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Some Improvements in the Method of the Weakly Conjugate Operator

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Abstract. We present some improvements in the method of the weakly conjugate operator, one variant of the Mourre theory. When applied to certain two-body Schrödinger operators, this leads to a limiting absorption principle that is uniform on the positive real axis.

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1. Introduction

Recently there has been an increasing interest in the study of two-body Schrödinger operators near the threshold at energy zero (see for example [5] or [8]). Since a positive commutator in the sense of Mourre does not exist at this energy, the usual method of the conjugate operator cannot be used in that particular situation. On the other hand, the method of the weakly conjugate operator gives the existence of the boundary values of resolvents also at thresholds but applies only to situations where the operators have no bound states at all. However the authors of [5] derive a limiting absorption principle at zero energy for a special class of two-body Schrödinger operators which have bound states below zero. In this context, an improvement of the method of the weakly conjugate operator that will cover the behaviour at thresholds of operators with bound states would be of interest. The purpose of this Letter is to describe such an extension and to give an application to two-body Schrödinger operators.

Let us recall the main idea of methods based on a conjugate operator. One way to obtain strong results for the spectral analysis of a self-adjoint operator H is to find an auxiliary self-adjoint operator A such that the commutator $[iH, A]$ is positive in a suitable sense. In the method of the conjugate operator one looks for intervals J of $\mathbb{R}$ such that

$$E(J)[iH,A]E(J) \geqslant aE(J) \tag{1}$$

for some strictly positive constant a that depends on J, where $E(J)$ denotes the spectral projection of H on the interval J. For the method of the weakly conjugate operator one assumes that $[iH,A] > 0$, i.e. the commutator is positive and injective. This requirement is closer to the initial Kato–Putnam theory, on which it improves.

The first approach has reached a very high degree of precision and abstraction in [1]. There also exists a huge number of applications based on an inequality of the form (1). The second approach was initiated in [3] and fully developed in [4]. Only a few papers contain applications, see for example [7], [11] or [12]. We also mention [5] and [6] that contain arguments that are very close to this method. Its main disadvantage is that if the method can be applied to H, then the spectrum of H is purely absolutely continuous, which limits drastically its range of applications. On the other hand, it leads to a limiting absorption principle that is uniform on $\mathbb{R}$ and to global H-smooth operators, that are of special interest. We refer to [14] and references therein for more information on that subject.

Motivated by some calculations borrowed from [5], we shall prove in this Letter that the fundamental assumption $[iH,A] > 0$ of the method of the weakly conjugate operator can be weakened. The main idea is that H itself can add some positivity. Surprisingly, the new requirement is that there exists a constant $c \geqslant 0$ such that

$$-cH + [iH,A] > 0 .$$

This inequality together with some technical assumptions lead to a limiting absorption principle that is either uniform on $\mathbb{R}$ if $c = 0$ or uniform on $[0,\infty)$ if $c > 0$. The absolute continuity of the spectrum and H-smooth operators are then standard by-products of that estimate.

In the next section, we introduce the framework and state the abstract result. Its proof is postponed to Section 4. In between, we give an application to two-body Schrödinger operators. We prove that under suitable conditions such operators admit a limiting absorption principle uniform on $[0,\infty)$. Since our approach applies to operators that may have discrete spectrum below zero, our abstract result is really an improvement of the method developed in [3] and [4].

We close this introduction with some comments on generality. As in the early papers on the method of the conjugate operator, our condition on the second commutator $\big[i[iH,A],A\big]$ can certainly be weakened. Also an approach divided into two stages (first by dealing with bounded operators and then by applying the result to the resolvent $(H-\lambda_0)^{-1}$ for a real λ_0 outside the spectrum of H) would certainly lead to some improvements. However, since such modifications would also lengthen and complicate our arguments, we decided not to take them into account in this Letter.

2. The Abstract Construction

Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We consider a self-adjoint operator H in $\mathcal{H}$ with its domain denoted by $\mathcal{G}^2$ and its form domain denoted by $\mathcal{G}^1$. Endowed with the corresponding graph norms, $\mathcal{G}^2$ and $\mathcal{G}^1$ are also Hilbert spaces. Their adjoint spaces (topological anti-duals) are denoted by $\mathcal{G}^{-2}$ and $\mathcal{G}^{-1}$, and by identifying $\mathcal{H}$ with its adjoint through the Riesz isomorphism one has the continuous dense embeddings:

$$\mathcal{G}^2 \hookrightarrow \mathcal{G}^1 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{G}^{-1} \hookrightarrow \mathcal{G}^{-2}.$$

Let $\{W_t\}_{t\in\mathbb{R}}$ be a strongly continuous unitary group in $\mathcal{H}$ with its self-adjoint generator denoted by A. We assume that for each $t\in\mathbb{R}$, W_t leaves $\mathcal{G}^2$ invariant. It is then a standard fact that $\{W_t\}_{t\in\mathbb{R}}$ induces a C_0-group in each space $\mathcal{G}^s$ introduced above [1, Section 6.3]. We keep the same notation for these C_0-groups.

