads to Q_1.

In this letter we shall revisit the question of reductions of the integrable lattice equations of the "Q list" to one-dimensional mappings. As we shall show this reduction does not necessarily lead to QRT mappings. In the examples we will exhibit below the mappings obtained belong rather to a type first discovered by Hirota et al. in [15] which we shall refer to in what follows as the HKY mapping.

2. The HKY Mapping

Before proceeding further let us summarise what is known about the HKY mappings. In [15] Hirota et al. have investigated the integrability of third order mappings. They postulated a functional form $x_{n+2}x_{n-1} = f(x_n, x_{n+1})$, where f is a rational function, and identified nine integrable cases, using the algebraic entropy integrability criterion. In a recent work Takahashi and Matsukidaira [16] have presented the reduction of most of them to a composition of two QRT mappings.

Another interesting related finding concerns second-order mappings. In [17] we presented the following system

$$(x_nx_{n+1}-1)(x_nx_{n-1}-1)=\frac{(x_n-a)(x_n-1/a)(x_n^2-1)}{p^2x_n^2-1} \tag{2.1}$$

The search for a conserved quantity of (2.1) led to the following result:

$$\mathcal{K}=\frac{\left((x_n-x_{n-1})^2-p^2(x_nx_{n-1}-1)^2\right)\left((x_n+x_{n-1}-a-1/a)^2-p^2(x_nx_{n-1}-1)^2\right)}{(x_nx_{n-1}-1)^2} \tag{2.2}$$

i.e. $\mathcal{K}$ is a ratio of two biquartic polynomials, i.e. quartic in x_n and x_{n-1} separately. This invariant was quite astonishing since for all integrable cases previously known the invariant was a ratio of biquadratic polynomials. In [17] we derived this mapping (and many more of the same type) through the appropriate autonomisation of q-discrete Painlevé equations. In [18] we showed that the solution of (2.1) is indeed a sampling of an elliptic function.

In [17] we presented a method for the construction of HKY-type mappings starting from an elliptic function solution. Here we shall adopt a different approach. The main building block for the HKY mapping in this paper will be the QRT invariant $K(x_n, x_{n-1})$. However, the mapping is not obtained by asking that $K(x_n, x_{n-1})$ be constant when $(x_n, x_{n-1})\to(x_{n+1}, x_n)$, i.e., $\overline{K}=K$, where $\overline{K}$ is the invariant computed with updated variables. Rather, one asks that the invariant be transformed to a homography of itself which moreover must be an involution.

The general form of the latter is

$$\overline{K}=\frac{\alpha K+\beta}{\gamma K-\alpha} \tag{2.3}$$

Two cases can be distinguished. If $\gamma = 0$ one can, by a translation, bring (2.3) to $\overline{K} = -K$. In this case, the quantity that is invariant from the mapping evolution is simply $\mathcal{K} = K^2$. If $\gamma \neq 0$ one can again perform a translation and with a scaling of K bring (2.3) to the form $\overline{K} = 1/K$. In this case, the true invariant is $\mathcal{K} = K + 1/K$.

It is of interest to study the invariant curves along which the mappings iterate. The invariant curve for HKY-type mappings is a product of two curves of QRT-type. Typically the latter is the product $(K - c)(K + c) = 0$ if $\mathcal{K} = K^2$ and $(cK - 1)(c/K - 1) = 0$ if $\mathcal{K} = K + 1/K$. Iteration on one of the factored QRT-curves, automatically means we iterate (with double step) on the invariant curve of the original mapping. Moreover, each of the two QRT-curves gives rise to a potentially interesting system in its own right. Thus, studying the QRT invariant curve, $(K - c) = 0$, provides the key to understanding the behaviour of the HKY mapping.

3. The ABS Lattice and its Reductions

In what follows we shall present the reductions of equations of the “Q list” to second-order mappings. We use two reductions on the lattice. One is the reduction $x_{n,m+1} = x_{n+1,m}$ and the other is $x_{n,m+1} = x_{n+2,m}$. Note that the second reduction necessarily gives third-order mappings on the line, except in the case of Q_1, for which we were able to integrate the resulting mapping. In section 3.1 we apply both reductions to Q_1, while in sections 3.2, 3.3 and 3.4, we only consider the first reduction applied to Q_2, Q_3, Q_4, respectively, since our focus lies on second-order mappings. All other reductions we could have considered lead to higher-order mappings.