Now, let us consider an operator $S\in\mathscr{B}(\mathcal{G}^1,\mathcal{G}^{-1})$ that satisfies $S>0$, i.e., $\langle f, Sf\rangle > 0$ for all $f\in\mathcal{G}^1\setminus\{0\}$. We have written $\mathscr{B}(\mathcal{G}^1,\mathcal{G}^{-1})$ for the set of bounded linear operators from $\mathcal{G}^1$ to $\mathcal{G}^{-1}$ and kept the notation $\langle\cdot,\cdot\rangle$ for the duality between $\mathcal{G}^1$ and $\mathcal{G}^{-1}$. Since S is positive we define the completion $\mathcal{S}$ of $\mathcal{G}^1$ with respect to the norm $\|f\|_{\mathcal{S}} := \langle f, Sf\rangle^{1/2}$. Its adjoint space $\mathcal{S}^*$ can then be identified with the completion of $S\mathcal{G}^1$ with respect to the norm $\|g\|_{\mathcal{S}^*} := \langle g, S^{-1}g\rangle^{1/2}$. One observes that S extends to an isometric element of $\mathscr{B}(\mathcal{S},\mathcal{S}^*)$. $\mathcal{S}$ and $\mathcal{S}^*$ are Hilbert spaces which are generally not comparable with $\mathcal{H}$. But since $\mathcal{G}^1\hookrightarrow\mathcal{S}$ and $\mathcal{S}^*\hookrightarrow\mathcal{G}^{-1}$ it makes sense to assume that $\{W_t\}_{t\in\mathbb{R}}$ restricts to a C_0-group in $\mathcal{S}^*$, or equivalently that it extends to a C_0-group in $\mathcal{S}$. Under this assumption (tacitly assumed in the sequel), we still keep the notation $\{W_t\}_{t\in\mathbb{R}}$ for these C_0-groups. Endowed with the graph norm, the domain of the generator of the C_0-group in $\mathcal{S}^*$ is denoted by $D(A,\mathcal{S}^*)$.

DEFINITION 1. For $j\in\{1,2\}$, let $\mathcal{T}_j$ be one of the spaces $\mathcal{H},\mathcal{G}^s,\mathcal{S}$ or $\mathcal{S}^*$ introduced above. An operator $T\in\mathscr{B}(\mathcal{T}_1,\mathcal{T}_2)$ belongs to $C^1(A;\mathcal{T}_1,\mathcal{T}_2)$ if the map

$$\mathbb{R}\ni t\mapsto W_{-t}TW_t\in\mathscr{B}(\mathcal{T}_1,\mathcal{T}_2)$$

is strongly differentiable. Its derivative at $t=0$ is denoted by $[iT,A]\in\mathscr{B}(\mathcal{T}_1,\mathcal{T}_2)$.

Before stating the main result of this section, let us recall some known facts. By duality and interpolation, any symmetric operator T in $\mathcal{H}$ with $T\in\mathscr{B}(\mathcal{G}^2,\mathcal{H})$ has a unique extension to a symmetric element of $\mathscr{B}(\mathcal{G}^1,\mathcal{G}^{-1})$, still denoted by T. Then, the assumption $T\in\mathscr{B}(\mathcal{S},\mathcal{S}^*)$ has an unambiguous meaning. It is equivalent to the requirement that $T(\mathcal{G}^1)\subset\mathcal{S}^*$ and $T:\mathcal{G}^1\to\mathcal{S}^*$ is continuous when $\mathcal{G}^1$ is provided with the topology induced by $\mathcal{S}$. In that case the unique extension to a continuous mapping from $\mathcal{S}$ to $\mathcal{S}^*$ is still denoted by T. On the other hand, if $\mathcal{E}$ is the Banach space $(D(A,\mathcal{S}^*),\mathcal{S}^*)_{1/2,1}$ defined by real interpolation (see for example

[1, Proposition 2.7.3]), then one has the natural continuous embeddings :

$$\mathscr{B}(\mathcal{G}^{-1}, \mathcal{G}^{1}) \subset \mathscr{B}(\mathcal{S}^{*}, \mathcal{S}) \subset \mathscr{B}(\mathcal{E}, \mathcal{E}^{*}) \ .$$

THEOREM 1. *Let H be a self-adjoint operator in $\mathcal{H}$ that belongs to $C^1(A; \mathcal{G}^2, \mathcal{H})$ and assume that there exist two constants $c_1 \geqslant 0$ and $c_2 > 0$ such that*

$$S := -c_1 H + [iH, A] \ > \ 0 \qquad \text{and} \qquad [iH, A] \ \geqslant \ -c_2 \ . \tag{2}$$

Assume furthermore that $[iH, A]$ extends to an element of $C^1(A; \mathcal{S}, \mathcal{S}^*)$. Then, there exists $c < \infty$ such that

$$|\langle f, (H - \lambda \mp i\mu)^{-1} f \rangle| \ \leqslant \ c\|f\|_{\mathcal{E}}^2 \tag{3}$$

for all $\lambda \in \mathbb{R}$ with $c_1\lambda \geqslant 0$, all $\mu > 0$ and all $f \in \mathcal{E}$.

We observe that the condition on λ splits into two cases. Either $c_1 = 0$ and then the result holds for all $\lambda \in \mathbb{R}$, or $c_1 > 0$ and then λ has to be restricted to the positive axis. Since the case $c_1 = 0$ was already treated in [3], we shall state two well-known corollaries only in the case $c_1 > 0$.

Corollary 1. *Assume that the assumptions of Theorem 1 hold for some $c_1 > 0$. Then,*

(i) *Any element of $\mathscr{B}\big((\mathcal{E}^*)^\circ, \mathcal{H}\big)$ is H-smooth on $[0, \infty)$, where $(\mathcal{E}^*)^\circ$ stands for the closure of $\mathcal{S}$ in $\mathcal{E}^*$,*
(ii) *The spectrum of H on $[0, \infty)$ is absolutely continuous.*

3. Application to Schrödinger Operators

In this section, we apply the abstract result to some Schrödinger operators in the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^n)$. Let us first recall that for $j \in \{1, \ldots, n\}$, Q_j is the operator of multiplication by the variable x_j, $P_j := -i\nabla_j$ is a component of the momentum operator and $-\Delta \equiv P^2$ is Laplace operator on $\mathbb{R}^n$. For each $s \in \mathbb{R}$, $\mathcal{H}^s$ denotes the usual Sobolev space of order s on $\mathbb{R}^n$.

Let V be a real and bounded $C^\infty(\mathbb{R}^n)$-function. We shall work under this smoothness assumption that is not essential but which simplifies our arguments. The Schrödinger operator

$$H := -\Delta + V$$

is self-adjoint in $\mathcal{H}$ with domain $\mathcal{G}^2 \equiv \mathcal{H}^2$. Obviously one has $\mathcal{G}^1 \equiv \mathcal{H}^1$, and by duality, $\mathcal{G}^{-2} \equiv \mathcal{H}^{-2}$ and $\mathcal{G}^{-1} \equiv \mathcal{H}^{-1}$. It is well known that all these spaces are invariant under the action of the dilation group $\{W_t\}_{t \in \mathbb{R}}$ whose generator A has the form $A := \frac{1}{2}(P \cdot Q + Q \cdot P)$.

Let us now assume that the map $\mathbb{R}^n \ni x \mapsto \widetilde{V}(x) := \sum_{j=1}^n x_j[\partial_j V](x) \in \mathbb{R}$ is bounded. It follows that $H \in C^1(A; \mathcal{H}^2, \mathcal{H})$ and that $[iH, A] = -2\Delta - \widetilde{V}$. In this situation the main positivity requirement of Theorem 1 is that there exists $c_1 \geqslant 0$ such that

$$-(2-c_1)\Delta - c_1 V - \widetilde{V} \ > \ 0 \ .$$

One observes that if there exists $c_1 \in [0,2)$ such that $-c_1 V - \widetilde{V} \geqslant 0$, then this inequality is obviously satisfied.

In the next proposition we use this idea and give a very simple and explicit application of Theorem 1. But let us also note that if $n \geqslant 3$, some additional positivity can be obtained from the inequality $-\Delta \geqslant (\frac{n-2}{2})^2 |Q|^{-2}$. For purposes of simplicity we do not take this improvement into account, and refer to [3] for an extensive use of this inequality in the special case $c_1 = 0$.

PROPOSITION 1. *Let V be a real and bounded $C^\infty(\mathbb{R}^n)$-function. Assume furthermore that the following three conditions are satisfied for all $x \in \mathbb{R}^n$:* (i) $V(x) \leqslant 0$, (ii) *there exists $c \in [0,2)$ such that $|\widetilde{V}(x)| \leqslant -cV(x)$,* (iii) *there exists $d \geqslant 0$ such that :*

$$\Big| \sum_{j,k=1}^{n} x_j \partial_j x_k \partial_k V(x) \Big| \ \leqslant \ -dV(x) \ .$$

Then for $c_1 \in (c,2)$ fixed and $S := -(2-c_1)\Delta - c_1 V - \widetilde{V}$, the limiting absorption principle (3) *is satisfied for all $\lambda \geqslant 0$, all $\mu > 0$ and all $f \in \mathcal{E}$. Furthermore, any element of $\mathscr{B}\big((\mathcal{E}^*)^\circ, \mathcal{H}\big)$ is H-smooth on $[0,\infty)$ and the spectrum of H on $[0,\infty)$ is absolutely continuous.*

Since the spaces $\mathcal{E}$, $\mathcal{E}^*$ and $(\mathcal{E}^*)^\circ$ are rather intricate, H-smooth operators are not so easily exhibited. But under one not too restrictive extra assumption on V, a large class of H-smooth operators can be constructed. For that purpose, let us set $M(x) := \min\big\{-V(x), \frac{1}{|x|^2}\big\}$ for any $x \in \mathbb{R}^n$.