3.1. THE Q_1 EQUATION

The simplest reduction $x_{n,m+1} = x_{n+1,m}$ leads to a trivial mapping. We find (omitting the second index) that $(\alpha + \beta)(x_{n+1} - x_n)(x_n - x_{n-1}) + \gamma = 0$. Putting $y_n = x_n - x_{n-1}$, we find the homographic mapping $(\alpha + \beta)y_{n+1}y_n + \gamma = 0$, which can be trivially solved because one iteration and substitution for y_{n+1} shows that $y_{n+2} = y_n$, i.e., the solutions are functions of period 2.

In order to find a more interesting mapping we introduce a more complicated reduction, i.e., $x_{n,m+1} = x_{n+2,m}$. We can integrate this once by setting $y_n = x_{n+2,m} - x_{n+1,m}$. The result is the mapping

$$a(y_{n+1} + y_n)(y_n + y_{n-1}) + by_{n+1}y_{n-1} + c = 0 \tag{3.1}$$

This mapping is a member of the QRT family. Its invariant can be readily computed:

$$K = \frac{a(a+b)(y_n + y_{n-1})^2 + b(a+b)y_n y_{n-1} - ac}{(y_n + y_{n-1})(by_n y_{n-1} + c)} \tag{3.2}$$

3.2. THE Q_2 EQUATION

The reduction $x_{n,m+1} = x_{n+1,m}$ leads to a second-order mapping (where we omit the second index)

$$(x_{n+1} - x_n)(x_n - x_{n-1}) - ab(x_{n+1} + 2x_n + x_{n-1}) + ab(a^2 - ab + b^2) = 0 \tag{3.3}$$

where the specific form of the parameters is chosen so as to simplify the form of the invariant. The mapping (3.3) does not belong to the QRT family: it is of HKY type. A somewhat lengthy calculation shows that (3.3) corresponds to the "invariant":

$$K = \frac{2(x_n + x_{n-1}) - (a-b)^2}{(x_n - x_{n-1})^2 - a^2b^2} \tag{3.4}$$

and the "conservation" equation $\overline{K} = g/K$ where $g = 1/(a^2b^2)$. The equation $\overline{K}K - g = 0$ factorises into two mappings one of which is (3.3) and the other is its "dual" mapping. It turns out that the dual mapping is obtained from (3.3) with $a \to -a$ (or equivalently $b \to -b$).

As described above, in section 2, the invariant curve associated with $\mathcal{K} = K + g/K$ is the product of two curves of the form $K - c = 0$ where K is the QRT invariant (3.4). The structure of the latter, which leads to a "standard" QRT mapping, gives information on the structure of the former. From the conservation of (3.4), $\overline{K} - K = 0$, we find the QRT mapping

$$x_{n+1}x_{n-1} + (x_{n+1} + x_{n-1})\big(x_n - (a-b)^2/2\big) - 3x_n^2 + (a-b)^2 x_n + a^2b^2 = 0 \tag{3.5}$$

In order to bring (3.5) under canonical form we translate x, $x_n = y_n + a^2/4 + b^2/4$, which leads to:

$$(y_{n+1} + y_n)(y_n + y_{n-1}) = 4y_n^2 - a^2b^2 \tag{3.6}$$

This equation has been identified in [19] as (in its full nonautonomous form) a special limit of the discrete Painlevé IV equation and moreover it was pointed out there that it is linearisable.

3.3. THE Q_3 EQUATION

In Q_3 once gain, the coefficients $\alpha, \beta, \gamma, \delta$ are not all independent. The reduction $x_{n,m+1} = x_{n+1,m}$ suffices in order to yield a second-order mapping. Omitting the second index we find

$$x_{n+1}x_{n-1} + x_n^2 + ax_n(x_{n+1} + x_{n-1}) + b = 0 \tag{3.7}$$

Here we are again in presence of a mapping of HKY type. It is obtained from the "invariant"

$$K = \frac{2x_nx_{n-1} + a(x_n^2 + x_{n-1}^2) - ab}{2ax_nx_{n-1} + x_n^2 + x_{n-1}^2 + b} \tag{3.8}$$

together with the conservation relation $\overline{K}=-K$. From the factorisation of the equation $\overline{K}+K=0$ we obtain (3.7) and its dual mapping which is obtained from (3.7) with $a\to 1/a$ and $b\to -b$.