Corollary 2. *Assume that V satisfies the assumptions of Proposition 1 with* (i) *replaced by $V(x) < 0$ for all $x \in \mathbb{R}^n$. If $L : \mathbb{R}^n \to \mathbb{R}$ is a Borel function that satisfies $|L(x)| \leqslant c M(x)^{\frac{1}{4}+\delta}\big(-V(x)\big)^{\frac{1}{4}-\delta}$ for some $\delta \in (0, \frac{1}{4})$, $c < \infty$ and all $x \in \mathbb{R}^n$, then the operator of multiplication by L is H-smooth on* $[0,\infty)$.

Remark 1. Assume for a while that V is of the form $V(x) = -d|x|^{-\mu}$ for $d > 0$, $\mu > 0$ and for x large enough. Then condition (ii) of Proposition 1 implies the upper bound $\mu < 2$. Thus, an implicit consequence of condition (ii) is to prevent a rapid decrease of V at infinity. Even if this restriction is not satisfactory, condition (ii) is of central importance in our approach and cannot be weakened. However, let us note that similar conditions that prevent V to decrease faster than

$|x|^{-2}$ already appear in the literature: In [13, Theorem 1.9] or in [5, Corollary 3.5] a result similar to Corollary 2 is obtained under the requirement $V(x) \leqslant -d(1+x^2)^{-\mu/2}$ for some $d>0$ and $\mu \in (0,2)$. On the other hand, the resolvent expansion around zero obtained in [8] holds only for potentials V that decrease faster than $|x|^{-2}$, and thus cannot be directly compared with our results.

Remark 2. In [5], an additional multiplication operator W is considered. This function is non sign-definite, may have local singularities, but has to decrease in a way controlled by V. By a perturbative argument, the authors succeed in proving that the limiting absorption principle still holds for H replaced by $H+W$. Since we have not been able to perform a similar improvement in the abstract framework developed in Section 2 we do include such a treatment for the particular situation considered in this section.

Before starting the proofs we want the reader to note that the same letter c or d may denote different constants from line to line.

Proof of Proposition 1. Let us write $\mathcal{D}$ for the set $C_c^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ with compact support. Because of our smoothness assumption on V, all calculations below are well justified on $\mathcal{D}$.

(a) One has already noticed that H is a self-adjoint operator in $\mathcal{H}$ with domain $\mathcal{H}^2$, and that H belongs to $C^1(A;\mathcal{H}^2,\mathcal{H})$. Let us now fix $c_1 \in (c,2)$ and observe that :

$$S \geqslant -(2-c_1)\Delta\ , \quad S \geqslant -(c_1-c)V \quad \text{and} \quad S > 0\ . \tag{4}$$

Furthermore the self-adjoint operator $[iH,A] = -2\Delta - \widetilde{V}$ is bounded from below. Thus, both conditions in (2) are satisfied.

(b) By performing some easy calculations on $\mathcal{D}$ and by taking into account hypotheses (ii) and (iii) and the inequalities (4) one obtains that there exists $d>0$ such that on $\mathcal{D}$ the following inequalities hold :

$$-dS \leqslant [iH,A] \leqslant dS\ , \tag{5}$$

$$-dS \leqslant \big[i[iH,A],A\big] \leqslant dS\ , \tag{6}$$

$$-dS \leqslant [iS,A] \leqslant dS\ . \tag{7}$$

It follows from (5) that $|\langle f,[iH,A]f\rangle| \leqslant d\langle f,Sf\rangle \equiv d\|f\|_{\mathcal{S}}^2$ for all $f\in\mathcal{D}$, and then from the density of $\mathcal{D}$ in $\mathcal{S}$ that $[iH,A]$ extends to an element of $\mathscr{B}(\mathcal{S},\mathcal{S}^*)$. Relation (6) leads to the same conclusion for the operator $\big[i[iH,A],A\big]$.

(c) Now we check that $\{W_t\}_{t\in\mathbb{R}}$ extends to a C_0-group in $\mathcal{S}$. This easily reduces to the proof that $\|W_t f\|_{\mathcal{S}} \leqslant c(t)\|f\|_{\mathcal{S}}$ for all $f\in\mathcal{D}$ and $t\in\mathbb{R}$. By (7) one has:

$$\|W_t f\|_{\mathcal{S}}^2 = \langle f,Sf\rangle + \int_0^t \langle W_\tau f,[iS,A]W_\tau f\rangle \mathrm{d}\tau \leqslant \|f\|_{\mathcal{S}}^2 + d\left|\int_0^t \|W_\tau f\|_{\mathcal{S}}^2 \mathrm{d}\tau\right|\ .$$

The function $(0,t)\ni\tau\mapsto\|W_\tau f\|_{\mathcal{S}}^2\in\mathbb{R}$ is bounded (since $\mathcal{G}^1\hookrightarrow\mathcal{S}$), and hence by a simple form of the Gronwall Lemma, we get the inequality $\|W_t f\|_{\mathcal{S}}\leqslant e^{\frac{d}{2}|t|}\|f\|_{\mathcal{S}}$. Thus $\{W_t\}_{t\in\mathbb{R}}$ extends to a C_0-group in $\mathcal{S}$, and by duality $\{W_t\}_{t\in\mathbb{R}}$ also defines a C_0-group in $\mathcal{S}^*$. This completes the proof that $[iH,A]$ extends to an element of $C^1(A;\mathcal{S},\mathcal{S}^*)$. All hypotheses of Theorem 1 have been checked, and the statements follow from this theorem and from its corollary. □

Proof of Corollary 2. Let $\mathcal{M}$ be the completion of $\mathcal{D}$ with respect to the norm $\|f\|_{\mathcal{M}}:=\|M^{-1/2}f\|$, and similarly let $\mathcal{N}$ be the completion of $\mathcal{D}$ with respect to the norm $\|f\|_{\mathcal{N}}:=\|(-V)^{-1/2}f\|$. We first observe that $\mathcal{M}\subset\mathcal{N}$, $\mathcal{M}\subset D(A,\mathcal{S}^*)$ and $\mathcal{N}\subset\mathcal{S}^*$. Indeed, the first continuous embedding follows directly from the inequality $\|f\|_{\mathcal{N}}\leqslant\|f\|_{\mathcal{M}}$ for all $f\in\mathcal{D}$. For the second we show that $\|f\|_{\mathcal{S}^*}^2+\|Af\|_{\mathcal{S}^*}^2\leqslant c\|f\|_{\mathcal{M}}^2$ for $c<\infty$ and all $f\in\mathcal{D}$. From Corollary 1 of [9] and (4) one gets

$$\|f\|_{\mathcal{S}^*}^2 \;\equiv\; \langle f,S^{-1}f\rangle \;\leqslant\; \tfrac{1}{c_1-c}\langle f,(-V)^{-1}f\rangle \;\leqslant\; d\|f\|_{\mathcal{N}}^2 \;\leqslant\; d\|f\|_{\mathcal{M}}^2\ . \tag{8}$$

Furthermore, it easily follows from (4) that for each $j\in\{1,\dots,n\}$ P_j extends to an element of $\mathscr{B}(\mathcal{H},\mathcal{S}^*)$, and therefore:

$$\|Af\|_{\mathcal{S}^*}\;\leqslant\; c\sum_{j=1}^{n}\|Q_jf\|+d\|f\|_{\mathcal{S}^*}\;\leqslant\; c'\||Q|f\|+d'\|f\|_{\mathcal{M}}\;\leqslant\; c''\|f\|_{\mathcal{M}}\ .$$

The third embedding is also obtained from (8). One may then apply [1, Corollary 2.6.3] and obtains the following relations between spaces defined by real interpolation:

$$\big(\mathcal{M},\mathcal{N}\big)_{\theta,p}\subset\big(D(A,\mathcal{S}^*),\mathcal{S}^*\big)_{\theta,p}\quad\forall\theta\in(0,1)\text{ and }p\in[1,\infty]\ . \tag{9}$$

In order to exhibit explicit norms on $(\mathcal{M},\mathcal{N})_{\theta,2}$ let us set $\Lambda:=\big(\frac{-V}{M}\big)^{1/2}$ and observe that $\Lambda\geqslant 1$. It is easily checked that the couple $(\mathcal{M},\mathcal{N})$ is quasi-linearizable in the sense of [1, Section 2.7] (with $V_\tau:=(1+\tau\Lambda)^{-1}$). By applying then Lemma 2.7.1 of the same reference, one obtains that an admissible norm on $(\mathcal{M},\mathcal{N})_{\theta,p}$ is given by the expression $\big(\int_1^\infty\big\|r^{1-\theta}\frac{\Lambda}{r+\Lambda}f\big\|_{\mathcal{N}}^p\frac{dr}{r}\big)^{1/p}$. Furthermore, by the same argument as in the proof of [1, Proposition 2.8.1] one gets that in the special case $p=2$ this norm is equivalent to the norm given by $\|\Lambda^{1-\theta}f\|_{\mathcal{N}}$. Altogether, one has obtained that the interpolation space $(\mathcal{M},\mathcal{N})_{\theta,2}$ is equal to the completion of $\mathcal{D}$ with respect to the norm $\|\Lambda^{1-\theta}(-V)^{-1/2}f\|$.