A QRT mapping is also obtained from (3.8) and the equation $\overline{K}-K=0$. We find

$$-x_n x_{n+1}x_{n-1}(1-a^2)+ab(x_{n+1}+x_{n-1})+x_n^3(1-a^2)+b(1+a^2)x_n=0 \tag{3.9}$$

We can bring (3.9) under canonical form by scaling $x_n = y_n\sqrt{ab/(1-a^2)}$. We obtain thus

$$(y_{n+1}y_n-1)(y_ny_{n-1}-1)=y_n^4+(a+1/a)y_n^2+1 \tag{3.10}$$

Under this form (3.10) is a special case of (the autonomous limit of) the discrete Painlevé V [19]. As we shall show in the next section (3.10) is a linearisable mapping.

3.4. THE Q_4 EQUATION

Q_4 is the generic equation from which all other equations can be obtained as special limits as explained above. In what follows we shall work with the Hietarinta parametrisation.

In order to obtain a second-order mapping we perform the reduction $x_{n,m+1}=x_{n+1,m}$. We find

$$\begin{aligned}&(\operatorname{sn}\alpha-\operatorname{sn}\beta)x_n(x_{n+1}+x_{n-1})-\operatorname{sn}(\alpha-\beta)(x_{n+1}x_{n-1}+x_n^2)+\\&\quad+\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)(1+k^2x_n^2x_{n+1}x_{n-1})=0\end{aligned} \tag{3.11}$$

Again we are in the presence of a non-QRT mapping. The HKY conservation is now $\overline{K}=-K$ where K is a QRT-type invariant

$$K=\frac{((1+k^2x_{n+1}^2x_n^2)\operatorname{sn}\alpha\operatorname{sn}\beta-x_{n+1}^2-x_n^2)\operatorname{sn}(\alpha-\beta)+2x_{n+1}x_n(\operatorname{sn}\alpha-\operatorname{sn}\beta)}{((1+k^2x_{n+1}^2x_n^2)\operatorname{sn}\alpha\operatorname{sn}\beta+x_{n+1}^2+x_n^2)(\operatorname{sn}\alpha-\operatorname{sn}\beta)+2x_{n+1}x_n\operatorname{sn}(\alpha-\beta)(k^2\operatorname{sn}^2\alpha\operatorname{sn}^2\beta-1)} \tag{3.12}$$

As in the cases of the Q_2 and Q_3 systems the evolution $\overline{K}+K=0$ factorises into two mappings one of which is (3.11) and its "dual" obtained with $\beta\to-\beta$.

Again from the QRT invariant we can obtain a QRT mapping through the standard conservation $\overline{K}-K=0$. It can be easily obtained with the help of computer algebra from the expression of the invariant above.

$$\begin{aligned}&\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}(\alpha-\beta)(\operatorname{sn}\alpha-\operatorname{sn}\beta)(x_{n+1}+x_{n-1})(k^2x_n^4-1)+\\&\quad+\operatorname{sn}\alpha\operatorname{sn}\beta(S-(\operatorname{sn}\alpha-\operatorname{sn}\beta)^2)x_n(k^2x_{n+1}x_{n-1}x_n^2-1)-\\&\quad-(S+(\operatorname{sn}\alpha-\operatorname{sn}\beta)^2)x_n(x_{n+1}x_{n-1}-x_n^2)=0\end{aligned} \tag{3.13}$$

where $S = \mathrm{sn}^2(\alpha-\beta)(k^2\mathrm{sn}^2\alpha\,\mathrm{sn}^2\beta - 1)$. Note that the invariant curve $K - c = 0$ is an elliptic curve in that case, for generic value of the invariant c (and generic values of the parameters α and β), because of the presence of a $x_n^2 x_{n+1}^2$ term. When we take $k = 0$, however, this term disappears and the invariant curve becomes just a conic section. This explains why when we take the limit from Q_4 to Q_3, the invariant curve, which is just the product of two curves of the type $K - c = 0$, also degenerates, and the mappings become linearisable.

4. Integration of the Q_2 and Q_3 Mappings

At this point, having established the integrability of the mappings obtained from the equations of the "Q list", one can wonder about their actual integration. As explained in the previous sections, the integration of a mapping with bi-quadratic invariant leads generically to a solution in terms of an elliptic function. While the procedure is algorithmic it cannot, in principle, be described in a global way since it depends on the value of the invariant. On the basis of this argument, we would not expect to be able to integrate explicitly the mappings we have obtained, with the exception of (3.1). However since in the cases of Q_2 and Q_3 the related QRT mappings were linearisable we expected the full mappings (3.3) and (3.7) to be linearisable too. This turned out to be true.

In order to assess the linearisability of the mappings at hand, we analysed them using the algebraic entropy approach. This method links integrability with the polynomial growth of the degree of the iterates of the mapping. (In order to obtain the degree one must introduce homogeneous coordinates and compute the homogeneity degree.) For second-order mappings integrable in terms of elliptic functions the growth is quadratic while a linear degree growth is an indication of linearisability [20] . It turned out that while for the mappings (3.1) and (3.11) the degree growth was quadratic, for (3.3) and (3.7) the growth of the degree was linear, as expected. We give below the explicit linearisation of these mappings.

We start with mapping (3.3) which we rewrite as

$$(x_{n+1} - x_n)(x_n - x_{n-1}) - \alpha(x_{n+1} + 2x_n + x_{n-1}) + \beta = 0 \tag{4.1}$$

We subtract (4.1) from its upshift so as to eliminate β. Moreover, we introduce $y_n = x_{n+1} - x_n$ and obtain for y the mapping

$$y_{n+1}y_n - y_n y_{n-1} + \alpha(y_{n+1} + 2y_n + y_{n-1}) = 0 \tag{4.2}$$

It is then straightforward to show that (4.2) is equivalent to the system

$$y_n = -\frac{y_{n-1}(z_n + \alpha)}{y_{n-1} + z_n} \tag{4.3}$$

$$z_{n+1} = z_n + \alpha \tag{4.4}$$

which provides the effective linearisation of (4.2). System (4.3) is in fact a very simple form of the discrete Gambier equation [21]. We should point out here that the QRT mapping (3.6) obtained from the invariant associated to (3.3) is also linearisable and, as pointed out in [19], a very simple form of the Gambier mapping. In fact the general solution of (3.6) is $y_n = pn^2 + qn + r$ where $p^2 + 4pr - q^2 + a^2b^2 = 0$.

Next we examine (3.7). Again we subtract (3.7) from its upshift. The resulting equation being homogeneous, we introduce the auxiliary variable $y_n = x_{n+1}/x_n$. We obtain for y the mapping

$$(ay_n + 1)y_n y_{n-1} y_{n+1} + (y_n^2 - 1)y_{n-1} - y_n - a = 0 \tag{4.5}$$

We can then show that (4.5) is equivalent to the system

$$y_n = \frac{1 + (a + y_{n-1})z_n}{y_{n-1}} \tag{4.6}$$

$$z_{n+1} = -\frac{z_n}{1 + az_n} \tag{4.7}$$

which linearises it. Again what we have here is a simple form of the Gambier mapping.

In analogy to the Q_2 case we expect the QRT mapping obtained from the invariant associated with the Q_3 system to be also linearisable. This is indeed the case. We start by solving for the quantity $(a + 1/a)$, and we eliminate it by subtracting the mapping from its upshifted counterpart. It turns out that the result factorises into a trivial term and one which is an exact difference. We integrate the latter and find

$$y_{n+1}y_{n-1} - y_n^2 + k^2 = 0 \tag{4.8}$$

This is a well-known linearisable subcase of the (autonomous form of the) discrete Painlevé III equation. Its linearisation is straightforward. The solution of (4.8) is simply $y_n = p\cos(qn + r)$ provided $k = p\sin q$. Clearly y obeys a linear equation: $y_{n+1} + y_{n-1} = 2\cos q\ y_n$.

5. Conclusion

In this paper we have examined one-dimensional mappings which are obtained as reductions of the integrable lattices derived by Adler et al. [12]. The particularity of three out of these four second-order mappings resides in the fact that they do not belong to the QRT class. For two of the mappings we could show that they are linearisable and provide their explicit linearisation.

The other family of lattices of ABS did not lead to any interesting results. The first lattice, H_1, is just the discrete KdV equation, the reductions of which have already been studied extensively [22]. For the remaining two lattices, the first non-trivial reduction leads to a third-order mapping which is outside the scope of the present study.

Finally, one can wonder what are the possible nonautonomous forms of the mappings obtained, since in the case of the discrete KdV the reduction leads to a special form of the discrete Painlevé I equation. This question is all the more justified since we have been able to show, (in all cases except for Q_4 for which the calculations become prohibitively cumbersome) using the singularity confinement integrability criterion, that the lattice parameters can be taken as free functions of the independent variable along the respective axes. However, it turned out that, at least for the second-order mappings studied, no nonautonomous form was acceptable. Still, since the lattice KdV equation does possess nonautonomous reductions we expect such forms to appear for the reductions which lead to higher-order mappings. We hope to come back to this question in some future study.

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