For each $\varepsilon\in(0,\frac12)$ let us set $\theta:=\frac12-\varepsilon$ and $\mathcal{F}_\varepsilon:=(\mathcal{M},\mathcal{N})_{\theta,2}$. One has $\mathcal{F}_\varepsilon\subset(\mathcal{M},\mathcal{N})_{1/2,1}$ [1, Proposition 2.4.1], and it follows then by (9) that $\mathcal{F}_\varepsilon\subset\mathcal{E}$. Thus any element of $\mathscr{B}(\mathcal{F}_\varepsilon^*,\mathcal{H})$ is H-smooth on $[0,\infty)$. It is then readily checked that the operator L belongs to $\mathscr{B}(\mathcal{F}_\varepsilon^*,\mathcal{H})$ with $\varepsilon=2\delta$, which completes the proof. □

4. Proof of the main theorem

This section is entirely devoted to the proof of the abstract result.

Proof of Theorem 1. (a) For $\lambda\in\mathbb{R}$ with $c_1\lambda\geqslant 0$, $\mu>0$ and $\varepsilon>0$, let us consider the operators

$$\big(H-\lambda\mp i\mu\mp i\varepsilon[iH,A]\big)\in\mathscr{B}(\mathcal{G}^2,\mathcal{H}) \ . \tag{10}$$

We first prove that there exists $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ these operators are isomorphisms from $\mathcal{G}^2$ to $\mathcal{H}$. As a consequence of the open mapping theorem, it is enough to prove that they are bijective. For that purpose, let us note that for any $f\in\mathcal{G}^2\setminus\{0\}$ and $T\in\mathscr{B}(\mathcal{G}^2,\mathcal{H})$, one has $\langle f,Tf\rangle=0$ if and only if $\mathrm{Re}\langle f,Tf\rangle=0$ and $\mathrm{Im}\langle f,Tf\rangle=0$. It follows that if there exist two finite numbers c and d such that $c\,\mathrm{Re}\langle f,Tf\rangle+d\,\mathrm{Im}\langle f,Tf\rangle\neq 0$, then $\langle f,Tf\rangle\neq 0$. Now, one observes that

$$\begin{aligned}&-c_1\,\mathrm{Re}\big\langle f,\big(H-\lambda\mp i\mu\mp i\varepsilon[iH,A]\big)f\big\rangle\mp\tfrac{1}{\varepsilon}\,\mathrm{Im}\big\langle f,\big(H-\lambda\mp i\mu\mp i\varepsilon[iH,A]\big)f\big\rangle\\&=\big\langle f,\big(-c_1H+[iH,A]\big)f\big\rangle+(c_1\lambda+\tfrac{\mu}{\varepsilon})\|f\|^2\ \geqslant\ \langle f,Sf\rangle\ >\ 0\ ,\end{aligned} \tag{11}$$

which implies that $\langle f,(H-\lambda\mp i\mu\mp i\varepsilon[iH,A])f\rangle\neq 0$ for all $f\in\mathcal{G}^2\setminus\{0\}$. Thus both operators in (10) are injective. Furthermore, for $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0:=(\|[iH,A]\|_{\mathcal{G}^2\to\mathcal{H}})^{-1}$ one easily deduces that they are closed operators in $\mathcal{H}$ and adjoint to each other. This immediately leads to their surjectivity [10, Section V.3.1] and thus to their bijectivity.

(b) For $\varepsilon\in(0,\varepsilon_0)$, let us set $G_\varepsilon^\pm:=(H-\lambda\mp i\mu\mp i\varepsilon[iH,A])^{-1}$. These operators belong to $\mathscr{B}(\mathcal{H},\mathcal{G}^2)$, and by duality and interpolation to $\mathscr{B}(\mathcal{G}^{-1},\mathcal{G}^1)\subset\mathscr{B}(\mathcal{S}^*,\mathcal{S})$. It is then easily shown that for all $f,g\in\mathcal{G}^{-1}$: $\langle f,G_\varepsilon^\pm g\rangle\ =\ \langle G_\varepsilon^\mp f,g\rangle$. By taking into account these equalities and the continuous extensions of the inequalities (11) valid for all $f\in\mathcal{G}^1\setminus\{0\}$, one observes that there exists $c<\infty$ such that for all $f\in\mathcal{G}^{-1}$:

$$\begin{aligned}\|G_\varepsilon^\pm f\|_{\mathcal{S}}^2&=\ \langle G_\varepsilon^\pm f,SG_\varepsilon^\pm f\rangle\\&\leqslant c_1\big|\,\mathrm{Re}\big\langle G_\varepsilon^\pm f,\big(H-\lambda\pm i\mu\pm i\varepsilon[iH,A]\big)G_\varepsilon^\pm f\big\rangle\big|+\\&\quad+\tfrac{1}{\varepsilon}\big|\,\mathrm{Im}\big\langle G_\varepsilon^\pm f,\big(H-\lambda\pm i\mu\pm i\varepsilon[iH,A]\big)G_\varepsilon^\pm f\big\rangle\big|\\&\leqslant c_1|\langle f,G_\varepsilon^\pm f\rangle|+\tfrac{1}{\varepsilon}|\langle f,G_\varepsilon^\pm f\rangle|\ \leqslant\ \tfrac{c}{\varepsilon}|\langle f,G_\varepsilon^\pm f\rangle|\ .\end{aligned}$$

Thus, one has obtained that for all $f\in\mathcal{G}^{-1}$:

$$\|G_\varepsilon^\pm f\|_{\mathcal{S}}\ \leqslant\ \sqrt{\tfrac{c}{\varepsilon}}\ \ |\langle f,G_\varepsilon^\pm f\rangle|^{1/2}\ , \tag{12}$$

and by using the inequality $|\langle f,g\rangle|\leqslant\|f\|_{\mathcal{S}^*}\|g\|_{\mathcal{S}}$ valid for all $f\in\mathcal{S}^*$ and $g\in\mathcal{S}$, it follows that:

$$\|G_\varepsilon^\pm\|_{\mathcal{S}^*\to\mathcal{S}}\ \leqslant\ \tfrac{c}{\varepsilon}\ . \tag{13}$$

(c) This part of the proof is similar to parts (ii)–(iv) of the proof of [4, Theorem 2.1]. For $\varepsilon>0$ and $f\in\mathcal{E}\equiv(D(A,\mathcal{S}^*),\mathcal{S}^*)_{1/2,1}$, let us set $f_\varepsilon:=\frac{1}{\varepsilon}\int_0^\varepsilon(W_tf)\mathrm{d}t$. Then, $f_\varepsilon\in D(A,\mathcal{S}^*)$, $\varepsilon\mapsto f_\varepsilon\in\mathcal{S}^*$ is C^1 in norm, $f_\varepsilon\to f$ in $\mathcal{S}^*$ as $\varepsilon\to 0$ and

$$\int_0^1\big(\|f_\varepsilon'\|_{\mathcal{S}^*}+\|Af_\varepsilon\|_{\mathcal{S}^*}\big)\,\varepsilon^{-1/2}\,\mathrm{d}\varepsilon\ \leqslant\ c\|f\|_{\mathcal{E}}^2\ , \tag{14}$$

where f_ε' denotes the derivative of the map $\varepsilon\mapsto f_\varepsilon$.

Now, for $\varepsilon \in (0, \varepsilon_1)$, with $\varepsilon_1 := \min\{\varepsilon_0, 1\}$, let us set $F^{\pm}_{\varepsilon} := \langle f_\varepsilon, G^{\pm}_{\varepsilon} f_\varepsilon\rangle$. A formal calculation leads to:

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon} F^{\pm}_{\varepsilon} \equiv (F^{\pm}_{\varepsilon})' = \langle f'_\varepsilon \mp A f_\varepsilon, G^{\pm}_{\varepsilon} f_\varepsilon\rangle + \langle G^{\mp}_{\varepsilon} f_\varepsilon, f'_\varepsilon \pm A f_\varepsilon\rangle - \\ -\varepsilon\langle G^{\mp}_{\varepsilon} f_\varepsilon, \big[i[iH, A], A\big] G^{\pm}_{\varepsilon} f_\varepsilon\rangle \ .\end{aligned}$$

A rigorous proof of these equalities can be derived similarly as in the usual Mourre theory, see for example [1, Section 7.3] or [2, Lemma 3.4]. By taking (12) into account, we obtain the fundamental differential inequalities:

$$\tfrac{1}{c}|(F^{\pm}_{\varepsilon})'| \leqslant \tfrac{1}{\sqrt{\varepsilon}}\big(\|f'_\varepsilon\|_{\mathcal{S}^*} + \|A f_\varepsilon\|_{\mathcal{S}^*}\big)|F^{\pm}_{\varepsilon}|^{1/2} + \big\|\big[i[iH, A], A\big]\big\|_{\mathcal{S}\to\mathcal{S}^*}|F^{\pm}_{\varepsilon}| \ .$$

Then, by an application of the Gronwall lemma [1, Lemma 7.A.1] together with the use of the inequality (14) one concludes that $F^{\pm}_0 = \lim_{\varepsilon\to 0} F^{\pm}_{\varepsilon}$ exist and satisfy $|F^{\pm}_0| \leqslant c(|F^{\pm}_{\varepsilon_1}| + \|f\|^2_{\mathcal{E}})$. Furthermore, one has by (13) that

$$|F^{\pm}_{\varepsilon_1}| \leqslant \|G^{\pm}_{\varepsilon_1}\|_{\mathcal{S}^*\to\mathcal{S}}\|f_{\varepsilon_1}\|^2_{\mathcal{S}^*} \leqslant c\left[\int_0^{\varepsilon_1} \|W_t f\|_{\mathcal{S}^*}\mathrm{d}t\right]^2 \leqslant d\|f\|^2_{\mathcal{S}^*} \leqslant d\|f\|^2_{\mathcal{E}} \ ,$$

which leads to the expected inequalities: $|F^{\pm}_0| \leqslant c\|f\|^2_{\mathcal{E}}$.

(d) It only remains to show that $F^{\pm}_0 = \langle f, (H - \lambda \mp i\mu)^{-1} f\rangle$, i.e., that the right objects have been obtained. For that purpose, let us set $G^{\pm}_0 := (H - \lambda \mp i\mu)^{-1}$ and observe that

$$\begin{aligned}|\langle f_\varepsilon, G^{\pm}_{\varepsilon} f_\varepsilon\rangle - \langle f, G^{\pm}_0 f\rangle| \leqslant \|f_\varepsilon - f\|_{\mathcal{S}^*}\ \|G^{\pm}_{\varepsilon}\|_{\mathcal{S}^*\to\mathcal{S}}\ \|f_\varepsilon\|_{\mathcal{S}^*} + \\ + \|f\|_{\mathcal{S}^*}\ \|G^{\pm}_{\varepsilon} - G^{\pm}_0\|_{\mathcal{S}^*\to\mathcal{S}}\ \|f_\varepsilon\|_{\mathcal{S}^*} + \\ + \|f\|_{\mathcal{S}^*}\ \|G^{\pm}_0\|_{\mathcal{S}^*\to\mathcal{S}}\ \|f_\varepsilon - f\|_{\mathcal{S}^*} \ .\end{aligned}$$

Since $\|f_\varepsilon - f\|_{\mathcal{S}^*} \to 0$ as $\varepsilon \to 0$ and $\|T\|_{\mathcal{S}^*\to\mathcal{S}} \leqslant d\|T\|_{\mathcal{G}^{-1}\to\mathcal{G}^1}$ for all $T \in \mathscr{B}(\mathcal{G}^{-1}, \mathcal{G}^1)$, it is enough to prove that for λ and μ fixed there exist $\varepsilon_2 > 0$ and $c < \infty$ such that $\|G^{\pm}_{\varepsilon}\|_{\mathcal{G}^{-1}\to\mathcal{G}^1} \leqslant c$ for all $\varepsilon \in [0, \varepsilon_2]$. Indeed, by using the second identity of the resolvent can then gets the inequalities:

$$\begin{aligned}\|G^{\pm}_{\varepsilon} - G^{\pm}_0\|_{\mathcal{G}^{-1}\to\mathcal{G}^1} \leqslant \|G^{\pm}_{\varepsilon}\|_{\mathcal{G}^{-1}\to\mathcal{G}^1}\|\varepsilon[iH, A]\|_{\mathcal{G}^1\to\mathcal{G}^{-1}}\|G^{\pm}_0\|_{\mathcal{G}^{-1}\to\mathcal{G}^1} \\ \leqslant \varepsilon c^2\|[iH, A]\|_{\mathcal{G}^1\to\mathcal{G}^{-1}} \ .\end{aligned}$$

So, let us set $\varepsilon_2 := \min\{\varepsilon_1, \frac{\mu}{2c_2}\}$ and observe that for $\varepsilon \in [0, \varepsilon_2]$ and all $f \in \mathcal{G}^2$, the inequality $\frac{\mu}{2}\|f\|^2 + \langle f, \varepsilon[iH, A]f\rangle \geqslant 0$ holds. It easily follows that for all $f \in \mathcal{G}^2$:

$$\big\|\big(H - \lambda \mp i\tfrac{\mu}{2} \mp i(\tfrac{\mu}{2} + \varepsilon[iH, A])\big)f\big\| \geqslant \big\|\big(H - \lambda \mp i\tfrac{\mu}{2}\big)f\big\| \ ,$$

and then that for all $g \in \mathcal{H}$:

$$\big\|(H - \lambda \mp i\tfrac{\mu}{2})G^{\pm}_{\varepsilon} g\big\| \equiv \big\|(H - \lambda \mp i\tfrac{\mu}{2})\big(H - \lambda \mp i\tfrac{\mu}{2} \mp i(\tfrac{\mu}{2} + \varepsilon[iH, A])\big)^{-1} g\big\| \leqslant \|g\|.$$

Therefore, one obtains that for all $g \in \mathcal{H}$:

$$\begin{aligned}\|G_\varepsilon^\pm g\|_{\mathcal{G}^2} &= \|(H+i)G_\varepsilon^\pm g\| = \|(H+i)(H-\lambda \mp i\tfrac{\mu}{2})^{-1}(H-\lambda \mp i\tfrac{\mu}{2})G_\varepsilon^\pm g\| \\ &\leqslant \|(H+i)(H-\lambda \mp i\tfrac{\mu}{2})^{-1}\|\,\|g\| \\ &\leqslant \big(1+|\lambda \pm i\tfrac{\mu}{2}+i|\tfrac{2}{\mu}\big)\|g\|,\end{aligned}$$

or equivalently that $\|G_\varepsilon^\pm\|_{\mathcal{H}\to\mathcal{G}^2} \leqslant c$ with c independent of $\varepsilon \in [0, \varepsilon_2]$. By duality and interpolation, one concludes that $\|G_\varepsilon^\pm\|_{\mathcal{G}^{-1}\to\mathcal{G}^1} \leqslant c$ for all $\varepsilon \in [0, \varepsilon_2]$.

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